Normality of monodromy group in generic Tannakian group

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Abstract

On an abelian variety A, sheaf convolution gives a Tannakian formalism for perverse sheaves. Let X be an irreducible algebraic variety with generic point η. Let K be a family of perverse sheaves (more precisely, a universally locally acyclic, relative perverse sheaf) on the constant abelian scheme $p_X : A \times X \to X$. We show that for uncountably many character sheaves L_χ on A, the monodromy groups of $K \otimes p_A^* L_\chi$ are normal in the Tannakian group $G(K|_{A_\eta})$ of $K|_{A_\eta}$.

This result is inspired from and could be compared to two other normality results: In the same setting, the Tannakian group $G(K|_{A_{\bar{\eta}}})$ is normal in $G(K|_{A_\eta})$ (due to Lawrence-Sawin). For a polarizable variation of Hodge structures, outside a meager locus, the connected monodromy group is normal in the derived Mumford-Tate group (due to André).

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1 Introduction

1.1 Background

Constructing local systems (or ℓ-adic lisse sheaves) with a prescribed monodromy group is an important problem having a long history.

In positive characteristics, Katz and his collaborators exhibit local systems whose monodromy groups are the simple algebraic group G_2 ([Kat88, 11.8]), $2.J_2$ ([KRL19]), the finite symplectic groups ([KT19b]), the special unitary groups ([KT19a]), *etc*. In particular, the exceptional Lie groups appear unexpectedly in algebraic geometry.

In characteristic 0, such constructions help to understand Galois groups of number fields. Dettweiler and Reiter [DR10] prove the existence of a local system on $\mathbf{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ whose monodromy group is G_2. It produces a motivic Galois representations with image dense in G_2. Their proof relies on Katz's middle convolution of perverse sheaves. Yun [Yun14] constructs local systems with some other exceptional groups as monodromy groups. As applications, he answers a long standing question of Serre, and solves new cases of the inverse Galois problem. His construction uses the geometric Langlands correspondence.

A new proof of Mordell's conjecture [LV20], and its potential generalization to higher dimensional varieties over number fields, rely on the existence of local systems with big monodromy over the variety in question. Lawrence and Sawin [LS20] use this technique to prove Shafarevich's conjecture for hypersurfaces in abelian varieties. Krämer and Maculan [KM23] apply roughly the same strategy to obtain an arithmetic finiteness result for very irregular varieties of dimension less than half the dimension of their Albanese variety. In both cases, the construction of local systems uses perverse sheaves.

In [LS20], that construction rests on comparing the monodromy group with the Tannakian group from Krämer-Weissauer's convolution theory [KW15b]. As [JKLM23, p.4] comments, this comparison is similar to the study of monodromy groups *via* Mumford-Tate groups in [And92].

We briefly outline their argument. On an abelian variety A, a quotient of the abelian category $\mathrm{Perv}(A)$ (of perverse sheaves) is a Tannakian category under sheaf convolution. Let X be an irreducible algebraic variety with generic point η. Let K be a universally locally acyclic, relative perverse sheaf on the constant abelian scheme $p_X : A \times X \to X$ (intuitively, a family of perverse shaves on A parameterized by X). The Tannakian group $G(K|_{A_{\bar{\eta}}})$ of $K|_{A_{\bar{\eta}}} \in \mathrm{Perv}(A_{\bar{\eta}})$ is normal in the Tannakian group $G(K|_{A_\eta})$ of $K|_{A_\eta} \in \mathrm{Perv}(A_\eta)$ ([LS20, Lem. 3.7], [JKLM23, Thm. 4.3]). This normality is used to prove that for most character sheaves L_χ on A, the monodromy groups $\mathrm{Mon}(K \otimes p_A^* L_\chi)$ contain $G(K|_{A_{\bar\eta}})$. Then Lawrence-Venkatesh's machinery works for these twists $K \otimes p_A^* L_\chi$.

1.2 Statements

In the main result (Theorem 1.2.2), we prove that the generic Tannakian group of a semisimple, relative perverse sheaf is reductive. Moreover, for many characters, the monodromy group is a normal subgroup of this reductive group. This normality puts further restriction on the monodromy group. Using Krämer's method ([Krä22, Thm. 6.2.1]), Lawrence and Sawin [LS20, Lem. 4.6] even show that the geometric generic Tannakian group is simple.

Setting 1.2.1. Let k be an algebraically closed field of characteristic 0. Let X be an integral algebraic variety over k with generic point η. Let A be an abelian variety over k. Denote by $p_X : A \times X \to X$ and $p_A : A \times X \to A$ the projections.

Let ℓ be a prime number. Let $\bar{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Let $D_c^b(A \times X)$ be the triangulated category of bounded constructible $\bar{\mathbb{Q}}_\ell$-sheaves on $A \times X$. Let $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X) \subset D_c^b(A \times X)$ be the full subcategory of p_X-universally locally acyclic (ULA, Definition 2.2.1) relative perverse sheaves (Definition 2.3.2). It is an abelian category. Let $\pi_1(A)$ be the étale fundamental group of A based at the geometric origin point. For every character $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$, let $\chi_\eta : \pi_1(A_\eta) \to \bar{\mathbb{Q}}_\ell^\times$ be the pullback of χ along $(p_A|_{A_\eta}) : \pi_1(A_\eta) \to \pi_1(A)$. Fix $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ which is a semisimple object of $D_c^b(A \times X)$ (in the sense of Definition 2.1.3). Let $\mathrm{Mon}(K, \chi_\eta)$ be the corresponding monodromy group (Definition 4.3.4). Let $G_{\omega_\chi}(K|_{A_\eta})$ be the Tannakian monodromy group (Definition 4.2.1) of $K|_{A_\eta}$, referred to as the *generic Tannakian group*.

Theorem 1.2.2 (Theorems 5.1.1, 5.3.1)**.** *Assume* $\dim A > 0$*. Then there are uncountably many characters* $\chi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$*, such that* $G_{\omega_\chi}(K|_{A_\eta})$ *is a well-defined reductive group. It contains* $\mathrm{Mon}(K, \chi_\eta)$ *as a closed, reductive,* normal *subgroup.*

The line of the proof of Theorem 1.2.2 is similar to that of André's normality theorem [And92, Thm. 1]. André proves that for a polarizable good variation of mixed Hodge structure, the connected monodromy group outside a meager locus is normal in the derived Mumford-Tate group. As [And92, p.10] explains, the normality is a consequence of the theorem of the fixed part due to Griffiths-Schmidt-Steenbrink-Zucker. In our case, an analog of the theorem of the fixed part is Theorem 1.2.3.

Let $\mathcal{C}(A)_\ell$ be the cotorus parameterizing pro-ℓ characters of $\pi_1(A)$ (Definition 3.2.2). For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$ and every $\chi_\ell \in \mathcal{C}(A)_\ell$, set $\chi = \chi_{\ell'}\chi_\ell$.

Theorem 1.2.3 (Theorem 5.2.1)**.** *Assume that* X *is smooth. Then there is a subobject* $K^0 \subset K$ *in* $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ *with the following property: For every character* $\chi_{\ell'} : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$ *of finite order prime to* ℓ*, there is a nonempty Zariski open subset* $U \subset \mathcal{C}(A)_\ell$*, such that for every* $\chi_\ell \in U$*, one has*

$$H^0(A_{\bar\eta}, K^0|_{A_{\bar\eta}} \otimes^L L_{\chi_\eta}) = H^0(A_{\bar\eta}, K|_{A_{\bar\eta}} \otimes^L L_{\chi_\eta})^{\Gamma_{k(\eta)}}.$$

The proof of Theorem 1.2.3 uses the projection $p_A : A \times X \to A$, which restricts our results to constant abelian schemes. We leave the question whether Theorem 1.2.2 has an analog for relative perverse sheaves on an arbitrary (non-constant) abelian scheme.

Notation and conventions

An object of an abelian category is *semisimple* if it is the direct sum of finitely many simple objects. An abelian category is *semisimple* if every object is semisimple. For a field k, its absolute Galois group is denoted by Γ_k. An algebraic variety means a scheme of finite type and separated over k. A linear algebraic group is *reductive*, if its identity component is reductive (in the sense of [Mil17, 6.46, p.135]). For a topological group, $\bar{\mathbb{Q}}_\ell$-characters are assumed to be continuous. For an irreducible algebraic variety X (on which ℓ is invertible) and a $\bar{\mathbb{Q}}_\ell$-character χ of its étale fundamental group $\pi_1(X)$, let L_χ be the corresponding rank one lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X.

2 Recollections on constructible sheaves

No originality is claimed in Section 2. Let k be a field. Let ℓ be a prime number invertible in k. Fix an algebraic closure $\bar{k}$ of k. For every algebraic variety X over k, denote by $D_c^b(X) := D_c^b(X, \bar{\mathbb{Q}}_\ell)$ the triangulated category of complexes of $\bar{\mathbb{Q}}_\ell$-sheaves on X with bounded constructible cohomologies defined in [BBDG82, p.74]. Let $\mathbb{D}_X : D_c^b(X) \to D_c^b(X)^{\mathrm{op}}$ be the Verdier duality functor. The heart of the standard t-structure on $D_c^b(X)$ is denoted by $\mathrm{Cons}(X)$, which is the category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on X. For $F \in \mathrm{Cons}(X)$, set $\mathrm{Supp}\, F := \{x \in X | F_x \neq 0\}$ to be its support. Then $\mathrm{Supp}\, F$ is a quasi-constructible subset of X in the sense of [Gro66, 10.1.1]. Let $\mathrm{Loc}(X) \subset \mathrm{Cons}(X)$ be the full subcategory of lisse $\bar{\mathbb{Q}}_\ell$-sheaves on X. For every integer n, let $\mathcal{H}^n : D_c^b(X) \to \mathrm{Cons}(X)$ be the functor taking the n-th cohomology sheaf.

For every subset $S \subset X$, let $\bar{S}$ be its Zariski closure. Let ${}^pD^{\leq 0}(X) \subset D_c^b(X)$ be the full subcategory of objects K with $\dim \overline{\mathrm{Supp}\, \mathcal{H}^n K} \leq -n$ for every integer n. Let ${}^pD^{\geq 0}(X) \subset D_c^b(X)$ be the full subcategory of objects K with $\mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. Then $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ defines the (absolute) perverse t-structure on $D_c^b(X)$, whose heart is denoted by $\mathrm{Perv}(X)$. The functor $\mathbb{D}_X$ interchanges ${}^pD^{\leq 0}(X)$ and ${}^pD^{\geq 0}(X)$. For every integer n, let ${}^p\mathcal{H}^n : D_c^b(X) \to \mathrm{Perv}(X)$ be the functor taking the n-th perverse cohomology sheaf. For a morphism $f : X' \to X$ of schemes and $K \in D_c^b(X)$, set $K|_{X'} := f^*K$.

2.1 Basics

Fact 2.1.1 (Projection formula, [FK88, Rk. (2), p.100], [Sta24, Tag 0F10 (1)]). *Let $f : X \to Y$ be a morphism of algebraic varieties over $\bar{k}$. Let $L \in D_c^b(Y)$ be an object with $\mathcal{H}^n L \in \mathrm{Loc}(X)$ for every integer n. Then there is a natural isomorphism $(Rf_*-) \otimes^L L \to Rf_*(- \otimes^L f^*L)$ of functors $D_c^b(X) \to D_c^b(Y)$.*

Let X be an algebraic variety over k.

Fact 2.1.2 ([FK88, Prop. 12.10]). *For every $F \in \mathrm{Cons}(X)$, there is a nonempty Zariski open subset $U \subset X$ with $F|_U \in \mathrm{Loc}(U)$.*

Definition 2.1.3 ([BC18, Def. 78]). An object $K \in D_c^b(X)$ is called semisimple if it is isomorphic to a finite direct sum of degree shifts of semisimple objects of $\mathrm{Perv}(X)$.

If $K \in D_c^b(X)$ is semisimple, then it is isomorphic to $\oplus_{n\in\mathbb{Z}} {}^p\mathcal{H}^n(K)[-n]$ in $D_c^b(X)$, and each ${}^p\mathcal{H}^n(K)$ is a semisimple object of $\mathrm{Perv}(X)$. A degree shift of a semisimple object of $D_c^b(X)$ is still semisimple. Lemma 2.1.4 is used in the proof of Theorem 5.1.1.

Lemma 2.1.4. *Let $U \subset X$ be an open subset of X. Then the functor $(-)|_U : \mathrm{Perv}(X) \to \mathrm{Perv}(U)$ sends every simple object of $\mathrm{Perv}(X)$ to a simple or zero object of $\mathrm{Perv}(U)$. In particular, the functor $(-)|_U : D_c^b(X) \to D_c^b(U)$ preserves semisimplicity.*

Proof. Let K be a simple object of $\mathrm{Perv}(X)$. By [BBDG82, Thm. 4.3.1 (ii)], there is an irreducible, locally closed and geometrically smooth subvariety $j : V \to X$ and a simple lisse $\bar{\mathbb{Q}}_\ell$-sheaf on V, such that K is isomorphic to $j_{!*}L[\dim V]$. If V is disjoint from U, then $K|_U = 0$. Otherwise, take a geometric point $\bar{x}$ on $V \cap U$. From [GR71, V, Prop. 8.2], the morphism $\pi_1(U \cap V, \bar{x}) \to \pi_1(V, \bar{x})$ is surjective. Thus, the composite representation $\pi_1(U \cap V, \bar{x}) \to \mathrm{GL}(L_{\bar{x}})$ is also simple, *i.e.*, the lisse $\bar{\mathbb{Q}}_\ell$-sheaf $L|_{U\cap V}$ is simple. Let $h : U \cap V \to U$ be the base change of $j : V \to X$ along the inclusion $U \to X$. Then $K|_U$ is isomorphic to $h_{!*}L|_{U\cap V}[\dim(U \cap V)]$, hence simple in $\mathrm{Perv}(U)$. □

When $k = \mathbb{C}$, Fact 2.1.5 1 follows from Kashiwara's conjecture for semisimple perverse sheaves and the paragraph following [BBDG82, Thm. 6.2.5]. Kashiwara's conjecture is formulated in [Kas98, Sec. 1]; see also [Dri01, Sec. 1.2, 1]. It is reduced to de Jong's conjecture by Drinfeld [Dri01], which in turn is proved in [BK06] and [Gai07]. The case of general k follows *via* Fact 2.1.6.

Fact 2.1.5. *Let k be an algebraically closed field of characteristic 0. Let $f : X \to Y$ be a proper morphism of algebraic varieties over k. Let K be a semisimple object of $D_c^b(X)$.*

1. (Decomposition theorem) *Then Rf_*K is a semisimple object of $D_c^b(Y)$.*
2. (Global invariant cycle theorem, [BBDG82, Cor. 6.2.8]) *Let i be an integer. By Fact 2.1.2, there is a nonempty connected open subset $V \subset Y$ such that $\mathcal{H}^i Rf_*K|_V$ is a lisse sheaf. Then for every $y \in V(k)$, the canonical map*

$$H^i(X, K) \to H^i(X_y, K|_{X_y})^{\pi_1(V,y)}$$

is surjective.

Fact 2.1.6. *Let E/F be an extension of algebraically closed fields. Let X be an algebraic variety over F. Then:*

1. ([JKLM23, proof of Lem. A.1]) *The functor* $(-)|_{X_E} : D^b_c(X) \to D^b_c(X_E)$ *is fully faithful. It induces an exact functor* $\mathrm{Perv}(X) \to \mathrm{Perv}(X_E)$.

2. ([BBDG82, Thm. 4.3.1 (ii)]) *An object of* $\mathrm{Perv}(X)$ *is simple if and only if its image under* $(-)|_{X_E} : \mathrm{Perv}(X) \to \mathrm{Perv}(X_E)$ *is simple.*

Lemma 2.1.7. *Let* L *be a lisse* $\bar{\mathbb{Q}}_\ell$*-sheaf of rank one on* X*. Then* $- \otimes^L L : D^b_c(X) \to D^b_c(X)$ *is an equivalence of categories. It is t-exact for the perverse t-structures.*

Proof. Let L^{-1} be the lisse sheaf dual to L. By associativity of the derived tensor product $\otimes^L$, the pair of functors $(- \otimes^L L, - \otimes^L L^{-1})$ is an equivalence.

1. Right t-exactness: The functor is t-exact for the standard t-structures. Thus, for every $K \in {}^pD^{\leq 0}(X)$ and every integer n, one has $\mathcal{H}^n(K \otimes^L L) = \mathcal{H}^n(K) \otimes^L L$. Therefore, one has $\mathrm{Supp}\,\mathcal{H}^n(K \otimes^L L) = \mathrm{Supp}\,\mathcal{H}^n(K)$. Thus, $K \otimes^L L \in {}^pD^{\leq 0}(X)$.

2. Left t-exactness: By Part 1, for every $K \in {}^pD^{\geq 0}(K)$, one has $L^{-1} \otimes^L \mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. By [KW01, II, Cor. 7.5 f)], one has isomorphisms
$$\mathbb{D}_X(K \otimes^L L) \to \mathrm{RHom}(L, \mathbb{D}_X K) \to L^{-1} \otimes^L \mathbb{D}_X K$$
in $D^b_c(X)$. Therefore, one gets $K \otimes^L L \in {}^pD^{\geq 0}(X)$.

□

2.2 Universal local acyclicity

In Section 2.2, all schemes are assumed to be quasi-compact and quasi-separated. For a scheme X and a geometric point $\bar{x}$ on X, denote by $O^{\mathrm{sh}}_{X,\bar{x}}$ the strict henselization (in the sense of [Sta24, Tag 04GQ (3)]) of $O_{X,\bar{x}}$. Set $X_{(\bar{x})} := \mathrm{Spec}\, O^{\mathrm{sh}}_{X,\bar{x}}$.

Let $f : X \to S$ be a separated morphism of finite presentation between $\mathbb{Z}[1/\ell]$-schemes.

Definition 2.2.1 ([Sta24, Tag 0GJM], [Bar23, Def. 1.2])**.** Let K be an object of $D^b_c(X)$.

- If for every geometric point $\bar{x}$ on X and every geometric point $\bar{t}$ on $S_{(\bar{s})}$ with $\bar{s} = f(\bar{x})$, the canonical morphism $R\Gamma(X_{(\bar{x})}, K) \to R\Gamma(X_{(\bar{x})} \times_{S_{(\bar{s})}} t, K)$ is an isomorphism, then K is called f-locally acyclic.

- If for every morphism $S' \to S$ of schemes, in the cartesian square
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array} \tag{1}$$
g'^*K is f'-locally acyclic, then K is called f-universally locally acyclic (f-ULA). Let $D^{\mathrm{ULA}}(X/S) \subset D^b_c(X)$ be the full subcategory of f-ULA objects.

By [HS23, Thm. 4.4], an object $K \in D_c^b(X)$ is f-ULA if and only if K is universally locally acyclic in the sense of [HS23, Def. 3.2]. Thus, the notation $D^{\mathrm{ULA}}(X/S)$ agrees with that in [HS23]. It is a triangulated subcategory of $D_c^b(X)$.

Fact 2.2.2.

1. ([Bar23, Lem. 3.4]) *If $S = \operatorname{Spec} k$, then $D^{\mathrm{ULA}}(X/k) = D_c^b(X)$.*

2. ([Bar23, Cor. 3.10 (i)]) *If $f : X \to S$ is an isomorphism, then $D^{\mathrm{ULA}}(X/S) \subset D_c^b(X)$ is the full subcategory of objects whose cohomology sheaves are lisse.*

3. ([HS23, Prop. 3.4 (i)]) *Let $g : S' \to S$ be a morphism of $\mathbb{Z}[1/\ell]$-schemes. Then in the notation of* (1)*, the functor $g'^* : D_c^b(X) \to D_c^b(X')$ restricts to a functor $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X'/S')$.*

4. ([Ric14, Lem. 3.15], [Bar23, Lem. 3.3 (i), (ii)]) *Let $f' : Y \to S$ be a separated morphism of finite presentation between $\mathbb{Z}[1/\ell]$-schemes. Let $h : X \to Y$ be a morphism of S-schemes. If h is smooth (resp. proper), then the functor $h^* : D_c^b(Y) \to D_c^b(X)$ (resp. $Rh_* : D_c^b(X) \to D_c^b(Y)$) restricts to a functor $D^{\mathrm{ULA}}(Y/S) \to D^{\mathrm{ULA}}(X/S)$ (resp. $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(Y/S)$).*

5. ([HS23, p.643]) *Let $g : S \to T$ be a smooth morphism of $\mathbb{Z}[1/\ell]$-schemes. Then $D^{\mathrm{ULA}}(X/S) \subset D^{\mathrm{ULA}}(X/T)$.*

6. ([Zhu17, Thm. A.2.5 (4)]) *Let $f_i : X_i \to S$ ($i = 1, 2$) be a separated morphism of finite presentation between $\mathbb{Z}[1/\ell]$-schemes. Let $K_i \in D^{\mathrm{ULA}}(X_i/S)$. Then $K_1 \boxtimes_S K_2 \in D^{\mathrm{ULA}}(X_1 \times_S X_2/S)$.*

Lemma 2.2.3. *Assume that S is irreducible with generic point η. Let $K \in D^{\mathrm{ULA}}(X/S)$. If $K|_{X_{\bar\eta}} = 0$ in $D_c^b(X_{\bar\eta})$, then $K = 0$.*

Proof. It suffices to prove that for every $s \in S$, one has $K|_{X_{\bar s}} = 0$ in $D_c^b(X_{\bar s})$. By [Gro61, Prop. 7.1.9], there is a discrete valuation ring R and a separated morphism $g : \operatorname{Spec}(R) = S' \to S$, sending the generic (resp. closed) point ξ (resp. r) of S' to η (resp. s). Let $i : R \to R^h$ be the henselization of R (in the sense of [Sta24, Tag 04GQ (1)]). By [Sta24, Tag 0AP3], R^h is a discrete valuation ring. From [Mil80, I, Exercise 4.9], the local morphism i is injective. Then $i^* : \operatorname{Spec}(R^h) \to S'$ preserves the generic (resp. closed) point. Replacing R by R^h, one may assume further that R is henselian.

Consider the following cartesian squares

$$\begin{array}{ccccc} X'_{\bar r} & \xrightarrow{\bar i} & X'_{(\bar r)} & \xleftarrow{\bar j} & X'_{\bar\xi} \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \bar r & \longrightarrow & S'_{(\bar r)} & \longleftarrow & \bar\xi, \end{array}$$

where every vertical morphism is a base change of $f : X \to S$. In the notation of (1), let $R\Phi : D^+(X') \to D^+(X'_{\bar{r}})$ be the vanishing cycle functor. Let $R\Psi : D^+(X') \to D^+(X'_{\bar{r}})$ be the nearby cycle functor. Set $K' = g'^* K$. By definition, one has $R\Psi(K') = \bar{i}^* R\bar{j}_*(K'|_{X'_{\bar{\xi}}})$. As R is henselian, from [Ill06, (1.1.3)], there is a natural exact triangle $K'|_{X'_{\bar{r}}} \to R\Psi(K') \to R\Phi(K') \xrightarrow{+1}$ in $D^+(X'_{\bar{r}})$. Since $K'|_{X'_{\bar{\xi}}}$ is a pullback of $K|_{X_{\bar{\eta}}} = 0$, one has $K'|_{X'_{\bar{\xi}}} = 0$ and $R\Psi(K') = 0$. By [Ill06, Cor. 3.5], the universal local acyclicity of K implies $R\Phi(K') = 0$. Therefore, one gets $K'|_{X'_{\bar{r}}} = 0$.

Since $K'|_{X'_{\bar{r}}}$ is the pullback of $K|_{X_{\bar{s}}}$ under the field extension $k(\bar{r})/k(\bar{s})$, by Fact 2.1.6 1, one gets $K|_{X_{\bar{s}}} = 0$. □

2.3 Relative perverse sheaves

Let $f : X \to S$ be a morphism of algebraic varieties over k. In particular, f is separated and of finite presentation. Set $K_{X/S} := Rf^! \bar{\mathbb{Q}}_\ell \in D^b_c(X)$ to be the relative dualizing complex. The functor

$$\mathbb{D}_{X/S}(-) = \mathrm{RHom}_{\bar{\mathbb{Q}}_\ell}(-, K_{X/S}) : D^b_c(X) \to D^b_c(X)^{\mathrm{op}}$$

is called the relative Verdier duality. There is a canonical morphism of functors $\mathrm{Id}_{D^b_c(X)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$ ([KL85, (1.1.5)]).

Fact 2.3.1 is stated for ∞-categories in [HS23], but holds for the underlying triangulated categories (described in [HRS23, Lem. 7.9]) by [HS23, Footnote 1].

Fact 2.3.1.

1. ([HS23, Thm. 1.1]) *There is a unique t-structure $({}^{p/S}D^{\leq 0}(X/S), {}^{p/S}D^{\geq 0}(X/S))$ on $D^b_c(X)$, called the relative perverse t-structure, with the following property: An object $K \in D^b_c(X)$ lies in ${}^{p/S}D^{\leq 0}(X/S)$ (resp. ${}^{p/S}D^{\geq 0}(X/S)$) if and only if for every geometric point $\bar{s} \to S$, the restriction $K|_{X_{\bar{s}}}$ lies in ${}^{p}D^{\leq 0}(X_{\bar{s}})$ (resp. ${}^{p}D^{\geq 0}(X_{\bar{s}})$). In particular, for every $s \in S$, the functor $(-)|_{X_s} : D^b_c(X) \to D^b_c(X_s)$ is t-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse t-structure.*

2. ([HS23, Thm. 1.9]) *The relative perverse t-structure on $D^b_c(X)$ restricts to a t-structure $({}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S), {}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S))$ on $D^{\mathrm{ULA}}(X/S)$.*

3. ([HS23, Prop. 3.4]) *The functor $\mathbb{D}_{X/S}$ preserves $D^{\mathrm{ULA}}(X/S)$, and the morphism $\mathrm{Id}_{D^{\mathrm{ULA}}(X/S)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$ of functors $D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X/S)$ is an isomorphism. The formation of $\mathbb{D}_{X/S} : D^{\mathrm{ULA}}(X/S) \to D^{\mathrm{ULA}}(X/S)^{\mathrm{op}}$ commutes with any base change in S, so $\mathbb{D}_{X/S}$ exchanges ${}^{p/S}D^{\mathrm{ULA},\leq 0}(X/S)$ with ${}^{p/S}D^{\mathrm{ULA},\geq 0}(X/S)$.*

Definition 2.3.2. Let $\mathrm{Perv}(X/S)$ (resp. $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$) be the heart of the relative perverse t-structure on $D^b_c(X)$ (resp. $D^{\mathrm{ULA}}(X/S)$).

By Fact 2.3.1 1, an object $K \in D_c^b(X)$ lies in $\mathrm{Perv}(X/S)$ if and only if for every geometric point $\bar{s} \to S$, one has $K|_{X_{\bar{s}}} \in \mathrm{Perv}(X_{\bar{s}})$.

Example 2.3.3.

1. ([HS23, p.632]) If $S = \mathrm{Spec}(k)$, then $\mathrm{Perv}(X/k) = \mathrm{Perv}(X)$.
2. If f is universally injective, then $\mathrm{Perv}(X/S) = \mathrm{Cons}(X)$.
3. ([Bar23, Cor. 3.10 (ii)]) If f is smooth of relative dimension r, then the functor $(-)[r] : \mathrm{Loc}(X) \to D_c^b(X)$ factors through $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Example 2.3.4. Let $i : Y \to X$ be a closed immersion of S-schemes, with $Y \to S$ smooth of relative dimension d and with geometrically connected fibers. If L is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on Y, then $i_*L[d] \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Indeed, by Fact 2.2.2 2, one has $L \in D^{\mathrm{ULA}}(Y/Y)$. From the smoothness of $Y \to S$ and Fact 2.2.2 5, one has $L \in D^{\mathrm{ULA}}(Y/S)$. Using the properness of $i : Y \to X$ and Fact 2.2.2 4, one has $i_*L[d] \in D^{\mathrm{ULA}}(X/S)$. For every geometric point $\bar{s} \to S$, let $i_{\bar{s}} : Y_{\bar{s}} \to X_{\bar{s}}$ be the base change of i along the morphism $X_{\bar{s}} \to X$. By the proper base change theorem, $i_*L[d]|_{X_{\bar{s}}} = (i_{\bar{s}})_*(L|_{Y_{\bar{s}}})[d] \in \mathrm{Perv}(X_{\bar{s}})$. Therefore, $i_*L[d] \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$.

Fact 2.3.5 ([HS23, Thm. 1.10 (ii)])**.** *Assume that S is geometrically unibranch and irreducible with generic point η. Then the functor*

$$(-)|_{X_\eta} : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X_\eta)$$

is exact and fully faithful, and its essential image is stable under subquotients.

Lemma 2.3.6. *If S is geometrically unibranch and irreducible, then $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is a Serre subcategory of $\mathrm{Perv}(X/S)$.*

Proof. By definition, $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is a strictly full subcategory of $\mathrm{Perv}(X/S)$. By Fact 2.3.1 2 and [BBDG82, Thm. 1.3.6], $\mathrm{Perv}^{\mathrm{ULA}}(X/S) \subset \mathrm{Perv}(X/S)$ is an abelian subcategory and closed under extensions in $D^{\mathrm{ULA}}(X/S)$. As $D^{\mathrm{ULA}}(X/S) \subset D_c^b(X)$ is a triangulated subcategory, $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is closed under extensions in $\mathrm{Perv}(X/S)$. Because S is geometrically unibranch, from the proof of [HS23, Thm. 6.8 (ii)], $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is closed under subquotients in $\mathrm{Perv}(X/S)$. By [Sta24, Tag 02MP], it is a Serre subcategory. $\square$

Lemma 2.3.7 is stated without proof for regular schemes S in [HS23, p.636].

Lemma 2.3.7. *Assume that S is smooth over k of equidimension d. Then the shifted inclusion*

$$(-)[d] : D^{\mathrm{ULA}}(X/S) \to D_c^b(X) \tag{2}$$

is t-exact, where $D^{\mathrm{ULA}}(X/S)$ (resp. $D_c^b(X)$) is equipped with the relative (resp. absolute) perverse t-structure. In particular, it induces an exact functor

$$(-)[d] : \mathrm{Perv}^{\mathrm{ULA}}(X/S) \to \mathrm{Perv}(X). \tag{3}$$

Proof. 1. The functor $(-)[d] : D^b_c(X) \to D^b_c(X)$ is right t-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse t-structure. For every geometric point $\bar{s}$ on S, the functor $(-)|_{X_{\bar{s}}} : D^b_c(X) \to D^b_c(X_{\bar{s}})$ is t-exact for the standard t-structures. Then for every integer n and every $K \in {}^{p/S}D^{\leq 0}(X/S)$, one has $\mathcal{H}^n(K[d])|_{X_{\bar{s}}} = \mathcal{H}^{n+d}(K|_{X_{\bar{s}}})$. Hence

$$X_{\bar{s}} \cap \operatorname{Supp} \mathcal{H}^n(K[d]) = \operatorname{Supp} \mathcal{H}^{n+d}(K|_{X_{\bar{s}}}).$$

As $K|_{X_{\bar{s}}} \in {}^pD^{\leq 0}(X_{\bar{s}})$, one has $\dim \operatorname{Supp} \mathcal{H}^{n+d}(K|_{X_{\bar{s}}}) \leq -n-d$. By Lemma 2.3.9 3, one has

$$\dim \operatorname{Supp} \mathcal{H}^n(K[d]) \leq -n.$$

From Lemma 2.3.9 1, the Zariski closure of $\operatorname{Supp} \mathcal{H}^n(K[d])$ in X has dimension at most $-n$. Hence $K[d] \in {}^pD^{\leq 0}(X)$.

2. The functor (2) is left t-exact. One may assume that k is algebraically closed. For every $K \in {}^{p/S}D^{\mathrm{ULA}, \geq 0}(X/S)$, by smoothness of S and the proof of [Bar23, Cor. 3.8], $\mathbb{D}_X(K[d])$ is (noncanonically) isomorphic to $(\mathbb{D}_{X/S}K)[d]$ in $D^b_c(X)$. From Fact 2.3.1 3, $\mathbb{D}_{X/S}K \in {}^{p/S}D^{\mathrm{ULA}, \leq 0}(X/S)$. By Part 1, one has $(\mathbb{D}_{X/S}K)[d] \in {}^pD^{\leq 0}(X)$. Hence $K[d] \in {}^pD^{\geq 0}(X)$.

□

Lemma 2.3.8. *If S is integral with generic point η and $\dim S = d$, then the functor $(-)|_{X_\eta}[-d] : D^b_c(X) \to D^b_c(X_\eta)$ is t-exact for the absolute perverse t-structures. In particular, it restricts to an exact functor*

$$(-)|_{X_\eta}[-d] : \operatorname{Perv}(X) \to \operatorname{Perv}(X_\eta). \tag{4}$$

Proof. 1. Right t-exactness: For every $K \in {}^pD^{\leq 0}(X)$ and every integer n, one has $\operatorname{Supp} \mathcal{H}^n(K|_{X_\eta}[-d]) = \operatorname{Supp} \mathcal{H}^{n-d}(K|_{X_\eta}) = X_\eta \cap \operatorname{Supp} \mathcal{H}^{n-d}(K)$. By Lemma 2.3.9 4, one has

$$\dim \operatorname{Supp} \mathcal{H}^n(K|_{X_\eta}[-d]) \leq \dim \operatorname{Supp}(\mathcal{H}^{n-d}(K)) - d \leq -n.$$

From Lemma 2.3.9 1, one has $K|_{X_\eta}[-d] \in {}^pD^{\leq 0}(X_\eta)$.

2. Left t-exactness: For every $K \in D^b_c(X)$ and every integer n, one has

$$\operatorname{Supp} \mathcal{H}^n(\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d])) = \operatorname{Supp} \mathcal{H}^n((\mathbb{D}_X K)|_{X_\eta}[-d]). \tag{5}$$

Indeed, from [DGIV77, Thm. 2.13, p.242], by shrinking S to a nonempty open subset, one may assume that $K \in D^{\mathrm{ULA}}(X/S)$. By the proof of [Bar23, Cor. 3.8], one has $\mathbb{D}_X K = (\mathbb{D}_{X/S}K)(d)[2d]$. From Fact 2.3.1 3, $(\mathbb{D}_X K)|_{X_\eta}[-d]$ is a Tate twist of $\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d])$, which proves (5).

Now assume $K \in {}^pD^{\geq 0}(X)$. Then $\mathbb{D}_X K \in {}^pD^{\leq 0}(X)$. From Part 1, one has $(\mathbb{D}_X K)|_{X_\eta}[-d] \in {}^pD^{\leq 0}(X_\eta)$. By (5), one has $\mathbb{D}_{X_\eta}(K|_{X_\eta}[-d]) \in {}^pD^{\leq 0}(X_\eta)$, or equivalently, $K|_{X_\eta}[-d] \in {}^pD^{\geq 0}(X_\eta)$.

□

By convention, the dimension of an empty space is $-\infty$.

Lemma 2.3.9. *Let X be a scheme of finite type over a field F. Let C be a quasi-constructible subset of X.*

1. *Then* $\dim C = \dim \bar{C}$.
2. *Let* $\{B_i\}_{i=1}^n$ *be finitely many locally closed subsets of* X *and* $B = \cup_{i=1}^n B_i$. *Then* $\dim B = \max_{i=1}^n \dim B_i$.

Let $f : X \to Y$ be a morphism between schemes of finite type over F.

3. *Let* $n \geq 0$ *be an integer such that* $\dim(C \cap f^{-1}(y)) \leq n$ *for every* $y \in Y$. *Then* $\dim C \leq \dim Y + n$.
4. *Assume that Y is integral with generic point η. Then* $\dim Y + \dim(C \cap X_\eta) \leq \dim C$.

Proof.

1. As X is a Noetherian scheme, the topological subspace C is Noetherian. Therefore, C is the union of finitely many irreducible components. Thus, one may assume further that C is nonempty and irreducible. Then the reduced induced closed subscheme $\bar{C}$ of X is integral and of finite type over F. By [Bor91, AG. Prop. 1.3], C contains a nonempty open subset of $\bar{C}$. By [Har77, II, Exercise 3.20 (e)], one has $\dim C = \dim \bar{C}$.

2. For every $1 \leq i \leq n$, since $B_i \subset B$, one has $\dim B_i \leq \dim B$. Then $\max_i \dim B_i \leq \dim B$. As B_i is quasi-constructible in X, by 1, one has $\dim B_i = \dim \overline{B_i}$. As $\{\overline{B_i}\}_{i=1}^n$ is a finite closed cover of $\bar{B}$, one gets $\dim B \leq \dim \bar{B} = \max_i \dim \overline{B_i} = \max_i \dim B_i$.

3. By 2, one may assume that C is locally closed in X. Taking irreducible components, one may assume further that C is irreducible. Let Z be the Zariski closure of $f(C)$ in Y. Then Z is irreducible. With reduced induced subscheme structures, one views C and Z as integral schemes of finite type over F. Moreover, $f : X \to Y$ induces a dominant morphism $g : C \to Z$ over F. By [Har77, II, Exercise 3.22 (b)], for every $y \in f(C) = g(C)$, one has
$$n \geq \dim C \cap f^{-1}(y) = \dim g^{-1}(y) \geq \dim C - \dim Z.$$
Hence $\dim C \leq \dim Z + n \leq \dim Y + n$.

4. As in the proof of 3, one may assume that C is an irreducible, locally closed subset of X and view C as an integral scheme of finite type over F. One may assume that $C \cap X_\eta$ is nonempty. As C_η is homeomorphic to $C \cap X_\eta$, the morphism $C \to Y$ induced by f is dominant. Thus, by [Har77, II, Exercise 3.22 (c)], one gets $\dim C \cap X_\eta = \dim C_\eta = \dim C - \dim Y$.

□

Lemma 2.3.10. *Assume that S is smooth over k, integral with generic point η and $\dim S = d$. Then:*

1. *Let $A \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$, and let $B[d]$ be a subquotient of $A[d]$ in $\mathrm{Perv}(X)$. If the image $B|_{X_\eta} \in \mathrm{Perv}(X_\eta)$ of $B[d]$ under the functor (4) is zero, then $B[d] = 0$ in $\mathrm{Perv}(X)$.*

2. *The functor* (3) *identifies $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ as a Serre subcategory of $\mathrm{Perv}(X)$.*

Proof.

1. By regularity of S and [HS23, Cor. 1.12], one has $B \in D^{\mathrm{ULA}}(X/S)$. Since $B|_{X_\eta} = 0$, by Lemma 2.2.3, one has $B = 0$.

2. It follows from the definition that the functor (3) is fully faithful. Its essential image is closed under extensions in $\mathrm{Perv}(X)$, because $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ is closed under extensions in the triangulated subcategory $D^{\mathrm{ULA}}(X/S)$ of $D^b_c(X)$.

   We claim that the essential image is closed under taking subobjects. Take $K \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$ and a subobject $L[d]$ of $K[d] \in \mathrm{Perv}(X)$. As S is integral, Lemma 2.3.8 shows that $L|_{X_\eta} \subset K|_{X_\eta}$ is a subobject in $\mathrm{Perv}(X_\eta)$. By smoothness of S and Fact 2.3.5, there is a subobject $L' \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ with $L'|_{X_\eta} = L|_{X_\eta}$. Set $M = K/L' \in \mathrm{Perv}^{\mathrm{ULA}}(X/S)$. Let $N[d]$ be the image of $L[d]$ under the morphism $K[d] \to M[d]$ in $\mathrm{Perv}(X)$. As the sequence
   $$0 \to L'[d] \cap L[d] \to L[d] \to N[d] \to 0$$
   is exact in $\mathrm{Perv}(X)$, by Lemma 2.3.8, the sequence
   $$0 \to L'|_{X_\eta} \cap L|_{X_\eta} \to L|_{X_\eta} \to N|_{X_\eta} \to 0$$
   is exact in $\mathrm{Perv}(X_\eta)$. Hence $N|_{X_\eta} = 0$. Since $N[d]$ is a subobject of $M[d] \in \mathrm{Perv}(X)$, by Part 1, one has $N[d] = 0$. Then $L[d] \subset L'[d]$ is a subobject in $\mathrm{Perv}(X)$. Since $(L'[d])/(L[d])$ is a quotient of $L'[d]$ in $\mathrm{Perv}(X)$ and $(L'|_{X_\eta})/(L|_{X_\eta}) = 0$ in $\mathrm{Perv}(X_\eta)$, one gets $(L'[d])/(L[d]) = 0$ in $\mathrm{Perv}(X)$. Therefore, $L[d] = L'[d]$. The claim is proved.

   Similarly, the essential image is closed under taking quotients. By [Sta24, Tag 02MP], the essential image is a Serre subcategory of $\mathrm{Perv}(X)$.

□

3 Cotori

We review the contents of [GL96, Sec. 3.2]. For a commutative ring R and an ideal $I \subset R$, let $V_R(I) = \operatorname{Spec} R/I (\subset \operatorname{Spec} R)$. For $r \in R$, let $V_R(r) = V_R(rR)$. For an integer $m \geq 1$, let μ_{ℓ^m} be the set of ℓ^m-roots of unity in $\bar{\mathbb{Q}}_\ell$. Set $\mu_{\ell^\infty} = \cup_{m\geq 1}\mu_{\ell^m}$. Let $\mathcal{M} = \cup_E m_E$, where E runs through all finite subextensions of $\mathbb{Q}_\ell \subset \bar{\mathbb{Q}}_\ell$, and m_E is the maximal ideal of the ring of integers of E.

3.1 ℓ-adic characters

By [Rob00, p.127], there is a canonical absolute value on $\bar{\mathbb{Q}}_\ell$ extending the discrete absolute value $|\cdot|_\ell$ on $\mathbb{Q}_\ell$. It induces a topology on $\bar{\mathbb{Q}}_\ell$ which is totally disconnected. A subset $A \subset \bar{\mathbb{Q}}_\ell$ is closed if and only if for every finite subextension $E/\mathbb{Q}_\ell$ of $\bar{\mathbb{Q}}_\ell$, the subset $A \cap E$ is closed in the discrete valuation field E.

Lemma 3.1.1.

1. *Let C be a compact subset of $\bar{\mathbb{Q}}_\ell$. Then there is a finite subextension E of $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_l$ with $C \subset E$.*

2. *Let $G \leq \bar{\mathbb{Q}}_\ell^\times$ be a compact subgroup. Then there is a finite subextension E of $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_l$ with $G \subset O_E^\times$.*

3. *In 2, let $G^{(\ell)}$ (resp. $G^{(\ell')}$) be the ℓ-Sylow subgroup (resp. maximal prime-to-ℓ quotient) of G. Then the topological group $G \xrightarrow{\sim} G^{(\ell)} \times G^{(\ell')}$, and $G^{(\ell')}$ is finite.*

Proof. 1. Otherwise, there is a sequence of elements $x_1, x_2, \ldots$ in C with $[\mathbb{Q}_l(x_{n+1}) : \mathbb{Q}_l] > [\mathbb{Q}_l(x_n) : \mathbb{Q}_l]$ for every integer $n > 0$. Let $B \subset C$ be the (infinite) set of elements of this sequence. For every subset $S \subset B$, every finite subextension $F/\mathbb{Q}_\ell$, the set $S \cap F$ is finite, so closed in F. Therefore, S is closed in $\bar{\mathbb{Q}}_\ell$. In particular, the set B is closed and hence compact in C. Every subset of B is closed in B, so B is discrete. Thus, B is finite, a contradiction.

2. By 1, there is a finite subextension E of $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_l$ containing G. By [Ser92, Thm. 1 2, p.122], one has $G \subset O_E^\times$.

3. By 2 and [Ser92, Cor., p.155], G is an ℓ-adic Lie group. From Lazard's theorem (see, *e.g.*, [GSK09, p.711]), there is a pro-ℓ open subgroup $U \leq G$. By [RZ10, Cor. 2.3.6 (b)], there is an ℓ-Sylow subgroup $H \leq G$ containing U. Since G is compact, $[G : U]$ is finite. Thus, the group G/H is finite of order prime to ℓ. By [RZ10, Prop. 2.3.8], G is isomorphic to $G/H \times H$. Since G is commutative, by [RZ10, Cor. 2.3.6 (c)], G has exactly one ℓ-Sylow subgroup.

□

For a profinite group G, let $\mathcal{C}(G)$ be the group of ℓ-adic characters, *i.e.*, continuous morphisms $G \to \bar{\mathbb{Q}}_\ell^\times$. Let $\mathcal{C}(G)_{\ell'}$ (resp. $\mathcal{C}(G)_\ell$) be the subgroup of characters of finite order prime to ℓ (resp. that are pro-ℓ). Then there is a canonical isomorphism $\mathcal{C}(G)_\ell \xrightarrow{\sim} \mathcal{C}\big((G^{(\ell)})^{\mathrm{ab}}\big)$. By Lemma 3.1.1 3, one has $\mathcal{C}(G) = \mathcal{C}(G)_{\ell'} \times \mathcal{C}(G)_\ell$. The group of ℓ-adic characters of $\mathbb{Z}_\ell$ is well-known.

Lemma 3.1.2. *There is a group isomorphism $\mathcal{C}(\mathbb{Z}_\ell) \to 1 + \mathcal{M}, \quad \chi \mapsto \chi(1)$.*

Proof. For every $1 \leq i \leq n$, when $m \to +\infty$, one has $\ell^m \to 0$ in $\mathbb{Z}_\ell$. For every character $\chi : \mathbb{Z}_\ell \to \bar{\mathbb{Q}}_\ell^\times$, by continuity, one has $\chi(\ell^m) = \chi(1)^{\ell^m} \to \chi(0) = 1$. Then $|\chi(1)|_\ell^{\ell^m} \to 1$. Hence $|\chi(1)|_\ell = 1$. There is a finite subextension $E/\mathbb{Q}_\ell$ with $\chi(1) \in O_E$. In the residue field of E, one has $(\chi(1) - 1)^{\ell^m} \equiv \chi(1)^{\ell^m} - 1$, which is zero when m is large. Hence $\chi(1) - 1 \in m_E$. The morphism is well-defined. Because 1 is a topological generator of $\mathbb{Z}_\ell$, the morphism is injective.

For every $u \in 1 + \mathcal{M}$, there is a finite subextension $E/\mathbb{Q}_\ell$ with $u - 1 \in m_E$. Every successive quotient of the filtration $1+m_E \supset 1+m_E^2 \supset \dots$ is isomorphic to the finite residue field of E, so the multiplicative group $1 + m_E$ is pro-ℓ. As $\mathbb{Z}_\ell$ is the pro-ℓ completion of $\mathbb{Z}$, the group morphism $\mathbb{Z} \to 1+m_E, \quad m \mapsto u^m$ extends to a unique $\bar{\mathbb{Q}}_\ell$-character of $\mathbb{Z}_\ell$. Therefore, the morphism is an isomorphism. □

3.2 Definition and basic properties

Fix an integer $n \geq 0$. Let A_n be a free $\hat{\mathbb{Z}}$-module of rank n. Let $\{\gamma_1, \dots, \gamma_n\}$ be a $\mathbb{Z}_\ell$-basis of $A_n^{(\ell)}$. Let $\mathcal{R} = \{O_E : E/\mathbb{Q}_\ell \text{ is a finite subextension of } \bar{\mathbb{Q}}_\ell\}$, which is a directed set under inclusion. For every $R \in \mathcal{R}$, let m_R be the maximal ideal of R. Let $R[[A_n^{(\ell)}]] := \varprojlim_{i,j \geq 1} (R/m_R^i)[A_n^{(\ell)}/\ell^j]$ be the completed group ring. There is a canonical injective morphism $A_n^{(\ell)} \to R[[A_n^{(\ell)}]]^\times$ of groups.

Fact 3.2.1 ([GL96, p.509])**.** *The ring $R[[A_n^{(\ell)}]]$ is a Noetherian, regular, complete, local domain of Krull dimension $1 + n$. There is an isomorphism of topological rings*

$$R[[A_n^{(\ell)}]] \to R[[X_1, \dots, X_n]], \quad \gamma_i \mapsto 1 + X_i. \tag{6}$$

Gabber and Loeser introduce a scheme of ℓ-adic characters.

Definition 3.2.2. Write$R_n = \bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[A_n^{(\ell)}]]$. Define the “cotorus" associated with A_n to be $\mathcal{C}_\ell := \operatorname{Spec}(R_n)$.

By [GL96, Prop. A.2.2.3 (ii)], the scheme $\mathcal{C}_\ell$ is integral and regular. Its set of closed points coincides with $\mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell)$, and is Zariski dense in $\mathcal{C}_\ell$. When $n > 0$, the $\bar{\mathbb{Q}}_\ell$-scheme $\mathcal{C}_\ell$ is **not** locally of finite type.

Lemma 3.2.3. *Every character $\chi : A_n^{(\ell)} \to \bar{\mathbb{Q}}_\ell^\times$ extends canonically to a surjective morphism $R_n \to \bar{\mathbb{Q}}_\ell$ of $\bar{\mathbb{Q}}_\ell$-algebras.*

Proof. There is a finite subextension $E/\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$ containing all the $\chi(\gamma_i)$. Then for every $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in \mathbb{Z}_\ell[[X_1, \dots, X_n]]$, by completeness of E, the series $\sum_{\alpha \in \mathbb{N}^n} c_\alpha \prod_{i=1}^n (\chi(\gamma_i) - 1)^{\alpha_i}$ converges in E. Denote its limit by $f(\chi(\gamma_1) - 1, \dots, \chi(\gamma_n) - 1)$. The composition $\mathbb{Z}_\ell[[A_n^{(\ell)}]] \to E$ of (6) followed by $\mathbb{Z}_\ell[[X_1, \dots, X_n]] \to E, \quad f \mapsto f(\chi(\gamma_1) - 1, \dots, \chi(\gamma_n) - 1)$ extends χ. It induces the stated surjection. The construction is independent of the choice of the $\mathbb{Z}_\ell$-basis of $A_n^{(\ell)}$. □

For every $\chi \in \mathcal{C}(A_n)_\ell$, by Lemma 3.2.3, the corresponding character $A_n^{(\ell)} \to \bar{\mathbb{Q}}_\ell^\times$ induces a surjection $R_n \to \bar{\mathbb{Q}}_\ell$. Let $\Psi(\chi)$ be the assigned element of $\mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell)$. Hence a map

$$\Psi : \mathcal{C}(A_n)_\ell \to \mathcal{C}_\ell(\bar{\mathbb{Q}}_\ell). \tag{7}$$

Fact 3.2.4 ([GL96, p.519]). *The map (7) is bijective.*

Set $S_n := \bar{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[X_1, \dots, X_n]])$. By [GL96, Prop. 3.2.2 (1)], the natural morphism $S_n \to \bar{\mathbb{Q}}_\ell[[X_1, \dots, X_n]]$ is injective. Then the isomorphism (6) identifies R_n with the $\bar{\mathbb{Q}}_\ell$-subalgebra $S_n \subset \bar{\mathbb{Q}}_\ell[[X_1, \dots, X_n]]$. Let $\mathcal{C}(A_n)_{\ell,\mathrm{tor}}$ be the torsion subgroup of $\mathcal{C}(A_n)_\ell$.

3.3 Cotori are Baire

The objective of Section 3.3 is Lemma 3.3.10, used in the proof of Theorem 5.3.1. We show that over an uncountable algebraically closed field, a reasonable scheme has uncountably many rational points outside a countable union of strict closed subsets. Fix an uncountable, algebraically closed field k.

Baire schemes

Definition 3.3.1. A k-scheme X is called k-Baire, if its dimension $\dim X$ is finite and $X(k) \setminus \cup_{i \geq 1} Z_i(k)$ is uncountable for every countable sequence $\{Z_i\}_{i \geq 1}$ of closed subschemes of X with $\dim Z_i < \dim X$ for all i. A k-algebra R is called k-Baire if $\mathrm{Spec}(R)$ is k-Baire.

The underlying reduced induced closed subscheme $X_{\mathrm{red}} \to X$ induces a bijection $X_{\mathrm{red}}(k) \to X(k)$, so X is k-Baire if and only X_{red} is k-Baire. A Noetherian k-scheme of dimension 1 with uncountably many k-points is k-Baire.

Remark 3.3.2. Let $k = \mathbb{C}$. Let X be a complex algebraic variety with $\dim X > 0$. The analytification X^{an} of X is locally compact Hausdorff. Then by the Baire category theorem (see, *e,g.*, [Wil70, Cor. 25.4 a)]), X is $\mathbb{C}$-Baire.

Lemma 3.3.3. *Let $f : X \to Y$ be a finite surjective morphism of k-schemes. If Y is k-Baire, then so is X.*

Proof. Let $\{Z_i\}_i$ be a sequence of closed subschemes of X with $\dim Z_i < \dim X$. Then for every integer $i \geq 1$, since f is a closed morphism, $Y_i := f(Z_i)$ is closed in Y. Endow each Y_i with the reduced induced structure. Let $Z_i' := f^{-1}(Y_i) = Y_i \times_Y X$. Then there is a canonical closed immersion $Z_i \to Z_i'$. The restriction $Z_i \to Y_i$ of f is a finite surjective morphism. By [Sta24, Tag 0ECG], one has $\dim X = \dim Y$ and $\dim Y_i = \dim Z_i$. In particular, $\dim X$ is finite and $\dim Y_i < \dim Y$.

As k is algebraically closed, the induced map $X(k) \to Y(k)$ is surjective. Then the induced map

$$X(k) \setminus (\cup_{i \geq 1} Z_i'(k)) \to Y(k) \setminus (\cup_i Y_i(k))$$

is surjective. Because Y is k-Baire, the target is uncountable. Then $X(k) \setminus (\cup_{i \geq 1} Z_i(k))$ is also uncountable, as it contains the source. □

Lemma 3.3.4. *Let X be a Noetherian k-scheme.*

1. *Then X is k-Baire if and only if X has an irreducible component C with $\dim C = \dim X$, such that the underlying reduced induced closed subscheme C is k-Baire.*

2. *Assume that $n := \dim X - 1$ is finite. If X has uncountably many (pairwise set-theoretically distinct) irreducible, k-Baire, closed subschemes of dimension n, then X is k-Baire.*

Proof. 1. Assume that there is such a component C. Consider a sequence of closed subschemes $\{Z_i\}_{i\geq 1}$ of X with $\dim Z_i < \dim X$ for all $i \geq 1$. Then for every $i \geq 1$, one has $\dim C \cap Z_i \leq \dim Z_i < \dim X = \dim C$. Since C is k-Baire, the set $C(k) \setminus \cup_i (C \cap Z_i)(k)$ is uncountable. Therefore, $X(k) \setminus \cup_i Z_i(k)$ is also uncountable.

Assume that every component of X of maximum dimension is not k-Baire. As X is Noetherian, one can write $X = \cup_{j=1}^n C_j$ as a finite union of the irreducible components. For every j with $\dim C_j = \dim X$, the scheme C_j is not k-Baire. Therefore, there is a sequence $\{Z_i^j\}_{i\geq 1}$ of closed subschemes of C_j such that $\dim Z_i^j < \dim C_j$ for all i and $C_j(k) \setminus \cup_i Z_i^j(k)$ is countable. The finite family of components C_k with $\dim C_k < \dim X$, joint with the sequences $\{Z_i^j\}_i$ for all j with $\dim C_j = \dim X$, gives a countable family $\{Z_s\}_s$ of closed subschemes of X with $\dim Z_s < \dim X$ for all s. Then $X(k) \setminus (\cup_s Z_s(k))$ is countable, so X is not k-Baire.

2. Consider a sequence of closed subschemes $\{Z_i\}_{i\geq 1}$ of X with $\dim Z_i < \dim X$ for all $i \geq 1$. Every Z_i is a Noetherian scheme, so it has only finitely many irreducible components. The set of irreducible components of the family $\{Z_i\}_i$ is countable. Thus, one may assume that every Z_i is irreducible. By assumption, X has an n-dimensional, irreducible, k-Baire closed subscheme X' which is set-theoretically distinct from any Z_i. For every $i \geq 1$, because $\dim X' = n \geq \dim Z_i$ and Z_i is irreducible, one has $X' \not\subset Z_i$ and $X' \cap Z_i \neq X'$. Since X' is irreducible, one has $\dim(X' \cap Z_i) < \dim X'$. As X' is k-Baire, the set $X'(k) \setminus \cup_{i\geq 1}(X' \times_X Z_i)(k)$ is uncountable, which is a subset of $X(k) \setminus \cup_{i\geq 1} Z_i(k)$. Therefore, X is k-Baire.

□

Lemma 3.3.5 is well-known.

Lemma 3.3.5. *If X is a finite type k-scheme with $\dim X > 0$, then X is k-Baire.*

Proof. Since X is of finite type over k, its dimension m is finite and X has only finitely many irreducible components. Replacing X with an irreducible component of dimension m, one may that assume X is irreducible. Then by [Har77, Exercise 3.20 (e), p.94], every nonempty open subset of X has dimension m. Replacing X by an affine open, one may assume that X is affine. By Noether's normalization lemma, there is a finite surjective morphism $p : X \to \mathbf{A}_k^m$ over k. By Lemma 3.3.3, one may assume $X = \mathbf{A}_k^m$.

By induction on $m > 0$, we prove that $\mathbf{A}_k^m$ is k-Baire. When $m = 1$, $\dim \mathbf{A}_k^1 = 1$ and $\mathbf{A}_k^1(k)$ is uncountable, so $\mathbf{A}_k^1$ is k-Baire. Assume the statement for $m-1$ with $m \geq 2$. The set of hyperplanes in $\mathbf{A}_k^m$ is uncountable. By the inductive hypothesis, every hyperplane is k-Baire. From Lemma 3.3.4 2, so is $\mathbf{A}_k^m$. The induction is completed. $\square$

Baireness of cotori

We show that every positive dimensional cotorus is $\bar{\mathbb{Q}}_\ell$-Baire.

Definition 3.3.6 ([BGR84, Def. 1, p.205])**.** Let A be a k-algebra, and let $A[X] \to B$ be an injective ring map. We say that B is *k-Rückert* over A if there is a nonempty family W of *monic* polynomials in $A[X]$ such that the following axioms are fulfilled:

1. If $f, g \in A[X]$ are monic polynomials with $fg \in W$, then $f, g \in W$.
2. For every $w \in W$, the A-algebra B/w is isomorphic to $A[X]/w$.
3. For every $b \in B \setminus \{0\}$, there is an automorphism σ of the k-algebra B and a unit $u \in B^\times$ such that $u\sigma(b) \in W$.

Remark 3.3.7. From Axiom 1, one gets $1 \in W$. If $W = \{1\}$, then by Axiom 3, for every $b \in B \setminus \{0\}$, one has $b \in B^\times$, *i.e.*, B is a field. Conversely, if B is a field, then B is k-Rückert over A with $W = \{1\}$.

If $W \neq \{1\}$, then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Indeed, take $w(\neq 1) \in W$. By Axiom 2, there is an A-isomorphism $B/w \to A[X]/w$, hence an isomorphism $\mathrm{Spec}(A[X]/w) \to \mathrm{Spec}(B/w)$ of $\mathrm{Spec}(A)$-schemes. Because w is a monic polynomial different from 1, the ring map $A \to A[X]/w$ is injective and finite. The induced morphism $\mathrm{Spec}(A[X]/w) \to \mathrm{Spec}(A)$ is surjective, so $\mathrm{Spec}(B/w) \to \mathrm{Spec}(A)$ is surjective.

Lemma 3.3.8 is used in the induction step of the proof of Lemma 3.3.10.

Lemma 3.3.8. *Let A be Noetherian k-algebra of dimension n. Let B be a domain, but not a field, containing $A[X]$. Assume that B is k-Rückert over A.*

1. *The ring B is Noetherian of dimension $n+1$.*
2. *Suppose that A is k-Baire. Let S be an uncountable subset of A such that for every $s \in S$, one has $\dim V_A(s) = n-1$. Suppose that the family $\{V_A(s)\}_{s\in S}$ is pairwise disjoint. Then B is k-Baire.*

Proof. For every $b \in B \setminus (B^\times \cup \{0\})$, by Axiom 3, there is an automorphism σ of the k-algebra B and a unit $u \in B^\times$ such that $w := u\sigma(b)$ is in W. Since b is not a unit, one has $w \neq 1$. By Axiom 2, the A-algebra B/w is isomorphic to $A[X]/w$. Since $w(\neq 1)$ is a monic polynomial over A, the ring map $A \to A[X]/w$ is injective finite.

1. One has
$$\dim B/b = \dim B/w = \dim A[X]/w \overset{(a)}{=} \dim A = n, \tag{8}$$
where (a) uses [Sta24, Tag 00OK]. The domain B is not a field, so $\dim B = n+1$. By [BGR84, Prop. 2, p.206], the ring B is Noetherian.

2. The morphism $\operatorname{Spec} A[x]/w \to \operatorname{Spec} A$ is finite surjective. Then by Lemma 3.3.3, the algebra $A[X]/w$ is k-Baire. As σ is over k, the k-algebra B/b is isomorphic to B/w. Then B/b is k-Baire.

   For every $s \in S$, one has $\dim V_A(s) < \dim A$, so $s \neq 0$. As B is not a field, from Remark 3.3.7, the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective. The preimage of $V_A(s)$ under the surjection $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is $V_B(s)$, so $V_B(s)$ is nonempty. In particular, $s \notin B^\times$ and B/s is k-Baire. Moreover, the family $\{V_B(s)\}_{s\in S}$ is pairwise disjoint. By (8), one gets $\dim V_B(s) = n$.

   By Part 1, B is Noetherian. Then for every $s \in S$, by Lemma 3.3.4 1, there is a k-Baire irreducible component $C_s \subset \operatorname{Spec}(B/s)$ of dimension n. The family $\{C_s\}_{s\in S}$ is pairwise disjoint. From Lemma 3.3.4 2, B is k-Baire.

□

Fact 3.3.9. *For every integer $n \geq 0$,*

1. ([GL96, Thm. A.2.1, Prop, A.2.2.1]) *the ring S_n is a Noetherian, regular, Jacobson domain of Krull dimension n;*

2. ([GL96, Prop A.2.2.2, proof of A.2.2.3 (ii)]) *S_{n+1} is $\bar{\mathbb{Q}}_\ell$-Rückert over S_n.*

Lemma 3.3.10. *For every integer $n \geq 1$, the algebra S_n is $\bar{\mathbb{Q}}_\ell$-Baire.*

Proof. Since $\bar{\mathbb{Q}}_\ell$ is a flat $\mathbb{Z}_\ell$-module, the injection $\mathbb{Z}_\ell[X_1,\ldots,X_n] \to \mathbb{Z}_\ell[[X_1,\ldots,X_n]]$ induces an injection $\bar{\mathbb{Q}}_\ell[X_1,\ldots,X_n] \to S_n$. The natural morphism
$$\operatorname{Spec}(\bar{\mathbb{Q}}_\ell[[X_1,\ldots,X_n]]) \to \mathbf{A}^n_{\bar{\mathbb{Q}}_\ell} \tag{9}$$
of $\bar{\mathbb{Q}}_\ell$-schemes factors through a morphism $p_n : \operatorname{Spec}(S_n) \to \mathbf{A}^n_{\bar{\mathbb{Q}}_\ell}$.

Then $\mathcal{M}$ is the maximal ideal of the integral closure $\overline{\mathbb{Z}_\ell}$ of $\mathbb{Z}_\ell$ inside $\bar{\mathbb{Q}}_\ell$. By [Rob00, Prop., p.128], the residue field $\overline{\mathbb{Z}_\ell}/\mathcal{M}$ is an algebraic closure of the finite field F_ℓ, so it is countable. As $\mathbb{Z}_\ell$ is uncountable, so is the set $\mathcal{M}$.

For every $(a_1,\ldots,a_n) \in \mathcal{M}^n$, there is a surjective morphism of $\bar{\mathbb{Q}}_\ell$-algebras:
$$\bar{\mathbb{Q}}_\ell[[X_1,\ldots,X_n]] \to \bar{\mathbb{Q}}_\ell, \quad f \mapsto f(a_1,\ldots,a_n).$$
Its kernel is a $\bar{\mathbb{Q}}_\ell$-point of $\operatorname{Spec}(\bar{\mathbb{Q}}_\ell[[X_1,\ldots,X_n]])$, whose image under (9) is $(a_1,\ldots,a_n) \in A^n_{\bar{\mathbb{Q}}_\ell}(\bar{\mathbb{Q}}_\ell)$. Hence $\mathcal{M}^n \subset p_n(\operatorname{Spec}(S_n)(\bar{\mathbb{Q}}_\ell))$. In particular, $\operatorname{Spec}(S_n)(\bar{\mathbb{Q}}_\ell)$ is uncountable.

By induction on $n > 0$, we prove that S_n is $\bar{\mathbb{Q}}_\ell$-Baire, and $\{V_{S_n}(X_1 - a)\}_{a\in\mathcal{M}}$ is a pairwise disjoint family of $(n-1)$-dimensional subsets. When $n = 1$, by Fact 3.3.9 1, S_1 is $\bar{\mathbb{Q}}_\ell$-Baire. Moreover, $\{V_{S_1}(X_1 - a)\}_{a\in\mathcal{M}}$ is a pairwise distinct family of closed point of $\operatorname{Spec}(S_1)$. The statement is proved for $n = 1$. Assume the statement for $n-1$ with $n \geq 2$. By Fact 3.3.9, (8), and Lemma 3.3.8 2, the statement holds for n. The induction is completed. □

4 Krämer-Weissauer theory

Let k be a field of characteristic 0. Let Vec_k be the category of finite dimensional k-vector spaces. Choose an algebraic closure $\bar{k}$ of k. Let $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ be the category of continuous, finite dimensional $\bar{\mathbb{Q}}_\ell$-representations of Γ_k. Let A be an abelian variety over k. Recall that $\pi_1(A_{\bar{k}})$ is a free $\hat{\mathbb{Z}}$-module of rank $2\dim A$. With the notation of Section 3, set

- $\mathcal{C}(A) = \mathcal{C}(\pi_1(A_{\bar{k}}))$: the group of characters $\pi_1(A_{\bar{k}}) \to \bar{\mathbb{Q}}_\ell^*$;
- $\mathcal{C}(A)_{\ell'} = \mathcal{C}(\pi_1(A_{\bar{k}}))_{\ell'}$: the group of characters of finite order prime to ℓ;
- $\mathcal{C}(A)_\ell$: the cotorus assigned to $\pi_1(A_{\bar{k}})$.

4.1 Generic vanishing theorem

For an object $K \in \mathrm{Perv}(A)$, set

$$\mathcal{S}(K) := \{\chi \in \mathcal{C}(A) | H^i(A_{\bar{k}}, K \otimes^L L_\chi) \neq 0 \text{ for some integer } i \neq 0\}.$$

Fact 4.1.1 ([KW15b, Thm. 1.1], [Wei16, Vanishing Theorem, p.561; Thm. 2])**.** *For every perverse sheaf $K \in \mathrm{Perv}(A)$ and every character $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, the set*

$$\{\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) | \chi_{\ell'}\chi_\ell \in \mathcal{S}(K)\}$$

is the set of $\bar{\mathbb{Q}}_\ell$-points of a strict Zariski closed subset of the scheme $\mathcal{C}(A)_\ell$.

We review [KW15a, p.725]. For every $K \in \mathrm{Perv}(A)$, its Euler characteristic satisfies

$$\chi(A, K) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\bar{\mathbb{Q}}_\ell} H^i(A_{\bar{k}}, K) \geq 0. \tag{10}$$

Let $N(A) \subset \mathrm{Perv}(A)$ be the full subcategory of objects K with $\chi(A, K) = 0$. From the additivity of the function $\chi(A, -) : \mathrm{Ob}(\mathrm{Perv}(A)) \to \mathbb{N}$ and (10), $N(A)$ is a Serre subcategory of $\mathrm{Perv}(A)$. Let $\bar{P}(A) := \mathrm{Perv}(A)/N(A)$ be the quotient abelian category. For every $\chi \in \mathcal{C}(A)$, set

$$\mathcal{E}^\chi(A_{\bar{k}}) = \{K \in \mathrm{Perv}(A_{\bar{k}}) | H^i(A_{\bar{k}}, K \otimes^L L_\chi) = 0, \quad \forall i \in \mathbb{Z} \setminus \{0\}\}.$$

Then $\mathcal{E}^\chi(A_{\bar{k}})$ is closed under extensions in $\mathrm{Perv}(A_{\bar{k}})$. Let $P^\chi(A) \subset \mathrm{Perv}(A)$ be the full subcategory of objects K with $Q \in \mathcal{E}^\chi(A_{\bar{k}})$ for every simple subquotient Q of $K|_{A_{\bar{k}}}$ in $\mathrm{Perv}(A_{\bar{k}})$.

By [BBDG82, Thm. 4.3.1 (i)], every object $K \in \mathrm{Perv}(A)$ is Noetherian and Artinian. For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, by Fact 4.1.1 and Lemma 4.1.2 1, the set $\{\chi_\ell \in \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) | K \in P^{\chi_{\ell'}\chi_\ell}(A)\}$ is the set of $\bar{\mathbb{Q}}_\ell$-points of a strict Zariski closed subset of $\mathcal{C}(A)_\ell$.

Lemma 4.1.2. *Let $\mathcal{A}$ be an abelian category, and let $X \in \mathcal{A}$ be a Noetherian and Artinian object.*

1. *Let Y be a simple subquotient of X. Then there is a composite series of X with one graded piece isomorphic to Y. In particular, up to isomorphism X has only finitely many simple subquotients.*

2. *If every subobject of X admits a direct complement, then X is semisimple.*

Proof.

1. There is a subobject $i : X_0 \subset X$ and a quotient $q : X_0 \to Y$ in $\mathcal{A}$. Let $N = \ker(q)$. By [Sta24, Tag 0FCH, Tag 0FCI], both N and X/X_0 are Noetherian and Artinian. From [Sta24, Tag 0FCJ], they admit composite series. A composite series of X/X_0 is equivalent to a filtration $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ by subobjects such that X_i/X_{i-1} is simple for every $1 \leq i \leq n$. This filtration and every composite series of N glue to a composite series of X with a step $N \subset X_0$, whose factor is isomorphic to Y. By the Jordan-Hölder lemma [Sta24, Tag 0FCK], up to isomorphism Y has finitely many choices.

2. One may assume that $X \neq 0$. Let $\mathcal{P}$ be the family of nonzero semisimple subobjects of X. By [Sta24, Tag 0FCJ], X has a nonzero simple subobject, so $\mathcal{P}$ is nonempty. Since X is Noetherian, the family $\mathcal{P}$ has a maximal element $i : X_0 \to X$. By assumption, there is a subobject $F \subset X$ with $X_0 \oplus F = X$. Then $F = 0$. (Otherwise, by [Sta24, Tag 0FCJ], F has a nonzero simple subobject F_0. Then $X_0 \oplus F_0 \in \mathcal{P}$ is strictly larger than X_0, which is a contradiction.) Therefore, i is an isomorphism and X is semisimple.

□

Remark 4.1.3. In a Noetherian and Artinian abelian category, an object may have infinitely many distinct (non semisimple) subobjects up to isomorphism.

Lemma 4.1.4. *Let $\mathcal{A}$ be a Noetherian and Artinian abelian category. Let $\mathcal{E}$ be a class of objects of $\mathcal{A}$ closed under isomorphisms. Let $\mathcal{S} \subset \mathcal{A}$ be the full subcategory of objects every nonzero simple subquotient of which is in $\mathcal{E}$.*

1. *Then $\mathcal{S}$ is a Serre subcategory of $\mathcal{A}$.*

2. *If further $\mathcal{E}$ is closed under extensions, then $\mathcal{S} \subset \mathcal{E}$.*

Proof.

1. (a) We prove that $\mathcal{S}$ is closed under subquotients. Let X be an object of $\mathcal{S}$ with a subquotient Y. Every simple subquotient of Y is that of X, hence in $\mathcal{E}$. Thus, $Y \in \mathcal{S}$.

   Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in $\mathcal{A}$ with $L, N \in \mathcal{S}$. Let Q be a nonzero simple subquotient of M. We prove that $Q \in \mathcal{E}$.

   (b) First, assume that Q is a quotient of M. The natural morphism $L \to Q$ is either an epimorphism or zero, in which case Q is a simple quotient of L or N respectively. Hence $Q \in \mathcal{E}$.

(c) Now assume that Q is general. There is a subobject $M_0 \subset M$ and an epimorphism $M_0 \to Q$. Then

$$0 \to f^{-1}(M_0) \to M_0 \to g(M_0) \to 0$$

is a short exact sequence in $\mathcal{A}$ with $f^{-1}(M_0)$ (resp. $g(M_0)$) a subobject of L (resp. N). From Part 1a, both $f^{-1}(M_0)$ and $g(M_0)$ are in $\mathcal{S}$. From Part 1b, one has $Q \in \mathcal{E}$.

From Part 1c, one has $M \in \mathcal{S}$ and $\mathcal{S}$ is closed under extensions. The result follows from [Sta24, Tag 02MP].

2. By [Sta24, Tag 0FCJ], every object $X \in \mathcal{S}$ admits a filtration in $\mathcal{A}$

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

by subobjects such that each X_i/X_{i-1} is a simple subquotient of X. Then $X_i/X_{i-1} \in \mathcal{E}$. As $\mathcal{E}$ is closed under extensions, one has $X \in \mathcal{E}$.

□

By Lemma 4.1.4 1, for every $\chi \in \mathcal{C}(A)$, $P^\chi(A) \subset \mathrm{Perv}(A)$ is a Serre subcategory. From Lemma 4.1.4 2, for every $K \in P^\chi(A)$ and every integer $i \neq 0$, one has

$$H^i(A_{\bar{k}}, K \otimes^L L_\chi) = 0. \tag{11}$$

From the proof of [LS20, Lem. 3.4 (3)], the functor

$$\omega_\chi : P^\chi(A) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}, \quad K \mapsto H^0(A_{\bar{k}}, K \otimes^L L_\chi) \tag{12}$$

is exact. Let $N^\chi(A)$ be the full subcategory of $P^\chi(A)$ of objects in $N(A)$. For every $K \in N^\chi(A)$, by [KW15b, Cor. 4.2], one has $\chi(A, K \otimes^L L_\chi) = 0$. From (11), one has $H^0(A_{\bar{k}}, K \otimes^L L_\chi) = 0$. By [Sta24, Tag 02MS], the functor ω_χ factors uniquely through an exact functor (still denoted by ω_χ)

$$P^\chi(A)/N^\chi(A) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}. \tag{13}$$

4.2 Tannakian groups

Let $(\mathcal{C}, \otimes)$ a neutral Tannakian category (in the sense of [DM22, Def. 2.19]) over an algebraically closed field Q of characteristic 0, with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vec}_Q$. Let $\mathrm{Aut}^\otimes(\mathcal{C}, \omega)$ be the corresponding affine group scheme over Q. By [Del90, Sec. 9.2, p.187], up to isomorphism of group schemes, $\mathrm{Aut}^\otimes(\mathcal{C}, \omega)$ is independent of the choice of ω. (See [Wib22, Thm. 1.2] for an elementary proof.)

For an object $K \in \mathcal{C}$, let $\iota : \langle K \rangle \hookrightarrow \mathcal{C}$ be the full subcategory whose objects are the subquotients of $\{(K \oplus K^\vee)^{\otimes n}\}_{n \geq 1}$. Then $(\langle K \rangle, \otimes)$ is a neutral Tannakian subcategory of $\mathcal{C}$ (in the sense of [Mil07, 1.7]), for which $\omega\iota : \langle K \rangle \to \mathrm{Vec}_Q$ is a fiber functor. The group scheme $\mathrm{Aut}^\otimes(\langle K \rangle, \omega\iota)$ is the image of the natural morphism $\mathrm{Aut}^\otimes(\mathcal{C}, \omega) \to \mathrm{GL}(\omega(K))$.

Definition 4.2.1. The algebraic group $\mathrm{Aut}^{\otimes}(\langle K\rangle, \omega\iota)$ is called the Tannakian monodromy group of K at ω and is denoted by $G_\omega(K)$.

By [Sim92, p.69], $G_\omega(K)$ is reductive if and only if K is semisimple in $\mathcal{C}$.

Example 4.2.2. With tensor product, $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ is a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$. The forgetful functor $\omega : \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k) \to \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ is a fiber functor. The Tannakian monodromy group of an object $\rho : \Gamma_k \to \mathrm{GL}(V)$ at ω is the Zariski closure of $\rho(\Gamma_k) \subset \mathrm{GL}(V)$.

4.3 Sheaf convolution

Let $m : A \times_k A \to A$ be the group law on A. Let $p_i : A \times_k A \to A$ be the projection to i-th factor ($i = 1, 2$). The bifunctor

$$* : D_c^b(A) \times D_c^b(A) \to D_c^b(A), \quad - * + := Rm_*(p_1^* - \otimes^L p_2^* +)$$

is called the *convolution* on A.

Example 4.3.1. For every closed reduced subvariety $i : X \to A$, let $\delta_X := i_*\bar{\mathbb{Q}}_{\ell,X} \in D_c^b(A)$. Then for every closed point $x \in A$, one has $\delta_x * \delta_X = \delta_{x+X}$.

By [Wei11] and [JKLM23, Sec. 3.1], the pair $(D_c^b(A), *)$ is a rigid, symmetric monoidal category, with unit δ_0. For every $K \in D_c^b(A)$, its adjoint dual is $K^\vee := [-1]_A^* \mathbb{D}_A K$.

Fact 4.3.2 ([KW15b, proof of Thm. 13.2], [LS20, Lem. 3.4 (4)], [JKLM23, Prop. 3.1]). *The convolution on $D_c^b(A)$ induces a bifunctor $\bar{P}(A) \times \bar{P}(A) \to \bar{P}(A), \quad (-, +) \mapsto {}^p\mathcal{H}^0(- * +)$ fitting into a commutative square*

$$\begin{array}{ccc} \mathrm{Perv}(A) \times \mathrm{Perv}(A) & \xrightarrow{\ *\ } & D_c^b(A) \\ \downarrow & & \downarrow {}^p\mathcal{H}^0 \\ \bar{P}(A) \times \bar{P}(A) & \dashrightarrow & \bar{P}(A). \end{array}$$

It makes $\bar{P}(A)$ a neutral Tannakian category over $\bar{\mathbb{Q}}_\ell$. For every $\chi \in \mathcal{C}(A)$, the subcategory $P^\chi(A)/N^\chi(A) \subset \bar{P}(A)$ is a Tannakian subcategory, on which (13) is a fiber functor.

Example 4.3.3. [KW15a, Example 7.1] Fix a closed point $x \in A$. Then $\delta_x \in \mathrm{Perv}(A)$. The spectrum $\mathcal{S}(\delta_x)$ is empty and for every $\chi \in \mathcal{C}(A)$, one has $\delta_x \in P^\chi(A)$. If x is a torsion point of order n, then $G_{\omega_\chi}(\delta_x)$ is isomorphic to $\mathbb{Z}/n$. If x is not a torsion point, then $G_{\omega_\chi}(\delta_x)$ is isomorphic to $\mathbb{G}_{m/\bar{\mathbb{Q}}_\ell}$.

Let $\psi : \pi_1(A) \to \bar{\mathbb{Q}}_\ell^\times$ be a character, and set $\psi' = \psi|_{\pi_1(A_{\bar{k}})}$. The functor

$$\omega_\psi : \mathrm{Perv}(A) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k), \quad K \mapsto H^0(A_{\bar{k}}, K \otimes^L L_\psi)$$

fits into a commutative square

$$\begin{array}{ccc} \mathrm{Perv}(A) & \xrightarrow{\omega_\psi} & \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k) \\ \uparrow & & \downarrow \omega \\ P^{\psi'}(A) & \xrightarrow{(12)} & \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell} \end{array}$$

The quotient functor $P^{\psi'}(A)/N^{\psi'}(A) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ of $\omega_\psi|_{P^{\psi'}(A)}$ induces a morphism of affine groups schemes

$$\omega_\psi^* : \mathrm{Aut}^{\otimes}(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k), \omega) \to \mathrm{Aut}^*(P^{\psi'}(A)/N^{\psi'}(A), \omega_{\psi'}). \tag{14}$$

Definition 4.3.4. For every $K \in \mathrm{Perv}(A)$, let $\mathrm{Mon}(K, \psi)$ be the Tannakian monodromy group of $\omega_\psi(K)$ in $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$.

For every $K \in P^{\psi'}(A)$, the functor $\omega_\psi|_{\langle K \rangle} : \langle K \rangle \to \langle \omega_\psi(K) \rangle$ induces a closed immersion of linear algebraic groups $\omega_\psi^* : \mathrm{Mon}(K, \psi) \to G_{\omega_{\psi'}}(K)$, which is the projection of (14) in $\mathrm{GL}(\omega_{\psi'}(K))$.

5 Main results

Consider Setting 1.2.1. For every character $\chi \in \mathcal{C}(A)$, denote the pullback of χ along $(p_A|_{A_\eta})_* : \pi_1(A_\eta) \to \pi_1(A)$ by $\chi_\eta : \pi_1(A_\eta) \to \bar{\mathbb{Q}}_\ell^\times$. Then the restriction $\chi_\eta|_{\pi_1(A_{\bar{\eta}})}$ is identified with χ *via* the isomorphism $(p_A|_{A_{\bar{\eta}}})_* : \pi_1(A_{\bar{\eta}}) \to \pi_1(A)$. Let $K \in \mathrm{Perv}(A \times X/X)$ be an object which is semisimple in $D_c^b(A \times X)$.

We shall prove that the monodromy group of K is normal in its Tannakian group. By the normality criterion (Lemma 5.0.1), it suffices to show that the monodromy is reductive, and to consider the monodromy fixed part of *all* the representations of the Tannakian group. Such representations are from perverse sheaves.

Lemma 5.0.1. *Let G be a linear algebraic group over an algebraically closed field C. Let H be a closed, reductive subgroup of G. If for every $V \in \mathrm{Rep}_C(G)$, the subspace V^H is G-stable, then H is normal in G.*

Proof. By [Gro97, Cor. 2.4] and reductivity, H is observable in G (in the sense of [BBHM63, p.134]). From [And21, Prop. C.3], H is normal in G. □

5.1 Reductivity

Theorem 5.1.1. *For every $\chi \in \mathcal{C}(A) \backslash \mathcal{S}(K|_{A_\eta})$, the monodromy group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ is reductive.*

Proof. By Lemma 2.1.4, when X is replaced by a nonempty open subset, the semisimplicity of K in $D_c^b(A \times X)$ is preserved. Moreover, the $\Gamma_{k(\eta)}$-representation $\omega_{\chi_\eta}(K|_{A_\eta})$ and hence the group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ remain unchanged. Thus, by [Sta24, Tag 056V], one may assume that X is smooth. As K is semisimple in

$D^b_c(A \times X)$, from Lemma 2.1.7, so is $K \otimes^L p_A^* L_\chi$. By Fact 2.1.5 1, the object $Rp_{X*}(K \otimes^L p_A^* L_\chi)$ is semisimple in $D^b_c(X)$.

By the proper base change theorem (see, *e.g.*, [Sta24, Tag 095T]), for every integer n, one has

$$\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi)_{\bar{\eta}} = H^n(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi).$$

Since $\chi \notin \mathcal{S}(K|_{A_\eta})$, when $n \neq 0$, one has $H^n(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi) = 0$. By Fact 2.1.2, there is a nonempty open subset U_0 (resp. U_n for every integer $n \neq 0$) of X such that $[\mathcal{H}^0 Rp_{X*}(K \otimes^L p_A^* L_\chi)]|_{U_0}$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf (resp. $[\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi)]|_{U_n} = 0$). The set

$$J := \{n \in \mathbb{Z} : \mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi) \neq 0\}$$

is finite and X is irreducible, so $U := U_0 \cap \cap_{n \in J} U_n$ is a nonempty open subset of X. Shrinking X to U, one may assume further that $\mathcal{H}^n Rp_{X*}(K \otimes^L p_A^* L_\chi) = 0$ for every integer $n \neq 0$, and that $\mathcal{H}^0 Rp_{X*}(K \otimes^L p_A^* L_\chi)$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X.

Thus, the semisimple object $Rp_{X*}(K \otimes^L p_A^* L_\chi)[\dim X]$ of $D^b_c(X)$ lies in $\mathrm{Perv}(X)$, so it is semisimple in $\mathrm{Perv}(X)$. By [Ach21, Prop. 3.4.1], the object $Rp_{X*}(K \otimes^L p_A^* L_\chi)$ of $\mathrm{Loc}(X)$ is semisimple. Therefore, the corresponding representation

$$\pi_1(X, \bar{\eta}) \to \mathrm{GL}(H^0(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi))$$

is semisimple. Because X is smooth, the natural morphism $\eta_* : \Gamma_{k(\eta)} \to \pi_1(X, \bar{\eta})$ is surjective. Then the composition $\Gamma_{k(\eta)} \to \mathrm{GL}(H^0(A_{\bar{\eta}}, K|_{A_{\bar{\eta}}} \otimes^L L_\chi))$, *i.e.*, the representation $\omega_{\chi_\eta}(K|_{A_\eta})$, is semisimple. Consequently, the algebraic group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ is reductive. □

Example 5.1.2. In Theorem 5.1.1, the algebraic group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ may not be semisimple. Let X be a smooth, projective, integral algebraic curve over k of genus 1. Then $\pi_1(X, \bar{\eta}) \cong \hat{\mathbb{Z}}^2$. There exists a character $\sigma : \pi_1(X, \bar{\eta}) \to \bar{\mathbb{Q}}_\ell^\times$ of *infinite* order. Let $A = \mathrm{Spec}(k)$. Then $\mathcal{C}(A) = \{1\}$ and $\mathrm{Mon}(L_\sigma|_{A_\eta}, 1) = \mathbb{G}_{m/\bar{\mathbb{Q}}_\ell}$ is an algebraic torus.

Remark 5.1.3. Let $i : Y \to A \times X$ be a closed subvariety, such that the induced morphism $f : Y \to X$ is smooth with connected fibers of dimension d:

$$\begin{array}{ccc} Y & \xhookrightarrow{i} & A \times X \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle p_X} & \downarrow{\scriptstyle p_A} \\ X & & A. \end{array}$$

By Example 2.3.4, one has $K := i_* \bar{\mathbb{Q}}_{\ell, Y}[d] \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X / X)$. By Fact 2.1.5 1, it is semisimple in $D^b_c(A \times X)$. Assume that X is smooth. Then for every $\chi \in \mathcal{C}(A) \setminus \mathcal{S}(K|_{A_\eta})$, the algebraic group $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ coincides with the Zariski closure of the image of the monodromy representation of the lisse $\bar{\mathbb{Q}}_\ell$-sheaf $R^d f_* i^* p_A^* L_\chi$ on X, which is studied in [KM23, Sec. 1.4] (but with coefficient $\mathbb{C}$ instead of $\bar{\mathbb{Q}}_\ell$).

5.2 Fixed part

Theorem 1.2.3 follows from Theorem 5.2.1 and Fact 4.1.1, because the union in Condition 1 of Theorem 5.2.1 is in fact a finite union.

Theorem 5.2.1. *Assume that X is smooth and $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Then there exists a subobject $K^0 \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ such that for every $\chi \in \mathcal{C}(A)$ with*

1. $\chi \notin \cup_{j \in \mathbb{Z}} \mathcal{S}({}^p\mathcal{H}^j(Rp_{A*}K))$,
2. $K|_{A_\eta} \in P^\chi(A_\eta)$ *and*
3. ${}^p\mathcal{H}^0(Rp_{A*}K) \in P^\chi(A)$,

one has $\omega_{\chi_\eta}(K^0|_{A_\eta}) = \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}}$.

Proof. By properness of $p_X : A \times X \to X$ and Fact 2.2.2 4, one has $Rp_{X*}K \in D^{\mathrm{ULA}}(X/X)$. Then from Fact 2.2.2 2, the sheaf $\mathcal{H}^0 Rp_{X*}K$ is lisse. Since X is smooth, by [Sta24, Tag 0BQM], the canonical morphism $\Gamma_{k(\eta)} \to \pi_1(X, \bar{\eta})$ is surjective. Thus, from Fact 2.1.5 2, the natural map

$$H^0(A \times X, K \otimes^L p_A^* L_\chi) \to \omega_{\chi_\eta}(K|_{A_{\bar{\eta}}})^{\Gamma_{k(\eta)}} \tag{15}$$

is surjective.

By Fact 2.1.1, one has

$$H^0(A \times X, K \otimes^L p_A^* L_\chi) = H^0(A, (Rp_{A*}K) \otimes^L L_\chi). \tag{16}$$

By Condition 1, for any integers $i \neq 0$ and j, one has

$$H^i(A, {}^p\mathcal{H}^j(Rp_{A*}K) \otimes^L L_\chi) = 0.$$

By Lemma 2.1.7, the spectral sequence in [Max19, Rk. 8.1.14 (6)] becomes

$$E_2^{i,j} = H^i(A, {}^p\mathcal{H}^j(Rp_{A*}K) \otimes^L L_\chi) \Rightarrow H^{i+j}(A, (Rp_{A*}K) \otimes^L L_\chi).$$

It degenerates at page E_2. Hence

$$H^0(A, (Rp_{A*}K) \otimes^L L_\chi) = H^0(A, ({}^p\mathcal{H}^0 Rp_{A*}K) \otimes^L L_\chi). \tag{17}$$

Set $K^1 := p_A^* {}^p\mathcal{H}^0(Rp_{A*}K) \in D_c^b(A \times X)$. By Fact 2.2.2 1, one has ${}^p\mathcal{H}^0(Rp_{A*}K) \in D^{\mathrm{ULA}}(A/k)$. From Fact 2.2.2 3, one gets $K^1 \in D^{\mathrm{ULA}}(A \times X/X)$. For every $x \in X(k)$, the restriction $p_A|_{A_x} : A_x \to A$ is an isomorphism of abelian varieties over k, so the functor $(p_A|_{A_x})^* : \mathrm{Perv}(A) \to \mathrm{Perv}(A_x)$ is an equivalence of abelian categories. It sends ${}^p\mathcal{H}^0(Rp_{A*}K)$ to $K^1|_{A_x}$, so $K^1|_{A_x} \in \mathrm{Perv}(A_x)$ and hence $K^1 \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. From $K^1|_{A_\eta} = (p_A|_{A_\eta})^{*p}\mathcal{H}^0(Rp_{A*}K)$ and Condition 3, one has $K^1|_{A_\eta} \in P^\chi(A_\eta)$. Then

$$\omega_\chi(K^1|_{A_\eta}) = H^0(A, {}^p\mathcal{H}^0(Rp_{A*}K) \otimes^L L_\chi). \tag{18}$$

Every fiber of $p_A : A \times X \to A$ has dimension $\dim X$, so by [BBDG82, 4.2.4], the functor

$$Rp_{A*}[-\dim X] : D^b_c(A \times X) \to D^b_c(A)$$

is left t-exact for the absolute perverse t-structures. From smoothness of X and Lemma 2.3.7, one has $K[\dim X] \in \mathrm{Perv}(A \times X)$ and so $Rp_{A*}K \in {}^pD^{\geq 0}(A)$. Taking the perverse truncation, one has ${}^p\tau^{\leq 0}(Rp_{A*}K) = {}^p\mathcal{H}^0(Rp_{A*}K)$. *Via* the adjunction formula (see, *e.g.*, [KW01, p.107]), the natural morphism

$${}^p\tau^{\leq 0}(Rp_{A*}K) \to Rp_{A*}K$$

in $D^b_c(A)$ (from the definition of t-structure) induces a morphism $h : K^1 \to K$ in $D^b_c(A \times X)$. Then h is a morphism in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Let K^0 be the image of h in the abelian category $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. By Fact 2.3.1 1, the functor $\mathrm{Perv}(A \times X/X) \to \mathrm{Perv}(A_\eta)$ is exact. Then $K^0|_{A_\eta}$ is the image of $h|_{A_\eta} : K^1|_{A_\eta} \to K|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$.

Because $P^\chi(A_\eta)$ is an abelian subcategory of $\mathrm{Perv}(A_\eta)$, by Condition 2, the image of $h|_{A_\eta}$ in $P^\chi(A_\eta)$ is still $K^0|_{A_\eta}$. As the functor (12) is exact, the image of $\omega_\chi(h|_{A_\eta}) : \omega_\chi(K^1|_{A_\eta}) \to \omega_\chi(K|_{A_\eta})$ is $\omega_\chi(K^0|_{A_\eta})$. Combining (15), (16), (17) with (18), one gets $\omega_\chi(K^0|_{A_\eta}) = \omega_{\chi_\eta}(K|_{A_\eta})^{\Gamma_{k(\eta)}}$. □

5.3 Normality

By [JKLM23, Thm. 4.3], for every character $\chi \in \mathcal{C}(A)$, the geometric generic Tannakian group $G_{\omega_\chi}(K|_{A_{\bar\eta}})$ is a normal closed subgroup of the generic Tannakian group $G_{\omega_\chi}(K|_{A_\eta})$. Theorem 5.3.1 shows that for uncountably many characters, the corresponding monodromy group is also a normal closed subgroup of the generic Tannakian group.

For every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$ and every $\chi_\ell \in \mathcal{C}(A)_\ell$, set $\chi = \chi_{\ell'}\chi_\ell$.

Theorem 5.3.1. *Assume $K \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ and $\dim A > 0$. Then for every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, there is an* uncountable *subset $E \subset \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell)$, such that for every $\chi_\ell \in E$,*

- *one has $K|_{A_\eta} \in P^\chi(A_\eta)$,*
- *the algebraic group $G_{\omega_\chi}(K|_{A_\eta})$ is reductive,*
- *and $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ is a* normal *closed subgroup of $G_{\omega_\chi}(K|_{A_\eta})$.*

We sketch the proof of Theorem 5.3.1. For every representation V of the Tannakian group $G(K|_{A_\eta})$ and every $\chi_{\ell'} \in \mathcal{C}(A)_{\ell'}$, by Theorem 1.2.3, there is a strict Zariski closed subset B_V of the cotorus $\mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in (\mathcal{C}(A)_\ell \setminus B_V)(\bar{\mathbb{Q}}_\ell)$, the monodromy invariant $V^{\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)}$ is a $G(K|_{A_\eta})$-subrepresentation. Choose $E = \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus \cup_V B_V(\bar{\mathbb{Q}}_\ell)$. From Lemma 5.0.1, normality holds when $\chi_\ell \in E$.

Proof. Both $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ and $G_{\omega_\chi}(K|_{A_\eta})$ depend only on the generic fiber of $p_X : A \times X \to X$. Therefore, shrinking X to a nonempty open subset does not change them. Thus, one may assume that X is smooth.

Claim 5.3.2. The object $K|_{A_\eta} \in \mathrm{Perv}(A_\eta)$ is semisimple.

From Claim 5.3.2 and Lemma 5.3.5 1, the object $K|_{A_\eta} \in \bar{P}(A_\eta)$ is also semisimple. Therefore, a (hence every) Tannakian group of the neutral Tannakian category $\langle K|_{A_\eta}\rangle (\subset \bar{P}(A_\eta))$ is a *reductive*, algebraic group over $\bar{\mathbb{Q}}_\ell$. Then by Lemma 5.3.4, there is a countable sequence of objects $\{\bar{K}_i\}_{i\geq 1}$, such that every object of $\langle K|_{A_\eta}\rangle$ is isomorphic to some $\bar{K}_i$. To apply Theorem 1.2.3, we need semisimple objects of $D^b_c(A \times X)$.

Claim 5.3.3. For every object $N \in \langle K|_{A_\eta}\rangle$, there is $L \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ that is semisimple in $D^b_c(A \times X)$, such that $L|_{A_\eta}$ isomorphic to N in $\bar{P}(A_\eta)$.

From Claim 5.3.3, for every integer $i \geq 1$, there is $K_i \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ that is semisimple in $D^b_c(A \times X)$ with $K_i|_{A_\eta}$ isomorphic to $\bar{K}_i$ in $\bar{P}(A_\eta)$. From smoothness of X and Theorem 1.2.3, there is a subobject $K_i^0 \subset K_i$ in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ and a strict Zariski closed subset $B_i \subset \mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in (\mathcal{C}(A)_\ell \setminus B_i)(\bar{\mathbb{Q}}_\ell)$, one has $K_i|_{A_\eta} \in P^\chi(A_\eta)$ and

$$\omega_{\chi_\eta}(K_i|_{A_\eta})^{\Gamma_{k(\eta)}} = \omega_{\chi_\eta}(K_i^0|_{A_\eta}). \tag{19}$$

Set $E := \mathcal{C}(A)_\ell(\bar{\mathbb{Q}}_\ell) \setminus \cup_{i\geq 1} B_i(\bar{\mathbb{Q}}_\ell)$. From Lemma 3.3.10 and the assumption $\dim A > 0$, the set E is uncountable. For every $\chi_\ell \in E$, one has $K|_{A_\eta} \in P^\chi(A_\eta)$. For every $i \geq 1$, by $\chi_\ell \notin B_i(\bar{\mathbb{Q}}_\ell)$ and (19), the subspace $\omega_{\chi_\eta}(K_i|_{A_\eta})^{\mathrm{Mon}(K|_{A_\eta},\chi_\eta)}$ is $G_{\omega_\chi}(K|_{A_\eta})$-stable. By Theorem 5.1.1 and Lemma 5.0.1, the subgroup $\mathrm{Mon}(K|_{A_\eta}, \chi_\eta)$ of $G_{\omega_\chi}(K|_{A_\eta})$ is *normal*. □

Proof of Claim 5.3.2. For every subobject $M \subset K|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$, by Fact 2.3.5 and the smoothness of X, there is a subobject $K' \subset K$ in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ with $K'|_{A_\eta} = M$. By Lemma 2.3.7, the morphism $K'[\dim X] \to K[\dim X]$ is a monomorphism in $\mathrm{Perv}(A \times X)$. Because K is semisimple in $D^b_c(A \times X)$, its shift $K[\dim X]$ is semisimple in $\mathrm{Perv}(A \times X)$. Thus, there is a subobject $N \subset K[\dim X]$ in $\mathrm{Perv}(A \times X)$ with

$$K[\dim X] = (K'[\dim X]) \oplus N.$$

Then $K = K' \oplus (N[-\dim X])$ in $D^b_c(A \times X)$. For every integer j, let ${}^{p/X}\mathcal{H}^j : D^b_c(A \times X) \to \mathrm{Perv}(A \times X/X)$ be the j-th cohomology functor associated with the relative perverse t-structure. If $j \neq 0$, then

$$0 = {}^{p/X}\mathcal{H}^j(K) = 0 \oplus {}^{p/X}\mathcal{H}^j(N[-\dim X])$$

in $\mathrm{Perv}(A \times X/X)$. Hence ${}^{p/X}\mathcal{H}^j(N[-\dim X]) = 0$ and

$$N[-\dim X] \in \mathrm{Perv}(A \times X/X).$$

Consequently, $K|_{A_\eta} = M \oplus (N|_{A_\eta}[-\dim X])$ in $\mathrm{Perv}(A_\eta)$. By [BBDG82, Thm. 4.3.1 (i)], the abelian category $\mathrm{Perv}(A_\eta)$ is Noetherian and Artinian. As every subobject of $K|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$ admits a direct complement, the semisimplicity follows from Lemma 4.1.2 2. □

Proof of Claim 5.3.3. From Lemma 5.3.4, the object $N \in \bar{P}(A_\eta)$ is semisimple. There is an integer $n \geq 0$ such that N is a subquotient of $(K|_{A_\eta} \oplus K|_{A_\eta}^\vee)^{*n}$ in $\bar{P}(A_\eta)$.

We "globalize" the fiberwise convolution functors as follows. Define a bifunctor

$$\begin{aligned} *_X : &D_c^b(A \times X) \times D_c^b(A \times X) \to D_c^b(A \times X), \\ &(-, +) \mapsto R(m \times \mathrm{Id}_X)_*(p_{13}^* - \otimes^L p_{23}^* +), \end{aligned} \tag{20}$$

where p_{ij} are the projections on $A \times A \times X$. By the proper base change theorem, for every $x \in X(k)$, one has $(- *_X +)|_{A_x} \xrightarrow{\sim} (-|_{A_x}) * (+|_{A_x})$ as bifunctors $D_c^b(A \times X) \times D_c^b(A \times X) \to D_c^b(A_x)$. Therefore, one has $(- *_X +)|_{A_\eta} \xrightarrow{\sim} (-|_{A_\eta}) * (+|_{A_\eta})$ as bifunctors $D_c^b(A \times X) \times D_c^b(A \times X) \to D_c^b(A_\eta)$.

The bifunctor (20) restricts to a bifunctor $D^{\mathrm{ULA}}(A \times X/X) \times D^{\mathrm{ULA}}(A \times X/X) \to D^{\mathrm{ULA}}(A \times X/X)$. Indeed, for any $K', K'' \in D^{\mathrm{ULA}}(A \times X/X)$, by Fact 2.2.2 6, one has

$$p_{13}^* K' \otimes^L p_{23}^* K'' \in D^{\mathrm{ULA}}(A \times A \times X/X).$$

By Fact 2.2.2 4, one gets $K' *_X K'' \in D^{\mathrm{ULA}}(A \times X/X)$.

Set $K^\vee := ([-1]_A \times \mathrm{Id}_X)^* \mathbb{D}_{A \times X/X} K$. By Fact 2.3.1 3, one has $K^\vee \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ and $(K^\vee)|_{A_\eta} = (K|_{A_\eta})^\vee$. Then

$$(K \oplus K^\vee)^{*_X n}) \in D^{\mathrm{ULA}}(A \times X/X).$$

Set $M := {}^{p/X}\mathcal{H}^0((K \oplus K^\vee)^{*_X n}) \in \mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$. Then $M|_{A_\eta} = {}^p\mathcal{H}^0([K|_{A_\eta} \oplus (K|_{A_\eta})^\vee]^{*n})$ in $\mathrm{Perv}(A_\eta)$. By Lemma 5.3.5 3, there is a semisimple subquotient L' of $M|_{A_\eta}$ in $\mathrm{Perv}(A_\eta)$, whose image in $\bar{P}(A_\eta)$ is N. By smoothness of X and Fact 2.3.5, there is a semisimple subquotient L of M in $\mathrm{Perv}^{\mathrm{ULA}}(A \times X/X)$ with $L|_{A_\eta} = L'$. By smoothness of X and Lemma 2.3.10 2, the object $L[\dim X]$ is semisimple in $\mathrm{Perv}(A \times X)$. Then L is semisimple in $D_c^b(A \times X)$. ☐

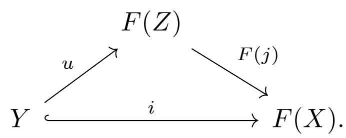
For a category $\mathcal{C}$, let $\mathcal{C}/\sim$ be the class of isomorphism classes of objects in $\mathcal{C}$.

Lemma 5.3.4. *Let $(\mathcal{C}, \otimes)$ be a neutral Tannakian category over k with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vec}_k$. Assume that $\mathrm{Aut}^\otimes(\mathcal{C}, \omega)$ is a reductive, algebraic group over k. Then the underlying abelian category is semisimple, and $\mathcal{C}/\sim$ is countable.*

Proof. Set $G = \mathrm{Aut}^\otimes(\mathcal{C}, \omega)$. Let $\mathrm{Rep}(G)$ be the category of k-rational representations of G. Then $\mathcal{C}$ is equivalent to $\mathrm{Rep}(G)$. Because k has characteristic zero, by [Mil17, Cor. 22.43], the abelian category $\mathrm{Rep}(G)$ is semisimple. As k is algebraically closed, by [AHR20, Thm. 2.16], there is an at most countable set X^+ and for every $\lambda \in X^+$, a unital k-algebra $\mathscr{A}^\lambda$ with the following property: The set $\mathrm{Irr}(G)$ of isomorphism classes of simple objects of $\mathrm{Rep}(G)$ is in bijection with the set of pairs (λ, E), where $\lambda \in X^+$ and E is an isomorphism class of simple left $\mathscr{A}^\lambda$-modules. For every $\lambda \in X^+$, from [AHR20, Lem. 2.19], the algebra $\mathscr{A}^\lambda$ is semisimple. Then by [Lan02, XVII, Thm. 4.3, Cor. 4.5], the set of isomorphism classes of simple left $\mathscr{A}^\lambda$-modules is finite. Therefore, $\mathrm{Irr}(G)$ is at most countable. Consequently, $\mathrm{Rep}(G)/\sim$ is countable. ☐

Lemma 5.3.5. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Consider the quotient functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{B}$.*

1. *Let $X \in \mathcal{A}$. Let $i : Y \to F(X)$ be a monomorphism in $\mathcal{A}/\mathcal{B}$. Then there is a monomorphism $j : Z \to X$ in $\mathcal{A}$ and an isomorphism $u : Y \to F(Z)$ in $\mathcal{A}/\mathcal{B}$ fitting into a commutative diagram in $\mathcal{A}/\mathcal{B}$*



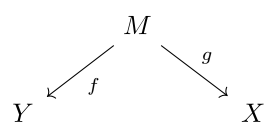
   Dually, up to isomorphism every quotient in $\mathcal{A}/\mathcal{B}$ lifts to a quotient in $\mathcal{A}$. In particular, if $X \in \mathcal{A}$ is a simple object, then $F(X)$ is either simple or zero in $\mathcal{A}/\mathcal{B}$.

2. *Let $V \in \mathcal{A}$ be a Noetherian and Artinian object. If $F(V)$ is simple in $\mathcal{A}/\mathcal{B}$, then there is a simple subquotient W of V in $\mathcal{A}$ such that $F(W)$ is isomorphic to $F(V)$ in $\mathcal{A}/\mathcal{B}$.*

3. *Assume that $\mathcal{A}$ is Noetherian and Artinian. Let $X \in \mathcal{A}$. If Y is a simple subquotient of $F(X)$ in $\mathcal{A}/\mathcal{B}$, then there is a simple subquotient W of X, with $F(W)$ isomorphic to Y in $\mathcal{A}/\mathcal{B}$.*

Proof.

1. By the construction in the proof of [Sta24, Tag 02MS] and the right calculus of fractions in [Sta24, Tag 04VB], there is a diagram



   in $\mathcal{A}$, such that $F(f)$ is an isomorphism and $F(g) = i \circ F(f)$ in $\mathcal{A}/\mathcal{B}$. Therefore, $F(g)$ is a monomorphism. Since F is exact, one has $F(\ker(g)) = \ker(F(g)) = 0$, so $\ker(g) \in \mathcal{B}$. Let $q : M \to M/\ker(g)$ be the epimorphism in $\mathcal{A}$, and let $j : M/\ker(g) \to X$ be the monomorphism in $\mathcal{A}$ induced by g. Then $F(q)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Set $u : Y \to F(M/\ker(g))$ to be the morphism $F(q) \circ F(f)^{-1}$ in $\mathcal{A}/\mathcal{B}$. Then u is an isomorphism with the stated property.

2. Let $\mathcal{P}$ be the family of subobjects V' of V in $\mathcal{A}$ with $V/V' \in \mathcal{B}$. Then $\mathcal{P}$ is nonempty since $V \in P$. As V is Artinian in $\mathcal{A}$, there is a minimal object $U \in \mathcal{P}$. Moreover, the morphism $F(U) \to F(V)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Let $\mathcal{Q}$ be the family of subobjects of $U \in \mathcal{A}$ lying in $\mathcal{B}$. Then $\mathcal{Q}$ is nonempty since $0 \in \mathcal{Q}$. As V is Noetherian in $\mathcal{A}$, so is U. Thus, $\mathcal{Q}$ has a maximal object U_0. Then $W := U/U_0$ is a subquotient of $V \in \mathcal{A}$ and the

morphism $F(U) \to F(W)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. In particular, $W \neq 0$ in $\mathcal{A}$.

We claim that W is simple in $\mathcal{A}$. Indeed, let $U' \to W$ be a subobject in $\mathcal{A}$. Then there is a subobject U'' of U in $\mathcal{A}$ containing U_0 with $U''/U_0 = U'$. As $F(U'')$ is a subobject of a simple object $F(U)$ in $\mathcal{A}/\mathcal{B}$, either the morphism $F(U'') \to F(U)$ is an isomorphism or $F(U'') = 0$. If $F(U'') = 0$, then $U'' \in \mathcal{B}$ and $U'' \in \mathcal{Q}$. Since U_0 is maximal in $\mathcal{Q}$, one has $U_0 = U''$, so $U' = 0$. If $F(U'') \to F(U)$ is an isomorphism, then $U/U'' \in \mathcal{B}$. Since the sequence

$$0 \to U/U'' \to V/U'' \to V/U \to 0$$

is exact in $\mathcal{A}$, and $\mathcal{B}$ is closed under extensions, one gets $V/U'' \in \mathcal{B}$ and $U'' \in \mathcal{P}$. Since U is minimal in $\mathcal{P}$, one has $U'' = U$. The morphism $U' \to W$ is thus an isomorphism in $\mathcal{A}$. The claim is proved.

3. By 1, there is a subquotient Z of X in $\mathcal{A}$ with $F(Z)$ isomorphic to Y. Then $F(Z)$ is simple in $\mathcal{A}/\mathcal{B}$. By assumption, Z is Noetherian and Artinian in $\mathcal{A}$. Thus from 2, there is a simple subquotient W of Z in $\mathcal{A}$ with $F(W)$ isomorphic to $F(Z)$ and to Y in $\mathcal{A}/\mathcal{B}$.

□

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