OPEN PROBLEMS

Metric Regularity Days Workshop
October 25-26, 2011

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Key words: metric regularity, distance to ill-posedness, variational analysis, implicit functions, nonconvex, prox-regularity, control systems under state constraints, neighboring feasible trajectory theorem

Abstract

We collect some open problems presented at the final panel of the Metric Regularity Days Workshop held at Université Pierre et Marie Curie CNRS - Institut de Mathématiques de Jussieu (UMR 7586), October 25-26, 2011.

1 Conditioning in Nonlinear Programming (Asen Dontchev)

Linear equation $Ax = y$: error in solution δx from error in input δy; then

$$\|\delta x\| \leq \|A^{-1}\| \|\delta y\|$$

$\|A^{-1}\|$ is called the *absolute condition number*. Main property: its reciprocal equals the distance to the set of singular matrices (radius theorem):

$$\inf\left\{ \|B\| \,\middle|\, A + B \text{ singular} \right\} = \frac{1}{\|A^{-1}\|}.$$

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Radius theorem for metric regularity. [A.D., A. Lewis, R. T. Rockafellar, Trans. AMS 2003] For Banach spaces X and Y, consider a mapping $F : X \rightrightarrows Y$ and $\bar{y} \in F(\bar{x})$ and let the graph of F be locally closed at $(\bar{x}, \bar{y})$. Suppose that F is metrically regular at $\bar{x}$ for $\bar{y}$. Then

$$\inf_{B \in \mathcal{L}(X,Y)} \left\{ \|B\| \,\middle|\, F + B \text{ is not metrically regular at } \bar{x} \text{ for } \bar{y} + B\bar{x} \right\}$$

$$\geq \frac{1}{\operatorname{reg}(F; \bar{x} \,|\, \bar{y})}. \qquad (1)$$

If, in addition, X and Y are finite-dimensional, then the inequality $\geq$ in (1) becomes equality; in this case the infimum is unchanged if taken with respect to linear mappings of rank 1.

Nonlinear programming: minimize $g_0(p, x)$ subject to

$$g_i(p, x) \leq 0 \ i \in [1, s],$$

$$g_i(p, x) = 0, i \in [s + 1, m]$$

We assume that all functions are C^2. Using the Lagrangian function

$$L(p, x, y) = g_0(p, x) + y_1 g_1(p, x) + \cdots + y_m g_m(p, x)$$

the first-order optimality condition is

$$f(p, x, y) + N_E(x, y) \ni (0, 0),$$

where

$$f(p, x, y) = (\nabla_x L(p, x, y), -\nabla_y L(p, x, y)),$$
$$E = I\!R^n \times [I\!R^s_+ \times I\!R^{m-s}].$$

Denote the set of solutions of the above Karush-Kuhn-Tucker (KKT) system by $S(p)$.

Fix reference parameter value $\bar{p}$ and an associated pair $(\bar{x}, \bar{y}) \in S(\bar{p})$

$$I = \left\{ i \in [1, m] \,\middle|\, g_i(\bar{p}, \bar{x}) = 0 \right\} \supset \{s + 1, \ldots, m\},$$
$$I_0 = \left\{ i \in [1, s] \,\middle|\, g_i(\bar{p}, \bar{x}) = 0 \text{ and } \bar{y}_i = 0 \right\} \subset I,$$
$$M^+ = \left\{ w \in I\!R^n \,\middle|\, w \perp \nabla_x g_i(\bar{p}, \bar{x}) \text{ for all } i \in I \backslash I_0 \right\},$$

Theorem [A.D., R. T. Rockafellar, SIOPT 1995]. Let $(\bar{x}, \bar{y}) \in S(\bar{p})$ and assume that the following conditions hold:

(a) the gradients $\nabla_x g_i(\bar{p}, \bar{x})$ for $i \in I$ are linearly independent, and
(b) $\langle w, \nabla^2_{xx} L(\bar{p}, \bar{x}, \bar{y}) w \rangle > 0$ for every $w \neq 0$ in the subspace M^+.

Then the solution mapping S has a Lipschitz continuous single-valued localization $s(\cdot)$ around $\bar{p}$ for $(\bar{x}, \bar{y})$ with the property that for every p in some neighborhood of $\bar{p}$, the x component of $s(p)$ furnishes a strong local minimum. Moreover, (a) and (b) are necessary for this conclusion when $n+m$ is the rank of the $(n+m) \times d$ matrix

$$\begin{pmatrix} \nabla^2_{xp} L(\bar{p}, \bar{x}, \bar{y}) \\ -\nabla_p g_1(\bar{p}, \bar{x}) \\ \vdots \\ -\nabla_p g_m(\bar{p}, \bar{x}) \end{pmatrix}.$$

Open Problem: by how much the problem can be perturbed so that the property of the solution mapping displayed in the theorem above is lost: the solution mapping has a Lipschitz continuous single-valued localization $s(\cdot)$ around $\bar{p}$ for $(\bar{x}, \bar{y})$ such that for every p in some neighborhood of $\bar{p}$, the x component of $s(p)$ furnishes a strong local minimum.

This property is a basic condition used in proving convergence of algorithms, e.g. the sequential programming method. The radius of validity of this property is a natural candidate to define a condition number in nonlinear programming.

Changing the problem by adding C^2 perturbing functions to the functions g_i leads to a variational inequality that is perturbed in a very specific way.

Simple case: no equality constraints and all inequality constraints are active. Denote

$$A = \nabla^2_{xx} L(\bar{p}, \bar{x}, \bar{y}), \;\; B = \nabla^2 L_{xy}(\bar{p}, \bar{x}, \bar{y}).$$

Also $\bar{y}_i > 0$ for $i = 1, 2, \ldots, r$ and $\bar{y}_i = 0$ for the rest of i in I.

Denoting by B_0 the submatrix of B composed by the first r rows of B and by B_1 the matrix composed by the remaining rows, the mapping of the associated linearized variational inequality whose strong regularity is involved has the form

$$(x, y) \mapsto M(x, y) := \begin{pmatrix} A & B_0^T & B_1^T \\ -B_0 & 0 & 0 \\ -B_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + N_E(x, y).$$

Thus, finding the radius of the desired property is the same as: Find the smallest perturbation of the matrix in the mapping M by a matrix *with the same structure* which preserves the symmetry so that either the positive definiteness of A on the kernel of B_0 or the surjectivity of B is lost.

2 Error estimates in control (Richard Vinter)

Distance estimates for controls systems with an accompanying state constraint are estimates of the distance of an arbitrary state trajectory from the set of state trajectories that satisfy the state constraint,expressed in terms of the state constraint violation of the arbitrary state trajectory. What kind of estimates (linear, superlinear, Hölder) are in general valid, w.r.t. either the $W^{1,1}$ or L^∞ distance functions, for arbitrary closed state constraint sets and for velocities sets that depend on (t,x)? For the most recent results on neighboring feasible trajectories theorems see

References

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3 Metrically subsmooth sets (Lionel Thibault)

Let C be a closed subset of an Asplund space X, let $x \in X$ and let $x^* \in \partial_F \operatorname{dist}(C, \cdot)(x)$. Then for every $\varepsilon > 0$ there is a $u \in C$ and $u^* \in \partial_F \operatorname{dist}(C, \cdot)(u)$ such that

$$\|u - x\| \leq \varepsilon + \operatorname{dist}(C, x) \quad \text{and} \quad \|u^* - x^*\| \leq \varepsilon.$$

Find a counter example when X is not an Asplund space.

4 Regularity of countable systems of sets (Russell Luke)

One characterization of strong regularity of the intersection of a collection of sets $\{\Omega_j\}_{j=1}^m \subset \mathbb{E}$ (Euclidean space) at $\overline{x} \in \cap_{j=1}^m \Omega_j$ is that $y_j = 0$ ($j = 1, 2, \ldots, m$) is the only solution to

$$\sum_j y_j = 0, \quad y_j \in N_{\Omega_j}(\overline{x}).$$

This is equivalent to metric regularity of the corresponding set-valued mapping $\Phi(\overline{x}) := (\Omega_1 - \overline{x}) \times (\Omega_2 - \overline{x}) \times \cdots \times (\Omega_m - \overline{x})$ at $\overline{x}$ for 0. What is a reasonable characterization of the regularity of the intersection of a system of countably many sets, $\{\Omega_j\}_{j=1}^\infty$?