Quasi-coherent GAGA

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1 Introduction

Let X be a complex algebraic variety.[1] Then the set of complex points $X(\mathbb{C})$ underlies a natural complex analytic space (in the sense of [Ser56, Déf. 1]) structure, denoted by X^{an}. When X is a projective variety, Serre [Ser56, Théorèmes 2 et 3] proves that the abelian category of (algebraic) coherent modules on X is naturally equivalent to that of (analytic) coherent modules on X^{an}. Hall [Hal23] extends the equivalence to the bounded derived category of coherent modules (Fact 2.1).

A natural question is to find analogous equivalences for the larger category of quasi-coherent sheaves on X. We show that good modules (in the sense of Kashiwara, Definition 2.2) is a analytic counterpart of quasi-coherent sheaves on algebraic varieties.

For a ringed space (X, O_X), let $\text{Mod}(O_X)$ be the abelian category of O_X-modules. Let $D(X)$ be its unbounded derived category.

For an algebraic variety (resp. a complex analytic space) X, let $\text{Qch}(X) \subset \text{Mod}(O_X)$ (resp. $\text{Good}(X) \subset \text{Mod}(O_X)$) be the full subcategory of quasi-coherent (resp. good) modules. Let $D_{\text{qc}}(X)$ (resp. $D_{\text{gd}}(X)$) be the full subcategory of $D(X)$ comprised of objects with quasi-coherent (resp. good) cohomologies.

Theorem (Proposition 3.2, Theorem 4.2)**.** *If X is proper over $\mathbb{C}$, then the analytification functor $D_{\text{qc}}(X) \to D_{\text{gd}}(X^{\text{an}})$ is an equivalence of triangulated categories.*

2 Review

We recall the work of Serre [Ser56] (known as "GAGA"), which gives an equivalence of algebraic coherent modules and analytic coherent modules on complex, projective varieties. The theory is extended to complex, proper algebraic varieties in [GR71, Exp. XII].

Let X be a complex algebraic variety. Let An (resp. Set) be the category of complex analytic spaces (resp. sets). Let Ψ_X be the functor $\text{An} \to \text{Set}$ sending a complex analytic space Y to the set $\text{Hom}_{\mathbb{C}}(Y, X)$ of morphisms of spaces

[1] An algebraic variety means a finite type, separated scheme over a field.

with a sheaf of $\mathbb{C}$-algebras. By [GR71, Exp. XII, Thm. 1.1], the functor Ψ_X is represented by a complex analytic space X^{an} (called the analytification of X) and a *flat* morphism $\psi_X \in \mathrm{Hom}_{\mathbb{C}}(X^{\mathrm{an}}, X)$. Because X is of finite type over $\mathbb{C}$, from [GR71, Exp. XII, Prop. 2.1 (viii)], the dimension of X^{an} is finite.

By [GR71, Exp. XII, 1.2], for every morphism $f : X \to Y$ of complex algebraic varieties, there is a commutative square

$$\begin{array}{ccc} X^{\mathrm{an}} & \xrightarrow{\psi_X} & X \\ \downarrow{\scriptstyle f^{\mathrm{an}}} & & \downarrow{\scriptstyle f} \\ Y^{\mathrm{an}} & \xrightarrow{\psi_Y} & Y \end{array} \tag{1}$$

in the category of ringed spaces. In other words, the analytification induces a functor $(\cdot)^{\mathrm{an}}$ from the category of complex algebraic varieties to An.

For a ringed space (Y, O_Y), let $\mathrm{Coh}(Y) \subset \mathrm{Mod}(O_Y)$ be the full subcategory comprised of coherent modules (in the sense of [Sta24, Tag 01BV]). Let $D_c(Y) \subset D(Y)$ be the full subcategory consisting of objects with coherent cohomologies. The pullback functor

$$\psi_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X^{\mathrm{an}}}), \quad F \mapsto F^{\mathrm{an}} \tag{2}$$

is exact and admits a right adjoint, so it commutes with colimits. It extends to a functor $D(X) \to D(X^{\mathrm{an}})$, which is t-exact relative to the standard t-structures. From [GR71, Exp. XII, 1.3], it restricts to a functor $D_c^b(X) \to D_c^b(X^{\mathrm{an}})$ and $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$.

Fact 2.1 can be retracted from [Hal23, Remark 1.1 and the proof of Theorem A]. Neeman [Nee21, Example A.2] modifies Hall's proof to some extent.

Fact 2.1. Assume that the complex algebraic variety X is proper. Then the functor (2) induces an equivalence $D_c^b(X) \to D_c^b(X^{\mathrm{an}})$ of triangulated categories. In particular, it restricts to an equivalence $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$ of abelian categories.

For algebraic quasi-coherent sheaves, an analytic analog is introduced by Kashiwara.

Definition 2.2. [Kas03, Def. 4.22] On a complex analytic space X, an O_X-module F called *good* if for every relatively compact open subset $U \subset X$, there exists a directed family $\{G_i\}_{i\in I}$ of coherent O_U-submodules of $F|_U$ such that $F|_U = \sum_{i\in I} G_i$, where $\{G_i\}_{i\in I}$ being a directed family means that for any $i, i' \in I$, there is $i'' \in I$ with $G_i + G_{i'} \subset G_{i''}$ (and hence $F|_U = \mathrm{colim}_{i\in I} G_i$).

By [Liu23, Lem. A.4.3], $\mathrm{Good}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$, and $D_{\mathrm{gd}}(X)$ is a triangulated subcategory of $D(X)$.

Lemma 2.3. For the complex algebraic variety X, the functor (2) restricts to a functor

$$\mathrm{Qch}(X) \to \mathrm{Good}(X^{\mathrm{an}}) \tag{3}$$

and induces a functor

$$D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(X^{\mathrm{an}}). \tag{4}$$

Proof. For every quasi-coherent O_X-module F, by Fact 2.4,

$$F = \sum_{i\in I} F_i \tag{5}$$

is the sum of a direct family of coherent O_X-submodules. As ψ_X^* commutes with colimits, one has

$$\psi_X^* F = \mathrm{colim}_{i\in I} \psi_X^* F_i \tag{6}$$

in the category $\mathrm{Mod}(O_{X^{\mathrm{an}}})$. Since ψ_X^* is exact, each $\psi_X^* F_i$ is a coherent $O_{X^{\mathrm{an}}}$-*submodule* of $\psi_X^* F$. Therefore, the $O_{X^{\mathrm{an}}}$-module $\psi_X^* F$ is good.

For every $G \in D_{\mathrm{qc}}(X)$ and every integer n, because (2) is an exact functor, the $O_{X^{\mathrm{an}}}$-module $H^n(\psi_X^* G) = \psi_X^*(H^n G)$ is good by last paragraph. Hence $\psi_X^* G \in D_{\mathrm{gd}}(X^{\mathrm{an}})$. □

Fact 2.4 ([Gro60, Cor. 9.4.9], [Sta24, Tag 01PG])**.** On a Noetherian scheme, every quasi-coherent sheaf is the sum of the directed family of all coherent submodules.

3 GAGA for quasi-coherent modules

Using Fact 2.4 and that ψ_X^* commutes with colimits, we extend GAGA from coherent O_X-modules to quasi-coherent O_X-modules. When $Y = \operatorname{Spec}\mathbb{C}$, Proposition 3.1 generalizes [Ser56, Thm. 1].

Proposition 3.1. *Let $f : X \to Y$ be a* proper *morphism of complex algebraic varieties. Then the base change natural transformation $(Rf_*\cdot)^{\mathrm{an}} \to Rf_*^{\mathrm{an}}(\cdot^{\mathrm{an}})$ (induced by the commutative square* (1)*) induces an isomorphism of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$.*

Proof. For every $F \in D_{\mathrm{qc}}(X)$, by [Lip60, Prop. 3.9.2], one has $Rf_* F \in D_{\mathrm{qc}}(Y)$. By Lemma 2.3, one has $F^{\mathrm{an}} \in D_{\mathrm{gd}}(X^{\mathrm{an}})$ and $(Rf_*F)^{\mathrm{an}} \in D_{\mathrm{gd}}(Y^{\mathrm{an}})$. Since f is proper, from [GR71, Exp. XII, Prop. 3.2 (v)], the morphism $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper. As X^{an} has finite dimension, by [Liu23, Thm. 3.1.6], one has $Rf_*^{\mathrm{an}} F^{\mathrm{an}} \in D_{\mathrm{gd}}(Y^{\mathrm{an}})$. Therefore, both functors $(Rf_*\cdot)^{\mathrm{an}}$ and $Rf_*^{\mathrm{an}}(\cdot^{\mathrm{an}})$ restrict to functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$.

We prove that the morphism $(Rf_*F)^{\mathrm{an}} \to Rf_*^{\mathrm{an}} F^{\mathrm{an}}$ is an isomorphism. By [Liu23, Lem. 3.1.10] (resp. [Lip60, Prop. 3.9.2]), the functor $Rf_*^{\mathrm{an}} : D(X^{\mathrm{an}}) \to D(Y^{\mathrm{an}})$ (resp. $Rf_* : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$) is bounded. From [Sta24, Tag 06YZ], the inclusion functor $\mathrm{Qch}(X) \to \mathrm{Mod}(O_X)$ exhibits $\mathrm{Qch}(X)$ as a weak Serre subcategory (in the sense of [Sta24, Tag 02MO]) of $\mathrm{Mod}(O_X)$. Then by (way-out argument) [Har66, I, Prop. 7.1 (iii)], one may assume $F \in \mathrm{Qch}(X)$. By [KS06, Prop. 13.1.5 (ii), p.320], it suffices to check that for every integer $n \geq 0$, the natural morphism $(R^n f_* F)^{\mathrm{an}} \to R^n f_*^{\mathrm{an}}(F^{\mathrm{an}})$ in $\mathrm{Mod}(O_{Y^{\mathrm{an}}})$ is an isomorphism.

By Fact 2.4, one can write $F = \sum_{i\in I} F_i$ as the sum of a direct family of coherent O_X-submodules of F. By [Sta24, Tag 07TB], one has

$$\mathrm{colim}_{i\in I} R^n f_* F_i \xrightarrow{\sim} R^n f_* F.$$

The analytification commutes with colimits, so

$$\mathrm{colim}_{i\in I}(R^n f_* F_i)^{\mathrm{an}} \xrightarrow{\sim} (R^n f_* F)^{\mathrm{an}}.$$

By [GR71, XII, Thm. 4.2], the natural morphisms $(R^n f_* F_i)^{\mathrm{an}} \to R^n f_*^{\mathrm{an}}(F_i^{\mathrm{an}})$ are isomorphisms. By [Liu23, Lem. 3.1.8], the natural morphism

$$\mathrm{colim}_{i\in I} R^n f_*^{\mathrm{an}}(F_i^{\mathrm{an}}) \to R^n f_*^{\mathrm{an}}(F^{\mathrm{an}})$$

is an isomorphism. □

Proposition 3.2 shows that goodness on complex analytic spaces is an analytic counterpart of quasi-coherence on complex algebraic varieties.

Proposition 3.2. *Suppose that the complex algebraic variety X is proper. Then (3) is an equivalence of abelian categories.*

Proof.

- The functor (3) is essentially surjective: Indeed, because X is proper over $\mathbb{C}$, by [GR71, Exp. XII, Prop. 3.2 (v)], the complex analytic spare X^{an} is compact. Then for every good $O_{X^{\mathrm{an}}}$-module G, one can write $G = \sum_{i\in I} G_i$ as the sum of a directed family of coherent $O_{X^{\mathrm{an}}}$-submodules. From the equivalence $\psi_X^* : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$ ([GR71, XII, Thm. 4.4]), there is a filtered inductive system $\{H_i\}_{i\in I}$ in $\mathrm{Coh}(X)$ whose analytification is the filtered inductive system $\{G_i\}_{i\in I}$. By [Sta24, Tag 01LA (4)], the colimit H of $\{H_i\}$ in $\mathrm{Mod}(O_X)$ exists and lies in $\mathrm{Qch}(X)$. Because ψ_X^* commutes with colimits, one has $H^{\mathrm{an}} = \mathrm{colim}_{i\in I} G_i$. In particular, H^{an} is isomorphic to G in $\mathrm{Good}(X^{\mathrm{an}})$.

- The functor (3) is fully faithful: For any quasi-coherent O_X-modules F and G, we have to show that the canonical morphism

$$\mathrm{Hom}_{O_X}(F, G) \to \mathrm{Hom}_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}) \tag{7}$$

  is an isomorphism. Assume first that F is coherent.

  - From [GW20, Exercise 7.20 (b)], one has

  $$[\mathcal{H}om_{O_X}(F, G)]^{\mathrm{an}} = \mathcal{H}om_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}).$$

  - As F is of finite presentation, the O_X-module $\mathcal{H}om_{O_X}(F, G)$ is quasi-coherent.

  Therefore, by Proposition 3.1, the canonical morphism

  $$H^0(X, \mathcal{H}om_{O_X}(F, G)) \to H^0(X^{\mathrm{an}}, \mathcal{H}om_{O_{X^{\mathrm{an}}}}(F^{\mathrm{an}}, G^{\mathrm{an}}))$$

  is an isomorphism, which is exactly (7).

  By (5) and (6), the general case follows.

□

4 Derived category of quasi-coherent sheaves

By [Sta24, Tag 0BKN], for every ringed space Y, the derived category $D(Y)$ has products and derived limits. This plays an essential role in step 4 of the proof of Theorem 4.2.

Definition 4.1. [Sta24, Tag 07LS] Let $\mathcal{A}$ be an additive category with arbitrary direct sums. An object $K \in \mathcal{A}$ is called compact, if $\mathrm{Hom}_{\mathcal{A}}(K, \cdot) : \mathcal{A} \to \mathrm{Ab}$ preserves direct sums.

Theorem 4.2. *If the complex algebraic variety X is proper, then the functor (4) is an equivalence of triangulated categories.*

Proof. Since X^{an} is compact, by Lemma 4.6, the perfect complex $O_{X^{\mathrm{an}}}$ is a compact object of $D(X^{\mathrm{an}})$. Then from the proof of [Hal23, Lem. 4.3], the functor $\psi_X^* : D_{\mathrm{qc}}(X) \to D(X^{\mathrm{an}})$ admits a right adjoint functor $R\psi_{\mathrm{qc},*} : D(X^{\mathrm{an}}) \to D_{\mathrm{qc}}(X)$ which preserves small coproducts.

1. The functor $\psi_X^* : D_{\mathrm{qc}}(X) \to D(X^{\mathrm{an}})$ is fully faithful.

From Fact 2.1, the unit of the adjunction $\eta : \mathrm{Id} \to R\psi_{\mathrm{qc},*}\psi_X^*$ (a natural transformation of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$) restricts to an isomorphism of functors $D_c^b(X) \to D_c^b(X)$. By [BB03, Thm. 3.1.1 1], the compact objects of $D_{\mathrm{qc}}(X)$ are precisely the perfect complexes. From [Nee96, Prop. 2.5], $D_{\mathrm{qc}}(X)$ is generated by a family of perfect complexes $\{E_i\}_{i\in I}$. By [Sta24, Tag 0FXU (1)], every perfect complex in $D(X)$ belongs to $D_c^b(X)$, so the η_{E_i} are isomorphisms. From Lemma 4.5, η is an isomorphism of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$. Thus, 1 is proved.

2. The functor (4) restricts to an equivalence $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{gd}}(X^{\mathrm{an}})$.

We prove that every $F \in D^b_{\mathrm{gd}}(X^{\mathrm{an}})$ is in the essential image of $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. Induction on the cohomological length of F. By Proposition 3.2, it holds when F has length zero. Suppose that it is true for objects of length $\leq n$ and F has length $n+1$. There is an integer i such that $\tau^{\leq i}F, \tau^{>i}F$ have length $\leq n$. There is a canonical exact triangle

$$\tau^{\leq i}F \to F \to \tau^{>i}F \xrightarrow{+1} \tau^{\leq i}F[1]$$

in $D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. By 1 and the inductive hypothesis, the morphism $+1 : \tau^{>i}F \to \tau^{\leq i}F[1]$ is in the essential image of $D^b_{\mathrm{qc}}(X) \to D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. Then so is F. The essential surjectivity together with 1 proves 2.

3. The functor $\psi_X^* : D^+_{\mathrm{qc}}(X) \to D^+_{\mathrm{gd}}(X^{\mathrm{an}})$ is an equivalence.

For every $F \in D^+_{\mathrm{gd}}(X^{\mathrm{an}})$, by Lemma 4.3, one has $\mathrm{hocolim}_{n>0}\, \tau^{\leq n}F \xrightarrow{\sim} F$. Every $\tau^{\leq n}F$ is in $D^b_{\mathrm{gd}}(X^{\mathrm{an}})$. From 2, there is a system $(K_n)_{n>0}$ of objects of $D^b_{\mathrm{qc}}(X)$, whose image under ψ_X^* is isomorphic to the system $(\tau^{\leq n}F)_{n>0}$. Since

(4) respects coproducts, it respects homotopy colimits. Since $\mathrm{Qch}(X)$ is closed under filtered colimits in $\mathrm{Mod}(O_X)$, the subcategory $D_{\mathrm{qc}}(X)$ is closed under homotopy colimits in $D(X)$.

Then F is isomorphic to the image of $K := \mathrm{hocolim}_{n>0} K_n \in D_{\mathrm{qc}}(X)$ under ψ_X^*. There is an integer q, such that $H^i(F) = 0$ for every integer $i < q$. Then $\psi_X^* H^i(K) = H^i(\psi_X^* K) = 0$. By Proposition 3.2, one has $H^i(K) = 0$. Hence $K \in D_{\mathrm{qc}}^+(X)$. Thus, the functor $\psi_X^* : D_{\mathrm{qc}}^+(X) \to D_{\mathrm{gd}}^+(X^{\mathrm{an}})$ is essentially surjective. By 1, it is an equivalence.

4. Every $Z \in D_{\mathrm{gd}}(X^{\mathrm{an}})$ is in the essential image of (4).

By Lemma 4.4, the canonical morphism $Z \to \mathrm{Rlim}_{n>0} \tau^{\geq -n} Z$ is an isomorphism in $D(X^{\mathrm{an}})$. By 3, there is an inverse system (Y^{-n}) of objects of $D_{\mathrm{qc}}^+(X)$, whose image is isomorphic to the inverse system $(\tau^{\geq -n} Z)_{n>0}$. Let Y be $\mathrm{Rlim}_{n>0} Y^{-n}$ in $D(X)$. For any integers $n \geq 1$ and q, the functor ψ_X^* transforms the morphism $H^q(Y^{-n-1}) \to H^q(Y^{-n})$ in $\mathrm{Qch}(X)$ to $H^q(\tau^{\geq -n-1} Z) \to H^q(\tau^{\geq -n} Z)$ in $\mathrm{Good}(X^{\mathrm{an}})$.

The morphism $H^q(\tau^{\geq -n-1} Z) \to H^q(\tau^{\geq -n} Z)$ is surjective, and when $n \geq -q$, it is an isomorphism. By Proposition 3.2, the morphism $H^q(Y^{-n-1}) \to H^q(Y^{-n})$ is surjective, and when $n \geq -q$, it is an isomorphism. By [Sta24, Tag 0A0J (1)], the canonical morphism $H^q(Y) \to H^q(Y^{\min(q,-1)})$ is an isomorphism. In particular, the O_X-module $H^q(Y)$ is quasi-coherent. Hence $Y \in D_{\mathrm{qc}}(X)$.

For every integer $m > 0$, the functor ψ_X^* transforms $\prod_{n>0} Y^{-n} \to Y^{-m}$ to $\psi_X^*(\prod_{n>0} Y^{-n}) \to \tau^{\geq -m} Z$. Hence a morphism $\psi_X^*(\prod_{n>0} Y^{-n}) \to \prod_{n>0} \tau^{\geq -n} Z$ in $D(X^{\mathrm{an}})$. It fits to a commutative diagram

$$\begin{array}{ccccccc}
\psi_X^*(\prod_{n>0} Y^{-n})[-1] & \longrightarrow & \psi_X^*(\prod_{n>0} Y^{-n})[-1] & \longrightarrow & \psi_X^* Y & \longrightarrow & \psi_X^*(\prod_{n>0} Y^{-n}) \\
\downarrow & & \downarrow & & \vdots & & \downarrow \\
\prod_{n>0} \tau^{\geq -n} Z[-1] & \longrightarrow & \prod_{n>0} \tau^{\geq -n} Z[-1] & \longrightarrow & Z & \longrightarrow & \prod_{n>0} \tau^{\geq -n} Z
\end{array}$$

in $D(X^{\mathrm{an}})$, where the rows are exact triangles. By TR3, it induces a morphism of triangles. Hence a commutative square

$$\begin{array}{ccc}
H^q(\psi_X^* Y) & \xrightarrow{\sim} & H^q(\psi_X^* Y^{\min(q,-1)}) \\
\vdots & & \downarrow \wr \\
H^q(Z) & \xrightarrow{\sim} & H^q(\tau^{\geq \min(q,-1)} Z)
\end{array}$$

in $\mathrm{Mod}(O_{X^{\mathrm{an}}})$. Therefore, for every integer q, the induced morphism $H^q(\psi_X^* Y) \to H^q(Z)$ is an isomorphism. Therefore, the morphism $\psi_X^* Y \to Z$ is an isomorphism in $D_{\mathrm{gd}}(X^{\mathrm{an}})$. Thus, 4 is proved. By 4 and 1, the functor (4) is an equivalence. □

Lemma 4.3. Let $\mathcal{A}$ be an abelian category, where colimits over $\mathbb{N}$ exist and are exact. Then the natural transformation $\mathrm{hocolim}_{n>0} \tau^{\leq n} \cdot \to \mathrm{Id}$ is an isomorphism of functors $\mathcal{A} \to \mathcal{A}$.

Proof. It follows from [Sta24, Tag 0949] and the construction of canonical truncations. □

Lemma 4.4. Let X be a complex analytic space. Then the natural transformation $\mathrm{Id} \to \mathrm{Rlim}_{n>0} \tau^{\geq -n} \cdot$ is an isomorphism of functors $D(X) \to D(X)$.

Proof. For every $x \in X$, there is an integer $d_x \geq 0$, and a fundamental system $\mathcal{U}_x$ of open neighborhoods of x, such that every $U \in \mathcal{U}_x$ is a closed complex subspace of a domain in $\mathbb{C}^{d_x}$. By [Liu23, Fact 3.1.9], for every $E \in D(X)$, any integers $p > 2d_x$ and q, one has $H^p(U, H^q(E)) = 0$. By [Sta24, Tag 0D63], the canonical morphism $E \to \mathrm{Rlim}_{n>0} \tau^{\geq -n} E$ is an isomorphism in $D(X)$. □

Lemma 4.5. Let $\mathcal{C}, \mathcal{D}$ be triangulated categories. Assume that $\mathcal{C}$ has direct sums. Let $\{E_i\}_{i \in I}$ be a family of compact objects of $\mathcal{C}$ such that $\oplus_{i \in I} E_i$ generates $\mathcal{C}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be triangulated functors preserving direct sums. Let $\eta : F \to G$ be a natural transformation. If for every $i \in I$, the morphism $\eta_{E_i} : F(E_i) \to G(E_i)$ is an isomorphism in $\mathcal{D}$, then η is an isomorphism.

Proof. From [Sta24, Tag 09SN], every object $X \in \mathcal{C}$ can be written as $X = \mathrm{hocolim}_{n>0} X_n$, where

- X_1 is a direct sum of shifts of the E_i,
- each transition morphism $X_n \to X_{n+1}$ fits into an exact triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$,
- and Y_n is a direct sum of shifts of the E_i.

Since F, G preserve direct sums, and the η_{E_i} are isomorphisms, so are the $\{\eta_{Y_n}\}_{n>0}$ and η_{X_1}. By [Sta24, Tag 014A] and induction on $n > 0$, one proves that the η_{X_n} are isomorphisms. By [BN93, Lem. 4.1], $F, G : \mathcal{C} \to \mathcal{D}$ preserve homotopy colimits. Therefore, η_X is an isomorphism. □

Lemma 4.6. Let X be a compact complex analytic space. Then every perfect object of $D(X)$ belongs to $D^b_c(X)$. It is a compact object of $D(X)$ and of $D_{\mathrm{gd}}(X)$.

Proof. Let $E \in D(X)$ be a perfect object. By definition, there is an open covering $X = \cup_{i \in I} U_i$, such that for each $i \in I$, there is a morphism of complexes $E_i^\bullet \to E|_U$ which is a quasi-isomorphism, with $E_i^j = 0$ for all but finite many integers j, and every E_i^j is a direct summand of a finite free O_X-module. Since X is compact, one has $E \in D^b(X)$. By [Sta24, Tag 01BY (1)], every E_i^j is coherent. Therefore, every $H^j(E)|_{U_i}$ is coherent over O_{U_i}. Thus, $H^j(E)$ is coherent over O_X for all j. Hence $E \in D^b_c(X)$. In particular, E is in $D_{\mathrm{gd}}(X)$.

Let $E^\vee := R\mathcal{H}om(E, O_X) \in D(X)$. From [Sta24, Tag 08DQ], there is a natural isomorphism of functors $\mathrm{Hom}_{D(X)}(E, \cdot) \to H^0(X, E^\vee \otimes^L_{O_X} \cdot) : D(X) \to \mathrm{Ab}$. The functor $E^\vee \otimes^L_{O_X} \cdot : D(X) \to D(X)$ commutes with direct sums. Since X is compact, $\dim X$ is finite. Then by Lemma 4.7, the functor $H^0(X, \cdot) : D(X) \to \mathrm{Ab}$ also commutes with direct sums. Therefore, E is a compact object of $D(X)$. By [Liu23, Lem. A.4.3 2], $D_{\mathrm{gd}}(X)$ is closed under direct sums in $D(X)$. Then E is also a compact object of $D_{\mathrm{gd}}(X)$. □

Lemma 4.7. Let $f : X \to Y$ be a proper morphism of complex analytic spaces. If $\dim X$ is finite, then the functor $Rf_* : D(X) \to D(Y)$ commutes with direct sums.

Proof. First, we prove that for every integer q, there is a natural isomorphism

$$R^q f_* \xrightarrow{\sim} R^q f_* \tau_{\geq q-2\dim X} : D(X) \to \mathrm{Mod}(O_Y). \tag{8}$$

Indeed, by [Sta24, Tag 08J5], for every object $E \in D(X)$, there is an exact triangle $\tau_{\leq q-2\dim X-1}E \to E \to \tau_{\geq q-2\dim X}E \to (\tau_{\leq q-2\dim X-1}E)[1]$. It induces an exact sequence

$$R^q f_* \tau_{\leq q-2\dim X-1}E \to R^q f_* E \to R^q f_* \tau_{\geq q-2\dim X}E \to R^{q+1} f_* \tau_{\leq q-2\dim X-1}E$$

in $\mathrm{Mod}(O_Y)$. From [Liu23, Lem. 3.1.10], one has

$$R^q f_* \tau_{\leq q-2\dim X-1}E = R^{q+1} f_* \tau_{\leq q-2\dim X-1}E = 0.$$

Hence an isomorphism $R^q f_* E \to R^q f_* \tau_{\geq -q-2\dim X}E$ functorial in E.

Let $\{E_i : i \in I\}$ be a family of objects of $D(X)$. Set $E = \oplus_{i\in I} E_i$. To prove that the canonical morphism $\oplus_{i\in I} Rf_* E_i \to Rf_* E$ in $D(Y)$ is an isomorphism, it suffices to show that for every integer q, the induced morphism $\oplus_{i\in I} R^q f_* E_i \to R^q f_* E$ in $\mathrm{Mod}(O_Y)$ is an isomorphism. Since $\tau_{\geq q-2\dim X}E = \oplus_{i\in I} \tau_{\geq q-2\dim X}E_i$, by (8), one may assume that E and all the E_i are in $D^{\geq q-2\dim X}(X)$. Then from [Sta24, Tag 015J], one has canonical spectral sequences

$$R^s f_* H^t(E) \Rightarrow R^{s+t} f_* E, \quad R^s f_* H^t(E_i) \Rightarrow R^{s+t} f_* E_i.$$

By [Liu23, Lem. 3.1.8], for any integers s and t, the canonical morphism $\oplus_{i\in I} R^s f_* H^t(E_i) \to R^s f_* H^t(E)$ in $\mathrm{Mod}(O_Y)$ is an isomorphism. Consequently, the canonical morphism $\oplus_{i\in I} R^q f_* E_i \to R^q f_* E$ is an isomorphism. □

Corollary 4.8. If the complex algebraic variety X is proper, then the functor $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is an equivalence of triangulated categories.

Proof. For every $F \in D_c(X)$ and every integer i, the $O_{X^{\mathrm{an}}}$-module $H^i(\psi_X^* F) = \psi_X^* H^i(F)$ is coherent. Thus, the functors $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is well-defined. By Theorem 4.2, the functor $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is fully faithful. For every $F \in D_c(X^{\mathrm{an}})$, by Theorem 4.2, there is $G \in D_{\mathrm{qc}}(X)$ with $\psi_X^* G$ isomorphic to F. Then $\psi_X^* H^i(G) = H^i(\psi_X^* G) \xrightarrow{\sim} H^i(F)$ is coherent over $O_{X^{\mathrm{an}}}$. By Fact 2.1 and Proposition 3.2, the O_X-module $H^i(G)$ is coherent. Hence $G \in D_c(X)$. Therefore, $\psi_X^* : D_c(X) \to D_c(X^{\mathrm{an}})$ is essential surjective and hence an equivalence. □

5 Compact objects

Corollary 5.1. Suppose that the complex algebraic variety X is proper. Then the compact objects of $D_{\mathrm{gd}}(X^{\mathrm{an}})$ are precisely the perfect complexes in $D(X^{\mathrm{an}})$.

Proof. By compactness of X^{an} and Lemma 4.6, prefect complexes are compact objects of $D_{\mathrm{gd}}(X^{\mathrm{an}})$. Conversely, let F be a compact object of $D_{\mathrm{gd}}(X^{\mathrm{an}})$. By Theorem 4.2, there is a compact object $G \in D_{\mathrm{qc}}(X)$ with $\psi_X^* G$ isomorphic to F. By [Sta24, Tag 09M1], G is a perfect complex in $D(X)$. By definition, F is a perfect complex in $D(X^{\mathrm{an}})$. □

Let X be a compact complex manifold.

Question 5.2. Does the full subcategory of $D_{\mathrm{gd}}(X)$ of compact objects coincide with $D_c^b(X)$?

Question 5.3. Is the category $D_{\mathrm{gd}}(X)$ compactly generated?

When X is the analytification of a smooth proper complex algebraic variety, Corollary 5.1 (resp. Theorem 4.2) answers Questions 5.2 (resp. 5.3) affirmatively.

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