

# Distortion elements in diffeomorphism groups of one-manifolds

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## Definition

In  $G = \langle S \rangle$  (with  $S$  finite), endowed with the word-length  $\ell_S$ , an element  $f$  is **distorted** if  $\lim_{n \rightarrow +\infty} \frac{\ell_S(f^n)}{n} = 0$ .

## Observation

- Only depends on  $G$ , not on  $S$ .
- “Distorted” because (if  $\langle f \rangle$  is infinite)  $(\mathbb{Z}, d_{\mathbb{Z}}) \hookrightarrow (G, [d_S])$  given by  $\mathbb{Z} \simeq \langle f \rangle$  is not quasi-isometric.

## Examples

- torsion elements ( $(\ell_S(f^n))_n$  is bounded);
- in  $G = \langle f, g \mid gfg^{-1} = f^k \rangle$  (with  $k > 1$ ),  $g^n fg^{-n} = f^{k^n}$  and thus, for  $S = \{f, g\}$ ,

$$0 \leq \frac{\ell_S(f^{k^n})}{k^n} \leq \frac{2n+1}{k^n} \xrightarrow{n \rightarrow \infty} 0.$$

## Definition

In a group  $G$ , an element  $f$  is **distorted** if it is distorted in some finitely generated subgroup of  $G$ .

Point: if  $H \hookrightarrow G$  and  $G$  has no distorted elements, neither does  $H$ .

Application: Polterovich, Franks, Handel

From now on:  $G = \text{Diff}_c^r(M)$ ,  $r \in \mathbb{N}$ ,  $M$  manifold.

- There are distorted elements (because there are  $f$  conjugated to  $f^2$ ).
- If  $r \geq 1$ , there are (many) **undistorted** elements (*hyperbolicity*).
- (Avila, Milson) In  $\text{Diff}_0^\infty(M)$ , if  $\text{Id} \in \overline{\{f^n, n \in \mathbb{N}^*\}}$ ,  $f$  is distorted.

*But how does one show that an element is (un)distorted?*

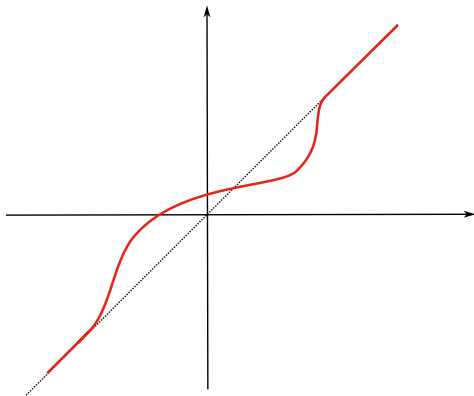
This is what we will study now for  $G = \text{Diff}_c^r(\mathbb{R})$ .

(It depends on the regularity  $r$ !)

$r = 0$  (homeos)

All elements are distorted.

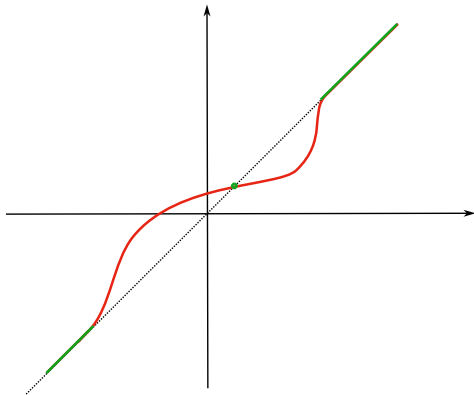
Because they are all conjugated to their square!



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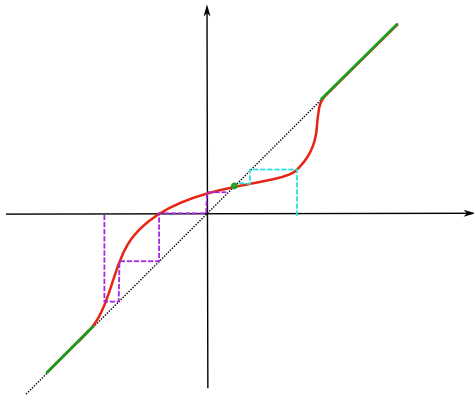
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## Theorem (with E. Militon)

In  $\text{Diff}_c^\infty(\mathbb{R})$ ,

$$f \text{ is } \textit{distorted} \iff f \in C^1 \text{ flow w/o hyperbolic zero}$$



## Theorem (with E. Militon)

In  $\text{Diff}_c^\infty(\mathbb{R})$ ,

*f is distorted*



*f ∈ C<sup>1</sup> flow  
w/o hyperbolic zero*



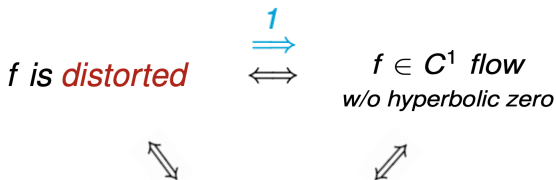
*f is quasi-reducible*

$$\exists (h_n)_n \subset G \text{ s.t.} \\ h_n f h_n^{-1} \rightarrow \text{Id}$$

( $\text{supp}(h_n) \subset \text{common } K$ )

## Theorem (with E. Milton)

In  $\text{Diff}_C^\infty(\mathbb{R})$ ,



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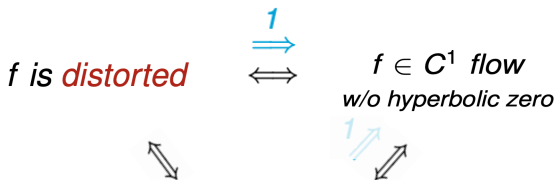
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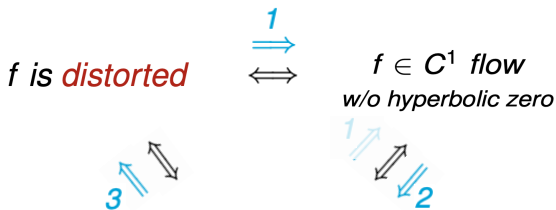
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## Theorem (with E. Milton)

In  $\text{Diff}_+^\infty(\mathbb{S}^1)$ ,

*f is distorted*  $\begin{matrix} \xRightarrow{1} \\ \iff \end{matrix}$  *f w/o periodic point or*  
 *$f^q \in C^1$  flow*  
*w/o hyperbolic zero*



*f is quasi-reducible:*

$\exists (h_n)_n \subset \mathbf{G}$  s.t.  
 $h_n f h_n^{-1} \rightarrow$  *rotation*

$\Leftarrow$  EB–Milton + Avila–Krikorian  
2

# 1. Obstructions to distortion

**Length function:**  $L : G \rightarrow \mathbb{R}_+$  s.t.  $\forall f, g \in G, L(fg) \leq L(f) + L(g)$ .

$$\rightsquigarrow L_\infty(f) := \lim_{n \rightarrow +\infty} \frac{L(f^n)}{n}.$$

## Observation

$f$  distorted  $\implies L_\infty(f) = 0$  for every length function  $L$ .

**Proof.**  $\forall S \subset G$  finite s.t.  $f \in \langle S \rangle$ ,  $0 \leq \frac{L(f^n)}{n} \leq \frac{\ell_S(f^n) \times \max_{g \in S} L(g)}{n} \rightarrow 0$ .

## Examples

- $G = \text{Diff}_c^1(\mathbb{R})$ ,  $L(f) = \|\log Df\|_\infty$ .

$$\rightsquigarrow L_\infty(f) = \max_{x \in \text{Fix}(f)} |\log Df(x)|.$$

$f$  distorted in  $\text{Diff}_c^1(\mathbb{R}) \implies f$  **without hyperbolic fixed point**.

- $G = \text{Diff}_c^r(\mathbb{R})$ ,  $r \geq 2$ ,  $V(f) = \text{var log } Df = \|D \log Df\|_{L^1} = \left\| \frac{D^2 f}{Df} \right\|_{L^1}$ .
- $\rightsquigarrow V_\infty(f) = \textit{asymptotic variation (Navas)}$ .

(EB – Navas)  $V_\infty(f) = 0 \Leftrightarrow f \in C^1$  flow without hyperbolic zero.

↑  
 $f$  distorted

Relation with quasi-reducibility:

- $V_\infty$  invariant by conjugacy;
- $V_\infty(f) \leq V(f)$ .

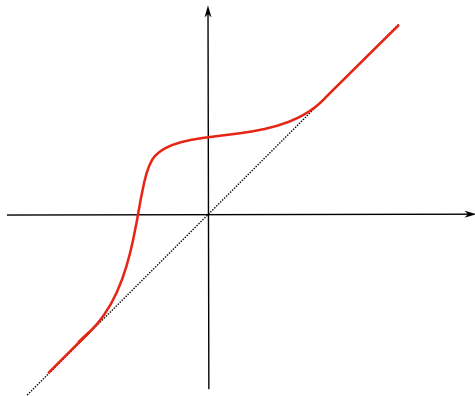
$$\implies V_\infty(f) \leq \inf_{h \in \text{Diff}_c^r(\mathbb{R})} \text{var log } D(hfh^{-1}).$$

$$f \text{ quasi-reducible} \implies V_\infty(f) = 0.$$

## 2. Criterion of quasi-reducibility

Let  $f$  be the time 1 of a  $C^1$  v.f.  $X$  without hyperbolic zero.  
We want to conjugate  $f$  close to Id.

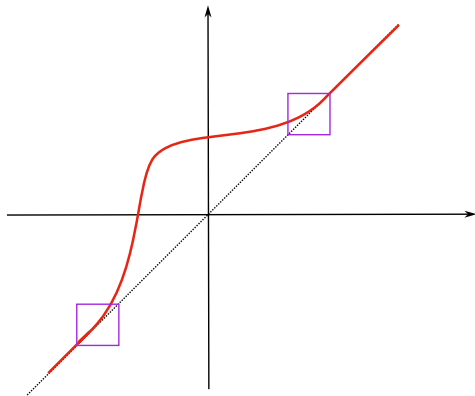
**Favorable case.**  $X$  is  $C^\infty$  and  $\mathbb{R} \setminus \text{Supp}(X)$  is an interval.



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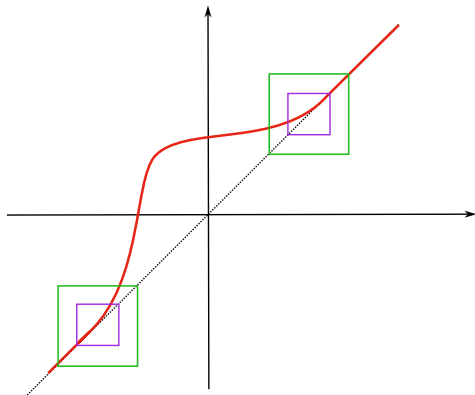
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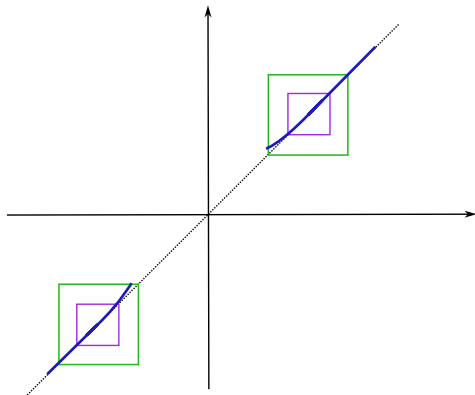
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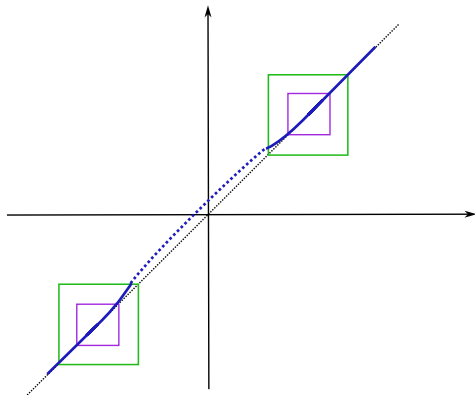
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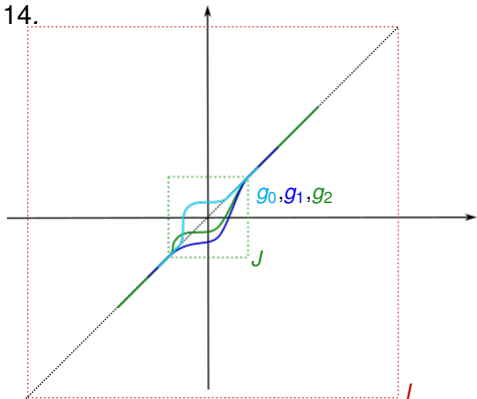


3. Quasi-reduc.  $\Rightarrow$  distortion

## Proposition (“Thurston’s trick”)

Let  $J \in \mathbb{R}$ .  $\forall (g_n)_n, (g'_n)_n \subset \text{Diff}_c^\infty(J)$  converging to Id fast enough,  
there exists a finite  $S \subset \text{Diff}_c^\infty(\mathbb{R})$  s.t.  $\forall n$ ,

- $[g_n, g'_n] \in \langle S \rangle$  ;
- $\ell_S([g_n, g'_n]) \leq 14n + 14$ .

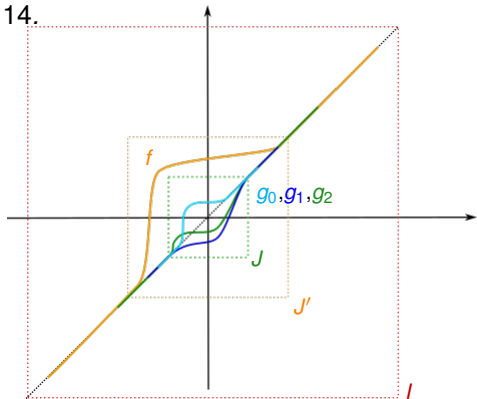


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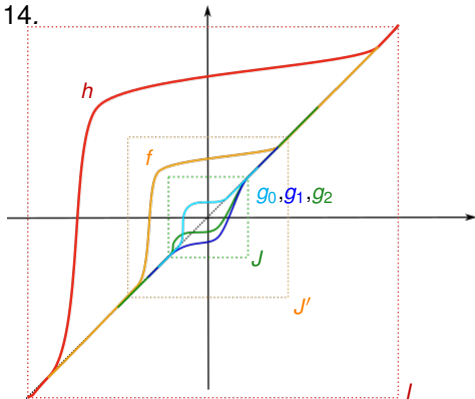


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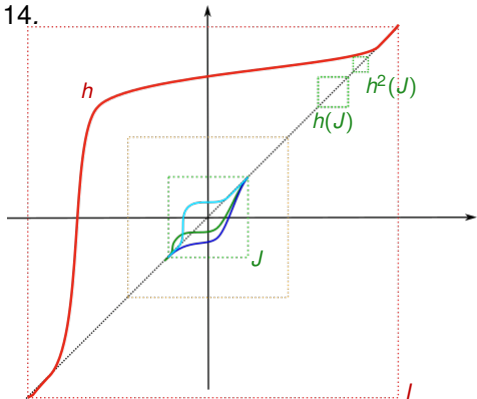


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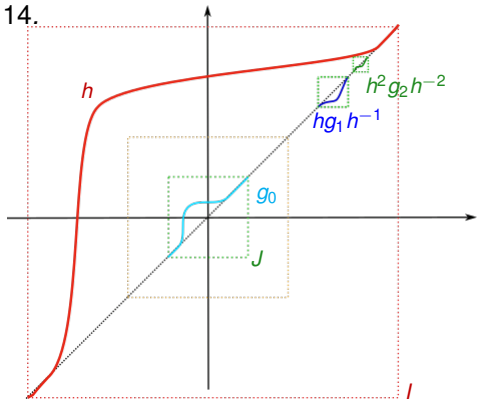


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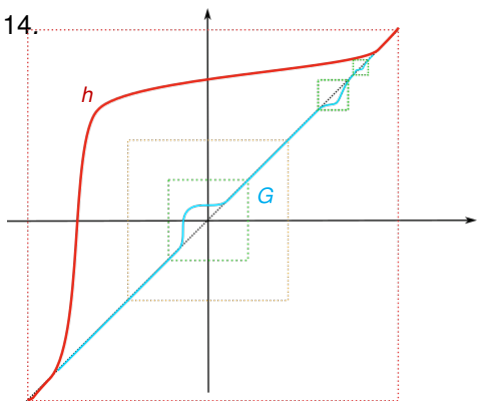


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$$[g_n, g'_n] = h^{-n} \circ [G, h^n f h^{-n}] \circ [G', h^n f h^{-n}] \circ [(G'G)^{-1}, h^n f h^{-n}] \circ h^n.$$

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+ “uniform local perfection”

## Proposition (“Controlled finite generation”)

Let  $J \in \mathbb{R}$ .  $\forall (f_n)_n \subset \text{Diff}_c^\infty(J)$  going to Id fast enough, there exists a finite family  $S \subset \text{Diff}_c^\infty(\mathbb{R})$  s.t.  $\forall n$ ,

- $f_n \in \langle S \rangle$  ;
- $\ell_S(f_n) \leq 70n + 70$ .

## (New) Theorem (Local Uniform Perfection)

Let  $J \in \mathbb{R}$ .  $\forall \varepsilon > 0, \exists \eta > 0$  s.t. every  $f$   $\eta$ -close to Id and supported in  $J$  writes as a product of **5** commutators of diffeos  $\varepsilon$ -close to Id.

Avila (Herman), Yoccoz (Mather), Bonatti-Crovisier-Vago-Wilkinson

Note:  $70 = 14 \times 5$  :)

## Theorem (Avila)

In  $\text{Diff}_+^\infty(\mathbb{S}^1)$ , if  $\text{Id} \in \overline{\{f^n, n \in \mathbb{N}^*\}}$ ,  $f$  is distorted.

## Proposition (“Controlled finite generation”)

$\forall (f_n)_n \subset \text{Diff}_+^\infty(\mathbb{S}^1)$  going to  $\text{Id}$  fast enough, there exists a finite family  $\mathcal{S} \subset \text{Diff}_+^\infty(\mathbb{S}^1)$  s.t.  $\forall n$ ,

- $f_n \in \langle \mathcal{S} \rangle$  ;
- $\ell_{\mathcal{S}}(f_n) \leq 56n + 56$ .

**Proof of Thm.** Apply prop. to  $f^{k_n}$  going fast to  $\text{Id}$  and with  $k_n \gg n$ .

## Theorem

In  $\text{Diff}_C^\infty(\mathbb{R})$ , *quasi-reducible*  $\implies$  *distorted*.

In fact, “quasi-conjugate” to distorted  $\implies$  distorted.

## Proposition (Distortion criterion)

*f is distorted*  $\iff \forall \delta > 0, \liminf_{n \rightarrow \infty} \min_{\substack{S \subset V_\delta(\text{Id}) \\ \#S \leq 15}} \left\{ \frac{\ell_S(f^n)}{n} \right\} = 0.$

**Proof.** “Controlled finite generation”.

## Proposition

*Quasi-reducible diffeomorphisms satisfy the criterion.*