Quasi-coherent sheaves on complex analytic spaces

Haohao LIU

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Abstract

We show that in the category of analytic sheaves on a complex analytic space, the full subcategory of quasi-coherent sheaves is an abelian subcategory.

1 Introduction

Let (X, O_X) be a ringed space. The category of O_X-modules is denoted by $\mathrm{Mod}(O_X)$.

Definition 1.1. An O_X-module F is called *quasi-coherent* if for every $x \in X$, there is an open neighborhood $U \subset X$ of x, two sets I, J and a morphism $O_U^{\oplus J} \to O_U^{\oplus I}$ whose cokernel is isomorphic to $F|_U$. The full subcategory of $\mathrm{Mod}(O_X)$ comprised of quasi-coherent modules is denoted by $\mathrm{Qch}(X)$.

According to [Sta23, Tag 01BD], in general $\mathrm{Qch}(X)$ is not an abelian category. If X is a scheme, then by [Sta23, Tag 06YZ], $\mathrm{Qch}(X)$ is a weak Serre subcategory (in the sense of [Sta23, Tag 02MO (2)]) of $\mathrm{Mod}(O_X)$. We show a complex analytic analog of this result.

Theorem 1.2. *If X is a complex analytic space, then the subcategory $\mathrm{Qch}(X) \subset \mathrm{Mod}(O_X)$ is weak Serre. In particular, it is an abelian subcategory.*

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2 Preliminaries

Example 2.1 ([Sta23, Tag 01BI])**.** Let $f : (X, O_X) \to (\{*\}, O_X(X))$ be the morphism of ringed spaces with $f : X \to \{*\}$ the unique map and with $f_*^\natural : O_X(X) \to O_X(X)$ the identity. Then f is flat. For an $O_X(X)$-module M, its pullback f^*M is called the sheaf associated with M. This O_X-module is quasi-coherent. The functor $f^* : \mathrm{Mod}(O_X(X)) \to \mathrm{Mod}(O_X)$ is called the localization and denoted by $\tilde{\cdot}$.

From [Gro60, 4.1.1], on a scheme the direct sum of any family of quasi-coherent modules is quasi-coherent. It fails for complex manifolds, shown by Example 2.2.

Example 2.2. [hs] Let $X \subset \mathbb{C}$ be the unit open disk. For every integer $n \geq 2$, Gabber ([Con06, Eg. 2.1.6]) constructs a locally free (hence quasi-coherent) O_X-module F_n of infinite rank, such that for every open subset $U \subset X$ containing $\{\pm 1/n\}$, one has $\Gamma(U, F_n) = 0$. We claim that $F := \oplus_{n\geq 2} F_n$ is not quasi-coherent.

Assume the contrary. Then there is an open neighborhood V of $0 \in X$, a set I and a quotient morphism $q : O_V^{\oplus I} \to F|_V$. There is an integer $N \geq 2$ with $\{\pm 1/N\} \subset V$. Let $p : F|_V \to F_N|_V$ be the quotient morphism. Because $\mathrm{Hom}_{\mathrm{Mod}(O_V)}(O_V, F_N|_V) = \Gamma(V, F_N) = 0$, the morphism $pq = 0$. However, it contradicts $F_N|_V \neq 0$. The claim is proved.

Let X be a complex analytic space in the sense of [GR04, p.18]. For an inclusion $i : K \to X$ of a compact subset, let $O_K = i^{-1}O_X$. Then O_K is naturally a sheaf of rings on K.

Definition 2.3. A compact subset $K \subset X$ is called a Stein compactum if K has a fundamental system of open neighborhoods that are Stein subspaces of X. A Stein compactum K is called Noetherian if $O_K(K)$ is a Noetherian ring.

Fact 2.4 ([Fri67, Thm. I, 9; Rem. I, 10])**.** *Every $x \in X$ admits a neighborhood which is a Noetherian Stein compactum in X.*

Lemma 2.5. *Let F be an O_X-module. Then the following conditions are equivalent:*

1. ([BBBP07, Def. 5.1]) *Every $x \in X$ admits a neighborhood K which is a Noetherian Stein compactum, such that $F|_K$ is associated with a $\Gamma(K, O_K)$-module.*

2. *The O_X-module F is quasi-coherent.*

Proof.

- Assume Condition 1. For every $x \in X$, take such a K and suppose that $F|_K$ is associated with a $\Gamma(K, O_K)$-module M. There is an exact sequence $\Gamma(K, O_K)^{\oplus I} \to \Gamma(K, O_K)^{\oplus J} \to M \to 0$ in the category of $\Gamma(K, O_K)$-modules. By [Sta23, Tag 01BH], it induces an exact sequence $O_K^{\oplus I} \to O_K^{\oplus J} \to F|_K \to 0$ in $\mathrm{Mod}(O_K)$. Then the O_{K°-module $F|_{K^\circ}$ is quasi-coherent. Thus, Condition 2 is proved.

- Assume Condition 2. Because X is locally compact Hausdorff, for every $x \in X$, by [Sta23, Tag 01BK], there is an open neighborhood $U \subset X$ of x

such that $F|_U$ is associated with a $\Gamma(U, O_X)$-module. From Fact 2.4, there is a neighborhood K of $x \in U$ which is a Noetherian Stein compactum. By [Sta23, Tag 01BJ] applied to the morphism $(K, O_K) \to (U, O_U)$ of ringed spaces, $F|_K$ is associated with a $\Gamma(K, O_K)$-module. Thus, Condition 1 is proved.

□

Lemma 2.6. *Let K be a Noetherian Stein compactum in X.*

1. *The natural transformation* $\mathrm{Id} \to \Gamma(K, \tilde{\cdot})$ *of functors* $\mathrm{Mod}(\Gamma(K, O_K)) \to \mathrm{Mod}(\Gamma(K, O_K))$ *is an isomorphism.*
2. *The localization functor* $\tilde{\cdot} : \mathrm{Mod}(O_K(K)) \to \mathrm{Mod}(O_K)$ *is exact, fully faithful.*
3. *For every $O_K(K)$-module M and every integer $q > 0$, one has $H^q(K, \tilde{M}) = 0$.*

Proof.

1. Let M be a $\Gamma(K, O_K)$-module. We prove that the morphism $M \to \Gamma(K, \tilde{M})$ is an isomorphism. Assume first that M is finitely generated. Then the result follows from [Tay02, p.299]. Assume now that M is arbitrary. Let $\{M_i\}_{i \in I}$ be the family of all finitely generated submodules of M. This family is directed in the inclusion relation and
$$M = \sum_{i \in I} M_i. \tag{1}$$
By [Sta23, Tag 01BH (4)], the localization functor preserves colimits. Therefore,
$$\tilde{M} = \mathrm{colim}_{i \in I} \tilde{M}_i. \tag{2}$$
By [God58, Thm. 4.12.1], one has
$$\Gamma(K, \tilde{M}) = \mathrm{colim}_{i \in I} \Gamma(K, \tilde{M}_i) = \mathrm{colim}_{i \in I} M_i = M.$$

2. The exactness is proved in [Tay02, Prop. 11.9.3 (ii)]. For any $M, N \in \mathrm{Mod}(O(K))$, we prove that the natural morphism
$$\mathrm{Hom}_{O(K)}(M, N) \to \mathrm{Hom}_{O_K}(\tilde{M}, \tilde{N}) \tag{3}$$
is an isomorphism.

   Assume first that M is finitely generated. As the ring $O(K)$ is Noetherian, the $O(K)$-module M is of finite presentation. Then by [GW20, Exercise 7.20 (b), p.205], one has $\mathrm{Hom}_{O(K)}(M, N) = \mathcal{H}om_{O_K}(\tilde{M}, \tilde{N})$. By Point 1, the morphism (3) is an isomorphism. Assume now that M is arbitrary. By (1) and (2), the morphism (3) is the inverse limit of the morphisms $\mathrm{Hom}_{O(K)}(M_i, N) \to \mathrm{Hom}_{O_K}(\tilde{M}_i, \tilde{N})$, each of which is an isomorphism.

3. When M is finitely generated, it follows from [Tay02, Prop. 11.9.2] and [Car57, Thm. 1 (B)]. Assume now that M is arbitrary. By (2) and [God58, Thm. 4.12.1], one has $H^q(K, \tilde{M}) = \mathrm{colim}_i H^q(K, \tilde{M_i}) = 0$.

□

3 Proof of Theorem 1.2

- For every morphism $f : F \to G$ in $\mathrm{Qch}(X)$, we prove that $\ker(f), \mathrm{coker}(f)$ in $\mathrm{Mod}(O_X)$ lie in $\mathrm{Qch}(X)$.

For every $x \in X$, by Lemma 2.5, there is a neighborhood A (resp. B) of $x \in X$ which is a Noetherian Stein compactum and an $O_A(A)$-module M (resp. $O_B(B)$-module N), such that $F|_A$ (resp. $G|_B$) is associated with M (resp. N). By Fact 2.4, there is a neighborhood C of $x \in A^\circ \cap B^\circ$ which is a Noetherian Stein compactum. From [Sta23, Tag 01BJ], $F|_C$ (resp. $G|_C$) is associated with $M \otimes_{O_A(A)} O_C(C)$ (resp. $N \otimes_{O_B(B)} O_C(C)$). By Lemma 2.6 2, there is a morphism $\phi : M \otimes_{O_A(A)} O_C(C) \to N \otimes_{O_B(B)} O_C(C)$ in $\mathrm{Mod}(O_C(C))$ whose localization is $f|_C : F|_C \to G|_C$. The restriction functor $\mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{C^\circ})$ is exact, so $\ker(f)|_{C^\circ}$ (resp. $\mathrm{coker}(f)|_{C^\circ}$) is the localization of $\ker(\phi \otimes_{O_C(C)} \mathrm{Id}_{O_X(C^\circ)})$ (resp. $\mathrm{coker}(\phi \otimes_{O_C(C)} \mathrm{Id}_{O_X(C^\circ)})$) in $\mathrm{Mod}(O_X(C^\circ))$. Therefore, the O_X-modules $\ker(f), \mathrm{coker}(f)$ are quasi-coherent.

- Let
$$0 \to F' \to F \to F'' \to 0 \tag{4}$$
be a short exact sequence in $\mathrm{Mod}(O_X)$, with F', F'' quasi-coherent. We prove that F is quasi-coherent.

For every $x \in X$, there is a neighborhood K' (resp. K'') of x which is a Noetherian Stein compactum, and an $O_{K'}(K')$-module M' (resp. $O_{K''}(K'')$-module M'') whose localization is $F'|_{K'}$ (resp. $F''|_{K''}$). By Fact 2.4, there is a neighborhood K of $x \in K'^\circ \cap K''^\circ$ that is a Noetherian Stein compactum. From [Sta23, Tag 01BJ], $F'|_K$ (resp. $F''|_K$) is associated with $M' \otimes_{O_{K'}(K')} O_K(K)$ (resp. $M'' \otimes_{O_{K''}(K'')} O_K(K)$).

Let $P = \Gamma(K, F)$. By Lemma 2.6 1 and 3, the sequence (4) induces a short exact sequence in $\mathrm{Mod}(O_K(K))$:

$$0 \to M' \otimes_{O_{K'}(K')} O_K(K) \to P \to M'' \otimes_{O_{K''}(K'')} O_K(K) \to 0.$$

From Lemma 2.6 2, its localization induces a shot exact sequence in $\mathrm{Mod}(O_K)$:

$$0 \to M' \widetilde{\otimes_{O_{K'}(K')} O_K}(K) \to \tilde{P} \to M'' \widetilde{\otimes_{O_{K''}(K'')} O_K}(K) \to 0.$$

By restriction to K° and [Sta23, Tag 01BJ], one has a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{M' \otimes_{O_{K'}(K')} O_X(K^\circ)} & \longrightarrow & \widetilde{P \otimes_{O_K(K)} O_X(K^\circ)} & \longrightarrow & \widetilde{M'' \otimes_{O_{K''}(K'')} O_X(K^\circ)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F'|_{K^\circ} & \longrightarrow & F|_{K^\circ} & \longrightarrow & F''|_{K^\circ} & \longrightarrow & 0
\end{array}$$

in $\mathrm{Mod}(O_{K^\circ})$, where the vertical morphisms are given by the adjunction of $\tilde{\cdot} : \mathrm{Mod}(O_X(K^\circ)) \to \mathrm{Mod}(O_{K^\circ})$ and $\Gamma(K^\circ, \cdot) : \mathrm{Mod}(O_{K^\circ}) \to \mathrm{Mod}(O_X(K^\circ))$. The rows are exact, and the two outside vertical arrows are isomorphisms. By the five lemma, the middle vertical morphism is an isomorphism. Therefore, $F|_{K^\circ}$ is quasi-coherent. Consequently, F is quasi-coherent.

By [Sta23, Tag 0754], $\mathrm{Qch}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$.

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