Quasi-coherent sheaves on complex analytic spaces

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Abstract

We show that in the category of analytic sheaves on a complex analytic space, the full subcategory of quasi-coherent sheaves is an abelian subcategory.

1 Introduction

Let (X, O_X) be a ringed space. The category of O_X -modules is denoted by $Mod(O_X)$.

Definition 1.1. An O_X -module F is called *quasi-coherent* if for every $x \in X$, there is an open neighborhood $U \subset X$ of x, two sets I, J and a morphism $O_U^{\oplus J} \to O_U^{\oplus I}$ with cokernel isomorphic to $F|_U$. The full subcategory of $Mod(O_X)$ of quasi-coherent modules is denoted by Qch(X).

According to [Sta24, Tag 01BD], in general Qch(X) is not an abelian category. By [Sta24, Tag 06YZ], if X is a scheme, then Qch(X) is a weak Serre subcategory (in the sense of [Sta24, Tag 02MO (2)]) of $Mod(O_X)$. We show a complex analytic analog of this result, contrary to a guess made in [hm].

Theorem 1.2. If X is a complex analytic space, then $Qch(X) \subset Mod(O_X)$ is a weak Serre subcategory. In particular, it is an abelian subcategory.

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2 Preliminaries

Let (X, O_X) be a ringed space. Every $O_X(X)$ -module induces naturally a quasicoherent O_X -module. **Example 2.1** ([Sta24, Tag 01BI]). Let $f : (X, O_X) \to (\{*\}, O_X(X))$ be the morphism of ringed spaces, with $f : X \to \{*\}$ the unique map and $f_*^{\natural} : O_X(X) \to O_X(X)$ the identity. Then f is flat. For an $O_X(X)$ -module M, its pullback f^*M is called the sheaf associated with M. This O_X -module is quasi-coherent. The functor $f^* : \operatorname{Mod}(O_X(X)) \to \operatorname{Mod}(O_X)$ is called *localization* and denoted by $\tilde{\cdot}$.

From [Gro60, 4.1.1], on a scheme the direct sum of any family of quasicoherent modules is quasi-coherent. It fails for complex manifolds, shown by Example 2.2.

Example 2.2. [hs] Let $X \subset \mathbb{C}$ be the unit open disk. For every integer $n \geq 2$, Gabber ([Con06, Eg. 2.1.6]) constructs a locally free (hence quasi-coherent) O_X -module F_n of infinite rank, such that for every open subset $U \subset X$ containing $\{\pm 1/n\}$, one has $\Gamma(U, F_n) = 0$. We prove that $F := \bigoplus_{n\geq 2} F_n$ is not quasi-coherent.

Assume the contrary. Then there is an open neighborhood V of $0 \in X$, a set I and a quotient morphism $q: O_V^{\oplus I} \to F|_V$. There is an integer $N \ge 2$ with $\{\pm 1/N\} \subset V$. Let $p: F|_V \to F_N|_V$ be the quotient morphism. Because $\operatorname{Hom}_{\operatorname{Mod}(O_V)}(O_V, F_N|_V) = \Gamma(V, F_N) = 0$, the morphism pq = 0. However, it contradicts $F_N|_V \neq 0$.

Let X be a complex analytic space in the sense of [GR04, p.18]. For an inclusion $i: K \to X$ of a compact subset, let $O_K = i^{-1}O_X$. Then O_K is naturally a sheaf of rings on K.

Definition 2.3. A compact subset $K \subset X$ is a *Stein compactum*, if K has a fundamental system of open neighborhoods that are Stein subspaces of X. A Stein compactum K is *Noetherian* if $O_K(K)$ is a Noetherian ring.

Fact 2.4 ([Fri67, Thm. I, 9; Rem. I, 10]). Every $x \in X$ admits a neighborhood which is a Noetherian Stein compactum in X.

Lemma 2.5. Let F be an O_X -module. Then the following conditions are equivalent:

- 1. ([BBBP07, Def. 5.1]) Every $x \in X$ admits a neighborhood K which is a Noetherian Stein compactum, such that $F|_K$ is associated with an $O_K(K)$ -module.
- 2. The O_X -module F is quasi-coherent.

Proof.

• Assume Condition 1. For every $x \in X$, take such a K and suppose that $F|_K$ is associated with an $O_K(K)$ -module M. There are sets I, J and an exact sequence $O_K(K)^{\oplus I} \to O_K(K)^{\oplus J} \to M \to 0$ in the category of $O_K(K)$ -modules. By [Sta24, Tag 01BH], it induces an exact sequence $O_{K}^{\oplus I} \to O_{K}^{\oplus J} \to F|_K \to 0$ in Mod (O_K) . Then the $O_{K^{\circ}}$ -module $F|_{K^{\circ}}$ is quasi-coherent. Thus, Condition 2 is proved.

• Assume Condition 2. Because X is locally compact Hausdorff, for every $x \in X$, by [Sta24, Tag 01BK], there is an open neighborhood $U \subset X$ of x such that $F|_U$ is associated with a $\Gamma(U, O_X)$ -module. From Fact 2.4, there is a neighborhood K of $x \in U$ which is a Noetherian Stein compactum. By [Sta24, Tag 01BJ] applied to the morphism $(K, O_K) \to (U, O_U)$ of ringed spaces, $F|_K$ is associated with an $O_K(K)$ -module. Thus, Condition 1 is proved.

Lemma 2.6. Let K be a Noetherian Stein compactum in X.

- 1. The natural transformation $\mathrm{Id} \to \Gamma(K,\tilde{\cdot})$ of functors $\mathrm{Mod}(O_K(K)) \to \mathrm{Mod}(O_K(K))$ is an isomorphism.
- 2. The localization functor $\tilde{\cdot}$: $\operatorname{Mod}(O_K(K)) \to \operatorname{Mod}(O_K)$ is exact, fully faithful.
- 3. For every $O_K(K)$ -module M and every integer q > 0, one has $H^q(K, \tilde{M}) = 0$.

Proof.

1. Let M be an $O_K(K)$ -module. We prove that the morphism $M \to \Gamma(K, M)$ is an isomorphism. Assume first that M is finitely generated. Then the result follows from [Tay02, p.299]. Assume now that M is arbitrary. Let $\{M_i\}_{i \in I}$ be the family of all finitely generated submodules of M. This family is directed in the inclusion relation and

$$M = \sum_{i \in I} M_i. \tag{1}$$

By [Sta24, Tag 01BH (4)], the localization functor preserves colimits. Therefore,

$$M = \operatorname{colim}_{i \in I} M_i. \tag{2}$$

By [God58, Thm. 4.12.1], one has

$$\Gamma(K, M) = \operatorname{colim}_{i \in I} \Gamma(K, M_i) = \operatorname{colim}_{i \in I} M_i = M.$$

2. The exactness is proved in [Tay02, Prop. 11.9.3 (ii)]. For any $M, N \in Mod(O_K(K))$, we prove that the natural morphism

$$\operatorname{Hom}_{O_K(K)}(M,N) \to \operatorname{Hom}_{O_K}(\tilde{M},\tilde{N})$$
(3)

is an isomorphism.

Assume first that M is finitely generated. As the ring $O_K(K)$ is Noetherian, the $O_K(K)$ -module M is of finite presentation. Then by [GW20, Exercise 7.20 (b)], one has $\operatorname{Hom}_{O_K(K)}(M, N) = \mathcal{H}om_{O_K}(\tilde{M}, \tilde{N})$. By Point 1, the morphism (3) is an isomorphism. Assume now that M is arbitrary. By (1) and (2), the morphism (3) is the inverse limit of the morphisms $\operatorname{Hom}_{O_K(K)}(M_i, N) \to$ $\operatorname{Hom}_{O_K}(\tilde{M}, \tilde{N})$, each of which is an isomorphism. 3. When M is finitely generated, it follows from [Tay02, Prop. 11.9.2] and [Car57, Thm. 1 (B)]. Assume now that M is arbitrary. By (2) and [God58, Thm. 4.12.1], one has $H^q(K, \tilde{M}) = \operatorname{colim}_i H^q(K, \tilde{M}_i) = 0$.

3 Proof of Theorem 1.2

1. For every morphism $f: F \to G$ in Qch(X), we prove that ker(f), coker(f) in $Mod(O_X)$ lie in Qch(X).

For every $x \in X$, by Lemma 2.5, there is a neighborhood A (resp. B) of $x \in X$ which is a Noetherian Stein compactum and an $O_A(A)$ -module M(resp. $O_B(B)$ -module N), such that $F|_A$ (resp. $G|_B$) is associated with M(resp. N). By Fact 2.4, there is a neighborhood C of $x \in A^\circ \cap B^\circ$ which is a Noetherian Stein compactum. From [Sta24, Tag 01BJ], $F|_C$ (resp. $G|_C$) is associated with $M \otimes_{O_A(A)} O_C(C)$ (resp. $N \otimes_{O_B(B)} O_C(C)$). By Lemma 2.6 2, there is a morphism

$$\phi: M \otimes_{O_A(A)} O_C(C) \to N \otimes_{O_B(B)} O_C(C)$$

in $\operatorname{Mod}(O_C(C))$ whose localization is $f|_C : F|_C \to G|_C$. The restriction functor $\operatorname{Mod}(O_X) \to \operatorname{Mod}(O_{C^\circ})$ is exact, so $\ker(f)|_{C^\circ}$ (resp. $\operatorname{coker}(f)|_{C^\circ}$) is the localization of $\ker(\phi \otimes_{O_C(C)} \operatorname{Id}_{O_X(C^\circ)})$ (resp. $\operatorname{coker}(\phi \otimes_{O_C(C)} \operatorname{Id}_{O_X(C^\circ)})$) in $\operatorname{Mod}(O_X(C^\circ))$. Therefore, the O_X -modules $\ker(f)$, $\operatorname{coker}(f)$ are quasi-coherent.

2. Let

$$0 \to F' \to F \to F'' \to 0 \tag{4}$$

be a short exact sequence in $Mod(O_X)$, with F', F'' quasi-coherent. We prove that F is quasi-coherent.

By Lemma 2.5, for every $x \in X$, there is a neighborhood K' (resp. K'') of x which is a Noetherian Stein compactum, and an $O_{K'}(K')$ -module M' (resp. $O_{K''}(K'')$ -module M'') whose localization is $F'|_{K'}$ (resp. $F''|_{K''}$). By Fact 2.4, there is a neighborhood K of $x \in K'^{\circ} \cap K''^{\circ}$ that is a Noetherian Stein compactum. From [Sta24, Tag 01BJ], $F'|_{K}$ (resp. $F''|_{K}$) is associated with the $O_{K}(K)$ -module $M' \otimes_{O_{K'}(K')} O_{K}(K)$ (resp. $M'' \otimes_{O_{K''}(K'')} O_{K}(K)$).

Let $P = \Gamma(K, F)$. By Lemma 2.6 1 and 3, the sequence (4) induces a short exact sequence in $Mod(O_K(K))$:

$$0 \to M' \otimes_{O_{K'}(K')} O_K(K) \to P \to M'' \otimes_{O_{K''}(K'')} O_K(K) \to 0.$$

From Lemma 2.6 2, by localization it induces a shot exact sequence in $Mod(O_K)$:

$$0 \to M' \otimes_{O_{K'}(K')} O_K(K) \to \tilde{P} \to M'' \otimes_{O_{K''}(K'')} O_K(K) \to 0$$

By restricting to K° and [Sta24, Tag 01BJ], one has a commutative diagram

in $\operatorname{Mod}(O_{K^{\circ}})$. The vertical morphisms are given by the canonical morphism $P \otimes_{O_K(K)} O_X(K^{\circ}) \to \Gamma(K^{\circ}, F)$ in $\operatorname{Mod}(O_X(K^{\circ}))$, and the adjunction of $\tilde{\cdot}$: $\operatorname{Mod}(O_X(K^{\circ})) \to \operatorname{Mod}(O_{K^{\circ}})$ and $\Gamma(K^{\circ}, \cdot) : \operatorname{Mod}(O_{K^{\circ}}) \to \operatorname{Mod}(O_X(K^{\circ}))$. The rows are exact, and the two outside vertical arrows are isomorphisms. By the five lemma, the middle vertical morphism is an isomorphism. By Example 2.1, the $O_{K^{\circ}}$ -module $F|_{K^{\circ}}$ is quasi-coherent. Consequently, F is quasi-coherent.

By 1, 2 and [Sta24, Tag 0754], Qch(X) is a weak Serre subcategory of $Mod(O_X)$.

References

- [BBBP07] Oren Ben-Bassat, Jonathan Block, and Tony Pantev. Noncommutative tori and Fourier–Mukai duality. Compositio Mathematica, 143(2):423–475, 2007.
- [Car57] Henri Cartan. Variétés analytiques réelles et variétés analytiques complexes. Bulletin de la Société Mathématique de France, 85:77– 99, 1957.
- [Con06] Brian Conrad. Relative ampleness in rigid geometry. Annales de l'institut Fourier, 56(4):1049–1126, 2006.
- [Fri67] Jacques Frisch. Points de platitude d'un morphisme d'espaces analytiques complexes. *Inventiones mathematicae*, 4:118–138, 1967.
- [God58] Roger Godement. Topologie algébrique et théorie des faisceaux. Publications de l'Institut de Mathématique de L'Université de Strasbourg XIII, 1, 1958.
- [GR04] Hans Grauert and Reinhold Remmert. *Theory of Stein spaces*, volume 236. Springer Science & Business Media, 2004.
- [Gro60] Alexander Grothendieck. Éléments de géométrie algébrique : I. Le langage des schémas. Publications Mathématiques de l'IHÉS, 4:5– 228, 1960.
- [GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes.* Springer, 2nd edition, 2020.
- [hm] Z. M (https://mathoverflow.net/users/176381/z m). Does quasicoherent modules over $(X^{an}, \mathcal{O}_{X^{an}})$ have enough injectives? https: //mathoverflow.net/q/421073 (version: 2022-04-25).

- [hs] Jason Starr (https://mathoverflow.net/users/13265/jason starr). Is an infinite direct sum of quasi-coherent \mathcal{O}_X -modules quasi-coherent on a complex manifold? https://mathoverflow.net/q/453047 (version: 2023-08-18).
- [Sta24] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2024.
- [Tay02] Joseph L Taylor. Several complex variables with connections to algebraic geometry and Lie groups, volume 46. American Mathematical Society Providence, 2002.