Normality of monodromy group in generic Tannakian group

Haohao Liu

January 4, 2024

Contents

1 Introduction 1

2 Cardinal argument 3
  2.1 Elementary facts .................................................. 3
  2.2 Cotori are good ................................................... 6
  2.3 Representations of a reductive group ......................... 8

3 Recollections on constructible sheaves 10
  3.1 Basics ............................................................. 10
  3.2 Universal local acyclicity ....................................... 14
  3.3 Relative perverse sheaves ...................................... 15

4 Cotori 20

5 Krämer-Weissauer’s vanishing theorem 23

6 Main results 27

1 Introduction

In Lawrence-Sawin’s work [LS20], the authors use the Lawrence-Venkatesh technique ([LV20]) to prove the Shafarevich conjecture for hypersurfaces in abelian varieties. Krämer-Maculan [KM23] apply roughly the same strategy to obtain an arithmetic finiteness result for very irregular varieties of dimension less than half the dimension of their Albanese variety. In both cases, a key is an estimation of monodromy groups associated with a moduli family.

In [LS20], the crucial estimation follows from a comparison to certain Tannakian group. As [JKLM23, p. 4] comments, this idea is similar to the study of monodromy groups via Mumford-Tate groups in [And92]. In Lawrence-Sawin’s case, the Tannakian group arises from Krämer-Weissauer’s convolution theory [KW15b] of perverse sheaves on the (geometric) generic fiber of a constant abelian scheme.
We recall (roughly) their argument. Both the monodromy group and the Tannakian group on the geometric generic fiber are embedded as closed subgroups in the Tannakian group on the generic fiber. The geometric generic Tannakian group is normal in the generic Tannakian group ([LS02, Lem. 3.7], [JKLM23, Thm. 4.3]). This normality is used to prove that for most characters, the corresponding monodromy group contains the geometric generic Tannakian group.

In the main result (Theorem 1.0.2), we prove that for many characters, the monodromy group is also normal in the generic Tannakian group.

**Setting 1.0.1.** Let $k$ be an algebraically closed field of characteristic 0. Let $X/k$ be an integral algebraic variety with generic point $\eta$. Let $A/k$ be an abelian variety. Denote by $\rho: A \times X \to X$ and $\pi: A \times X \to A$ the projections.

Let $\ell$ be a prime number. Let $\Lambda$ be an algebraic closure of $Q\ell$. Let $\text{Perv}^{\text{ULA}}(A \times X/X)$ be the abelian category of $\rho$-universally locally acyclic (ULA) perverse sheaves,\(^1\) inside the triangulated category $D^b(A \times X)$ of bounded constructible $\Lambda$-sheaves on $A \times X$. Let $\pi^\Lambda(A)$ be the étale fundamental group of $A$ based at the geometric origin point. For every character\(^2\) $\chi: \pi^\Lambda(A) \to \Lambda^*$, let $\psi: \pi^\Lambda(A) \to \Lambda^*$ be the pullback of $\chi$ along the morphism $\pi|_{A_\eta}: A_\eta \to A$. For every $K \in D^b(A \times X)$, every point $x$ of $X$, set $K_x := K|_{A_x}$. Fix $K \in \text{Perv}^{\text{ULA}}(A \times X/X)$ which is a semisimple object\(^3\) in $D^b(A \times X)$. The monodromy group $G_{\text{mon}}(K, \psi)$ and the generic Tannakian group $G_{\omega^\Lambda}(K_\eta)$ are defined in Section 5.

**Theorem 1.0.2** (Theorems 6.0.1, 6.0.6). Assume $\dim A > 0$. Then there are at least uncountably many characters $\chi: \pi^\Lambda(A) \to \Lambda^*$, such that the Tannakian group $G_{\omega^\Lambda}(K_\eta)$ is a well-defined reductive group and contains the monodromy group $G_{\text{mon}}(K, \psi)$ as a closed, reductive, normal subgroup.

The line of the proof of Theorem 1.0.2 is similar to that of André’s normality theorem [And92, Thm. 1]. It proves that for almost all stalks of a polarizable good variation of mixed Hodge structure, the connected monodromy group is a normal subgroup of the derived Mumford-Tate group. As [And92, p.10] explains, the normality is a consequence of the theorem of the fixed part due to Griffiths-Schmidt-Steenbrink-Zucker. In our case, an analog of the theorem of the fixed part is Theorem 1.0.3.

**Theorem 1.0.3** (Theorem 6.0.5). There is a subobject $K^0 \subset K$ in $\text{Perv}^{\text{ULA}}(A \times X/X)$, such that for every character $\chi^\Lambda: \pi^\Lambda(A) \to \Lambda^*$ of finite order prime to $\ell$, there is a nonempty Zariski open subset $U$ of the scheme $\mathcal{C}(A)_\ell$ of $\ell$-adic characters,\(^4\) such that for every $\chi^\Lambda \in U$, the $\Lambda$-vector space $H^0(A_\eta, K^0_\eta \otimes^L (L_\psi)_\eta)$ is the subspace of $\Gamma_{k(\eta)}$-invariants of the $\Gamma_{k(\eta)}$-representation $H^0(A_\eta, K_\eta \otimes^L (L_\psi)_\eta)$, where $\chi = \chi^\Lambda_\ell$, $\psi: \pi^\Lambda(A_\eta) \to \Lambda^*$ is the pullback of $\chi$ and $L_\psi$ is the rank 1 lisse $\Lambda$-sheaf on $A_\eta$ corresponding to $\psi$.

\(^1\)Relative perverse sheaves and universal local acyclicity are reviewed in Section 3.

\(^2\)Characters are assumed to be continuous.

\(^3\)In the sense of Definition 3.1.3

\(^4\)defined in Section 4
Notation and conventions

By an algebraic variety, we mean a scheme of finite type, separated over a field. An algebraic group $G$ is called **reductive** if its identity component $G^0$ is reductive in the sense of [Mil17, 6.46, p.135]. In an abelian category, an object is called semisimple if it is the direct sum of finitely many simple objects. The abelian category is called semisimple if every object is semisimple. For a field $k$, its absolute Galois group is denoted by $\Gamma_k$. Let $\text{Vec}_k$ be the category of finite dimensional $k$-vector spaces.

Acknowledgments

I am grateful to my advisor, Prof. Anna Cadoret, for her listening to my oral reports on this work and pointing out a flaw therein. She is always tolerant and patient. I also benefited from her various constructive advice. Prof. Peter Scholze kindly answered my question, to whom Lemma 3.3.9 is due. I appreciate the patience and detailed replies of Prof. Owen Barrett, Prof. Marco Maculan and Prof. Will Sawin to my questions on their respective work. The example in Remark 6.0.7 is also due to Prof. Will Sawin. I thank Emiliano Ambrosi for his hospitality during my visit to the Université de Strasbourg. During the preparation, I also received the help of many friends: Chenyu Bai, Arnaud Eteve, Arnab Kundu, Junbang Liu, Long Liu, Kai Mao, Keyao Peng, Mingchen Xia, Junsheng Zhang and Xiaoxiang Zhou. I thank Gabriel Ribeiro for noting a mistake in a previous version. Hui Zhang helped me out multiple times with his admirable knowledge, especially in algebraic geometry. All remaining errors are mine.

2 Cardinal argument

The objective of Section 2 is Lemma 2.2.5, used in the proof of Theorem 6.0.6. We show that over an uncountable algebraically closed field, a reasonable scheme has uncountably many rational points outside a countable union of strict closed subsets. For this, we need some elementary facts.

2.1 Elementary facts

**Lemma 2.1.1.** Let $k$ be a field. Let $X$ be a Noetherian $k$-scheme of dimension 0. Then the set underlying $X$ is finite.

**Proof.** By assumption, the scheme $X = \sqcup_{x \in X} \text{Spec}(O_{X,x})$, where each $O_{X,x}$ is an Artinian local ring. Then $X$ is quasi-compact and discrete. Thus, the set $X$ is finite.

**Lemma 2.1.2.** Let $f : X \to Y$ be a dominant integral morphism of schemes. Then $\dim X = \dim Y$. (It is possible that both sides are infinite.)
Proof. The induced morphism \( f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}} \) between reductions is still dominant and integral. Thereby, one may assume that \( X, Y \) are reduced. By [GD71, Prop. 5.4.3], for every affine open subset \( V = \text{Spec}(A) \subset Y \) with \( f^{-1}(V) = \text{Spec}(B) \), the induced ring map \( A \to B \) is injective. It is also integral as \( f \) is integral. From [Sta23, Tag 00OK], one has \( \dim A = \dim B \), i.e., \( \dim V = \dim f^{-1}(V) \). For \( V \) runs through all affine open subsets of \( Y \), one has \( \dim X = \sup_V \dim f^{-1}(V) = \sup_V \dim V = \dim Y \). \qed

Lemma 2.1.3. Let \( A \) be an integral domain that is not a field. Let \( n \geq 0 \) be an integer. If for every \( a \in A \setminus (A^* \cup \{0\}) \), one has \( \dim A/a = n \), then \( \dim A = n + 1 \).

Proof. Since \( A \) is not a field, there is \( a_0 \in A \setminus (A^* \cup \{0\}) \). Then \( Z(a_0) \) is a strict Zariski closed subset of the irreducible space \( \text{Spec}(A) \). Therefore, \( \dim A > \dim Z(a_0) = n \). Assume the contrary that \( \dim A = n + 2 \), then there is a chain

\[
0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_{n+2}
\]

of prime ideals of \( A \). Take \( f \in p_1 \setminus \{0\} \). Then

\[
p_1/(f) \subsetneq \cdots \subsetneq p_{n+2}/(f)
\]

is a chain of prime ideals of \( A/f \), which contradicts the assumption \( \dim A/f = n \). \qed

Fix an uncountable, algebraically closed field \( k \). A \( k \)-scheme \( X \) is called "good", if \( \dim X \) is finite and \( X(k) \setminus \bigcup_{i \geq 1} Z_i(k) \) is uncountable for every sequence \( \{Z_i\}_{i \geq 1} \) of closed subschemes of \( X \) with \( \dim Z_i < \dim X \) for all \( i \). A \( k \)-algebra \( R \) is called good if the \( k \)-scheme \( \text{Spec}(R) \) is good.

Lemma 2.1.4. If \( X/k \) is a Noetherian scheme of dimension 1 with \( X(k) \) uncountable, then \( X \) is good.

Proof. Let \( \{Z_i\}_i \) be a sequence of closed subschemes of \( X \) with \( \dim Z_i < \dim X \) for all \( i \). By Lemma 2.1.1, each \( Z_i(k) \) is finite. Therefore \( X(k) \setminus \bigcup_{i \geq 1} Z_i(k) \) is uncountable. \qed

Lemma 2.1.5. If \( f : X \to Y \) is a finite surjective morphism of \( k \)-schemes and \( Y \) is good, then \( X \) is also good.

Proof. By Lemma 2.1.2, \( \dim X = \dim Y \) is finite. The induced map \( X(k) \to Y(k) \) is surjective. Indeed, for every \( y \in Y(k) \), the fiber \( X_y \) is a nonempty finite \( k \)-scheme. As \( k \) is algebraically closed, \( X_y(k) \) is nonempty, so there is \( x \in X(k) \) lying over \( y \).

Let \( \{Z_i\}_i \) be a sequence of closed subschemes of \( X \) with \( \dim Z_i < \dim X \). Then for every integer \( i \geq 1 \), since \( f \) is a closed morphism, \( Y_i := f(Z_i) \) is closed in \( Y \). Endow each \( Y_i \) with the reduced induced structure. Let \( Z'_i := f^{-1}(Y_i) = Y_i \times_Y X \). Then there is a canonical closed immersion \( Z_i \to Z'_i \) and (the restriction of \( f \)) \( Z_i \to Y_i \) is a finite surjective morphism. By Lemma
2.1.2, one has \( \dim Y_i = \dim Z_i < \dim X = \dim Y \). From the surjectivity of \( X(k) \rightarrow Y(k) \), the induced map

\[
X(k) \setminus (\cup_{i \geq 1} Z_i(k)) \rightarrow Y(k) \setminus (\cup_i Y_i(k))
\]

is surjective. Because \( Y \) is good, the target is uncountable. Then \( X(k) \setminus (\cup_{i \geq 1} Z_i(k)) \) is also uncountable as it contains the source. \( \square \)

Lemma 2.1.6 should be well known.

**Lemma 2.1.6.** If \( X/k \) is a finite type scheme with \( \dim X > 0 \), then \( X \) is good.

**Proof.** Since \( X \) is of finite type over \( k \), its dimension \( m \) is finite and \( X \) has only finitely many irreducible components. Replacing \( X \) with an irreducible component of dimension \( m \), one may that assume \( X \) is irreducible. Then by [Har77, Exercise 3.20 (e), p.94], every nonempty open subset of \( X \) has dimension \( m \). Replacing \( X \) by an affine open, one may assume that \( X \) is affine. By Noether’s normalization lemma, there is a finite surjective \( k \)-morphism \( p : X \rightarrow \mathbb{A}^m_{\mathbb{K}} \). By Lemma 2.1.5 one may assume \( X = \mathbb{A}^m_{\mathbb{K}} \).

By induction on \( m \), one proves that \( \mathbb{A}^m_{\mathbb{K}} \) is good. The case \( m = 1 \) is treated by Lemma 2.1.4. Assume the statement for \( m-1 \) with \( m \geq 2 \). Let \( \{Z_i\}_i \) be a sequence of closed subschemes of \( \mathbb{A}^m_{\mathbb{K}} \) with \( \dim Z_i < m \). Each \( Z_i \) is a Noetherian scheme, so it has only finitely many irreducible components. The set of irreducible components of the family \( \{Z_i\}_i \) is at most countable. Thus, one may assume that each \( Z_i \) is irreducible. Since \( Z_i(k) = Z_{i,\text{red}}(k) \), we may assume each \( Z_i \) is reduced.

The set of hyperplanes in \( \mathbb{A}^m \) is uncountable, so there is a hyperplane \( H \subset \mathbb{A}^m \) with \( H \neq Z_i \) for all integers \( i \geq 1 \). As the \( Z_i \) are irreducible and \( \dim H \geq \dim Z_i \), one gets \( H \not\subset Z_i \) for all \( i \), so \( H \cap Z_i \neq H \). Since \( H \) is irreducible, one gets \( \dim(H \cap Z_i) < \dim H \) for all \( i \). By the inductive hypothesis, the set \( H(k) \setminus \bigcup_{i \geq 1}(H \cap Z_i)(k) \) is uncountable, which is a subset of \( X(k) \setminus \bigcup_{i \geq 1} Z_i(k) \). The induction is completed. \( \square \)

**Lemma 2.1.7.** Let \( X/k \) be a Noetherian scheme. Then \( X \) is good if and only if \( X \) has an irreducible component \( C \) with \( \dim C = \dim X \) such that \( C \) is good in the reduced induced structure.

**Proof.** Assume that there is such a component \( C \). Consider a sequence of closed subschemes \( \{Z_i\}_i \) of \( X \) with \( \dim Z_i < \dim X \) for all \( i \). Then for every \( i \geq 1 \), \( \dim C \cap Z_i \leq \dim Z_i < \dim X = \dim C \). Since \( C \) is good, the set \( C(k) \setminus \bigcup_i(C \cap Z_i)(k) \) is uncountable. Therefore, \( X(k) \setminus \bigcup_i Z_i(k) \) is also uncountable.

Assume that every component of \( X \) of maximum dimension is not good. As \( X \) is Noetherian, one can write \( X = \bigcup_{j=1}^n C_j \) as a finite union of the irreducible components. For every \( j \) with \( \dim C_j = \dim X \), the scheme \( C_j \) is not good. Therefore, there is a sequence \( \{Z^j_i\}_{i \geq 1} \) of closed subschemes of \( C_j \) such that \( \dim Z^j_i < \dim C_j \) for all \( i \) and \( C_j(k) \setminus \bigcup_i Z^j_i(k) \) is at most countable. The finite family of components \( C_k \) with \( \dim C_k < \dim X \), joint with the sequences \( \{Z^j_i\}_i \) for all \( j \) with \( \dim C_j = \dim X \), gives a countably family \( \{Z_i\} \) of closed
sub schemes of $X$ with $\dim Z_s < \dim X$ for all $s$. Then $X(k) \setminus (\cup_s Z_s(k))$ is countable, so $X$ is not good. \qed

2.2 Cotori are good

As in Lemma 2.1.6, we show that every positive dimensional "cotorus" defined in Section 4 is good. This cotorus is not locally of finite type over the base field.

Definition 2.2.1. Let $A$ be a $k$-algebra, and let $A[X] \to B$ be an injective ring map. We say that $B$ is Rücker over $A/k$ if there is a nonempty family $W$ of monic polynomials in $A[X]$ such that the following axioms are fulfilled:

1. If $f, g \in A[X]$ are monic polynomials with $fg \in W$, then $f, g \in W$.

2. For every $w \in W$, the $A$-algebra $B/\omega B$ is isomorphic to $A[X]/\omega A[X]$.

3. For every $f \in B \setminus \{0\}$, there is an automorphism $\sigma$ of the $k$-algebra $B$ and a unit $u \in B^*$ such that $u \sigma(f) \in W$.

Remark 2.2.2. From Axiom 1, one gets $1 \in W$. If $W = \{1\}$, then by Axiom 3, for every $f \in B \setminus \{0\}$, one has $f \in B^*$, i.e., $B$ is a field. Conversely, if $B$ is a field, then $B$ is Rücker over $A/k$ with $W = \{1\}$.

If $W \neq \{1\}$, take $w(\neq 1) \in W$. Then by Axiom 2, there is an $A$-isomorphism $B/\omega B \to A[X]/\omega$, hence an isomorphism $\text{Spec}(A[X]/\omega) \to \text{Spec}(B/\omega)$ of $\text{Spec}(A)$-schemes. Because $w$ is a monic polynomial different from 1, the ring map $A \to A[X]/\omega$ is injective finite. The induced morphism $\text{Spec}(A[X]/\omega) \to \text{Spec}(A)$ is surjective, so $\text{Spec}(B/\omega) \to \text{Spec}(A)$ is surjective. In particular, the morphism $\text{Spec}(B) \to \text{Spec}(A)$ is surjective.

Lemma 2.2.3 is used in the proof of Lemma 2.2.5.

Lemma 2.2.3. Let $A$ be Noetherian good $k$-algebra of Krull dimension $n$. Let $B$ be a domain but not a field containing $A[X]$ and Rücker over $A/k$. Assume that $S$ is an uncountable subset of $A$ such that for every $f \in S$, the subset $Z_A(f) \subset \text{Spec}(A)$ is of dimension $n - 1$. Suppose that the family $\{Z_A(f)\}_{f \in S}$ is pairwise disjoint. Then $B$ is a Noetherian good $k$-algebra of Krull dimension $n + 1$. Moreover, for every $f \in S$, the subset $Z_B(f)$ of $\text{Spec}(B)$ is of dimension $n$, and the family $\{Z_B(f)\}_{f \in S}$ is pairwise disjoint.

Proof. By [BGR84, Prop. 2, p.206], the ring $B$ is Noetherian. As $B$ is not a field, from Remark 2.2.2, the morphism $\text{Spec}(B) \to \text{Spec}(A)$ is surjective.

For every $b \in B \setminus (B^* \cup \{0\})$, by Axiom 3, there is an automorphism $\sigma$ of the $k$-algebra $B$ and a unit $u \in B^*$ such that $w := u \sigma(b)$ is in $W$. Since $b$ is not a unit, one has $w \neq 1$. By Axiom 2, the $A$-algebra $B/w$ is isomorphic to $A[X]/w$. Since $w \neq 1$ is a monic polynomial over $A$, the ring map $A \to A[X]/w$ is injective finite. By [Sta23, Tag 00OK], one has

$$\dim B/w = \dim A[X]/w = \dim A = n.$$
By Lemma 2.1.5, the $k$-algebra $A[X]/w$ is good. As $\sigma$ is over $k$, the $k$-algebra $B/b$ is isomorphic to $B/w$. Therefore, for every $b \in B \setminus (B^* \cup \{0\})$, the $k$-algebra $B/b$ is good and
\[ \dim B/b = n. \] (1)

**Fact 2.2.4.** For every integer $n \geq 0$, consider the $\Lambda$-algebra
\[ A_n := (\mathbb{Z}_\ell[[X_1, \ldots, X_n]]) \otimes_{\mathbb{Z}_\ell} \Lambda. \] (2)

By [GL96, Prop. 3.2.2 (1)], the natural morphism $A_n \to A[[X_1, \ldots, X_n]]$ is injective.

**Fact 2.2.4. For every integer $n \geq 0$,**

1. ([GL96, Thm. A.2.1, Prop. A.2.2.1] the ring $A_n$ is a Noetherian, regular, Jacobson domain of Krull dimension $n$;

2. ([GL96, Prop A.2.2.2, proof of A.2.2.3 (ii)]) $A_{n+1}$ is Rückercor over $A_n/\Lambda$.

**Lemma 2.2.5.** For every integer $n \geq 1$, the $\Lambda$-algebra $A_n$ is good.

**Proof.** Since $\Lambda$ is a flat $\mathbb{Z}_\ell$-module, the injection $\mathbb{Z}_\ell[X_1, \ldots, X_n] \to A_\ell[[X_1, \ldots, X_n]]$ induces an injection $\Lambda[X_1, \ldots, X_n] \to A_n$. The natural morphism
\[ \text{Spec}(\Lambda[[X_1, \ldots, X_n]]) \to A^n_\Lambda \] (3)
of $\Lambda$-schemes factors through a morphism $p_n : \text{Spec}(A_n) \to A^n_\Lambda$.

Let $\mathcal{M} = \bigcup_{E \in \mathcal{E} \subseteq E}$, where $E$ runs through all finite extensions of $\mathbb{Q}_\ell$ inside $\Lambda$ and $m_E$ is the maximal ideal of the ring of integers of $E$. Then $\mathcal{M}$ is the maximal ideal of the integral closure $\mathbb{Z}_\ell$ of $\mathbb{Z}_\ell$ inside $\Lambda$. The residue field $\mathbb{Z}_\ell/\mathcal{M}$ is the algebraic closure of the finite field $F_{\ell}$, hence countable. As $\mathbb{Z}_\ell$ is uncountable, so is the set $\mathcal{M}$.
For every \((a_1, \ldots, a_n) \in \mathcal{M}^n\), there is a surjective morphism of \(\Lambda\)-algebras:

\[
\Lambda[[X_1, \ldots, X_n]] \to \Lambda, \quad f \mapsto f(a_1, \ldots, a_n).
\]

Its kernel is a \(\Lambda\)-point of \(\text{Spec}(\Lambda[[X_1, \ldots, X_n]])\), whose image under the map (3) is \((a_1, \ldots, a_n) \in A^n_\Lambda(\Lambda)\). Therefore, \(\mathcal{M}^n\) is contained in \(p_n(\text{Spec}(A_n)(\Lambda))\). In particular, \(\text{Spec}(A_n)(\Lambda)\) is uncountable.

By induction on \(n\), we prove that \(A_n\) is good and \(\{Z_{A_n}(X_1 - a)\}_{a \in \mathcal{M}}\) is a pairwise disjoint closed subsets with dimension \(n - 1\) of \(\text{Spec}(A_n)\).

When \(n = 1\), by Lemma 2.1.4 and Fact 2.2.4, \(A_1\) is good. Moreover, \(\{Z_{A_n}(X_1 - a)\}_{a \in \mathcal{M}}\) is a family of closed point in \(\text{Spec}(A_1)\) and these points are pairwise different. The statement is proved for \(n = 1\). Assume the statement for \(n - 1\) with \(n \geq 2\). By Fact 2.2.4 and Lemma 2.2.3, the statement holds for \(n\). The induction is completed. \(\square\)

### 2.3 Representations of a reductive group

We review some basics of representation theory. Let \(k\) be a field. For a group scheme \(G/k\), let \(\text{Rep}(G)\) be the category of (rational) representations of \(G\). Let \(H \leq G\) be an open subgroup of finite index. Let \(\rho : H \to \text{GL}(W)\) be an object of \(\text{Rep}(H)\). Let \(V\) be the set of functions \(f : G \to W\) which satisfy \(f(h g) = \rho(h) f(g)\) for all \(h \in H, g \in G\). Then \(V\) is naturally a \(k\)-vector space of dimension \([G : H] \dim_k W\).

**Definition 2.3.1.** Define a morphism \(\sigma : G \to \text{GL}(V)\) by \(\sigma(g)(f)(x) = f(xg)\) for all \(g, x \in G\) and \(f \in V\). Then \(\sigma\) is an object of \(\text{Rep}(G)\), called the representation induced by \(\rho\) and denoted by \(\text{Ind}^G_H \rho\) or \(\text{Ind}^G_H \rho\).

An equivalent reformulation is as follows. Take a set of left representatives \(\{g_1, \ldots, g_n\}\) of \(G/H\). As a \(k\)-vector space \(V = \bigoplus_{i=1}^n g_i W\). Here each \(g_i W\) is an isomorphic copy of \(W\) whose elements are written as \(g_i w\) with \(w \in W\). For every \(g \in G\), there exist a permutation \(\pi\) in the symmetric group \(S_n\) and a subset \(\{h_1, \ldots, h_n\}\) of \(H\) with \(gg_i = g_{\pi(i)} h_i\). The \(G\)-action on \(V\) is

\[
\sigma(g) \sum_{i=1}^n g_i w_i := \sum_{i=1}^n g_{\pi(i)} \rho(h_i) w_i.
\]

Thus, one gets an additive functor \(\text{Ind}^G_H : \text{Rep}(H) \to \text{Rep}(G)\).

**Proposition 2.3.2** (Frobenius reciprocity). The functor \(\text{Ind}^G_H : \text{Rep}(H) \to \text{Rep}(G)\) is right adjoint to the restriction functor \(\text{Res}^G_H : \text{Rep}(G) \to \text{Rep}(H)\): For \(W \in \text{Rep}(H)\), let

\[
\text{ev}_W : \text{Ind}^G_H W \to W, \quad f \mapsto f(1)
\]

be the evaluation \((k\text{-linear})\) map. For \(V \in \text{Rep}(G)\), consider the maps

\[
\text{Hom}_{\text{Rep}(G)}(V, \text{Ind}^G_H W) \to \text{Hom}_{\text{Rep}(H)}(\text{Res}^G_H V, W), \quad \phi \mapsto \text{ev}_W \circ \phi;
\]

\[
\text{Hom}_{\text{Rep}(H)}(\text{Res}^G_H V, W) \to \text{Hom}_{\text{Rep}(G)}(V, \text{Ind}^G_H W), \quad \psi \mapsto \psi_*,
\]

\[
\text{ev}_W : \text{Ind}^G_H W \to W, \quad f \mapsto f(1)
\]

be the evaluation \((k\text{-linear})\) map. For \(V \in \text{Rep}(G)\), consider the maps

\[
\text{Hom}_{\text{Rep}(G)}(V, \text{Ind}^G_H W) \to \text{Hom}_{\text{Rep}(H)}(\text{Res}^G_H V, W), \quad \phi \mapsto \text{ev}_W \circ \phi;
\]

\[
\text{Hom}_{\text{Rep}(H)}(\text{Res}^G_H V, W) \to \text{Hom}_{\text{Rep}(G)}(V, \text{Ind}^G_H W), \quad \psi \mapsto \psi_*.
\]

\[
\text{ev}_W : \text{Ind}^G_H W \to W, \quad f \mapsto f(1)
\]
where $\psi_*(v)(g) = \psi(gv)$. The two maps are inverse to each other and functorial in $V$ and $W$.

Proof. By assumption, the map $ev_W$ is $H$-equivariant. Hence, it gives a natural transformation $ev : Res^G_H \text{Ind}^G_H \to \text{Id}_{\text{Rep}(H)}$ of functors $\text{Rep}(H) \to \text{Rep}(H)$. Therefore, the two maps in the statement are functorial in $V$ and $W$.

For every $\phi \in \text{Hom}_{\text{Rep}(G)}(V, \text{Ind}^G_H W)$, every $v \in V$ and every $g \in G$, one has

$$(ev_W \circ \phi)_*(v)(g) = (ev_W \circ \phi)(gv) = \phi(gv)(1) = [g\phi(v)](1) = \phi(v)(g),$$

so $(ev_W \circ \phi)_*(v) = \phi(v)$. Therefore, one has $(ev_W \circ \phi)_* = \phi$.

Conversely, for every $\psi \in \text{Hom}_{\text{Rep}(H)}(V, W)$, $(ev_W \circ \psi_*)(v) = \psi_*(v)(1) = \psi(v)$ for all $v \in V$. Therefore, one has $ev_W \circ \psi_* = \psi$. Hence, the two maps are inverse to each other.

Lemma 2.3.3. Assume that $H$ is normal in $G$. Let $\pi : G \to \text{GL}(V)$ be a simple representation. Then there is a simple representation $\rho : H \to \text{GL}(W)$ such that $\pi$ is a $G$-subrepresentation of $\text{Ind}^G_H \rho$.

Proof. Since in the abelian category $\text{Rep}(H)$ is Noetherian and Artinian, by [Sta23, Tag 0FCJ], there is a simple quotient $\text{Res}^G_H V \to W$ in $\text{Rep}(H)$. By Proposition 2.3.2, it induces a nonzero morphism $V \to \text{Ind}^G_H(W)$ in $\text{Rep}(G)$. As $V$ is a simple object of $\text{Rep}(G)$, this morphism identifies $V$ as a $G$-subrepresentation of $\text{Ind}^G_H(W)$.

Lemma 2.3.4. Assume that $k$ is algebraically closed of characteristic 0. Let $G/k$ be a reductive algebraic group. Then up to isomorphism, $G$ has at most countably many representations.

Proof. By Lemma 2.3.3, for every simple representation $W$ of $G$, there is a simple representation $U$ of $G^o$ such that $W$ is isomorphic to a $G$-subrepresentation of $\text{Ind}^G_{G^o} U$. By [Mil17, 22.3], up to isomorphism in $\text{Rep}(G^o)$, there are at most countably many such $U$. As the abelian category $\text{Rep}(G)$ is Noetherian and Artinian, by Lemma 2.3.5.1, each $\text{Ind}^G_{G^o} U$ has only finitely many simple subobjects up to isomorphism. Therefore, up to isomorphism $\text{Rep}(G)$ has at most countably many simple objects. By [Mil17, Cor. 22.43], the abelian category $\text{Rep}(G)$ is semisimple. Consequently, $\text{Rep}(G)$ has at most countably many objects up to isomorphism.

In a Noetherian and Artinian abelian category, an object may have infinitely many distinct subobjects up to isomorphism.

Lemma 2.3.5. Let $A$ be an abelian category, and let $X \in A$ be a Noetherian and Artinian object.

1. Let $Y$ be a simple subquotient of $X$. Then there is a composite series of $X$ with one graded piece isomorphic to $Y$. In particular, up to isomorphism $X$ has only finitely many simple subquotients.
2. If every subobject of \( X \) admits a direct complement, then \( X \) is semisimple.

**Proof.**

1. There is a subobject \( i : X_0 \subset X \) and a quotient \( q : X_0 \to Y \) in \( A \). Let \( N = \ker(q) \). By [Sta23, Tag 0FCH, Tag 0FCI], both \( N \) and \( X/X_0 \) are Noetherian and Artinian. From [Sta23, Tag 0FCJ], they admit composite series. A composite series of \( X/X_0 \) is equivalent to an alteration \( X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X \) by subobjects such that \( X_i/X_{i-1} \) is simple for \( 1 \leq i \leq n \). This filtration and every composite series of \( N \) glue to a composite series of \( X \) with a step \( N \subset X_0 \), whose factor is isomorphic to \( Y \). By the Jordan-Hölder lemma [Sta23, Tag 0FCK], up to isomorphism \( Y \) has finitely many choices.

2. One may assume that \( X \neq 0 \). Let \( P \) be the family of nonzero semisimple subobjects of \( X \). By [Sta23, Tag 0FCJ], \( X \) has a nonzero simple subobject, so \( P \) is nonempty. Since \( X \) is Noetherian, the family \( P \) has a maximal element \( i : X_0 \to X \). By assumption, there is a subobject \( F \subset X \) with \( X_0 \oplus F = X \). We claim that \( F = 0 \). Otherwise, by [Sta23, Tag 0FCJ], \( F \) has a nonzero simple subobject \( F_0 \), so \( X_0 \oplus F_0 \in P \) is strictly larger than \( X_0 \), a contradiction. From the claim, \( i \) is an isomorphism and \( X \) is semisimple.

\[ \square \]

3 **Recollections on constructible sheaves**

No originality is claimed in Section 3. Let \( k \) be a field where \( \ell \) is invertible. For every algebraic variety \( X/k \), denote by \( D^b_c(X) := D^b_c(X, \Lambda) \) the triangulated category of complexes of \( \Lambda \)-sheaves on \( X \) with bounded constructible cohomologies defined in [BBDG82, p.74]. For a closed subvariety \( i : Z \to X \) and \( K \in D^b_c(X) \), set \( K|_Z := i^*K \).

### 3.1 Basics

The general strategy in [KW01, p.110] shows that Fact 3.1.1 follows from [KW01, p.344].

**Fact 3.1.1** (Projection formula). Let \( f : X \to Y \) be a morphism of algebraic varieties over an algebraically closed field where the prime \( \ell \) is invertible. Let \( L \) be a bounded complex of lisse \( \Lambda \)-sheaves on \( Y \). Then there is a natural isomorphism \( (Rf_\ast \cdot) \otimes^L L \to Rf_\ast (\cdot \otimes^L f^*L) \) of functors \( D^b_c(X) \to D^b_c(Y) \).

**Fact 3.1.2** ([FK88, Prop. 12.10]). For every constructible \( \Lambda \)-sheaf \( F \) on \( X \), there is a nonempty Zariski open subset \( U \subset X \) such that \( F|_U \) is a lisse sheaf. In particular, when \( X \) is integral with generic point \( \eta \), there is a natural \( \Lambda \)-representation \( \Gamma_{k(\eta)} \to \GL(F_\eta) \).
The heart of the standard (resp. perverse) t-structure on $D^b_c(X)$ is denoted by $\text{Cons}(X)$ (resp. $\text{Perv}(X)$). The corresponding six-functor formalism is reviewed in [CES, 2.5]. For every integer $q$, let $H^q : D^b_c(X) \to \text{Cons}(X)$ (resp. $p^q H^q : D^b_c(X) \to \text{Perv}(X)$) be the functor taking the $q$-th usual (resp. perverse) cohomology sheaf.

Definition 3.1.3 ([BC18, Def. 78]). An object $K \in D^b_c(X)$ is called semisimple if it is isomorphic to a finite direct sum of degree shifts of semisimple objects of $\text{Perv}(X)$.

If $K \in D^b_c(X)$ is semisimple, then there is a non-canonical isomorphism $K \to \bigoplus_{q \in \mathbb{Z}} p^q H^q(K)[-q]$ in $D^b_c(X)$, where each $p^q H^q(K)$ is a semisimple object of $\text{Perv}(X)$. A degree shift of a semisimple object of $D^b_c(X)$ is still semisimple.

Example 3.1.4. Every perverse cohomology sheaf of a semisimple object of $D^b_c(X)$ is semisimple. By contrast, its cohomology sheaves may be no longer semisimple in $D^b_c(X)$.

Consider $k = \mathbb{C}$ and $X = \mathbb{A}^1$. Let $j : U = \mathbb{A}^1 \setminus \{0, 1\} \to X$ be the inclusion. Then the topological fundamental group $\pi_1(U^{\text{an}}, -1)$ is the free group generated by two loops $a$ and $b$, surrounding 0 and 1 respectively. There is a unique morphism

$$\pi_1(U^{\text{an}}, -1) \to \text{SL}_2(\mathbb{Z})$$

(4)

saying $a, b$ to

$$A = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$$

respectively. By Grauert-Remmert’s theorem (see, e.g., [Sza09, Thm. 5.7.4]), the étale fundamental group $\pi^{\text{et}}_1(U, -1)$ is the profinite completion of $\pi_1(U^{\text{an}}, -1)$. Since $\text{SL}_2(\mathbb{Z})$ is a profinite group, the morphism (4) extends naturally to a continuous morphism

$$\pi^{\text{et}}_1(U, -1) \to \text{SL}_2(\mathbb{Z}).$$

(5)

It defines a 2-dimensional $\Lambda$-representation of $\pi^{\text{et}}_1(U, -1)$.

The representation (5) is irreducible. Otherwise, assume that $v := (x, y)^T \neq 0$ in $\mathbb{A}^2$ generates a 1-dimensional subrepresentation. Then $Av = (-x - 2y, -y)^T$ is parallel to $v$. Therefore, $y = 0$. Similarly, $Bv = (-x, -2x - y)^T$ is parallel to $v$, then $x = 0$, a contradiction.

Let $L$ be the rank 2 simple lisse $\Lambda$-sheaf on $U$ corresponding to (5). Then $L^{\text{an}}$ is the local system on $U^{\text{an}}$ corresponding to (4). For every small open ball $B_0 \subset X^{\text{an}}$ centered at 0, the $\mathbb{C}$-vector space $H^0(B_0, j^{\text{an}}_* L^{\text{an}})$ is the kernel of the linear operator $A - 1$ on the stalk $L_0^{\text{an}}$. Since $A - 1$ is invertible, one has $H^0(B_0, j^{\text{an}}_* L^{\text{an}}) = 0$. Therefore, the stalk $(j^{\text{an}}_* L^{\text{an}})_0 = 0$. Similarly, the stalk $(j^{\text{an}}_* L^{\text{an}})_1 = 0$. In conclusion, the natural morphism $j_!^{\text{an}} L^{\text{an}} \to j^{\text{an}}_* L^{\text{an}}$ is an isomorphism in $\text{Cons}(X^{\text{an}})$.

We prove that $H^1(U^{\text{an}}, L^{\text{an}}) = H^1(\pi_1(U^{\text{an}}, -1), L_0^{\text{an}})$ is nonzero. Define a map $f : \pi_1(U^{\text{an}}, -1) \to \mathbb{A}^2$ inductively. Set $f(e) = 0$, $f(a) = f(b) = (1, 0)^T$, $f(a^{-1}) = -A^{-1} f(a)$, and $f(b^{-1}) = -B^{-1} f(b)$. Once $f$ is defined for every
element of $\pi_1(U_{an}, 1)$ with length $n \geq 1$, we define it on elements of length $n + 1$ as follows. For every element $g \in \pi_1(U_{an}, -1)$ of length $n$, set
\[
f(ay) = Af(g) + f(a), \quad f(by) = Bf(g) + f(b),
\]
\[
f(a^{-1}g) = A^{-1}f(g) + f(a^{-1}), \quad f(b^{-1}g) = B^{-1}f(g) + f(b^{-1}).
\]
The map $f$ is a crossed homomorphism. It is not a boundary, because the equation $(A - 1)x = (B - 1)x = (1, 0)^T$ admits no solution in $A^2$.

Therefore, $L^{an}$ is in the cohomology support loci of $U^{an}$ (in the sense of [BLSW17, p.295]). From [HT07, Example 8.1.35 (ii)], one has $j_{!*}L[1] \in \text{Perv}(X)$. By [BLSW17, p.299], $j^{an}_{!*}L^{an}[1]$ is not semisimple in $\text{Perv}(X^{an})$.

By [BBDG82, Thm. 4.3.1 (ii)], the intermediate extension $K := j_{!*}L[1]$ is a simple object of $\text{Perv}(X)$. We claim that $H^{-1}K$ is not semisimple in $D^b_c(X)$. From [HT07, Prop. 8.2.11], $K$ is isomorphic to $\tau^{\leq -1}Rj_{!*}L[1]$, where $\tau^{\leq -1} : D^b_c(X) \rightarrow D^b_c(X)$ is the truncation functor with respect to the standard t-structure. Thus, $H^{-1}K$ is isomorphic to $H^{-1}(Rj_{!*}L[1]) = j_{!*}L$ in $\text{Cons}(X)$. Then $(H^{-1}K)^{an}$ is isomorphic to $j^{an}_{!*}L^{an}$ in $\text{Cons}(X^{an})$. From [Kat90, p.375], one has $(H^{-1}K)^{an}[1] \in \text{Perv}(X)$. Since $(H^{-1}K)^{an}[1]$ is not semisimple in $\text{Perv}(X^{an})$, by [Kat90, Lem. 12.7.1.1], $(H^{-1}K)[1]$ is not semisimple in $\text{Perv}(X)$. The claim is proved.

Lemma 3.1.5. Let $U \subset X$ be an open subset of $X$. Then the restriction functor $\text{Perv}(X) \rightarrow \text{Perv}(U)$ sends every simple object of $\text{Perv}(X)$ to a simple or zero object of $\text{Perv}(U)$. In particular, the restriction function $D^b_c(X) \rightarrow D^b_c(U)$ preserves semisimplicity.

Proof. Let $K$ be a simple object of $\text{Perv}(X)$. By [BBDG82, Thm. 4.3.1 (ii)], there is an irreducible, locally closed and geometrically smooth subvariety $j : V \rightarrow X$ and a simple lisse $\mathcal{A}$-sheaf on $V$ such that $K$ is isomorphic to $j_{!*}L[\dim V]$. If $V$ is disjoint from $U$, then $K|_U = 0$. Otherwise, take a geometric point $x$ of $V \cap U$. From [Sta23, Tag 0BQI], the morphism $\pi^G_\mathcal{A}(U \cap V, x) \rightarrow \pi^G_\mathcal{A}(V, x)$ is surjective. Thus, the composite representation
\[
\pi^G_\mathcal{A}(U \cap V, x) \rightarrow \text{GL}(L_x)
\]
is also simple, i.e., the lisse $\mathcal{A}$-sheaf $L|_{U \cap V}$ is simple. Let $h : U \cap V \rightarrow U$ be the base change of $j$. Then $K|_U$ is isomorphic to $h_{!*}L|_{U \cap V}[\dim(U \cap V)]$, hence simple in $\text{Perv}(U)$.

When $k = \mathbb{C}$, Fact 3.1.6 follows from Kashiwara’s conjecture for semisimple perverse sheaves and the paragraph following [BBDG82, Thm. 6.2.5]. Kashiwara’s conjecture is formulated in [Kas98, Sec. 1]; see also [Dri01, Sec. 1.2, 1]. It is reduced to de Jong’s conjecture by Drinfeld [Dri01], which in turn is proved in [BK06] and [Ga07]. The case of general $k$ follows via Fact 3.1.7.

Fact 3.1.6. Let $k$ be an algebraically closed field of characteristic $0$. Let $f : X \rightarrow Y$ be a proper morphism of algebraic varieties over $k$. Let $K$ be a semisimple object of $D^b_c(X)$.
1. (Decomposition theorem) Then $Rf_*K$ is a semisimple object of $D^b_c(Y)$.

2. (Global invariant cycle theorem, [BBDG82, Cor. 6.2.8]) Let $i$ be an integer. Let $V \subset Y$ be a nonempty connected open subset such that $H^iRf_*K|_V$ is a lisse sheaf. Then for every $y \in V(k)$, the canonical map

$$H^i(X, K) \to H^i(X_y, K|_{X_y})^{\pi_1^i(V, y)}$$

is surjective.

**Fact 3.1.7.** Let $E/F$ be an extension of algebraically closed fields. Let $X$ be an algebraic variety over $F$. Then:

1. ([JKLM23, proof of Lem. A.1]) The base change functor

$$(\cdot)_E : D^b_c(X) \to D^b_c(X_E)$$

is fully faithful. It induces an exact functor $\text{Perv}(X) \to \text{Perv}(X_E)$.

2. ([BBDG82, Thm. 4.3.1 (ii)]) An object of $\text{Perv}(X)$ is simple if and only if it is simple in $\text{Perv}(X_E)$.

For the $k$-algebraic variety $X$ and $F \in \text{Cons}(X)$, set $\text{Supp} F := \{x \in X(k^a) : F_x \neq 0\}$ to be its support. Then $\text{Supp} F$ is a quasi-constructible subset of $X$ in the sense of [Gro66, 10.1.1]. For $K \in D^b_c(X)$, set $\text{Supp} K := \cup_{j \in \mathbb{Z}} \text{Supp} H^j K$.

**Lemma 3.1.8.** Let $L$ be a lisse $\Lambda$-sheaf of rank $1$ on $X$. Then

$$(\cdot \otimes L) : D^b_c(X) \to D^b_c(X)$$

(6)

is an equivalence of categories that is t-exact relative to perverse t-structures, with a quasi-inverse $\cdot \otimes L^\vee$.

**Proof.** By associativity of the derived tensor product $\otimes^L$, the pair of functors $(\cdot \otimes^L L, \cdot \otimes^L L^\vee)$ is an equivalence. The functor (6) is right t-exact relative to perverse t-structures. In fact, it is t-exact relative to the standard t-structures. Thus, for every $K \in pD^{\leq 0}(X)$, every integer $q$, one has $\mathcal{H}^q(K \otimes^L L) = \mathcal{H}^q(K) \otimes^L L$. Therefore, one has $\text{Supp} \mathcal{H}^0(K \otimes^L L) = \text{Supp} \mathcal{H}^0(K)$. Thus, $K \otimes^L L \in pD^\leq 0(X)$.

The functor (6) is also left t-exact relative to perverse t-structures. Indeed, by [KW01, II, Cor. 7.5 f]], the functor $\mathbb{D}_X (\cdot \otimes^L L) = \mathbb{R}\text{Hom}(L, \mathbb{D}_X \cdot) = L^\vee \otimes^L \mathbb{D}_X$ on $D^b_c(X)$. For every $K \in pD^{\geq 0}(K)$, one has $\mathbb{D}_X K \in pD^{\leq 0}(X)$. By last paragraph, $L^\vee \otimes^L \mathbb{D}_X K \in pD^{\leq 0}(X)$. Therefore, one gets $K \otimes^L L \in pD^{\geq 0}(X)$. $\square$
3.2 Universal local acyclicity

All schemes in Section 3.2 are assumed to be qcqs. For a scheme $X$ and a geometric point $\bar{x}$ of it, set $X(\bar{x}) := \text{Spec} O^{\text{sh}}_{X, \bar{x}}$ for the strict henselization (in the sense of [Sta23, Tag 04GQ (3)]) of $X$ at $\bar{x}$. Let $f : X \to S$ be a separated morphism having finite presentation between schemes over $\mathbb{Z}[1/\ell]$.

**Definition 3.2.1** ([Sta23, Tag 0GJM], [Bar23, Def. 1.2]). Let $K$ be an object of $D^b_c(X)$.

- If for every geometric point $\bar{x}$ of $X$ and every geometric point $\bar{s}$ of $S(\bar{s})$ with $\bar{s} = f(\bar{x})$, the canonical morphism $Rf^*(\mathcal{H}_X, K) \to Rf^*(\mathcal{H}_{X \times S(\bar{s})}, K)$ is an isomorphism, then $f$ is called locally acyclic relative to $K$.

- If for every morphism $S' \to S$ of schemes, the base change $f' : X' \to S'$ of $f$ in the diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}
$$

is locally acyclic relative to the pullback of $K$ to $X'$, then $K$ is called universally locally acyclic ($f$-ULA). Let $D^{\text{ULA}}_c(X/S)$ be the full subcategory of $D^b_c(X)$ comprised of $f$-ULA objects.

By [HS23, Thm. 4.4], an object $K \in D^b_c(X)$ is $f$-ULA if and only if $K$ is universally locally acyclic in the sense of [HS23, Def. 3.2]. Thus, the notation $D^{\text{ULA}}_c(X/S)$ agrees with that in [HS23]. It is a triangulated subcategory of $D^b_c(X)$. Let $\text{Loc}(X) \subset \text{Cons}(X)$ be the full subcategory of lisse $\Lambda$-sheaves on $X$.

**Fact 3.2.2.**

1. ([Bar23, Lem. 3.4]) Let $f : X \to \text{Spec}(k)$ be the structure morphism of an algebraic variety. Then every object of $D^b_c(X)$ is $f$-ULA.

2. ([Bar23, Cor. 3.10 (i)]) $D^{\text{ULA}}_c(X/X) = \text{Loc}(X)$.

3. ([HS23, Prop. 3.4 (i)]) Let $g : S' \to S$ be a morphism of schemes. Then in the notation of (7), the functor $g'^* : D^b_c(X) \to D^b_c(X')$ restricts to a functor $D^{\text{ULA}}_c(X/S) \to D^{\text{ULA}}_c(X'/S')$.

4. ([Ric14, Lem. 3.15], [Bar23, Lem. 3.3 (i), (ii)]) Let $f : X \to Y$ be an $S$-morphism. If $f$ is smooth (resp. proper), then the functor $f^* : D^b_c(Y) \to D^b_c(X)$ (resp. $Rf_* : D^b_c(X) \to D^b_c(Y)$) restricts to a functor $D^{\text{ULA}}_c(Y/S) \to D^{\text{ULA}}_c(X/S)$ (resp. $D^{\text{ULA}}_c(X/S) \to D^{\text{ULA}}_c(Y/S)$).

5. ([HS23, p.643]) Let $g : S \to T$ be a smooth morphism of schemes. If $K \in D^b_c(X)$ is $f$-ULA, then $K$ is $g \circ f$-ULA.
Lemma 3.2.3. Assume that $S$ is irreducible with generic point $\eta$. Let $K \in D^{ULA}(X/S)$. If $K_\eta = 0$ in $D^b_c(X_\eta)$, then $K = 0$.

Proof. It suffices to prove that for every $s \in S$, one has $K_s = 0$ in $D^b_c(X_s)$.

By [Gro61, Prop. 7.1.9], there is a discrete valuation ring $R$ and a separated morphism $g : \text{Spec}(R) = S' \to S$, sending the generic (resp. closed) point $\xi$ (resp. $r$) of $S'$ to $\eta$ (resp. $s$). Let $i : R \to R^h$ be the henselization of $R$ in the sense of [Sta23, Tag 04GQ (1)]. By [Sta23, Tag 0AP3], $R^h$ is a discrete valuation ring. From [Mil80, I, Exercise 4.9], the local morphism $i$ is injective. Then $i^* : \text{Spec}(R^h) \to S'$ preserves the generic (resp. closed) point. Replacing $R$ by $R^h$, one may assume further that $R$ is henselian.

Let $f' : X' \to S'$ be the base change of $f$. Consider the following cartesian squares

$$
x'_i \xrightarrow{i} X'_{(f)} \xleftarrow{j} X'_\xi
\downarrow \Box \downarrow \Box
\downarrow \Box
\xrightarrow{\bar{r}} S'_{(f)} \xleftarrow{\xi},
$$

where every vertical morphism is a base change of $f' : X' \to S'$. Let $R\Psi : D^+(X') \to D^+(X'_\xi)$ be the vanishing cycle functor. Let $R\Phi : D^+(X') \to D^+(X'_\xi)$ be the nearby cycle functor. Set $F = g^* K$. By definition, one has $R\Phi(F) = \bar{r}^* R\phi_{\eta}(F|_{X'_\xi})$. From [Ill06, (1.1.3)], there is a natural exact triangle $F|_{X'_\xi} \to R\Psi(F) \to R\Phi(F) \xrightarrow{+1}$ in $D^+(X'_\xi)$. Since $F|_{X'_\xi}$ is a base change of $K_\eta$, one has $F|_{X'_\xi} = 0$ and $R\Phi(F) = 0$. By [Ill06, Cor. 3.5], the universal local acyclicity of $K$ implies $R\Phi(F) = 0$. Therefore, one gets $F|_{X'_\xi} = 0$.

By assumption, the characteristic of $k(\bar{s})$ is 0. Since $F|_{X'_\xi}$ is the base change of $K_\eta$ under the field extension $k(\bar{s})/k(s)$, by Fact 3.1.7 1, one gets $K_\bar{s} = 0$. \qed

3.3 Relative perverse sheaves

Let $f : X \to S$ be a morphism of algebraic varieties over $k$. For every point $s$ of $S$ and $F \in D^b_c(X)$, set $F_s := F|_{X_s}$. Set $K_{X/S} := Rf^! \Lambda_s \in D^b_c(D_X)$ to be the relative dualizing complex. The contravariant functor

$$D_{X/S} = R\text{Hom}_{\Lambda_X}(\cdot, K_{X/S}) : D^b_c(X) \to D^b_c(X)$$

is called the relative Verdier duality. There is a canonical natural transformation $\text{Id}_{D^b_c(X)} \to D_{X/S} \circ D_{X/S}$ ([KL85, (1.1.5)]).

Fact 3.3.1 is stated for $\infty$-categories in [HS23], but holds for the underlying triangulated categories (described in [HR23, Lem. 7.9]) by [HS23, Footnote 1].

Fact 3.3.1.

1. ([HS23, Thm. 1.1]) There is a unique t-structure $(p^! S D^{\leq 0}, p^! S D^{>0})$ on $D^b_c(X)$, called the relative perverse t-structure, with the following property:
An object $K \in D^b_p(X)$ lies in $p/S D^{\leq 0}$ (resp. $p/S D^{\geq 0}$) if and only if for all geometric points $\bar{s} \to S$, the restriction $K_{\bar{s}}$ lies in $\bar{s} D^{\leq 0}$ (resp. $\bar{s} D^{\geq 0}$), for the absolute perverse $t$-structure on $D^b_p(X)$. In particular, for every $s \in S$, the restriction functor $D^b_p(X) \to D^b_p(X_s)$ is $t$-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse $t$-structure. It induces an exact functor $\text{Perv}(X/S) \to \text{Perv}(X_s)$.

2. ([HS23, Thm. 1.9]) The relative perverse $t$-structure on $D^b_p(X)$ restricts to a $t$-structure $(p/S D^{ULA, \leq 0}(X/S), p/S D^{ULA, \geq 0}(X/S))$ on $D^{ULA}(X/S)$.

3. ([HS23, Prop. 3.4]) The functor $D_{X/S}$ preserves $D^{ULA}(X/S)$, and the restriction $\text{Id}_{D^{ULA}(X/S)} \to D_{X/S} \circ D_{X/S}$ is an isomorphism between functors $D^{ULA}(X/S) \to D^{ULA}(X/S)$. The formation of $D_{X/S} : D^{ULA}(X/S) \to D^{ULA}(X/S)$ commutes with any base change in $S$, so $D_{X/S}$ exchanges $p/S D^{ULA, \leq 0}(X/S)$ with $p/S D^{ULA, \geq 0}(X/S)$.

Let $\text{Perv}(X/S)$ (resp. $\text{Perv}^{ULA}(X/S)$) be the heart of the relative perverse $t$-structure on $D^b_p(X)$ (resp. $D^{ULA}(X/S)$). By Fact 3.3.11, an object $K \in D^b_p(X)$ lies in $\text{Perv}(X/S)$ if and only if for every geometric point $\bar{s} \to S$, one has $K|_{X_s} \in \text{Perv}(X_s)$.

Example 3.3.2.

1. ([HS23, p.632] If $S = \text{Spec}(k)$, then Fact 3.3.11 gives the absolute perverse $t$-structure and $\text{Perv}(X/k) = \text{Perv}(X)$. If $f$ is universally injective, then it gives the standard $t$-structure and $\text{Perv}(X/S) = \text{Cons}(X)$.

2. Let $L$ be a lisse $A$-sheaf on $X$. If $f$ is smooth of relative dimension $r$, then by [Bar23, Cor. 3.10 (ii)], one has $L[r] \in \text{Perv}^{ULA}(X/S)$.

Example 3.3.3. Let $i : Y \to X$ be a closed immersion of $S$-schemes. Assume that the morphism $Y \to S$ is smooth of relative dimension $d$ with geometrically connected fibers. If $L$ is a lisse $A$-sheaf on $Y$, then $i_* Y[d] \in \text{Perv}^{ULA}(X/S)$.

Indeed, by Fact 3.2.2 2, one has $L \in D^{ULA}(Y/Y)$. From the smoothness of the morphism $Y \to S$ and Fact 3.2.2 5, one has $L \in D^{ULA}(Y/S)$. Using the properness of $i$ and Fact 3.2.2 4, one gets $i_* Y[d] \in D^{ULA}(X/S)$. For every closed point $s \in S$, let $i_s : Y_s \to X_s$ be the base change of $i$. By the proper base change theorem, $i_* Y[d]|_{X_s} = (i_s)_*(L|_Y)[d] \in \text{Perv}(X_s)$. Therefore, $i_* Y[d] \in \text{Perv}^{ULA}(X/S)$.

Lemma 3.3.4. If $S$ is irreducible and geometrically unibranch, then the category $\text{Perv}^{ULA}(X/S)$ is a Serre subcategory of $\text{Perv}(X/S)$.

Proof. By [BBDG82, Thm. 1.3.6], $\text{Perv}^{ULA}(X/S)$ is a fully faithful, abelian subcategory of $\text{Perv}(X/S)$ and closed under extensions. From the proof of [HS23, Thm. 6.8 (ii)], it is closed under subquotients. By [Sta23, Tag 02MP], it is a Serre subcategory. 

Lemma 3.3.5 is stated more generally for regular schemes $S$ in [HS23, p.636].
Lemma 3.3.5. If $S$ is smooth over $k$ of equidimension $d$, then the shifted inclusion
\[ \bullet [d] : D^{ULA}(X/S) \to D^{b}_c(X) \] (8)
is $t$-exact, where $D^{ULA}(X/S)$ (resp. $D^b_c(X)$) is equipped with the relative (resp. absolute) perverse $t$-structure. It induces an exact functor
\[ \bullet [d] : \text{Perv}^{ULA}(X/S) \to \text{Perv}(X). \] (9)

Proof. We claim that the functor
\[ \bullet [d] : D^{b}_c(X) \to D^{b}_c(X) \] (10)
is right $t$-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse $t$-structure. For every geometric point $s$ of $S$, let $i_s : X_s \to X$ be the inclusion. For every integer $q$ and every $K \in p/S D^{\leq 0}$, the functor $i^*_s : D^{b}_c(X) \to D^{b}_c(X_s)$ is $t$-exact relative to the standard $t$-structures. Thus, $H^q(K[d])|_{X_s} = H^{q+d}(K_s)$. Then
\[ X_s \cap \text{Supp} \mathcal{H}^q(K[d]) = \text{Supp} \mathcal{H}^{q+d}(K_s). \]
As $K_s \in pD^{\leq 0}(X_s)$, one has $\dim \text{Supp} \mathcal{H}^{q+d}(K_s) \leq -q - d$. By Lemma 3.3.8 3, one has
\[ \dim \text{Supp} \mathcal{H}^q(K[d]) \leq -q. \]
From Lemma 3.3.8 1, the Zariski closure of $\text{Supp} \mathcal{H}^q(K[d])$ in $X$ has dimension at most $-q$. Therefore, $K[d] \in pD^{0}(X)$ (defined in [Max19, p.133]). The claim is proved.

It remains to show that the functor (8) is left $t$-exact. One may assume that $k$ is algebraically closed. For every $M \in p/S D^{ULA, \geq 0}(X/S)$, by the proof of [Bar23, Cor. 3.8], $\mathbb{D}_X(M[d])$ is (noncanonically) isomorphic to $(\mathbb{D}_{X/S}M)[d]$ in the category $D^{b}_c(X)$. From Fact 3.3.1 3, $\mathbb{D}_{X/S}M \in p/S D^{ULA, \leq 0}(X/S)$. By the claim, $(\mathbb{D}_{X/S}M)[d] \in pD^{\leq 0}(X)$. Thus, $M[d] \in pD^{\geq 0}(X)$.

Remark 3.3.6. The functor (10) may not send $\text{Perv}(X/S)$ to $\text{Perv}(X)$. Indeed, let $k = \mathbb{C}$, and let $f : X = 0 \to S = A^1_k$ be the inclusion of the origin. By Example 3.3.2 1, the relative perverse $t$-structure on $D^b_c(X)$ coincides with the standard one (which is also the absolute perverse $t$-structure). Then $\text{Perv}(X/S) = \text{Perv}(X)$.

Lemma 3.3.7 seems to be used in the proof of [SFFK23, Lem. 3.11].

Lemma 3.3.7. If $S$ is integral with generic point $\eta$ and $\dim S = d$, then the functor
\[ D^{b}_c(X) \to D^{b}_c(X_{\eta}), \quad K \mapsto K_{\eta}[-d] \] (11)
is $t$-exact relative to absolute perverse structures. It induces an exact functor
\[ \text{Perv}(X) \to \text{Perv}(X_{\eta}), \quad K \mapsto K_{\eta}[-d]. \] (12)
Proof. The functor (11) is right t-exact. Indeed, take $K \in {}^pD^{\leq 0}(X)$. For every integer $q$, one has $\text{Supp }H^q(K_\eta[-d]) = \text{Supp }H^{q-d}(K_\eta) = X_\eta \cap \text{Supp }H^{q-d}(K)$. By Lemma 3.3.8 4, one has

$$\dim \text{Supp }H^q(K_\eta[-d]) \leq \dim \text{Supp}(H^{q-d}(K)) - d \leq -q.$$  

From Lemma 3.3.8 1, one has $H^q(K_\eta[-d]) \in {}^pD^{\leq 0}(X_\eta)$. 

For every $M \in D^b_c(X)$, by [DGIV77, Thm. 2.13, p. 242], there is a nonempty open subset $U \subset S$ with $M|_U \in D^{UL\text{-}A}(X_U/U)$. By the proof of [Bar23, Cor. 3.8], one has $\mathbb{D}_X M = (\mathbb{D}_X/S M)(d)[2d]$. From Fact 3.3.1 3, $(\mathbb{D}_X M)_\eta[-d]$ is a Tate twist of $\mathbb{D}_X(M_\eta[-d])$. For every integer $q$, one has 

$$\text{Supp }H^q(\mathbb{D}_X(M_\eta[-d])) = \text{Supp }H^q((\mathbb{D}_X M)_\eta[-d]). \quad (13)$$

The functor (11) is left t-exact. Indeed, take $L \in {}^pD^{\geq 0}(X)$. Then $\mathbb{D}_X L \in {}^pD^{\leq 0}(X)$. From the first paragraph, $(\mathbb{D}_X L)_\eta[-d] \in {}^pD^{\leq 0}(X_\eta)$. By (13), $\mathbb{D}_X(M_\eta[-d]) \in {}^pD^{\leq 0}(X_\eta)$. Therefore, $L_\eta[-d] \in {}^pD^{\leq 0}(X_\eta)$. \[\square\]

**Lemma 3.3.8.** Let $f : X \to Y$ be an $F$-morphism between schemes of finite type over a field $F$. Let $A$ be a quasi-constructible subset of $X$. (By convention, the dimension of an empty space is $-\infty$.)

1. Then its Krull dimension $\dim A = \dim \bar{A}$.

2. Let $\{B_i\}_{i=1}^n$ be finitely many locally closed subsets of $X$ and $B = \cup_{i=1}^n B_i$. Then $\dim B = \max_{i=1}^n \dim B_i$.

3. Let $n \geq 0$ be an integer such that $\dim A \cap f^{-1}(y) \leq n$ for every $y \in Y$. Then $\dim A \leq \dim Y + n$.

4. Assume that $Y$ is integral with generic point $\eta$. Then $\dim Y + \dim A \cap X_\eta \leq \dim A$.

**Proof.**

1. As $X$ is a Noetherian scheme, the topological space $A$ is Noetherian. Therefore, $A$ is the union of finitely many irreducible components. Thus, one may assume further that $A$ is nonempty and irreducible. Then the reduced induced closed subscheme $\bar{A}$ of $X$ is integral and of finite type over $F$. By [Har77, II, AG Prop. 1.3], $A$ contains a nonempty open subset of $\bar{A}$. By [Har77, II, Exercise 3.20 (e)], one has $\dim A = \dim \bar{A}$.

2. For every $1 \leq i \leq n$, since $B_i \subset B$, one has $\dim B_i \leq \dim B$. Then $\max_i \dim B_i \leq \dim B$. As $B_i$ is quasi-constructible in $X$, by 1, one has $\dim B_i = \dim \overline{B_i}$. As $\{\overline{B_i}\}_{i=1}^n$ is a finite closed cover of $\bar{B}$, one gets $\dim B \leq \dim \bar{B} = \max_i \dim \overline{B_i} = \max_i \dim B_i$.

3. By 2, one may assume that $A$ is locally closed in $X$. Taking irreducible components, one may assume further that $A$ is irreducible. Let $Z$ be the Zariski closure of $f(A)$ in $Y$. Then $Z$ is irreducible. With reduced induced
subscheme structures, one views $A$ and $Z$ as integral schemes of finite type over $F$. Moreover, $f$ induces a dominant $F$-morphism $g : A \to Z$. By [Har77, II, Exercise 3.22 (b)], for every $y \in f(A) = g(A)$, one has

$$n \geq \dim A \cap f^{-1}(y) = \dim g^{-1}(y) \geq \dim A - \dim Z.$$  

Thus, one gets $\dim A \leq \dim Z + n \leq \dim Y + n$.

4. As in the proof of 3, one may assume that $A$ is an irreducible, locally closed subset of $X$ and view $A$ as an integral scheme of finite type over $F$. One may assume that $A \cap X_\eta$ is nonempty. As $A_\eta$ is homeomorphic to $A \cap X_\eta$, the morphism $A \to Y$ induced by $f$ is dominant. Thus, by [Har77, II, Exercise 3.22 (c)], one gets $\dim A \cap X_\eta = \dim A_\eta = \dim A - \dim Y$.

\[\square\]

Lemma 3.3.9 (Scholze). Assume that $S$ is regular, integral with generic point $\eta$ and $\dim S = d$. Then:

1. Let $A \in \text{Perv}^{\text{ULA}}(X/S)$, and let $B[d] \in \text{Perv}(X)$. If the image $B_\eta \in \text{Perv}(X_\eta)$ of $B[d]$ under the functor (12) is zero, then $B[d] = 0$ in $\text{Perv}(X)$.

2. The functor (9) identifies $\text{Perv}^{\text{ULA}}(X/S)$ as a Serre subcategory of $\text{Perv}(X)$.

Proof.

1. By regularity of $S$ and [HS23, Cor. 1.12], one has $B \in D^{\text{ULA}}(X/S)$. Since $B_\eta = 0$, by Lemma 3.2.3, one has $B = 0$.

2. The functor (9) is exact, fully faithful of kernel 0. By Fact 3.3.12 and [BBDG82, Thm. 1.3.6], $\text{Perv}^{\text{ULA}}(X/S)$ is closed under extensions in $D^{\text{ULA}}(X/S)$. Thus, the essential image of (9) is closed under extensions in $\text{Perv}(X)$. We claim that the essential image is closed under taking subobjects.

Take $K \in \text{Perv}^{\text{ULA}}(X/S)$ and a subobject $L[d]$ of $K[d] \in \text{Perv}(X)$. Then $L_\eta$ is a subobject of $K_\eta \in \text{Perv}(X_\eta)$. By [HS23, Thm. 1.10 (ii)], there is a subobject $L' \to K$ in $\text{Perv}^{\text{ULA}}(X/S)$ with $L'_\eta = L_\eta$. Set $M = K/L' \in \text{Perv}^{\text{ULA}}(X/S)$. Let $N[d]$ be the image of $L[d]$ under the morphism $K[d] \to M[d]$ in $\text{Perv}(X)$. As the sequence

$$0 \to L'[d] \cap L[d] \to L[d] \to N[d] \to 0$$

is exact in $\text{Perv}(X)$, by Lemma 3.3.7, the sequence

$$0 \to L'_\eta \cap L_\eta \to L_\eta \to N_\eta = 0$$

is exact in $\text{Perv}(X_\eta)$. Thus, one gets $N_\eta = 0$. Since $N[d]$ is a subobject of $M[d] \in \text{Perv}(X)$, by 1, one has $N[d] = 0$. Then $L[d] \subset L'[d]$ in $\text{Perv}(X)$.
Since \((L'[d])/(L[d])\) is a quotient of \(L'[d]\) and \(L'/L = 0\), by 1 again, one gets \((L'[d])/L[d]) = 0\) in \(\text{Perv}(X)\). Therefore, \(L[d] = L'[d]\). The claim is proved.

Similarly, the essential image is closed under taking quotients. By [Sta23, Tag 02MP], the essential image is a Serre subcategory of \(\text{Perv}(X)\).

\[
\square
\]

4 Cotori

We review the contents of [GL96, Sec. 3.2]. For a compact group \(G\), let \(C(G)(\Lambda)\) be the group of characters, i.e., continuous morphisms \(G \to \Lambda^*\). Let \(C(G)_f(\Lambda)\) (resp. \(C(G)_c(\Lambda)\)) be the subgroup of characters of finite order prime to \(\ell\) (resp. that are pro-\(\ell\)). We shall review the \(\Lambda\)-scheme whose set of \(\Lambda\)-points is naturally identified with \(C(G)_c(\Lambda)\). Fix an integer \(n \geq 1\).

**Lemma 4.0.1.** Let \(M\) be a finitely generated free module of rank \(r\) over a commutative ring \(A\). Let \(B = A[[T_1, \ldots, T_n]]\). Then the canonical morphism \(M \otimes A B \to M[[T_1, \ldots, T_n]]\) of \(B\)-modules is an isomorphism.

**Proof.** We claim that the map is surjective. Fix an \(A\)-basis \(\{e_1, \ldots, e_r\}\) of \(M\). Then for every \(f = \sum_{\alpha \in \mathbb{N}^n} x_{\alpha} T^\alpha \in M[[T_1, \ldots, T_n]]\), one may write \(x_{\alpha} = \sum_{i=1}^r a_{\alpha,i} e_i\) with \(a_{\alpha,i} \in A\). Define \(g_i \in B\) by \(g_i = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha,i} T^\alpha\). Then

\[
\sum_{i=1}^r e_i \otimes g_i \in M \otimes A B
\]

is mapped to \(f\). The claim is proved.

The \(B\)-module \(M \otimes A B\) is free of rank \(r\). Moreover, \(M[[T_1, \ldots, T_n]]\) is also a free \(B\)-module of same rank. By [Vas09, Prop. 1.2, p.506] and the claim, the map is an isomorphism.

**Remark 4.0.2.** Let \(\mathcal{R} = \{O_E : E/\mathbb{Q}\ell\) is a finite subextension of \(\Lambda\}\), which is a directed set under inclusion. We explain the isomorphism in the proof of [GL96, Prop. 3.2.2]. In fact, \(\Lambda = \lim_{\to \mathcal{R} \in \mathcal{R}} R[\ell^{-1}]\) and direct limits commute with tensor product, so

\[
\Lambda \otimes \mathbb{Z}_\ell \left[\mathbb{Z}[T_1, \ldots, T_n]\right] = \lim_{\mathcal{R} \in \mathcal{R}} (R[\ell^{-1}] \otimes \mathbb{Z}_\ell \mathbb{Z}[T_1, \ldots, T_n])
\]

\[
= \lim_{\mathcal{R} \in \mathcal{R}} (R \otimes \mathbb{Z}_\ell \mathbb{Z}[T_1, \ldots, T_n]) \left[\mathbb{Z}[T_1, \ldots, T_n]\right][\ell^{-1}] \xrightarrow{(a)} \lim_{\mathcal{R} \in \mathcal{R}} (R[[T_1, \ldots, T_n]][\ell^{-1}]),
\]

where (a) uses Lemma 4.0.1.

For an integer \(m \geq 1\), let \(\mu_{\ell^m}\) be the set of \(\ell^m\)-roots of unity in \(\Lambda\). Set \(\mu_{\ell^\infty} = \cup_{m \geq 1} \mu_{\ell^m}\).
Lemma 4.0.3. Let $E/\mathbb{Q}_\ell$ a finite extension. If $f \in O_E[[X_1, \ldots, X_n]]$ satisfies $f(\zeta_1 - 1, \ldots, \zeta_n - 1) = 0$ for all $(\zeta_1, \ldots, \zeta_n) \in \mu_{\ell^m}$, then $f = 0$.

Proof. Induction on $n$. When $n = 1$, for every integer $m \geq 1$, by Weierstrass’s division lemma (see, e.g., [Ouy, Lem. 4.0.4]), one may write

$$f(t) = ((1 + t)^{\ell^m} - 1)g(t) + r(t),$$

with $r \in O_E[t]$, deg($r$) $< \ell^m$ and $g \in O_E[[t]]$. For every $\zeta \in \mu_{\ell^m}$, by assumption $f(\zeta - 1) = 0$, so $r(\zeta) = 0$. Then $r$ has at least $\ell^m$ distinct roots, so $r = 0$. Or equivalently, the element $f$ is in the ideal $((1 + t)^{\ell^m} - 1) \subset O_E[[t]]$. By the last paragraph of the proof of [Ouy, Thm. 2.8], the intersection of ideals $\bigcap_{m \geq 1}((1 + t)^{\ell^m} - 1) = 0$, so $f = 0$. The case $n = 1$ is proved for all finite extensions $E$ of $\mathbb{Q}_\ell$.

Now assume that $n \geq 2$ and the statement is proved for $n - 1$ for every finite extension $E/\mathbb{Q}_\ell$. One may write

$$f(X_1, \ldots, X_n) = \sum_{i \geq 0} g_i(X_1, \ldots, X_{n-1})X_i^n,$$

where $g_i \in O_E[[X_1, \ldots, X_{n-1}]]$. For each $(\zeta_1, \ldots, \zeta_{n-1}) \in \mu_{\ell^m}$, set

$$h(X_n) := f(\zeta_1 - 1, \ldots, \zeta_{n-1} - 1, X_n) \in O_E[[X_n]],$$

where $E' = E(\zeta_1, \ldots, \zeta_{n-1})$ is another finite extension of $\mathbb{Q}_\ell$. For every $\zeta_n \in \mu_{\ell^m}$, one has $h(\zeta_n - 1) = 0$. By the proved case of $n = 1$ for $E'$, one has $h = 0$. Therefore, for every integer $i \geq 0$, one has $g_i(\zeta_1 - 1, \ldots, \zeta_{n-1} - 1) = 0$ for all $(\zeta_1, \ldots, \zeta_{n-1}) \in \mu_{\ell^m}$. By the inductive hypothesis (the case of $n - 1$ for $E$), one has $g_i = 0$ for every integer $i \geq 0$, so $f = 0$. The induction is completed. \(\square\)

There is a unique absolute value on $\Lambda$ extending the discrete absolute value $| \cdot |_{\ell}$ on $\mathbb{Q}_\ell$. It induces a topology on $\Lambda$ that is totally disconnected. A subset $A \subset \Lambda$ is closed if and only if for every finite subextension $E/\mathbb{Q}_\ell$ of $\Lambda$, the subset $A \cap E$ is closed in the discrete valuation field $E$.

Lemma 4.0.4.

1. Let $C$ be a compact subset of $\Lambda$. Then there is a finite subextension $E$ of $\Lambda/\mathbb{Q}_\ell$ containing $C$.

2. Let $G \leq \Lambda^*$ be a compact subgroup. Then there is a finite subextension $E$ of $\Lambda/\mathbb{Q}_\ell$ with $G \subset O_E$.

3. In 2, the topological group $G$ is isomorphic to the direct product of a finite group of order prime to $\ell$ with a pro-$\ell$ group.

Proof. 1. Otherwise, there is a sequence of elements $x_1, x_2, \ldots$ in $C$ with $[\mathbb{Q}_\ell(x_{n+1}) : \mathbb{Q}_\ell] > [\mathbb{Q}_\ell(x_n) : \mathbb{Q}_\ell]$ for every integer $n > 0$. Let $B \subset C$ be the (infinite) set of elements of this sequence. For every subset $S \subset B$, every
finite subextension $F/\mathbb{Q}_\ell$, the set $S \cap F$ is finite, so closed in $F$. Therefore, $S$ is closed in $\Lambda$. In particular, the set $B$ is closed and hence compact in $C$. Every subset of $B$ is closed in $B$, so $B$ is discrete. Thus, $B$ is finite, a contradiction.

2. By 1, there is a finite subextension $E$ of $\Lambda/\mathbb{Q}_\ell$ containing $G$. By [Ser92, Thm. 1.2, p.122], one has $G \subset O_E$.

3. By 2 and [Ser92, Cor., p.155], $G$ is an $\ell$-adic Lie group. From Lazard’s theorem (see, e.g., [GSK09, p.71]), there is a pro-$\ell$ open subgroup $U \leq G$. By [RV98, Thm. 1.23], there is an $\ell$-Sylow subgroup $H \leq G$ containing $U$. Since $G$ is compact, $[G : U]$ is finite. Thus, the group $G/H$ is finite of order prime to $\ell$. By [RV98, Cor. 1.24 (iii)], $G$ is isomorphic to $G/H \times H$.

By Lemma 4.0.4 3, for every compact group $G$, one has

$$C(G)(\Lambda) = C(G)_f(\Lambda) \times C(G)_{\ell}(\Lambda), \quad C(G)_f(\Lambda) \cap C(G)_{\ell}(\Lambda) = \{1\}.$$  

Fix $G = \mathbb{Z}_n^\ell$. For each $R \in \mathcal{R}$, the completed group ring $R[[G]]$ is a Noetherian, regular, complete, local domain of Krull dimension $1+n$, and there is a canonical injective morphism $G \to R[[G]]^*$ of groups. A $\mathbb{Z}_\ell$-basis

$$\{\gamma_1, \ldots, \gamma_n\}$$

of $G$ defines an isomorphism of topological rings

$$R[[G]] \to R[[t_1, \ldots, t_n]], \quad \gamma_i \mapsto 1 + t_i.$$  

and an injective morphism of group

$$\Gamma : C(G)_\ell(\Lambda) \to (\Lambda^*)^n, \quad \chi \mapsto (\chi(\gamma_1), \ldots, \chi(\gamma_n)).$$

Let $C(G)_{\ell,\text{tor}}$ be the torsion subgroup of $C(G)_\ell(\Lambda)$.

**Lemma 4.0.5.** Under the map (16), $C(G)_{\ell,\text{tor}}$ is identified with $\mu_{\ell,\infty}^n$.

**Proof.** If $\chi \in C(G)_{\ell,\text{tor}}$, then there is an integer $m \geq 1$ with $\chi^{\ell^m} = 1$. So $\chi$ takes value in $\mu_{\ell^m}$ and $\Gamma(\chi) \in \mu_{\ell^m}^n$. Therefore, $\Gamma(C(G)_{\ell,\text{tor}}) \subset \mu_{\ell,\infty}^n$.

For every $(a_1, \ldots, a_n) \in \mu_{\ell,\infty}^n$, there is an integer $m \geq 1$ with $a_i^{\ell^m} = 1$ for every $1 \leq i \leq n$. For every $g \in G$, there exists a unique $n$-tuple $(b_1, \ldots, b_n) \in \mathbb{Z}_\ell^n$ with $g = \sum_{i=1}^n b_i \gamma_i$. Define

$$\chi : G \to \Lambda^*, \quad g \mapsto \prod_{i=1}^n a_i^{b_i} \mod \ell^m.$$  

Then $\chi \in C(G)_\ell$ and $\Gamma(\chi) = (a_1, \ldots, a_n)$. Therefore, one has $\Gamma(C(G)_{\ell,\text{tor}}) = \mu_{\ell,\infty}^n$. \qed
Define a $\Lambda$-algebra $S = S(G) = \Lambda \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[G]]$. Let $C_\ell$ be the affine scheme Spec$(S)$. It is called a “cotorus”. In general, $C_\ell$ is not locally of finite type over $\Lambda$. By [GL96, Prop. A.2.2.3 (ii)], the scheme $C_\ell$ is integral and regular, $C_\ell(\Lambda)$ coincides with the set of closed points of $C_\ell$, and $C_\ell(\Lambda)$ is dense in $C_\ell$.

Every character $\chi : G \to \Lambda^*$ defines a continuous morphism of $\mathbb{Z}_l$-algebras

$$ev_\chi : \mathbb{Z}_l[[G]] \to \Lambda, \quad g \mapsto \chi(g), \quad \forall g \in G.$$  \hspace{1cm} (17)

It extends to a surjective morphism of $\Lambda$-algebras:

$$S \to \Lambda.$$ \hspace{1cm} (18)

Let $m_\chi \subset S$ be the kernel of (18). Then $m_\chi$ is a maximal ideal of $S$ with residue field $\Lambda$. Let $\Psi(\chi)$ be the corresponding element of $C_\ell(\Lambda)$. Hence a map

$$\Psi : C(G)_{\ell}(\Lambda) \to C_\ell(\Lambda).$$ \hspace{1cm} (19)

**Fact 4.0.6** ([GL96, p.519]). The map (19) is bijective.

**Lemma 4.0.7.** The subset $C_{\ell, \text{tor}} := \Psi(C(G)_{\ell, \text{tor}})$ is Zariski dense in $C_\ell$.

**Proof.** With the chosen basis (14), the morphism (17) induces a morphism of $\mathbb{Z}_l$-algebras $\mathbb{Z}_l[[X_1, \ldots, X_n]] \to \Lambda$ given by the formula

$$f \mapsto f(\gamma_1 - 1, \ldots, \gamma_n - 1)$$ \hspace{1cm} (20)

via the isomorphism (15). The isomorphism (15) identifies $S$ with the subalgebra $A_n$ of $\Lambda[[X_1, \ldots, X_n]]$ defined in (2). Under this identification, the map (18) is also given by (20).

Let $C \subset C_\ell$ be a Zariski closed subset containing $C_{\ell, \text{tor}}$. Then there is an ideal $I \subset S$ with $C = Z(I) \subset \text{Spec}(S)$. Fix $f \in I$. We show $f = 0$.

One can write $f = \sum_{i=1}^n a_i \otimes f_i$, $a_i \in \Lambda$, $f_i \in \mathbb{Z}_l[[G]]$. There is an integer $N > 0$ and a finite subextension $E/\mathbb{Q}_l$ such that $\ell^N a_i \in O_E$ for every $1 \leq i \leq n$. By [Bou99, Ch. III, §3, n. 4, Cor. 3], the injection $O_E \to \Lambda$ induces an injection $O_E \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[G]] \to S$ of rings. By Lemma 4.0.1, its image is $O_E[[G]]$. Since $\ell^N \in \Lambda^* \subset S^*$, one may replace $f$ by $\ell^N f$ and then $f \in O_E[[G]]$. One may view $f$ as an element of $O_E[[X_1, \ldots, X_n]]$ via the isomorphism (15). The regular function $f : C_\ell(\Lambda) \to \Lambda$ vanishes on $C(\Lambda)$. From (20) and Lemma 4.0.5, one has $f(\zeta_1 - 1, \ldots, \zeta_n - 1) = 0$ for all $(\zeta_1, \ldots, \zeta_n) \in \mu^\infty$. By Lemma 4.0.3, one gets $f = 0$.

Therefore, the ideal $I = 0$ and whence $C = C_\ell$. \hfill $\square$

### 5 Krämer-Weissauer’s vanishing theorem

Let $k$ be a field of characteristic 0. Let $A/k$ be an abelian variety. Choose an algebraic closure $k^a$ of $k$. For an abelian variety $A/k$ set $C(A)(\Lambda) = C(\pi_1^0(A_{k^a}, 0))(\Lambda)$, $C(A)_1(\Lambda) = C(\pi_1^0(A_{k^a}, 0))_1(\Lambda)$. Let $C(A)_1$ be the cotorus assigned to the Tate module $T_1A$ (which is isomorphic to $\mathbb{Z}_l^{\dim A}$). For every $\chi \in C(A)(\Lambda)$, let $L_\chi$ be the corresponding rank 1 lisse $A$-sheaf on $A_{k^a}$. 

23
**Definition 5.0.1.** For $K \in \text{Perv}(A)$, the set

$$S(K) := \{ \chi \in \mathcal{C}(A)(\Lambda) : H^i(A_{k^a}, K_{k^a} \otimes L_\chi) \neq 0 \text{ for some integer } i \neq 0 \}$$

is called the spectrum of $K$.

**Fact 5.0.2** ([KW15b, Thm. 1.1], [Wei16, Vanishing Theorem, p.561; Thm. 2]). For every $K \in \text{Perv}(A)$, every character $\chi_f : \pi^{\text{ét}}_1(A_{k^a}) \to \Lambda^*$ of finite order prime to $\ell$, the set $\{ \chi \in \mathcal{C}(A)(\Lambda) : \chi_f \chi \in S(K) \}$ is the set of $\Lambda$-points of a strict Zariski closed subset of the scheme $\mathcal{C}(A)_f$.

Let $m : A \times_k A \to A$ be the group law on $A$. Let $p_i : A \times_k A \to A$ be the projection to $i$-th factor ($i = 1, 2$). The bifunctor

$$* : D^b_c(A) \times D^b_c(A) \to D^b_c(A), \quad - * + := Rm_*(p_i^* - \otimes L^fp_2^+)$$

is called the convolution on $A$.

**Example 5.0.3.** For every closed reduced subvariety $i : X \to A$, let $\delta_X := i_\ast X \subset D^b_c(A)$. Then for every closed point $x \in X$, one has $\delta_x * \delta_X = \delta_{x+X}$.

By [Wei11] and [JKLM23, Sec. 3.1], the pair $(D^b_c(A), *)$ is a rigid, symmetric, monoidal category, with unit the skyscraper sheaf $\delta_0$ of rank 1 supported at the origin. For every $K \in D^b_c(A)$, its adjoint dual is $K^\vee := [-1]^\ast D_A K$. Moreover, for every $K \in \text{Perv}(A)$, the Euler characteristic

$$\chi(A, K) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_\Lambda H^i(A, K) \geq 0. \quad (21)$$

Let $N(A) \subset \text{Perv}(A)$ be the full subcategory of objects $K$ with $\chi(A, K) = 0$. From (21) and the additivity of the function $\chi(A, \cdot) : \text{Ob}(\text{Perv}(A)) \to \mathbb{N}$, the subcategory $N(A)$ is Serre in $\text{Perv}(A)$ ([KW15a, p.725]). Let $P(A) := \text{Perv}(A)/N(A)$ be the quotient abelian category. Fix $\chi \in \mathcal{C}(A)(\Lambda)$, and set

$$\mathcal{E}^\chi(A_{k^a}) = \{ K \in \text{Perv}(A_{k^a}) : H^i(A_{k^a}, K \otimes L_\chi) = 0, \ \forall i \in \mathbb{Z} \setminus \{0\} \}.$$

Then $\mathcal{E}^\chi(A_{k^a})$ is closed under extensions in $\text{Perv}(A_{k^a})$. Let $P^\chi(A) \subset \text{Perv}(A)$ be the full subcategory of objects $K$ with $Q \in \mathcal{E}^\chi(A_{k^a})$ for every simple subquotient $Q$ of $K_{k^a} \in \text{Perv}(A_{k^a})$.

By [BBDG82, Thm. 4.3.1 (i)], every $K \in \text{Perv}(A)$ is Noetherian and Artinian. For every character $\chi_f : \pi^{\text{ét}}_1(A, 0) \to \Lambda^*$ of finite order prime to $\ell$, by Fact 5.0.2 and Lemma 2.3.5 1, the set $\{ \chi \in \mathcal{C}(A)(\Lambda) : K \in P^{\chi_f}(A) \}$ is the set of $\Lambda$-points of a strict Zariski closed subset of $\mathcal{C}(A)_f$.

**Lemma 5.0.4.** Let $A$ be a Noetherian and Artinian abelian category. Let $\mathcal{E}$ be a class of objects of $A$ closed under isomorphisms. Let $S \subset A$ be the full subcategory of objects whose every nonzero simple subquotient is in $\mathcal{E}$. Then

1. $S$ is a Serre subcategory of $A$.  

24
2. If $E$ is closed under extensions, then $S \subset E$.

Proof.

1. Let $X$ be an object of $S$ with a subquotient $Y$. Every simple subquotient of $Y$ is that of $X$, hence in $E$. Thus, $Y \in S$. Therefore, $S$ is closed under subquotients.

Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $A$ with $L, N \in S$. Let $Q$ be a nonzero simple subquotient of $M$. We claim that $Q \in E$. First, assume that $Q$ is a quotient of $M$. The natural morphism $L \to Q$ is either an epimorphism or zero, in which case $Q$ is a simple quotient of $L$ or $N$ respectively. Thus, $Q \in E$.

Now assume that $Q$ is general. There is a subobject $M_0 \subset M$ and an epimorphism $M_0 \to Q$. Then $0 \to f^{-1}(M_0) \to M_0 \to g(M_0) \to 0$ is a short exact sequence in $A$ with $f^{-1}(M_0)$ (resp. $g(M_0)$) a subobject of $L$ (resp. $N$). From the first paragraph, $f^{-1}(M_0)$ and $g(M_0)$ are in $E$. From the second paragraph, $Q \in E$. The claim is proved.

From the claim, one has $M \in S$ and $S$ is closed under extensions. The result follows from [Sta23, Tag 02MP].

2. By [Sta23, Tag 0FCJ], every $X \in S$ admits a filtration in $A$

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

by subobjects such that each $X_i/X_{i-1}$ is a simple subquotient of $X$. Then $X_i/X_{i-1} \in E$. As $E$ is closed under extensions, one gets $X \in E$.

\[ \square \]

By Lemma 5.0.4.1, $P^\chi(A)$ is a Serre subcategory of $\text{Perv}(A)$. From Lemma 5.0.4.2, for every $K \in P^\chi(A)$ and every integer $i \neq 0$, one has

$$H^i(A_{k^a}, K_{k^a} \otimes^L L_\chi) = 0.$$  \hfill (22)

From the proof of [LS20, Lem. 3.4 (3)], the functor

$$\omega_\chi : P^\chi(A) \to \text{Vec}_A, \quad K \mapsto H^0(A_{k^a}, K_{k^a} \otimes^L L_\chi)$$  \hfill (23)

is exact. Let $N^\chi(A)$ be the full subcategory of $P^\chi(A)$ of objects in $N(A)$. For every $K \in N^\chi(A)$, by [KW15b, Cor. 4.2], one has $\chi(A, K \otimes^L L_\chi) = 0$. From (22), one has $H^0(A_{k^a}, K_{k^a} \otimes^L L_\chi) = 0$. By [Sta23, Tag 02MS], the functor $\omega_\chi$ factors uniquely through an exact functor (still denoted by $\omega_\chi$)

$$P^\chi(A)/N^\chi(A) \to \text{Vec}_A.$$  \hfill (24)
Fact 5.0.5 ([KW15b, proof of Thm. 13.2], [JKLM23, Prop. 3.1]). The convolution
on $D^0_\omega(A)$ induces a bifunctor
\[ \tilde{P}(A) \times \tilde{P}(A) \to \tilde{P}(A), \quad (-,+) \mapsto \mathcal{H}^0(-*+). \]

It defines the structure of a neutral Tannakian category (in the sense of [DM22, Def. 2.19]) over $\Lambda$ on $\tilde{P}(A)$. The subcategory $P^\chi(A)/N^\chi(A) \subset \tilde{P}(A)$ is a Tannakian subcategory (in the sense of [Mil07, 1.7]) on which (24) is a fiber
functor.

Let $(C, \otimes)$ a neutral Tannakian category over an algebraically closed field of characteristic 0 with a fiber functor $\omega : C \to \text{Vec}$. Let $\text{Aut}^\otimes(C, \omega)$ be the corresponding affine group scheme. By [Del90, Sec. 9.2, p.187], up to isomorphism of group schemes, $\text{Aut}^\otimes(C, \omega)$ is independent of the choice of $\omega$.

For an object $K$ of $C$, let $K^\ast$ be its adjoint dual. Let $\iota : (K) \to C$ be the full subcategory of subquotients of $\{(K \oplus K^\ast)^\otimes n\}_{n \geq 1}$. Then $((K), \otimes)$ is neutral Tannakian subcategory with a fiber functor $\omega_{\iota}$, with $G_{\omega_{\iota}}((K)) = \text{im}([\text{Aut}^\otimes(C, \omega) \to \text{GL}(\omega(K))])$. It is called the Tannakian monodromy group of $K$ at $\omega$ and denoted by $G_{\omega}(K)$. It is reductive if and only if $K$ is semisimple in $C$ ([Sim92, p.69]).

Let $\text{Rep}_A(\Gamma_k)$ be the category of continuous finite dimensional $A$-representations of $\Gamma_k$. Then with tensor product and the forgetful fiber functor $\omega : \text{Rep}_A(\Gamma_k) \to \text{Vec}_A$, it is a neutral Tannakian category over $A$. For an object $\rho : \Gamma_k \to \text{GL}(V)$ in it, the corresponding Tannakian monodromy group is the Zariski closure of $\rho(\Gamma_k)$ inside $\text{GL}(V)$.

Example 5.0.6. [KW15a, Example 7.1] Fix a closed point $x \in A$. Then $\delta_x \in \text{Perv}(A)$. The spectrum $\mathcal{S}(\delta_x)$ is empty and for every $\chi \in \mathcal{C}(A)(\Lambda)$, one has $\delta_x \in P^\chi(A)$. If $x$ is a torsion point of order $n$, then its Tannakian monodromy group $G(\delta_x)$ is isomorphic to $\mathbb{Z}/n$. If $x$ is not a torsion point, then $G(\delta_x)$ is isomorphic to $G_m/\Lambda$.

Let $\psi : \pi_1^A(A) \to \Lambda^*$ be a continuous character and set $\chi = \psi\mid_{\pi_1^A(A_k^{\ast})}$. Let $L_\psi$ denote the corresponding rank 1 lisse $\Lambda$-sheaf on $A$. The functor
\[ \omega_\psi : \text{Perv}(A) \to \text{Rep}_A(\Gamma_k), \quad K \mapsto H^0(A_k^{\ast}, (K \otimes^L L_\psi)_{k^{\ast}}) \]
restricts to an exact functor $P^\chi(A) \to \text{Rep}_A(\Gamma_k)$. The composition of the restricted functor with $\omega : \text{Rep}_A(\Gamma_k) \to \text{Vec}_A$ is (23). The composed functor induces a morphism of affine groups schemes
\[ \omega^*_\psi : \text{Aut}^\otimes(\text{Rep}_A(\Gamma_k), \omega) \to \text{Aut}^\ast(P^\chi(A)/N^\chi(A), \omega_\chi). \quad (25) \]

For $K \in \text{Perv}(A)$, let $G_{\text{mon}}(K, \psi)$ be the Tannakian monodromy group of $\omega_\psi(K)$ in $\text{Rep}_A(\Gamma_k)$. The restricted functor $\omega_\psi : (K) \to (\omega_\psi(K))$ induces a closed immersion of linear algebraic groups $\omega^*_\psi : G_{\text{mon}}(K, \psi) \to G_{\omega_\psi}(K)$, which is taking the image of (25) in $\text{GL}(\omega_\chi(K))$.
6 Main results

Consider Setting 1.0.1. For every \( \chi \in \mathcal{C}(A)(\Lambda) \), denote the pullback of \( \chi \) along \((\pi|_{A_\eta})_* : \pi^\et_1(A_\eta) \to \pi^\et_1(A) \) by \( \psi : \pi^\et_1(A_\eta) \to \Lambda^* \). Then the restriction \( \psi|_{\pi^\et_1(A_\eta)} \) is identified with \( \chi \) via the isomorphism \((\pi|_{A_\eta})_* : \pi^\et_1(A_\eta) \to \pi^\et_1(A) \). Let \( K \in \text{Perv}(A \times X/X) \) be an object which is semisimple in \( D^b_c(A \times X) \).

**Theorem 6.0.1.** For every \( \chi \in \mathcal{C}(A)(\Lambda) \setminus \mathcal{S}(K_\eta) \), the monodromy group \( G_{\text{mon}}(K_\eta, \psi) \) is reductive.

**Proof.** If \( X \) is replaced by a nonempty open subset, then by Lemma 3.1.5, the semisimplicity of \( K \) in \( D^b_c(A \times X) \) is preserved. Moreover, the representation \( \omega_\psi(K_\eta) \) and hence the group \( G_{\text{mon}}(K_\eta, \psi) \) remain unchanged. By [Sta23, Tag 056V], one may assume that \( X \) is smooth. As \( K \) is semisimple in \( D^b_c(A \times X) \), from Lemma 3.1.8, so is \( K \otimes^L \pi^*L_\chi \). By Fact 3.1.6 1, the object \( R\rho_* (K \otimes^L \pi^*L_\chi) \) is semisimple in \( D^b_c(X) \).

By proper base change theorem (see, e.g., [Sta23, Tag 095T]), for every integer \( j \), the geometric generic stalk \([\mathcal{H}^j R\rho_* (K \otimes^L \pi^*L_\chi)]_q \) is \( H^j(A_\eta, K_\eta \otimes^L L_\chi) \). Since \( \chi \notin \mathcal{S}(K_\eta) \), for every integer \( q \neq 0 \), one has

\[
H^q(A_\eta, K_\eta \otimes^L L_\chi) = 0.
\]

By Fact 3.1.2, there is a nonempty open subset \( U_0 \) (resp. \( U_q \) for every integer \( q \neq 0 \)) of \( X \) such that \([\mathcal{H}^0 R\rho_* (K \otimes^L \pi^*L_\chi)]|_{U_0} \) is a lisse \( \Lambda \)-sheaf (resp. \([\mathcal{H}^q R\rho_* (K \otimes^L \pi^*L_\chi)]|_{U_q} \) is finite and \( X \) is irreducible, so \( U := U_0 \cap \bigcap_{q \in J} U_q \) is a nonempty open subset of \( X \). Shrinking \( X \) to \( U \), one may assume further that \( \mathcal{H}^0 R\rho_* (K \otimes^L \pi^*L_\chi) = 0 \) for every integer \( q \neq 0 \) and \( \mathcal{H}^0 R\rho_* (K \otimes^L \pi^*L_\chi) \) is a lisse \( \Lambda \)-sheaf on \( X \).

Thus, the semisimple object \( R\rho_* (K \otimes^L \pi^*L_\chi)|_X \) of \( D^b_c(X) \) lies in \( \text{Perv}(X) \), so it is semisimple in \( \text{Perv}(X) \). By [Ach21, Prop. 3.4.1], the object \( R\rho_* (K \otimes^L \pi^*L_\chi) \) of \( \text{Loc}(X) \) is semisimple. Therefore, the corresponding representation

\[
\pi^\et_1(X, \bar{\eta}) \to \text{GL}(H^0(A_\eta, K_\eta \otimes^L L_\chi))
\]

is semisimple. Because \( X \) is smooth, the natural morphism \( \eta_* : \Gamma_{k(\eta)} \to \pi^\et_1(X, \bar{\eta}) \) is surjective. Then the composition \( \Gamma_{k(\eta)} \to \text{GL}(H^0(A_\eta, K_\eta \otimes^L L_\chi)) \), i.e., the representation \( \omega_\psi(K_\eta) \), is semisimple. By [Mil17, Cor. 19.18], the algebraic group \( G_{\text{mon}}(K_\eta, \psi) \) is reductive. \( \Box \)

**Example 6.0.2.** Let \( X/k \) be a smooth projective integral curve of genus 1. Then \( \pi^\et_0(X, \bar{\eta}) = \mathbb{Z}^2 \). There exists a character \( \sigma : \pi^\et_1(X, \bar{\eta}) \to \Lambda^* \) of infinite order. Let \( A = \text{Spec}(k) \) and \( K \in \text{Loc}(X) = \text{Perv}_{U^A_{\mathbb{L}}}(A \times X/X) \) be the lisse \( \Lambda \)-sheaf of rank 1 on \( X \) corresponding to \( \sigma \). Then \( \mathcal{C}(A)(\Lambda) = \{1\} \) and \( G_{\text{mon}}(K_\eta, 1) = \mathbb{G}_m/\Lambda \) is an algebraic torus. Here the algebraic group \( G_{\text{mon}}(K_\eta, 1) \) is not semisimple.
Remark 6.0.3. In view of Example 3.1.4, the semisimplicity of $H^0 R^\rho_s(K \otimes^{L} \pi^* L_\chi)$ in $D^b_c(X)$ is not clear \textit{a priori}. That is why we exclude characters in the spectrum $S(K_\eta)$ in Theorem 6.0.1.

Remark 6.0.4. Let $i : Y \to A \times X$ be a closed subvariety, such that the induced morphism $f : Y \to X$ is smooth with connected fibers of dimension $d$:

$$
\begin{array}{ccc}
Y & \longrightarrow & A \times X \\
\downarrow f & & \downarrow \pi \\
X & \leftarrow & A.
\end{array}
$$

By Example 3.3.3, $K := i_* C_Y[d]$ belongs to $\text{Perv}^{\text{ULA}}(A \times X/X)$. By Fact 3.1.6 1, it is semisimple in $D^b_c(A \times X)$. Assume that $X/\mathbb{C}$ is smooth. Then for every $\chi \in \mathcal{C}(A)(\Lambda) \setminus S(K_\eta)$, the algebraic group $G_{\text{mon}}(K_\eta, \psi)$ coincides with the Zariski closure of the image of the monodromy representation of the lisse $\Lambda$-sheaf $R^d f_* \pi^* L_\chi$ on $X$, which is studied in [KM23, Sec. 1.4] (but with coefficient $\mathbb{C}$ instead of $\Lambda$).

Theorem 1.0.3 follows from Theorem 6.0.5 and Fact 5.0.2, because (26) is in fact a finite union.

**Theorem 6.0.5.** Assume $K \in \text{Perv}^{\text{ULA}}(A \times X/X)$. Then there exists a subobject $K^0 \to K$ in $\text{Perv}^{\text{ULA}}(A \times X/X)$ such that for every $\chi \in \mathcal{C}(A)(\Lambda)$ satisfying the three conditions

- $\chi$ is not in $
\bigcup_{j \in \mathbb{Z}} S(\mathcal{H}^j(R\pi_* K)),
$ (26)
- $K_\eta \in P^X(A_\eta)$ and $\mathcal{P}\mathcal{H}^0(R\pi_* K) \in P^X(A),$

one has $\omega_\psi(K_\eta^0) = \omega_\psi(K_\eta)^{\Gamma_k(\eta)}$.

**Proof.** By Fact 3.1.2, there is a nonempty open subset $V \subset X$ such that $(\mathcal{H}^0 Rf_* K)|_V$ is a lisse sheaf. From [Har77, Lem. 10.5, p.271], one may assume that $V$ is smooth. By [Sta23, Tag 0BQM], the canonical morphism $\Gamma_k(\eta) \to \pi^0_1(V, \bar{\eta})$ is surjective. Thus, from Fact 3.1.6 2, the natural map

$$
H^0(A \times X, K \otimes^{L} \pi^* L_\chi) \to \omega_\psi(K_\eta)^{\Gamma_k(\eta)}
$$

is surjective.

By Fact 3.1.1, one has

$$
H^0(A \times X, K \otimes^{L} \pi^* L_\chi) = H^0(A, (R\pi_* K) \otimes^{L} L_\chi).
$$

(28)

For any integers $i \neq 0$ and $j$, as $\chi \notin S(\mathcal{P}\mathcal{H}^j(R\pi_* K))$, one has

$$
H^i(A, \mathcal{P}\mathcal{H}^j(R\pi_* K) \otimes^{L} L_\chi) = 0.
$$

(28)
By Lemma 3.1.8, the spectral sequence in [Max19, Rk. 8.1.14 (6)] becomes
\[ E_2^{i,j} = H^i(A, \mathcal{P}H^j(R\pi_*K) \otimes^L L_\chi) \Rightarrow H^{i+j}(A, (R\pi_*K) \otimes^L L_\chi). \]

It degenerates at page \( E_2 \). Hence
\[ H^0(A, (R\pi_*K) \otimes^L L_\chi) = H^0(A, (\mathcal{P}H^0 R\pi_*K) \otimes^L L_\chi). \tag{29} \]

Set \( K^1 := \pi^* \mathcal{P}H^0(R\pi_*K) \in D^b_c(A \times X) \). By Fact 3.2.2.1, one has \( \mathcal{P}H^0(R\pi_*K) \in D^{ULA}(A/k) \). From Fact 3.2.2.3, one gets \( K^1 \in D^{ULA}(A \times X/X) \). For every \( x \in X(k) \), the restriction \( \pi|_{A_x} : A_x \to A \) is an isomorphism of \( k \)-abelian varieties, so the functor
\[(\pi|_{A_x})^* : \text{Perv}(A) \to \text{Perv}(A_x)\] is an isomorphism of abelian categories. It sends \( \mathcal{P}H^0(R\pi_*K) \) to \( K^1_x \), so \( K^1_x \in \text{Perv}(A_x) \) and hence \( K^1 \in \text{Perv}^{ULA}(A \times X/X) \). Similarly,
\[ K^1_\eta = (\pi|_{A_\eta})^* \mathcal{P}H^0(R\pi_*K) \]
is the \( k(\eta)/k \)-scalar extension of \( \mathcal{P}H^0(R\pi_*K) \), so
\[ \omega_\chi(K^1_\eta) = H^0(A, \mathcal{P}H^0(R\pi_*K) \otimes^L L_\chi). \tag{31} \]

Every fiber of the morphism \( \pi \) is of dimension \( \text{dim} \ X \), so by [BBDG82, 4.2.4], the functor
\[ R\pi_\varnothing[-\text{dim} \ X] : D^b_c(A \times X) \to D^b_c(A) \]
is left \( t \)-exact with respect to the absolute perverse \( t \)-structures. From Lemma 3.3.5, one has \( K[\text{dim} \ X] \in \text{Perv}(A \times X) \) and so \( R\pi_\varnothing K \in \mathcal{P}D^{\leq 0}(A) \). Then the perverse truncation \( \mathcal{P}\mathcal{T}^{\leq 0}(R\pi_*K) = \mathcal{P}H^0(R\pi_*K) \). Via the adjunction formula (see, e.g., [KW01, p.107]), the natural morphism
\[ \mathcal{P}\mathcal{T}^{\leq 0}(R\pi_*K) \to R\pi_*K \]
in \( D^b_c(A) \) (from the definition of \( t \)-structure) induces a morphism \( h : K^1 \to K \) in \( D^b_c(A \times X) \). Then \( h \) is a morphism in \( \text{Perv}^{ULA}(A \times X/X) \). Let \( K^0 \) be the image of \( h \) in the abelian category \( \text{Perv}^{ULA}(A \times X/X) \). By Fact 3.3.1.1, the functor \( \text{Perv}(A \times X/X) \to \text{Perv}(A_\eta) \) is exact. Then \( K^0_\eta \) is the image of \( h_\eta : K^1_\eta \to K^0_\eta \) in \( \text{Perv}(A_\eta) \).

By assumption, both \( K^0_\eta, K^1_\eta \) are in \( P^\chi(A_\eta) \). Because \( P^\chi(A_\eta) \) is a Serre subcategory of \( \text{Perv}(A_\eta) \), the image of \( h_\eta \) in \( P^\chi(A_\eta) \) is still \( K^0_\eta \). As the functor (23) is exact, the image of \( \omega_\chi(h_\eta) : \omega_\chi(K^1_\eta) \to \omega_\chi(K^0_\eta) \) is \( \omega_\chi(K^1_\eta) \). Combining (27), (28), (29) with (31), one gets \( \omega_\chi(K^0_\eta) = \omega_\psi(K^1_\eta)^{\varnothing^{\eta(\varnothing)}}. \)

If \( K \in \text{Perv}(A \times X/X) \), then by [JKLM23, Thm. 4.3], for every character \( \chi \in C(A)(A) \), the geometric generic Tannakian group \( G_{\omega_\chi}(K^0_\eta) \) is a normal closed subgroup of the generic Tannakian group \( G_{\omega_\chi}(K^0_\eta) \). Theorem 6.0.6 shows that for uncountably many \( \chi \), the corresponding monodromy group is also a normal closed subgroup of the generic Tannakian group.
**Theorem 6.0.6.** Setting as in Theorem 6.0.5. Assume further that \( \dim A > 0 \).

For every character \( \chi_f : \pi_{1 \ell}^{\text{ét}}(A) \to \Lambda^* \) of finite order prime to \( \ell \), there is an uncountable subset \( E \subset C(A)_f(\Lambda) \) with the following property: For every \( \chi \in E \), set \( \chi = \chi_f \chi_t \) and denote the pullback of \( \chi \) by \( \psi : \pi_{1 \ell}^{\text{ét}}(A) \rightarrow \Lambda^* \). Then \( K_\eta \in P^X(A_\eta) \), the algebraic group \( G_{\omega_\chi}(K_\eta) \) is reductive, and \( G_{\text{mon}}(K_\eta, \psi) \) is a normal closed subgroup of \( G_{\omega_\chi}(K_\eta) \).

**Proof.** Both \( G_{\text{mon}}(K_\eta, \psi) \) and \( G_{\omega_\chi}(K_\eta) \) depend only on the generic fiber of \( \rho \).

Thus, shrinking \( X \) to a nonempty open subset does not change them. Therefore, one may assume that \( X \) is smooth. We claim that \( K_\eta \in \text{Perv}(A_\eta) \) is a semisimple object.

For every subobject \( M \subset K_\eta \) in \( \text{Perv}(A_\eta) \), by [HS23, Thm. 1.10 (ii)] and the smoothness of \( X \), there is a subobject \( K' \subset K \) in \( \text{Perv}^{\text{ULA}}(A \times X/X) \) with \( K'_\eta = M \). By Lemma 3.3.5, the morphism \( K'[\dim X] \rightarrow K[\dim X] \) is a monomorphism in \( \text{Perv}(A \times X) \). Because \( K \) is semisimple in \( D^b_c(A \times X) \), its shift \( K[\dim X] \) is semisimple in \( \text{Perv}(A \times X) \). Thus, there is a subobject \( N \subset K[\dim X] \) in \( \text{Perv}(A \times X) \) with

\[
K[\dim X] = (K'[\dim X]) \oplus N.
\]

Then \( K = K' \oplus (N[\dim X]) \) in \( D^b_c(A \times X) \). For an integer \( j \), let \( F^{\text{X}}H^j : D^b_c(A \times X) \rightarrow \text{Perv}(A \times X/X) \) be the \( j \)-th cohomology functor associated with the relative perverse t-structure. If \( j \neq 0 \), then

\[
0 = F^{\text{X}}H^j(K) = F^{\text{X}}H^j(K') \oplus F^{\text{X}}H^j(N[\dim X])
\]

in \( \text{Perv}(A \times X/X) \), so \( F^{\text{X}}H^j(N[\dim X]) = 0 \) and hence

\[
N[\dim X] \in \text{Perv}(A \times X/X).
\]

Consequently, \( K_\eta = M \oplus \langle N_\eta[-\dim X] \rangle \) in \( \text{Perv}(A_\eta) \). By [BBDG82, Thm. 4.3.1 (i)], the abelian category \( \text{Perv}(A_\eta) \) is Noetherian and Artinian. As every subobject of \( K_\eta \) in \( \text{Perv}(A_\eta) \) admits a direct complement, the claim follows from Lemma 2.3.5.

From the claim and Lemma 6.0.8.1, the object \( K_\eta \) of \( \hat{P}(A_\eta) \) is also semisimple. Therefore, there is a reductive group \( G/\Lambda \) and an equivalence of neutral Tannakian categories \( \text{Rep}(G) \rightarrow \langle K_\eta \rangle(\subset \hat{P}(A_\eta)) \). By Lemma 2.3.4, there is a sequence \( \{V_i \}_{i \geq 1} \) of objects in \( \text{Rep}(G) \), such that every object of \( \text{Rep}(G) \) is isomorphic to one \( V_i \). For every integer \( i \geq 1 \), let \( K_i \) be the object of \( \{K_\eta \} \) corresponding to \( V_i \).

We affirm that for every object \( N \in \langle K_\eta \rangle \), there is \( L \in \text{Perv}^{\text{ULA}}(A \times X/X) \) that is semisimple in \( D^b_c(A \times X) \) with \( L \) isomorphic to \( N \) in \( \hat{P}(A_\eta) \).

From [Mil17, Cor. 22.43], the abelian category \( \text{Rep}(G) \) is semisimple, so \( N \) is isomorphic in \( \hat{P}(A_\eta) \). There is an integer \( n \geq 0 \) such that \( N \) is a subquotient of \( (K_\eta \oplus K_\eta^n)\infty \) in \( \hat{P}(A_\eta) \).

We “globalize” the fiberwise convolution functors as follows. Define a bifunctor

\[
* : D^b_c(A \times X) \times D^b_c(A \times X) \rightarrow D^b_c(A \times X),
\]

\[
(-, +) \mapsto R(m \times \text{Id}_X)_*(p_{13}^* - \otimes Lp_{23}^* +),
\]

(32)
where $p_{ij}$ are the projections on $A \times A \times X$. By the proper base change theorem, for every $x \in X(k)$, one has $(- * x +)_x \sim (- x) * (+ x)$ as bifunctors $D^b_c(A \times X) \times D^b_c(A \times X) \to D^b_c(A_x)$. Therefore, one has $(- * x +)_\eta = (- \eta) * (+ \eta)$ as bifunctors $D^b_c(A \times X) \times D^b_c(A \times X) \to D^b_c(A_\eta)$.

The bifunctor (32) restricts to a bifunctor $D^{ULA}(A \times X/X) \times D^{ULA}(A \times X/X) \to D^{ULA}(A \times X/X)$. Indeed, for any $K, L \in D^{ULA}(A \times X/X)$, by [Zhu17, Thm. A.2.5 (4)], one has

$$p_{13}^! K \otimes^L p_{23}^! L \in D^{ULA}(A \times A \times X/X).$$

By Fact 3.2.24, one gets $K \ast X L \in D^{ULA}(A \times X/X)$.

Set $K^\vee := ([| - |_A \times \text{Id}_X]^* \mathbb{D}_{A \times X/X} K)$. By Fact 3.3.1, one has $K^\vee \in \text{Perv}^{ULA}(A \times X/X)$ and $(K^\vee)_\eta = (K_\eta)^\vee$. From last paragraph,

$$(K \oplus K^\vee)^{\ast \times n} \in D^{ULA}(A \times X/X).$$

Set $M := p^{/X}(K \oplus K^\vee)^{\ast \times n} \in \text{Perv}^{ULA}(A \times X/X)$. Then $M_\eta = p^{/X}(\mathbb{K}_\eta \oplus (K_\eta)^\vee)^{\ast \times n}$ in $\text{Perv}(A_\eta)$. By Lemma 6.0.8.3, there is a semisimple subquotient $L'$ of $M_\eta$ in $\text{Perv}(A_\eta)$, whose image in $\mathcal{P}(A_\eta)$ is $N$. By [HS23, Thm. 1.10 (ii)], there is a semisimple subquotient $L$ of $M$ in $\text{Perv}^{ULA}(A \times X/X)$ with $L_0 = L'$. By Lemma 3.3.9.2, $L[\text{dim } X]$ is a semisimple in $\text{Perv}(A \times X)$ and hence in $D_b^c(A \times X)$. The affirmation is proved.

From the affirmation, for every integer $i \geq 1$, there is $K_i \in \text{Perv}^{ULA}(A \times X/X)$ that is semisimple in $D^b_c(A \times X/X)$ with $K_{i,\eta}$ isomorphic to $K_i$ in $\mathcal{P}(A_\eta)$. From Theorem 6.0.5 and Fact 5.0.2, there is a subobject $K_i^0 \subset K_i$ in $\text{Perv}^{ULA}(A \times X/X)$ and a strict Zariski closed subset $B_i$ of the scheme $\mathcal{C}(A)_\eta$ such that for every $\chi \in \text{C}(A)_\eta \setminus B_i(\Lambda)$ and $\chi = \chi_{\ell} \chi_{\ell}$, one has $K_{i,\eta}^0 \in \text{P}^{X}(A_\eta)$ and

$$\omega_{\psi}(K_{i,\eta})^{\Gamma_{\psi}(n)} = \omega_{\psi}(K_{i,\eta}^0).$$

Set $E := (\mathcal{C}(A)_\eta \setminus \cup_{i \geq 1} B_i)(\Lambda)$. From Lemma 2.2.5 and $\text{dim } A > 0$, the set $E$ is uncountable. By Theorem 6.0.1 and [Gro97, Cor. 2.4], the algebraic subgroup $G_{\text{mon}}(K_\eta, \psi)$ of $G_{\omega_\psi}(K_\eta)$ is observable in the sense of [BBHM63, p.134]. Since $K_{i,\eta}^0$ is a subobject of $K_i$ in $(K_\eta)$, by [DE22, Prop. A.12], for every $\chi \in E$, the corresponding group $G_{\text{mon}}(K_\eta, \psi)$ is a normal closed subgroup of $G_{\omega_\psi}(K_\eta)$.

**Remark 6.0.7.** When $A = \text{Spec}(k)$, the bifunctor (32) becomes $\otimes^L : D^b_c(X) \times D^b_c(X) \to D^b_c(X)$. The derived tensor product may not preserve semisimplicity in the category $D^b_c(X)$. That is why we need semisimplicity in $\text{Perv}^{ULA}$ in the last paragraph of the proof of the affirmation.

In fact, consider $k = \mathbb{C}$, $X = A^1$ and $U = X \setminus \{0\}$. Let $j : U \to X$ be the inclusion. Then $\pi^{A^1}_1(U, 1) = \mathbb{Z}$. The unique surjective morphism $\pi^{A^1}_1(U, 1) \to \mathbb{Z}/2$ corresponds to a rank 1 lisse $A$-sheaf $L$ on $U$. Then $L \otimes^L L$ is the rank 1 constant lisse $A$-sheaf $\Lambda_U$, and $L^{an}$ is an $A$-local system on $U^{an} = \mathbb{C}^*$. Let $U_0$ be a punctured ball in $X^{an} = \mathbb{C}$ centered at 0 containing 1. One has $H^0(U_0, L^{an}) = (L^{an})_{\pi_1(U_0, 1)} = 0$, and $H^1(U_0, L^{an})$ coincides with the group cohomology $H^1(\pi_1(U_0, 1), L^{an}_\eta)$, where the $\pi_1(U_0, 1) = \mathbb{Z}$-action on the stalk $L^{an}_\eta$.
is the monodromy. For every crossed homomorphism \( f : Z \to L^{an}_1 \), every integer \( j \), one has \( f(1 + j) = f(1) - f(j) \). Therefore, when \( j \) is even (resp. odd), \( f(j) \) is 0 (resp. \( f(1) \)). In particular, \( f \) is a boundary and hence \( H^1(\pi_1(U_0), L^{an}_1) = 0 \). Thus, \( L^{an}_1 \) is not in the cohomology support loci of \( U_0 \). From [BLSW17, p.299], \( j^{an}_1 L^{an}[1] \) is a simple object of \( \text{Perv}(X^{an}) \). By [BBDG82, p.150], \( M := j_! L[1] \) is a simple object of \( \text{Perv}(X) \).

From [KW01, II, Cor. 7.5 g]], one has

\[
N := M \otimes^L M = j_!(L \otimes^L j^* j_! L)[2] = j_! \Lambda_U[2].
\]

By [HT07, Example 8.1.35 (ii)], one has \( N[-1] \in \text{Perv}(X) \). Let \( i : 0 \to A^1 \) be the inclusion. From the short exact sequence

\[
0 \to j_! \Lambda_U \to \Lambda_X \to i_!(\Lambda_0) \to 0
\]

in \( \text{Cons}(X) \), one gets an exact sequence

\[
\begin{align*}
\mathcal{P}H^0(\Lambda_X) &\to \mathcal{P}H^0(i_!(\Lambda_0)) \\
&\to \mathcal{P}H^1(j_! \Lambda_U) \\
&\to \mathcal{P}H^1(\Lambda_X) \\
&\to \mathcal{P}H^1(i_!(\Lambda_0))
\end{align*}
\]

in \( \text{Perv}(X) \). Since \( i_!(\Lambda_0), \Lambda_X[1] \in \text{Perv}(X) \), it gives a short exact sequence

\[
0 \to i_!(\Lambda_0) \to N[-1] \to \Lambda_X[1] \to 0
\]

in \( \text{Perv}(X) \). This sequence does not split as \( N[-1] \) is supported on \( U \). Therefore, \( N[-1] \) is not a semisimple object of \( \text{Perv}(X) \). It follows that \( M \otimes^L M \) is not semisimple in \( D^b_c(X) \).

**Lemma 6.0.8.** Let \( A \) be an abelian category. Let \( B \subset A \) be a Serre subcategory. Consider the quotient functor \( F : A \to A/B \).

1. Let \( X \in A \). Let \( i : Y \to F(X) \) be a monomorphism in \( A/B \). Then there is a monomorphism \( j : Z \to X \) in \( A \) and an isomorphism \( u : Y \to F(Z) \) in \( A/B \) fitting to a commutative diagram in \( A/B \)

\[
\begin{array}{ccc}
F(Z) & \xrightarrow{F(j)} & F(X) \\
\downarrow u & & \downarrow i \\
Y & \xrightarrow{i} & F(X).
\end{array}
\]

Dually, up to isomorphism every quotient in \( A/B \) lifts to a quotient in \( A \). In particular, if \( X \in A \) is a simple object, then \( F(X) \) is either simple or zero in \( A/B \).

2. Let \( V \in A \) be a Noetherian and Artinian object. If \( F(V) \) is simple in \( A/B \), then there is a simple subquotient \( W \) of \( V \) in \( A \) such that \( F(W) \) is isomorphic to \( F(V) \) in \( A/B \).

3. Assume that \( A \) is Noetherian and Artinian. Let \( X \in A \). If \( Y \) is a simple subquotient of \( F(X) \) in \( A/B \), then there is a simple subquotient \( W \) of \( X \), with \( F(W) \) isomorphic to \( Y \) in \( A/B \).
Proof.

1. By the construction in the proof of [Sta23, Tag 02MS] and the right calculus of fractions in [Sta23, Tag 04VB], there is a diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{g} \\
X & & X
\end{array}
\]

in \(\mathcal{A}\), such that \(F(f)\) is an isomorphism and \(F(g) = i \circ F(f)\) in \(\mathcal{A}/\mathcal{B}\). Therefore, \(F(g)\) is a monomorphism. Since \(F\) is exact, one has \(F(\ker(g)) = \ker(F(g)) = 0\), so \(\ker(g) \in \mathcal{B}\). Let \(q : M \to M/\ker(g)\) be the epimorphism in \(\mathcal{A}\), and let \(j : M/\ker(g) \to X\) be the monomorphism in \(\mathcal{A}\) induced by \(g\). Then \(F(q)\) is an isomorphism in \(\mathcal{A}/\mathcal{B}\). Set \(u : Y \to F(M/\ker(g))\) to be the morphism \(F(q) \circ F(f)^{-1}\) in \(\mathcal{A}/\mathcal{B}\). Then \(u\) is an isomorphism with the stated property.

2. Let \(\mathcal{P}\) be the family of subobjects \(V'\) of \(V\) in \(\mathcal{A}\) with \(V/V' \in \mathcal{B}\). Then \(\mathcal{P}\) is nonempty since \(V \in \mathcal{P}\). As \(V\) is Artinian in \(\mathcal{A}\), there is a minimal object \(U \in \mathcal{P}\). Moreover, the morphism \(F(U) \to F(V)\) is an isomorphism in \(\mathcal{A}/\mathcal{B}\). Let \(Q\) be the family of subobjects of \(U \in \mathcal{A}\) lying in \(\mathcal{B}\). Then \(Q\) is nonempty since \(0 \in Q\). As \(V\) is Noetherian in \(\mathcal{A}\), so is \(U\). Thus, \(Q\) has a maximal object \(U_0\). Then \(W := U/U_0\) is a subquotient of \(V \in \mathcal{A}\) and the morphism \(F(U) \to F(W)\) is an isomorphism in \(\mathcal{A}/\mathcal{B}\). In particular, \(W \neq 0\) in \(\mathcal{A}\).

We claim that \(W\) is simple in \(\mathcal{A}\). Indeed, let \(U' \to W\) be a subobject in \(\mathcal{A}\). Then there is a subobject \(U''\) of \(U\) in \(\mathcal{A}\) containing \(U_0\) with \(U''/U_0 = U'\). As \(F(U'')\) is a subobject of a simple object \(F(U)\) in \(\mathcal{A}/\mathcal{B}\), either the morphism \(F(U'') \to F(U)\) is an isomorphism or \(F(U'') = 0\). If \(F(U'') = 0\), then \(U'' \in \mathcal{B}\) and \(U'' \in Q\). Since \(U_0\) is maximal in \(Q\), one has \(U_0 = U''\), so \(U' = 0\). If \(F(U'') \to F(U)\) is an isomorphism, then \(U/U'' \in \mathcal{B}\). Since the sequence

\[
0 \to U/U'' \to V/U'' \to V/U \to 0
\]

is exact in \(\mathcal{A}\), and \(\mathcal{B}\) is closed under extensions, one gets \(V/U'' \in \mathcal{B}\) and \(U'' \in \mathcal{P}\). Since \(U\) is minimal in \(\mathcal{P}\), one has \(U'' = U\). The morphism \(U' \to W\) is thus an isomorphism in \(\mathcal{A}\). The claim is proved.

3. By 1, there is a subquotient \(Z\) of \(X\) in \(\mathcal{A}\) with \(F(Z)\) isomorphic to \(Y\). Then \(F(Z)\) is simple in \(\mathcal{A}/\mathcal{B}\). By assumption, \(Z\) is Noetherian and Artinian in \(\mathcal{A}\). Thus from 2, there is a simple subquotient \(W\) of \(Z\) in \(\mathcal{A}\) with \(F(W)\) isomorphic to \(F(Z)\) and to \(Y\) in \(\mathcal{A}/\mathcal{B}\).

\[\Box\]

**Example 6.0.9.** Let \(s : X \to A \times X\) be a section to \(\rho\). Let \(F\) be a lisse \(\Lambda\)-sheaf on \(X\), and let \(\sigma : \pi_1(X, \bar{\eta}) \to \text{GL}(F_{\bar{\eta}})\) be the corresponding monodromy
representation. By Fact 3.2.2 2, one has $F \in D^{ULA}(X/X)$. Then from Fact 3.2.2 4, one has $K := Rs_*F \in D^{ULA}(A \times X/X)$. For every $x \in X(k)$, by the proper base change theorem, $K_x \in D^b_{\text{c}}(A_x)$ is the skyscraper supported at the closed point $s(x) \in A_x$ with stalk $F_x$. Thus, $K_x \in \text{Perv}(A_x)$ and $K \in \text{Perv}^{ULA}(A \times X/X)$. Moreover, $K_{\bar{\eta}}$ is the skyscraper supported at $s(\bar{\eta}) \in A_{\bar{\eta}}$ with stalk $F_{\bar{\eta}}$. Therefore, the generic and the geometric generic Tannakian groups agree and are computed in Example 5.0.6.

For every $\chi \in C(A)(\Lambda)$, by Fact 3.1.1, one has $K \otimes^L \pi^*L \chi = Rs_*(F \otimes^L s^*\pi^*L \chi)$. Thus, $R\rho_*(K \otimes^L \pi^*L \chi) = F \otimes^L s^*\pi^*L \chi$ is a lisse $\Lambda$-sheaf on $X$. The corresponding $\pi_1^G(X, \bar{\eta})$-representation is the tensor product of $\sigma$ with the character

$$\pi_1^G(X, \bar{\eta}) \xrightarrow{(\pi_\sigma)} \pi_1^G(A, \pi s(\bar{\eta})) \xrightarrow{\sim} \pi_1^G(A, 0) \xrightarrow{\lambda^*} \Lambda^*.$$

The $\Gamma_{k(\eta)}$-representation induced by pulling back along $\eta_* : \Gamma_{k(\eta)} \to \pi_1^G(X, \bar{\eta})$ is $\omega_0(K_{\eta})$.

If $F$ is semisimple in $\text{Loc}(X)$, then $F[\dim X]$ is a semisimple object of $\text{Perv}(X)$. By Fact 3.1.6 1, $K$ is semisimple in $D^b_{\text{c}}(A \times X)$.

References


