Deforming foliations and commuting diffeomorphisms

Hélène Eynard-Bontemps
Université Grenoble Alpes

La Sapienza, 11/06/25

Let M be an n dimensional manifold.

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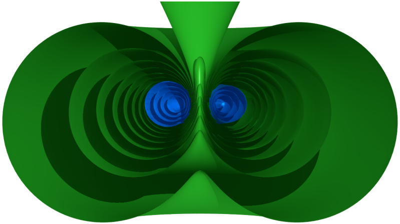
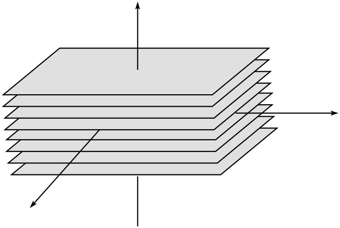
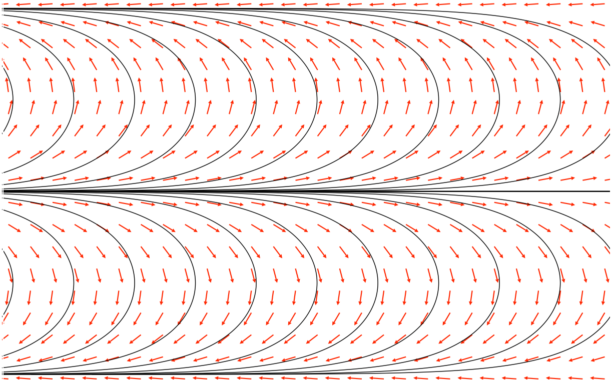
Definition

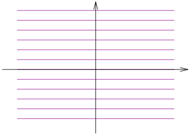
A *codimension* 1 *foliation* of M is a partition of M by injectively immersed hypersurfaces (called the *leaves*) which is locally modeled on the partition of $\mathbb{R}^n$ by the parallel affine hyperplanes $\mathbb{R}^{n-1} \times \{\cdot\}$.

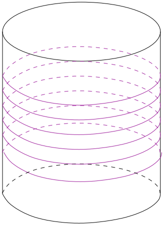






Linear foliations of $\mathbb{R}^2$ and $\mathbb{T}^2$





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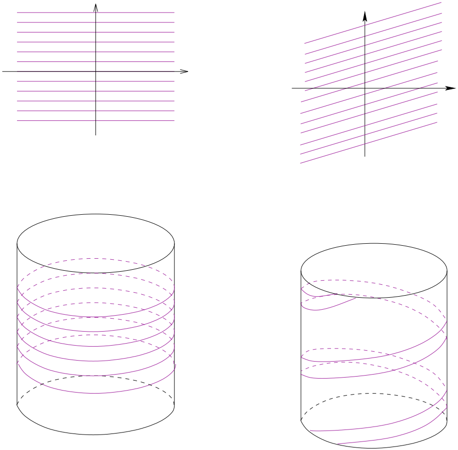

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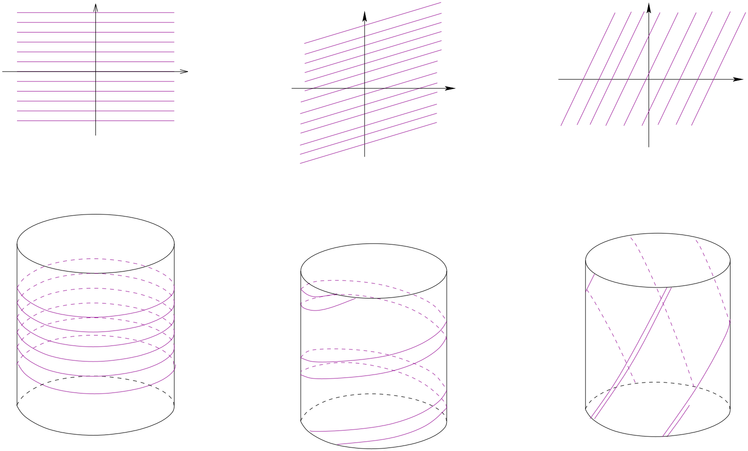
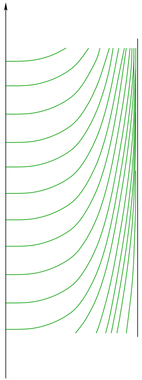


Reeb foliation of the solid torus and the 3-sphere $\mathbb{S}^3$



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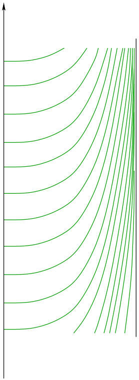

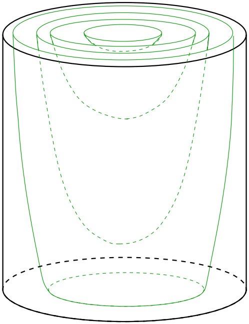

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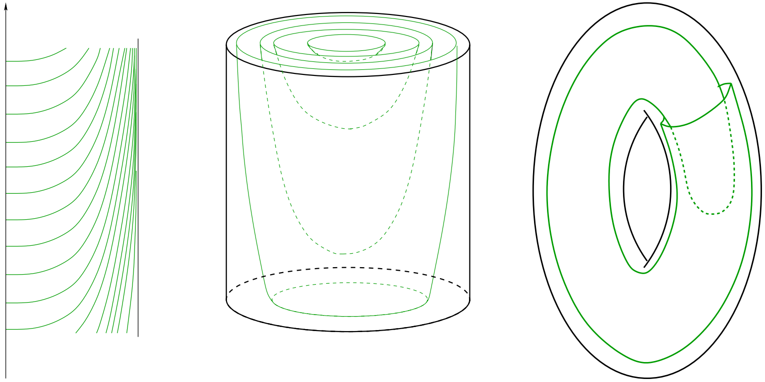
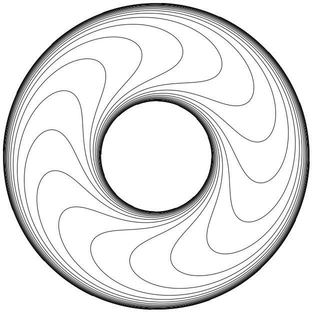
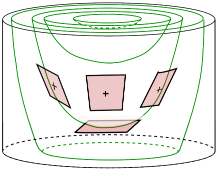




• Foliation $\rightsquigarrow$ tangent hyperplane field



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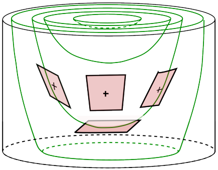

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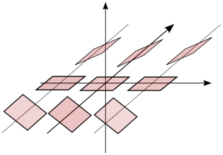

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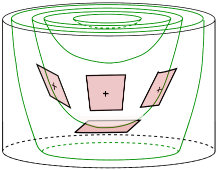

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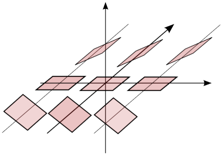

$\xi = \ker(dz - ydx)$ sur $\mathbb{R}^3$

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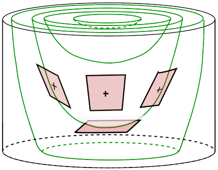

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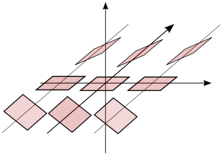

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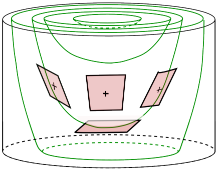

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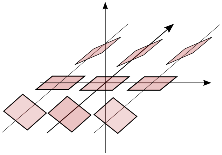

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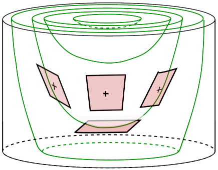

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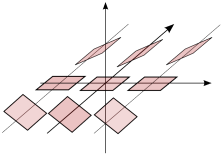

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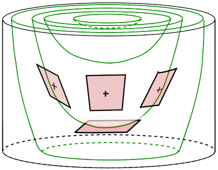

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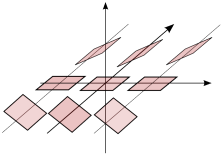

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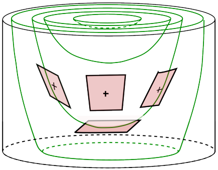

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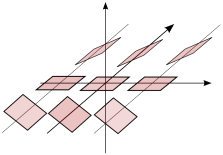

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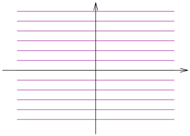
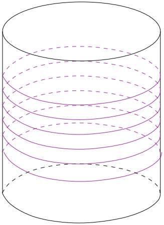
Theorem (Wood)

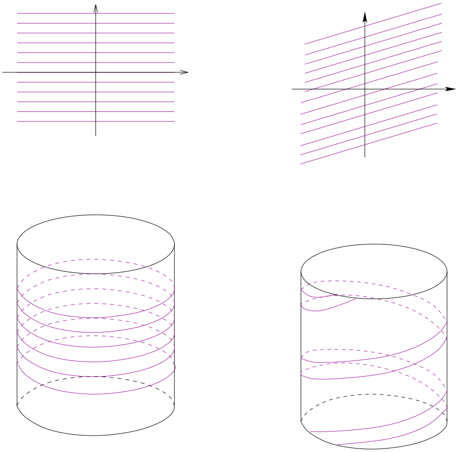
Every plane field on a closed 3-manifold can be deformed to an integrable one.

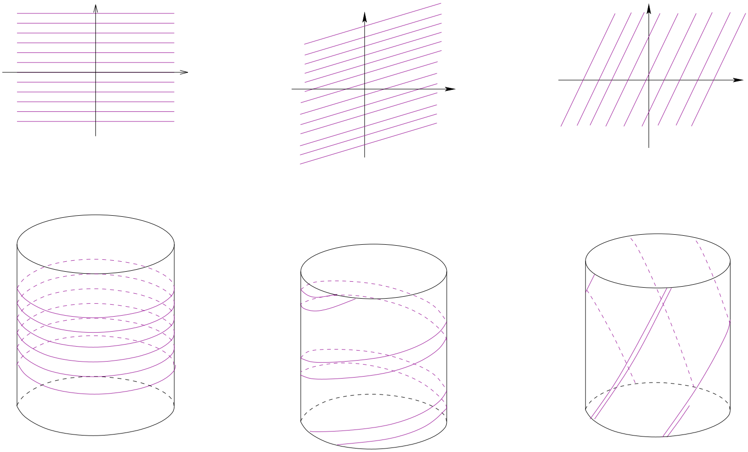
Question

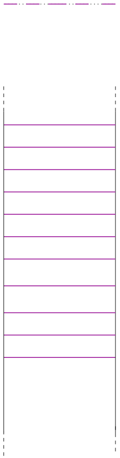
If two foliations have homotopic tangent plane fields, can they be connected by a path of foliations?

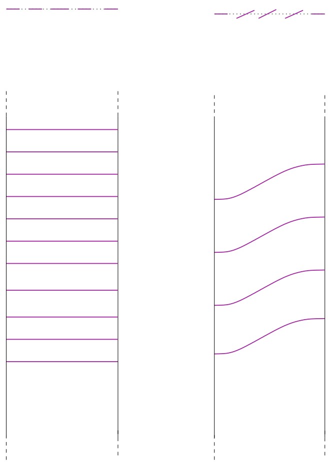


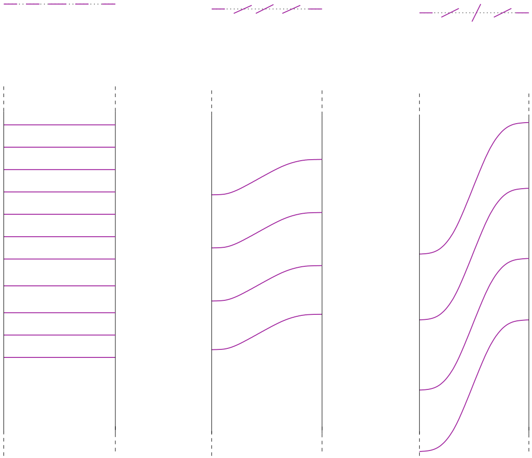


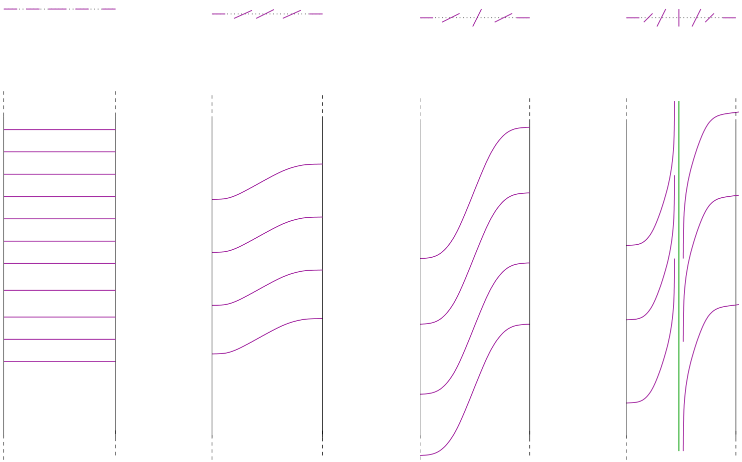


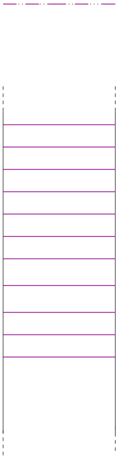


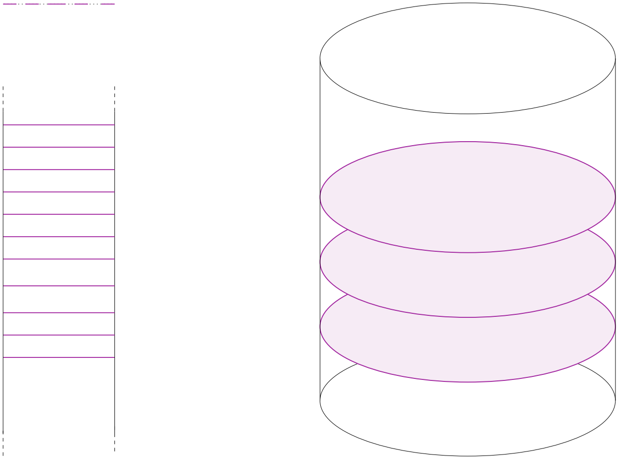


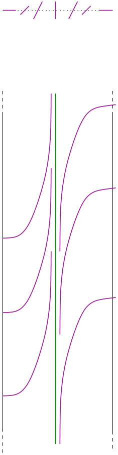


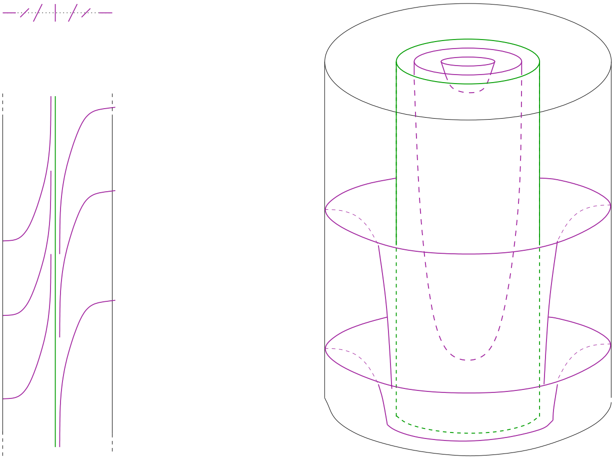












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EB, 2015: the maps $\pi_k\mathcal{F}(M) \to \pi_k\mathcal{P}(M)$ are surjective.

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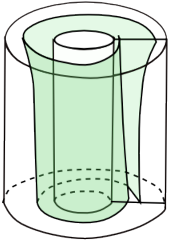

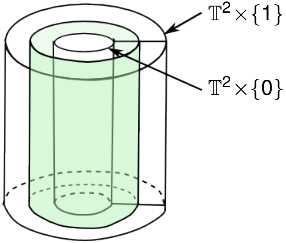

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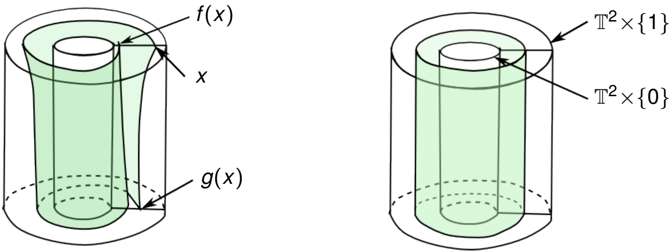

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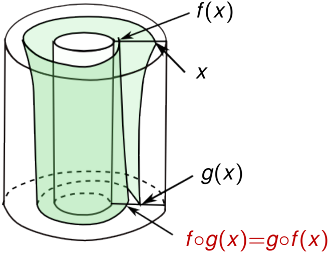

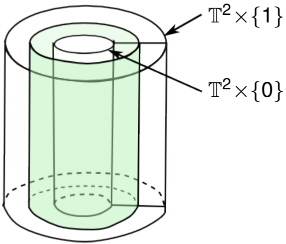

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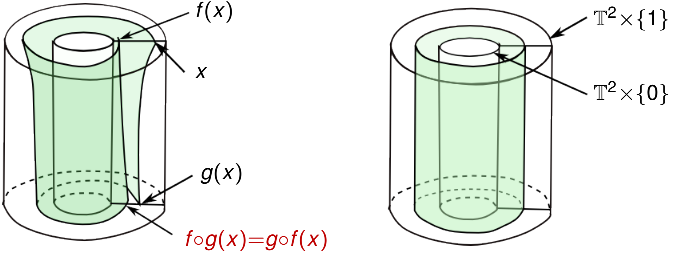

Equivalent problem : deform (f, g) to $(\mathrm{Id}, \mathrm{Id})$ among pairs of commuting diffeomorphisms.

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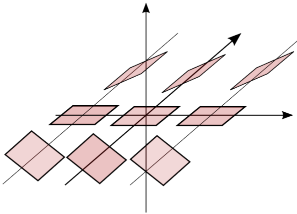
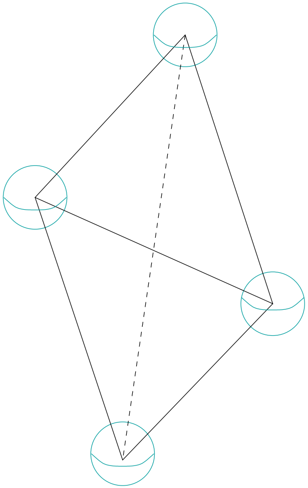
Theorem (EB – Navas, 2023)

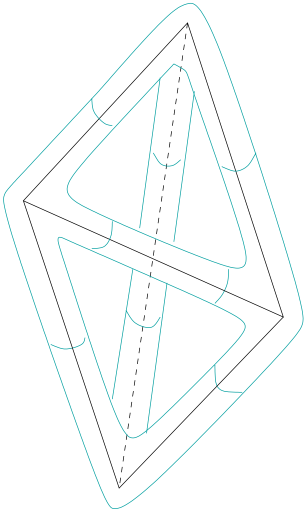
In regularity C^{1+ac}*, this space is path-connected.*

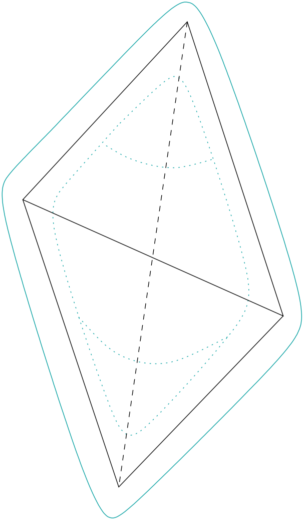
Theorem (Wood, Thurston)

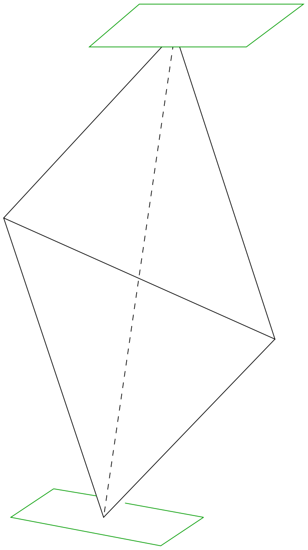
Every plane field is homotopic to a foliation.

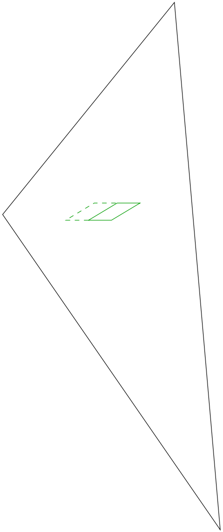


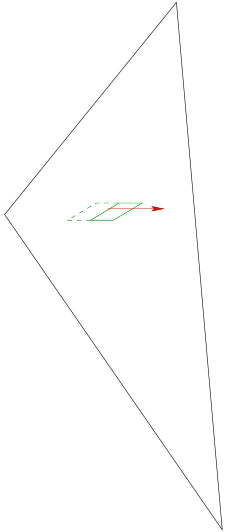


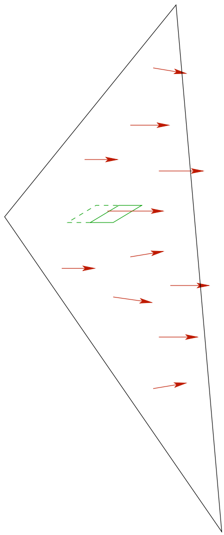


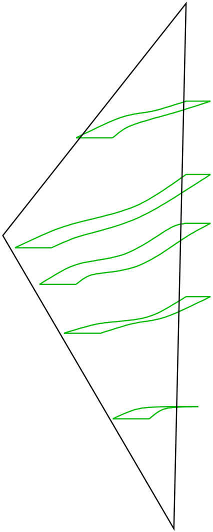


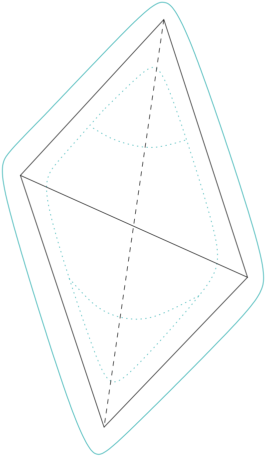


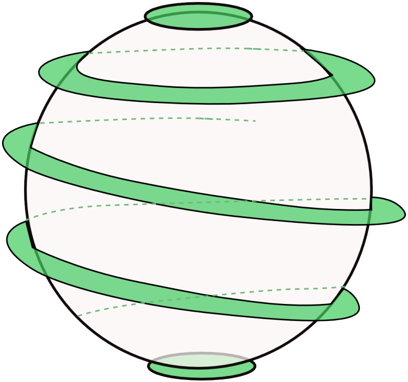


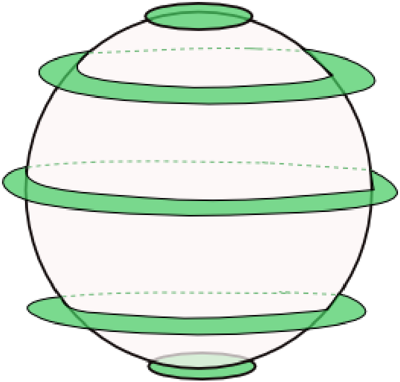


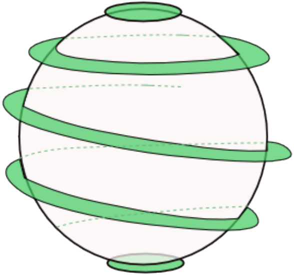


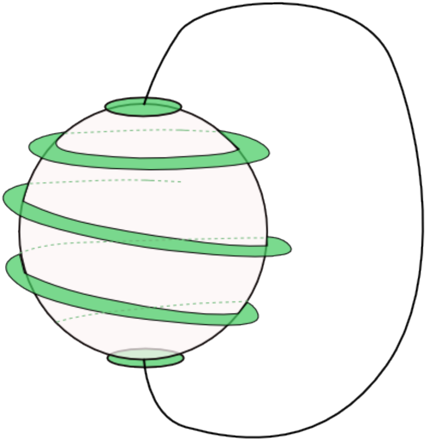


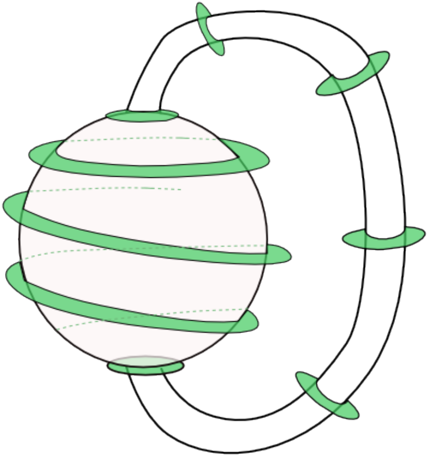


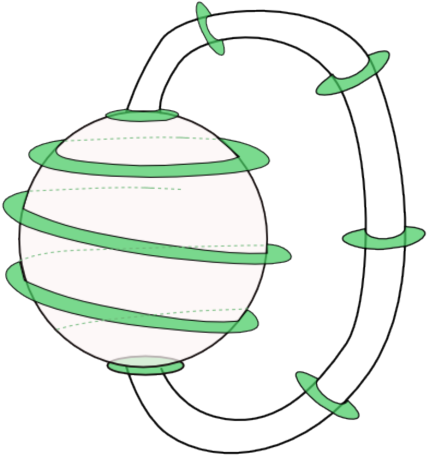






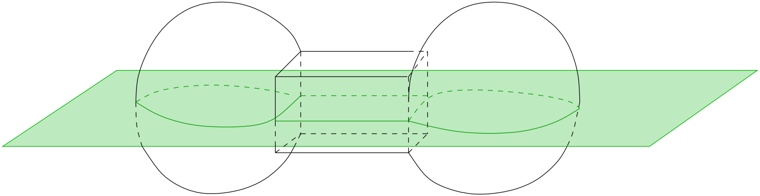


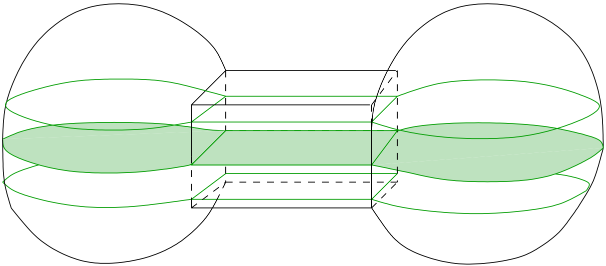


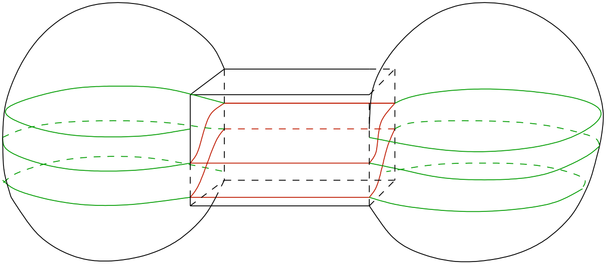


Proposition (Thurston)

Every foliation of $\partial\mathbb{D}^2 \times \mathbb{S}^1$ transverse to $\mathbb{S}^1$ extends to a foliation of $\mathbb{D}^2 \times \mathbb{S}^1$.







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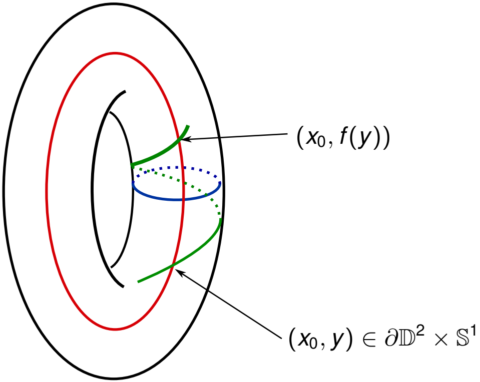
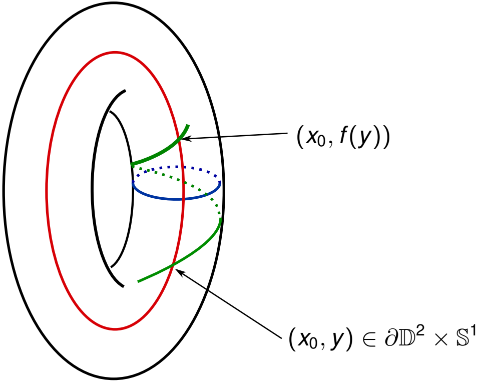
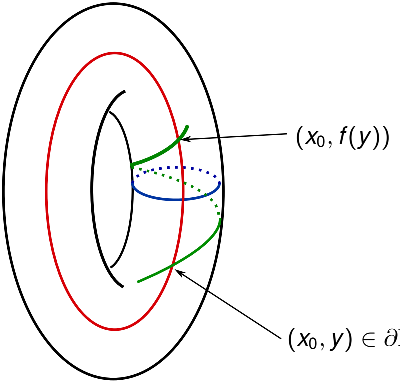

$(x_0, f(y))$
$(x_0, y) \in \partial\mathbb{D}^2 \times \mathbb{S}^1$

$f : \mathbb{S}^1 \to \mathbb{S}^1$ first return map.





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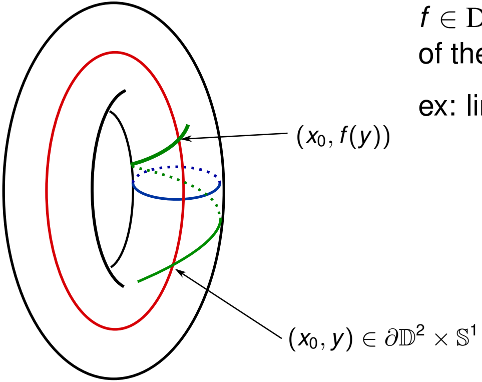

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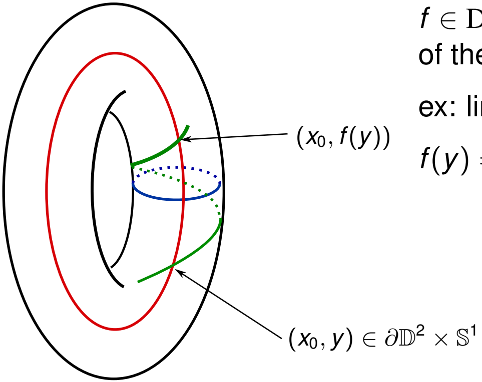

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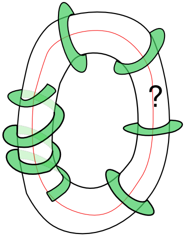
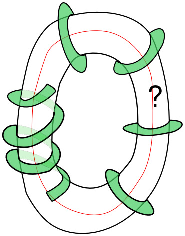
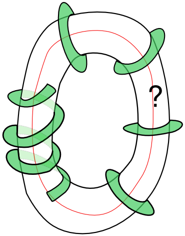
$f(y) = y + \alpha \pmod{\mathbb{Z}}$.





Proof 1: If the holonomy on the boundary is a product of commutators, there exists an extension.



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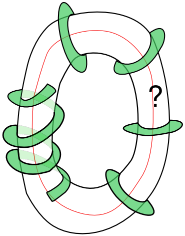
+ $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ is perfect (Herman).



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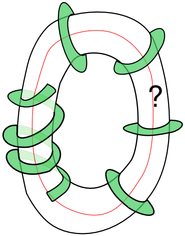

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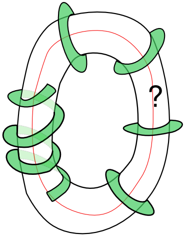

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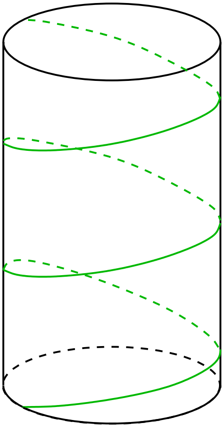
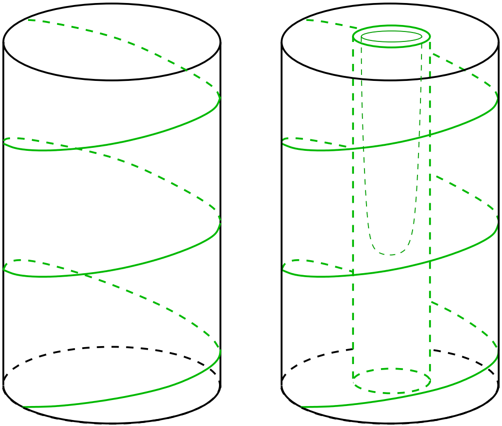
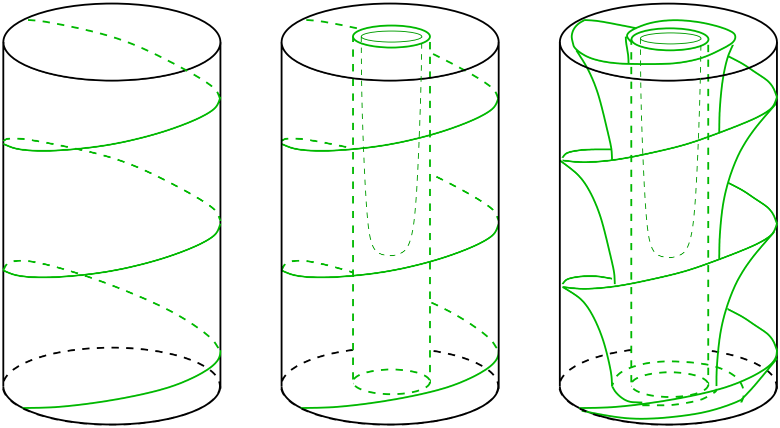
Proof 3 : (Schweitzer, Larcanché) gives in addition continuity w.r.t. boundary condition.







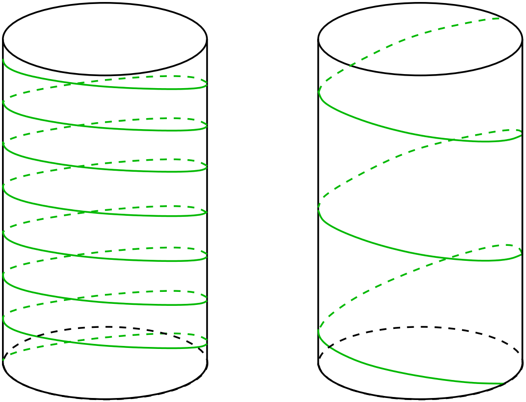
Theorem (Herman)

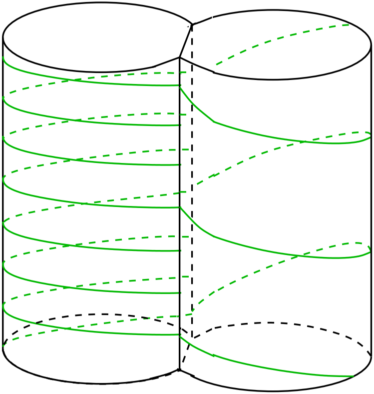
Let μ be a diophantine number (e.g. the golden ratio). Then every $f \in \mathrm{Diff}_+(\mathbb{S}^1)$ can be decomposed as

$$f = R_{\lambda_f} \circ (g_f^{-1} \circ R_\mu \circ g_f)$$

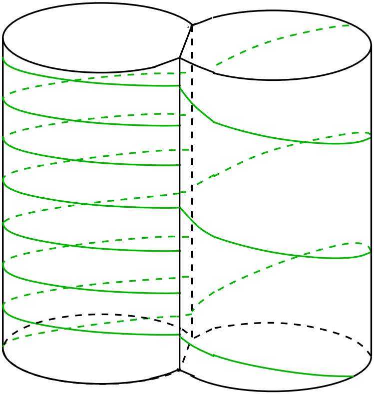
with $(\lambda_f, g_f) \in \mathbb{R} \times \mathrm{Diff}_+(\mathbb{S}^1)$ depending continuously on f.

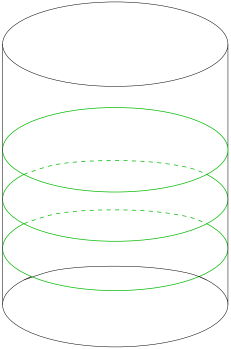
R_{λ_f} $\qquad$ $g_f^{-1} \circ R_\mu \circ g_f$

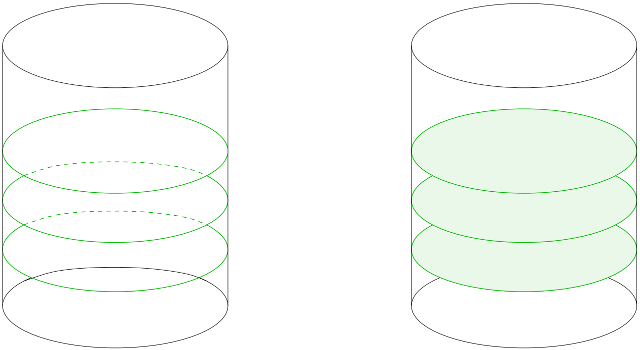


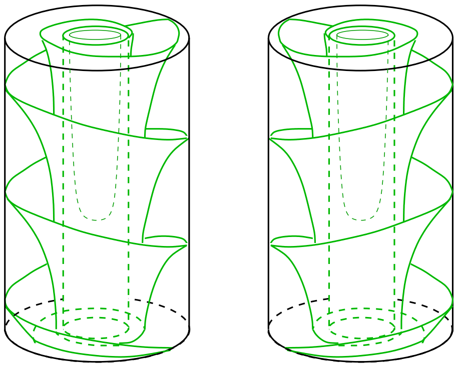


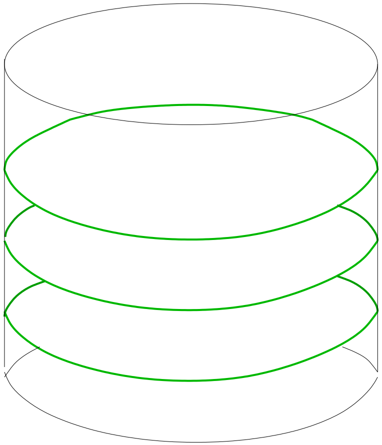
$$R_{\lambda_f} \circ (g_f^{-1} \circ R_\mu \circ g_f) = f$$

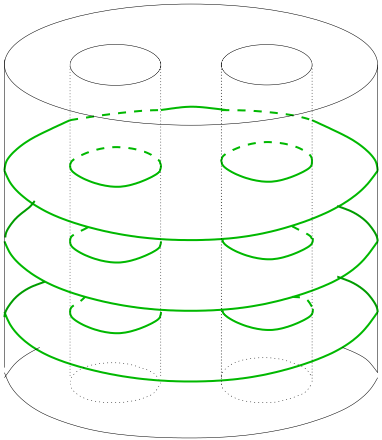


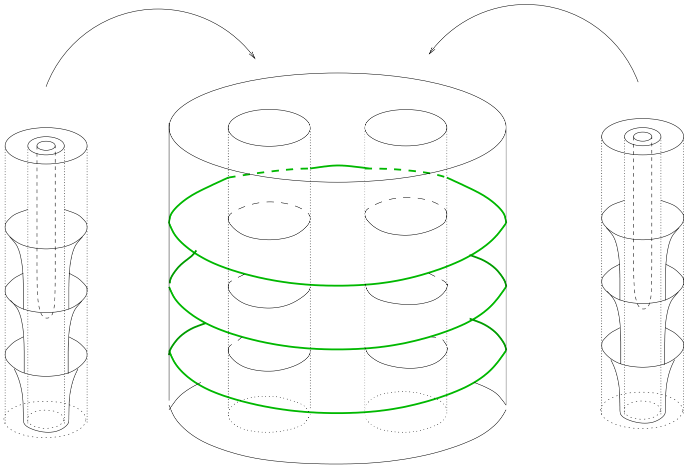


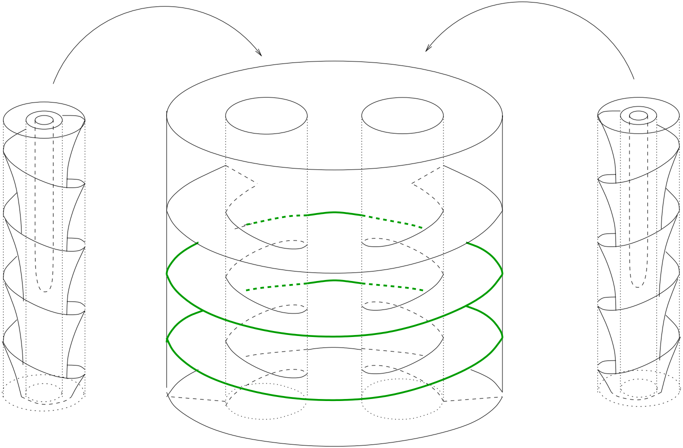












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There are problems at every stage! Because of the parameter, and of the relative character of the pb ($t = 0, 1$)

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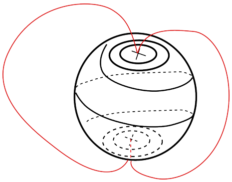

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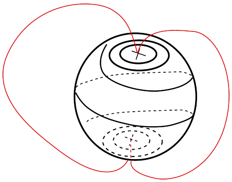

Siphon Lemma

The two foliations obtained by digging along one arc or the other are connected by a path of foliations.

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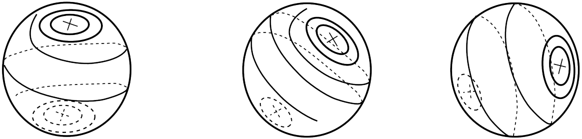

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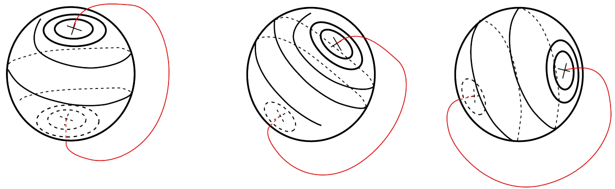

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Actually works for a bigger class: foliations called *malleable* (every leaf meets a closed loop, *except* maybe finitely many torus leaves which lie in "Schweitzer solid tori").

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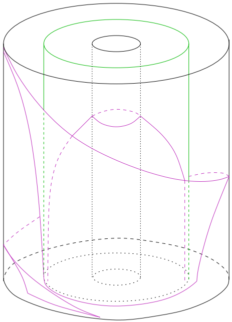
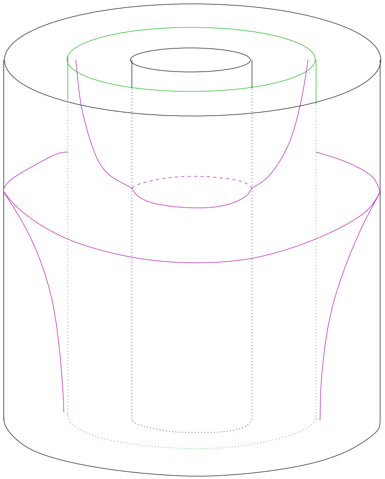
Question

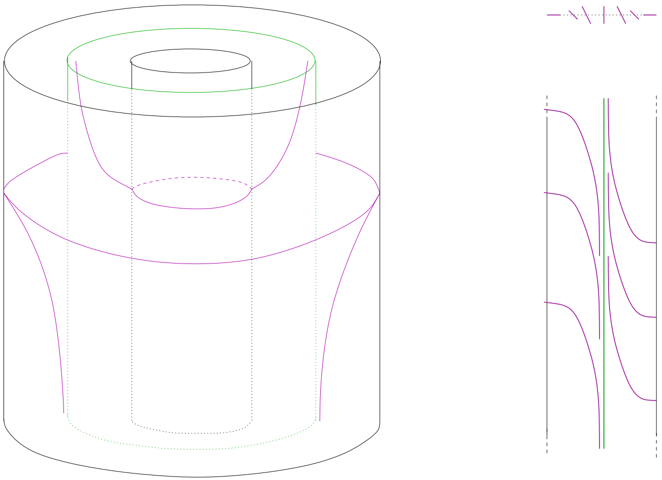
Can any foliation be deformed to a neat one?

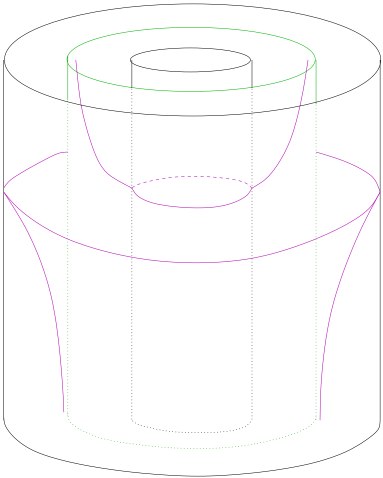
Theorem

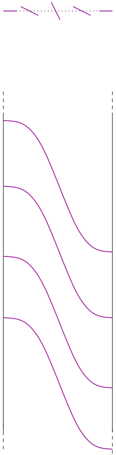
Deux feuilletages nets homotopes comme champs de plans le sont parmi les feuilletages nets.

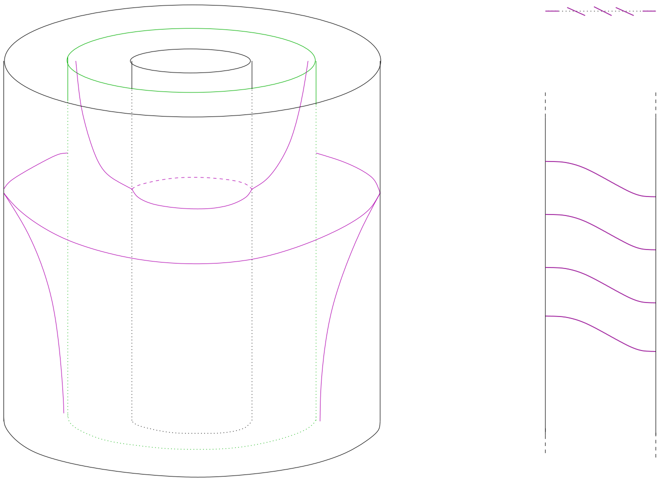


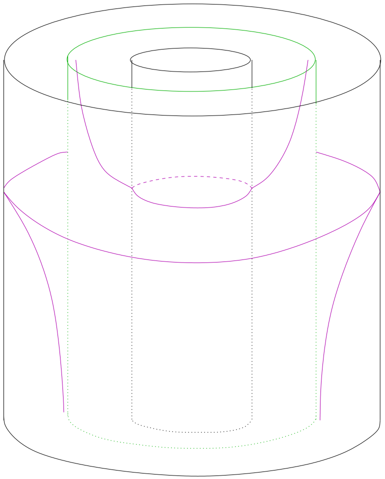


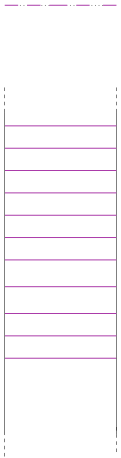


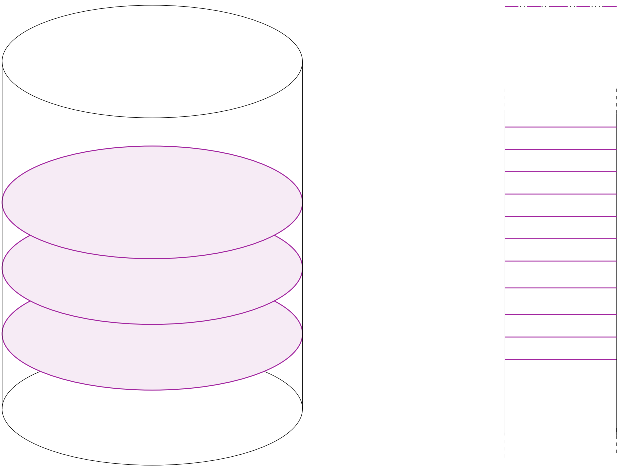


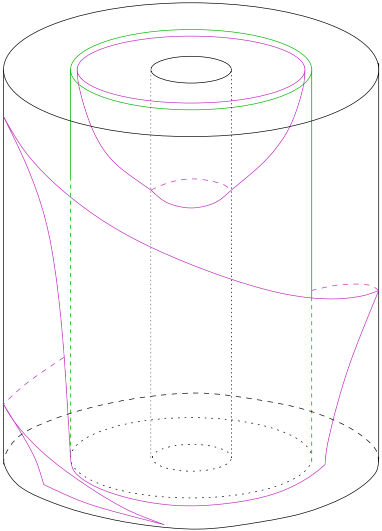


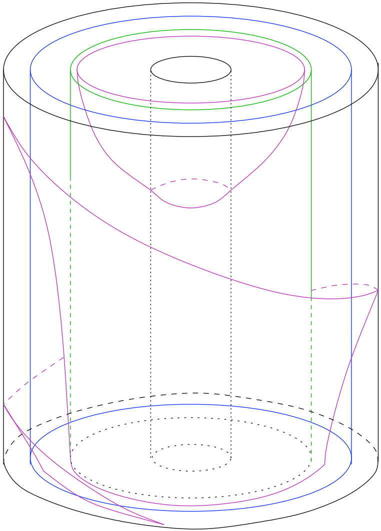


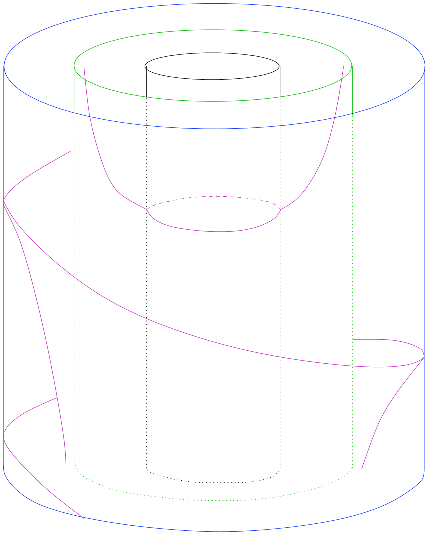


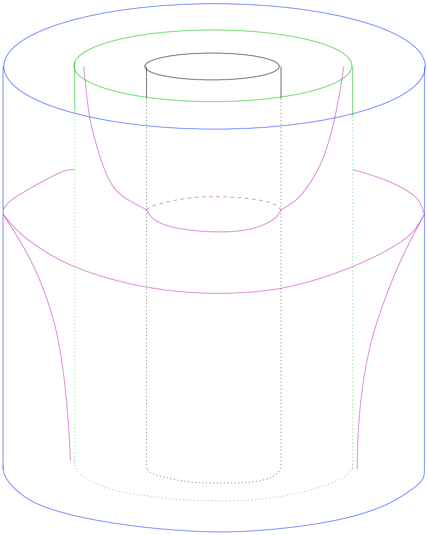


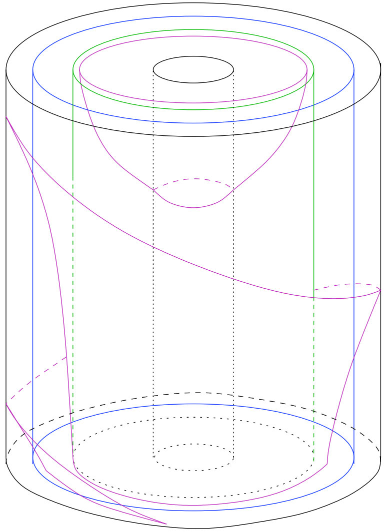


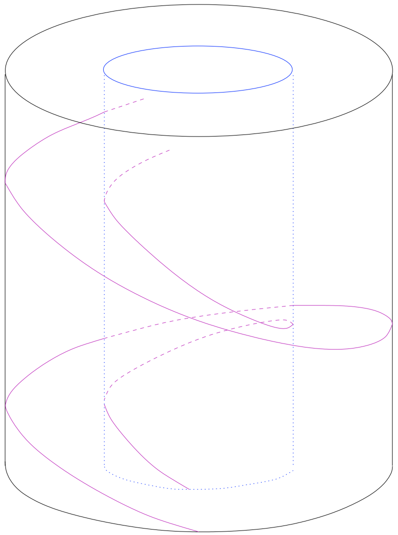


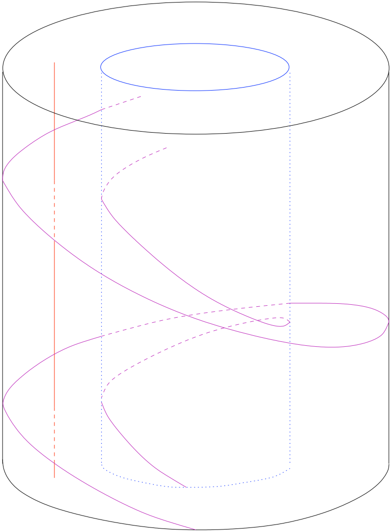


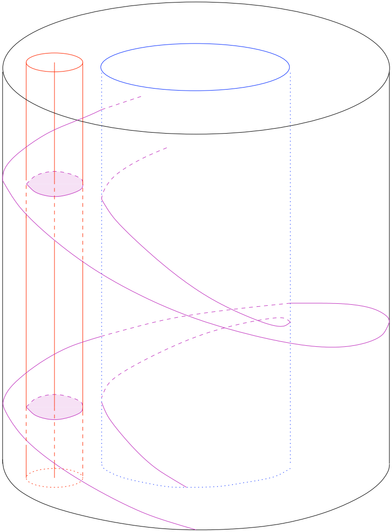












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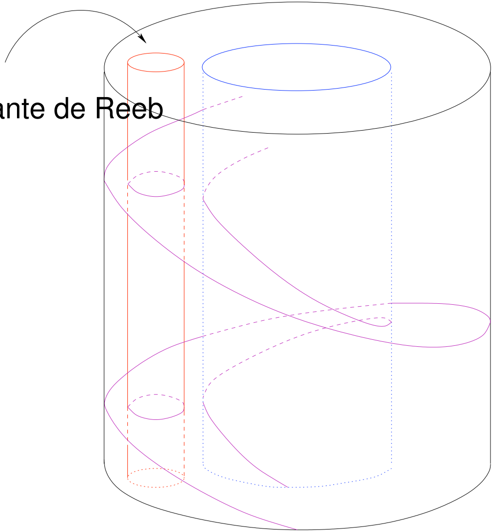

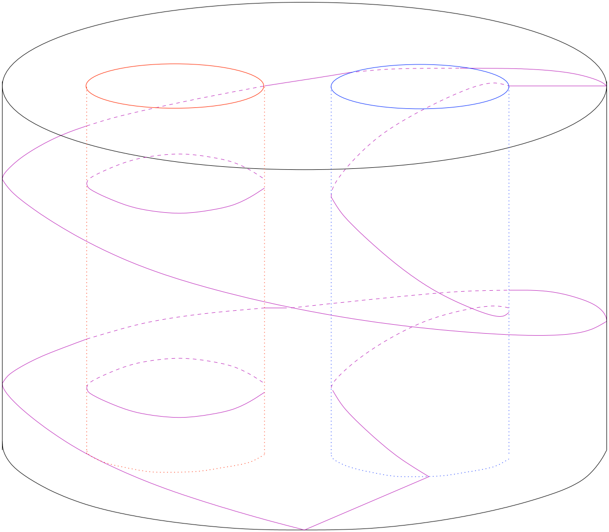

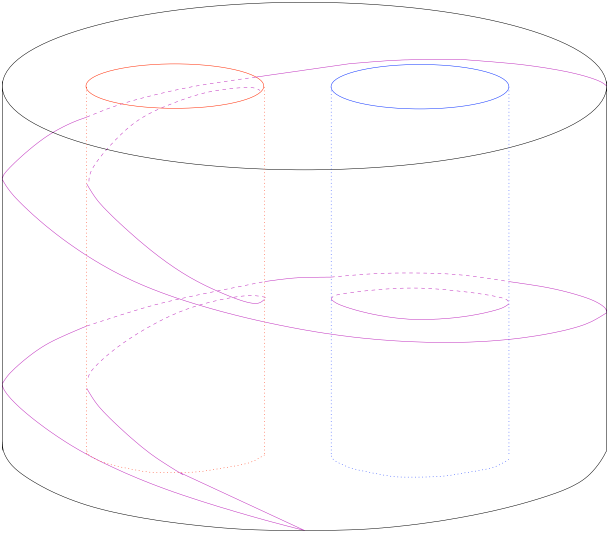

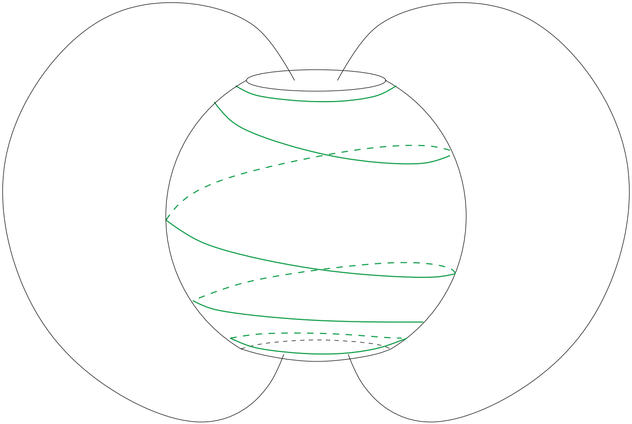

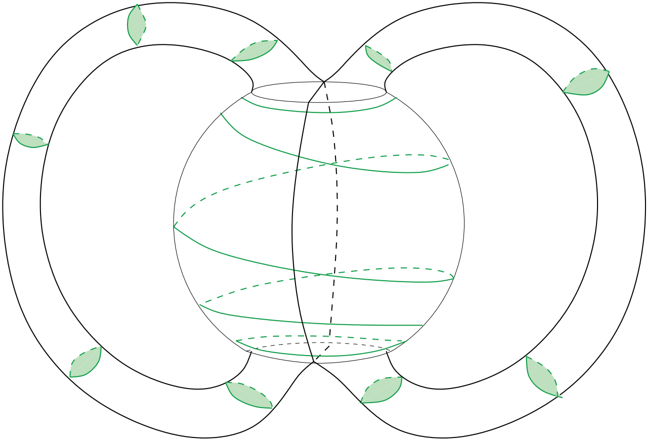

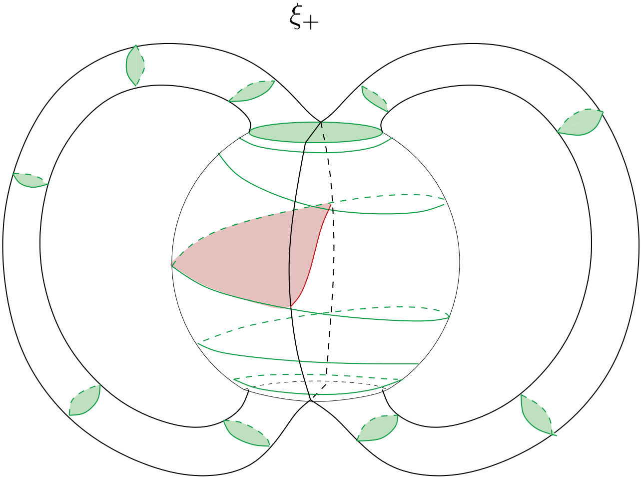
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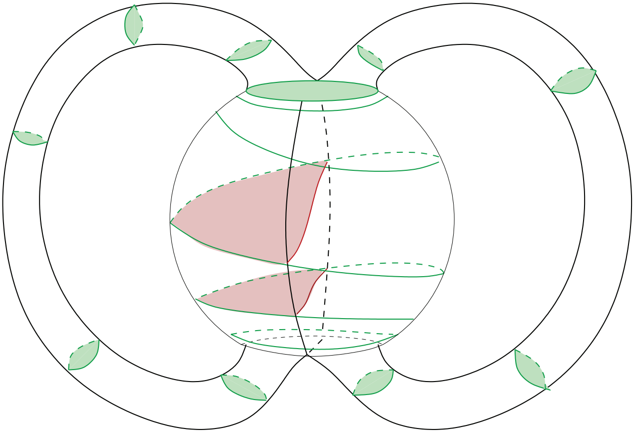
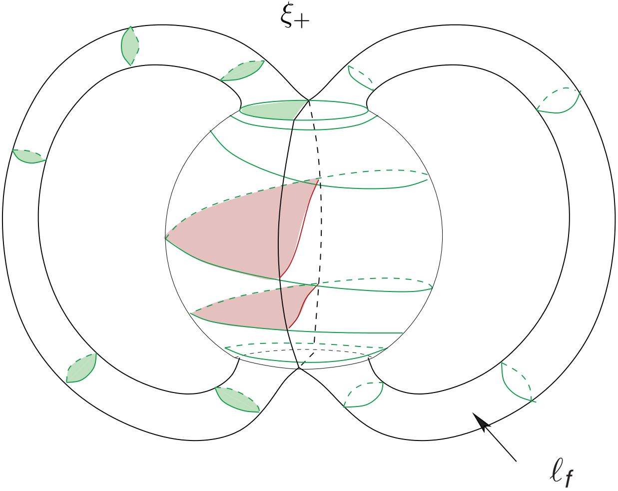
ξ_+

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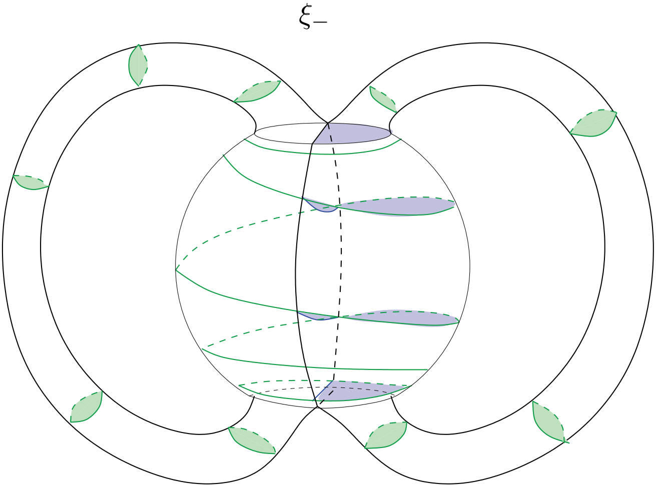
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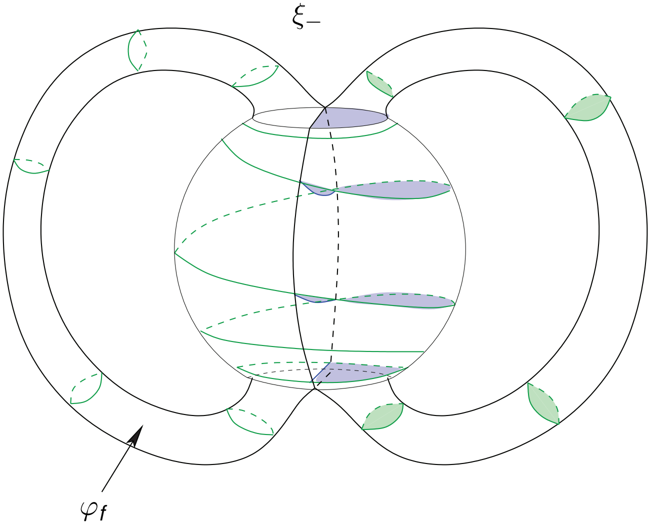
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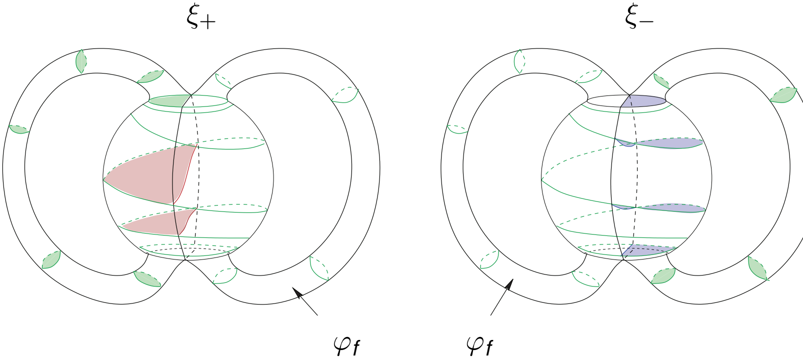
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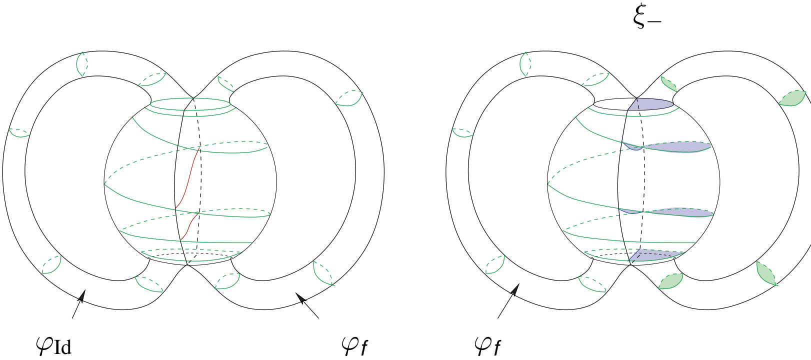
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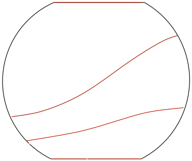

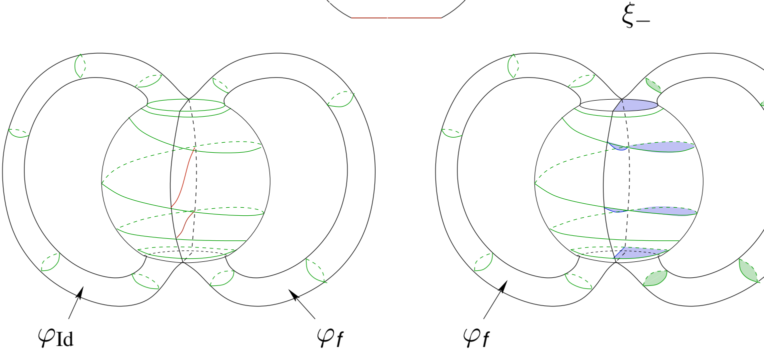
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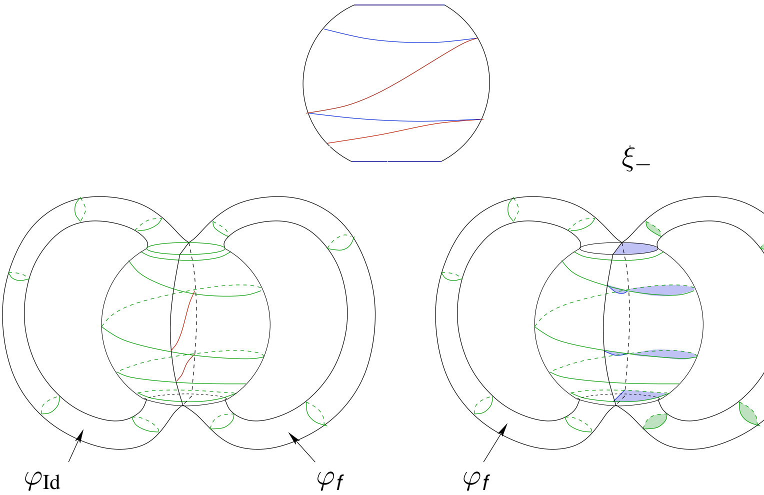
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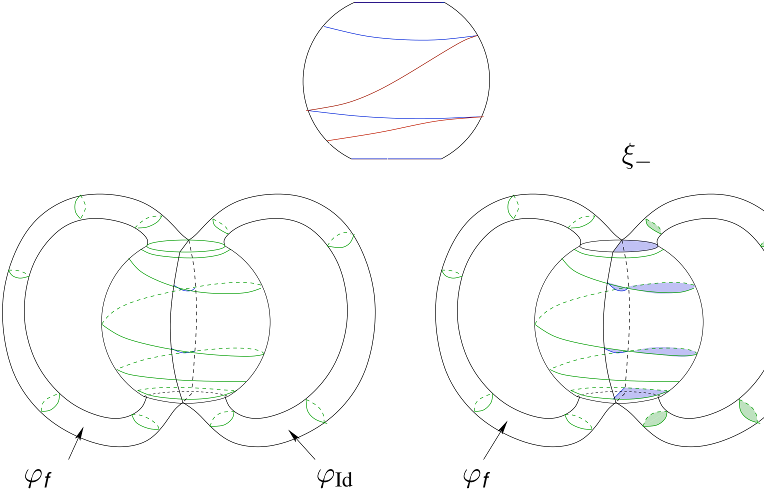
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