

Computing the Gromov hyperbolicity of a discrete metric space

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Abstract

We give exact and approximation algorithms for computing the Gromov hyperbolicity of an n -point discrete metric space. We observe that computing the Gromov hyperbolicity from a fixed base-point reduces to a (max,min) matrix product. Hence, using the (max,min) matrix product algorithm by Duan and Pettie, the fixed base-point hyperbolicity can be determined in $O(n^{2.69})$ time. It follows that the Gromov hyperbolicity can be computed in $O(n^{3.69})$ time, and a 2-approximation can be found in $O(n^{2.69})$ time. We also give a $(2 \log_2 n)$ -approximation algorithm that runs in $O(n^2)$ time, based on a tree-metric embedding by Gromov. We also show that hyperbolicity at a fixed base-point cannot be computed in $O(n^{2.05})$ time, unless there exists a faster algorithm for (max,min) matrix multiplication than currently known.

Keywords. Algorithms design and analysis; Approximation algorithms; Discrete metric space; Hyperbolic space; (max,min) matrix product.

1 Introduction

Gromov introduced a notion of metric-space hyperbolicity [2, 9] using a simple four point condition. (See Section 1.1.) This definition is very attractive from a computer scientist point of view as the hyperbolicity of a finite metric space can be easily computed by brute force, by simply checking the four point condition at each quadruple of points. However, this approach takes $\Theta(n^4)$ time for an n -point metric space, which makes it impractical for some applications to networking [6]. Knowing the hyperbolicity is important, as the running time and space requirements of previous algorithms designed for Gromov hyperbolic spaces are often analyzed in terms of their Gromov hyperbolicity [4, 5, 10]. So far, it seems that no better algorithm than brute force was known for computing the Gromov hyperbolicity [3]. In this note, we give faster exact and approximation algorithms based on previous work on (max-min) matrix products by Duan and Pettie [7], and the tree-metric embedding by Gromov [9].

The exponent of matrix multiplication μ is the infimum of the real numbers $\omega > 0$ such that two $n \times n$ real matrices can be multiplied in $O(n^\omega)$ time, exact arithmetic operations being performed in

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one step [11]. Currently, μ is known to be less than 2.373 [12]. In the following, ω is a real number such that we can multiply two $n \times n$ real matrices in $O(n^\omega)$ time.

Our algorithm for computing the Gromov hyperbolicity runs in $O(n^{(5+\omega)/2})$ time, which is $O(n^{3.69})$. (See Section 2.1.) For a fixed base-point, this improves to $O(n^{(3+\omega)/2})$, which also yields a 2-factor approximation for the general case within the same time bound. (See Section 2.2.) We also give a quadratic-time $(2 \log_2 n)$ -approximation algorithm. (See Section 2.3.) Finally, we show that hyperbolicity at a fixed base-point cannot be computed in time $O(n^{3(\omega-1)/2}) = O(n^{2.05})$, unless (max,min) matrix product can be computed in time $O(n^\tau)$ for $\tau < (3 + \omega)/2$. (See Section 3.) The currently best known algorithm runs in $O(n^{(3+\omega)/2})$ time [7].

1.1 Gromov hyperbolic spaces

An introduction to Gromov hyperbolic spaces can be found in the article by Bonk and Schramm [2], and in the book by Ghys and de la Harpe [8]. Here we briefly present some definitions and facts that will be needed in this note.

A metric space (M, d) is said to be δ -hyperbolic for some $\delta \geq 0$ if it obeys the so-called *four point condition*: For any $x, y, z, t \in M$, the largest two distance sums among $d(x, y) + d(z, t)$, $d(x, z) + d(y, t)$, and $d(x, t) + d(y, z)$, differ by at most 2δ . The *Gromov hyperbolicity* δ^* of (M, d) is the smallest $\delta^* \geq 0$ such that (M, d) is δ^* -hyperbolic.

For any $x, y, r \in M$, the *Gromov product* of x, y at r is defined as

$$(x|y)_r = \frac{1}{2} (d(x, r) + d(r, y) - d(x, y)).$$

The point r is called the *base point*. Gromov hyperbolicity can also be defined in terms of the Gromov product, instead of the four point condition above. The two definitions are equivalent, with the same values of δ and δ^* . So a metric space (M, d) is δ -hyperbolic if and only if, for any $x, y, z, r \in M$

$$(x|z)_r \geq \min\{(x|y)_r, (y|z)_r\} - \delta.$$

The Gromov hyperbolicity δ^* is the smallest value of δ that satisfies the above property. In other words,

$$\delta^* = \max_{x, y, z, r} \{\min\{(x|y)_r, (y|z)_r\} - (x|z)_r\}.$$

The hyperbolicity δ_r at base point r is defined as

$$\delta_r = \max_{x, y, z} \{\min\{(x|y)_r, (y|z)_r\} - (x|z)_r\}. \tag{1}$$

Hence, we have

$$\delta^* = \max_r \delta_r. \tag{2}$$

2 Algorithms

In this section, we consider a discrete metric space (M, d) with n elements, that we denote x_1, \dots, x_n . Our goal is to compute exactly, or approximately, its hyperbolicity δ^* , or its hyperbolicity δ_r at a base point r .

2.1 Exact algorithms

The (max,min)-product $A \otimes B$ of two real matrices A, B is defined as follows:

$$(A \otimes B)_{ij} = \max_k \min\{A_{ik}, B_{kj}\}.$$

Duan and Pettie [7] gave an $O(n^{(3+\omega)/2})$ -time algorithm for computing the (max,min)-product of two $n \times n$ matrices.

Let r be a fixed base-point. By Equation (1), if A is the matrix defined by $A_{ij} = (x_i | x_j)_r$ for any i, j , then δ_r is simply the largest coefficient in $(A \otimes A) - A$. So we can compute δ_r in $O(n^{(3+\omega)/2})$ time. Maximizing over all values of r , we can compute the hyperbolicity δ^* in $O(n^{(5+\omega)/2})$ time, by Equation (2).

2.2 Factor-2 approximation

The hyperbolicity δ_r with respect to any base-point is known to be a 2-approximation of the hyperbolicity δ^* [2]. More precisely, we have $\delta_r \leq \delta^* \leq 2\delta_r$. So, using the algorithm of Section 2.1, we can pick an arbitrary base-point r and compute δ_r in $O(n^{(3+\omega)/2})$ time, which gives us a 2-approximation of δ^* .

2.3 Logarithmic factor approximation

Gromov [9] (see also the article by Chepoi et al. [4, Theorem 1] and the book by Ghys and de la Harpe [8, Chapter 2]) showed that any δ -hyperbolic metric space (M, d) can be embedded into a weighted tree T with an additive error $2\delta \log_2 n$, and this tree can be constructed in time $O(n^2)$. In particular, if we denote by d_T the metric corresponding to such a tree T , then

$$d(a, b) - 2\delta^* \log_2 n \leq d_T(a, b) \leq d(a, b) \text{ for any } a, b \in M. \quad (3)$$

This construction can be performed without prior knowledge of δ^* .

We compute $D = \max_{a, b \in M} d(a, b) - d_T(a, b)$ in time $O(n^2)$. We claim that:

$$\delta^* \leq D \leq 2\delta^* \log_2 n. \quad (4)$$

So we obtain a $(2 \log_2 n)$ -approximation D of δ^* in time $O(n^2)$.

We still need to prove the double inequality (4). It follows from Equation (3) that $d(a, b) - d_T(a, b) \leq 2\delta^* \log_2 n$ for any a, b , and thus $D \leq 2\delta^* \log_2 n$. In the following, we prove the other inequality.

For any x, y, z, t , we denote by $\delta(x, y, z, t)$ the difference between the two largest distance sums among $d(x, y) + d(z, t)$, $d(x, z) + d(y, t)$, and $d(x, t) + d(y, z)$. Thus, if for instance $d(x, y) + d(z, t) \geq d(x, z) + d(y, t) \geq d(x, t) + d(y, z)$, we have $\delta(x, y, z, t) = d(x, y) + d(z, t) - d(x, z) - d(y, t)$. We also need to introduce the difference $\delta_T(x, y, z, t)$ between the two largest sums among $d_T(x, y) + d_T(z, t)$, $d_T(x, z) + d_T(y, t)$, and $d_T(x, t) + d_T(y, z)$.

For any $a, b \in M$, we have $d(a, b) - D \leq d_T(a, b) \leq d(a, b)$, so $\delta(x, y, z, t) - \delta_T(x, y, z, t) \leq 2D$, because in the worst case, the largest sum with respect to d is the same as the largest sum with respect to d_T , and the second largest sum with respect to d_T is equal to the second largest sum with respect to d minus $2D$. But by construction, d_T is a tree metric [4], so $\delta_T(x, y, z, t) = 0$ for any x, y, z, t . Therefore $\delta(x, y, z, t) \leq 2D$ for any x, y, z, t , which means that $\delta^* \leq D$.

3 Conditional lower bounds

We show that computing hyperbolicity at a fixed base-point is intimately connected with (max,min)-product. From the previous section, any improvement on the complexity of (max,min)-product yields an improvement on our algorithm to compute hyperbolicity. We show that a partial converse holds: Any improvement on the complexity of computing hyperbolicity at a fixed base-point below $n^{3(\omega-1)/2}$ would give an improved algorithm for computing the (max,min)-product.

We consider the following decision problem.

Problem 1 (HYP: Fixed-base hyperbolicity) *Given a metric on n points, a point r and $\alpha \geq 0$, decide if the hyperbolicity δ_r at base point r is larger than α .*

Note that we do not ask to check whether the input is indeed a metric (no subcubic algorithm is known for this problem [13]). A tree metric is a metric such that $\delta_r = 0$ for some base point r or, equivalently, such that $\delta_r = 0$ for any base point r , so the special case $\alpha = 0$ can be solved in $O(n^2)$ time as a tree metric can be recognized in $O(n^2)$ time [1]. In this section, we show that we cannot get a quadratic time algorithm (or even an $O(n^{2.05})$ time algorithm) unless some progress is made on the complexity of (max,min) matrix product. Our main tool is a result from Vassilevska and Williams [13] stated below for the special case of (min, max) structures. We first define the tripartite negative triangle problem (also known as the IJ -bounded triangle problem [13]).

Problem 2 (TNT: Tripartite negative triangle) *Given a tripartite graph $G = (I \cup J \cup K, E)$ with weights $w : E \rightarrow \mathbb{R}$, a triangle $(i, j, k) \in I \times J \times K$ is called negative if $\max\{w_{i,k}, w_{k,j}\} + w_{i,j} < 0$. The tripartite negative triangle problem is to decide whether there exists a negative triangle in G .*

Note that the above definition is not symmetric in I, J, K : Only I and J are interchangeable. Vassilevska and Williams proved the following reduction ([13, Theorem V.2] for the special case where $\odot = \max$).

Theorem 1 [13] *Let $T(n)$ be a function such that $T(n)/n$ is nondecreasing. Suppose the tripartite negative triangle problem in an n -node graph can be solved in $T(n)$ time. Then the (min,max)-product of two $n \times n$ matrices can be performed in $O(n^2 T(n^{1/3}) \log W)$, where W is the absolute value of the largest integer in the output.*

We now show how Problem 2 reduces to Problem 1.

Lemma 2 *The tripartite negative triangle problem (TNT) reduces to fixed-base hyperbolicity (HYP) in quadratic time.*

Proof: Besides TNT and HYP, we define two intermediate problems:

- Tripartite positive triangle (TPT). Given a tripartite graph $G = (I \cup J \cup K, E)$ with weights $w : E \rightarrow \mathbb{R}$, decide if there exists a triangle $(i, j, k) \in I \times J \times K$ such that $\min\{w_{i,k}, w_{k,j}\} - w_{i,j} > 0$.
- Positive triangle (PT). Given a complete undirected graph $G = (V, E)$ with weights $w : E \rightarrow \mathbb{R}$ and $\alpha \geq 0$, decide if there are three distinct vertices $i, j, k \in V$ such that $\min\{w_{i,k}, w_{k,j}\} - w_{i,j} > \alpha$.

We will denote by \leq_2 the quadratic-time reductions. The lemma is obtained by the following sequence of reductions: $\text{TNT} \leq_2 \text{TPT} \leq_2 \text{PT} \leq_2 \text{HYP}$.

Let us show that $\text{TNT} \leq_2 \text{TPT}$. Let $G = (I \cup J \cup K, E)$ and $w : E \rightarrow \mathbb{R}$ an instance of TNT. Consider the same graph with the following weights w' : $w'_{i,j} = w_{i,j}$ for $(i, j) \in I \times J$ and $w'_{\ell,k} = -w_{\ell,k}$ for $(\ell, k) \in (I \cup J) \times K$. The graph G with weights w' has a tripartite positive triangle if there exists $(i, j, k) \in I \times J \times K$ such that $\min\{w'_{i,k}, w'_{k,j}\} - w'_{i,j} > 0$. This is equivalent to $\max\{w_{i,k}, w_{k,j}\} + w_{i,j} < 0$.

We now show that $\text{TPT} \leq_2 \text{PT}$. Let $G = (I \cup J \cup K, E)$ and $w : E \rightarrow \mathbb{R}$ an instance of TPT. Let $G' = (V', E')$ be the complete graph on the set of $3n$ nodes $V' = I \cup J \cup K$. We shall define a symmetric weight function $w' : E' \rightarrow \mathbb{R}$ and $\alpha \geq 0$ such that G', w', α is a positive instance of PT if and only if G has a tripartite positive triangle. Let $\lambda = 1 + \max_{e \in E} |w(e)|$. For $(i, j) \in I \times J$, let $w'_{i,j} = w_{i,j}$. Let $w'_{\ell_1, \ell_2} = 2\lambda$ for $(\ell_1, \ell_2) \in (I \times I) \cup (J \times J) \cup (K \times K)$. For $(\ell, k) \in (I \cup J) \times K$, let $w'_{\ell,k} = w_{\ell,k} + 5\lambda$. Finally, let $\alpha = 5\lambda$. Assume that G contains a tripartite positive triangle (i, j, k) , and thus $\min\{w_{i,k}, w_{k,j}\} - w_{i,j} > 0$. It means that $\min\{w_{i,k} + 5\lambda, w_{k,j} + 5\lambda\} - w_{i,j} > 5\lambda$. So $\min\{w'_{i,k}, w'_{k,j}\} - w'_{i,j} > \alpha$, and thus G' has a positive triangle.

Conversely, assume G' has a positive triangle (i, j, k) :

$$\min\{w'_{i,k}, w'_{k,j}\} - w'_{i,j} > 5\lambda. \quad (5)$$

Let us determine the location of the triangle (i, j, k) .

- Inequality (5) implies that $6\lambda > \min\{w'_{i,k}, w'_{k,j}\} > 5\lambda + w'_{i,j}$. Hence, $\lambda > w'_{i,j}$. It follows that $(i, j) \in (I \times J) \cup (J \times I)$.
- Inequality (5) also implies that $\min\{w'_{i,k}, w'_{k,j}\} > 5\lambda + w'_{i,j} > 4\lambda$. Hence, it must be that both (i, k) and (j, k) belong to $(I \times K) \cup (J \times K)$.

From the two conditions above, it follows that $k \in K$ and $(i, j) \in (I \times J) \cup (J \times I)$. Without loss of generality, let us assume that $(i, j) \in I \times J$. Then $\min\{w_{i,k} + 5\lambda, w_{j,k} + 5\lambda\} - w_{i,j} > 5\lambda$ which means that (i, j, k) is a tripartite positive triangle in G .

Let us show that $\text{PT} \leq_2 \text{HYP}$. Consider a weighted complete graph G on the vertices $\{1, \dots, n\}$ and $\alpha \geq 0$ an instance of PT. Let W be the (symmetric) weight matrix of G , so for any i, j the coefficient W_{ij} is the weight $w_{i,j}$ of the edge (i, j) , and $W_{ii} = 0$. Let $\lambda' = |\max_{i,j} w_{i,j}|$ be the absolute value of the largest weight. Let I_n be the $n \times n$ identity matrix and E_n be the all-one matrix of size $n \times n$. Let P be the matrix defined by $P = W + 2\lambda'E_n + 4\lambda'I_n$. We will show that (M, d) is a metric space, where $M = \{r, x_1, x_2, \dots, x_n\}$, and the metric d is defined as follows.

$$\begin{aligned} d(x_i, x_j) &= P_{ii} + P_{jj} - 2P_{ij} && \text{for any } x_i, x_j \in M \setminus \{r\}, \\ d(x_i, r) &= d(r, x_i) = P_{ii} && \text{for any } x_i \in M \setminus \{r\}, \\ d(x, x) &= 0 && \text{for any } x \in M. \end{aligned}$$

We now prove that (M, d) is indeed a metric space. The function d is symmetric because the matrix P is symmetric. For any $x_i, x_j \in M \setminus \{r\}$, $i \neq j$, we have $d(x_i, x_j) = 8\lambda' - 2W_{ij}$ and thus $6\lambda' \leq d(x_i, x_j) \leq 10\lambda'$. We also have $d(x_i, r) = P_{ii} = 6\lambda'$, so $6\lambda' \leq d(x, y) \leq 10\lambda'$ for any two distinct points $x, y \in M$, which implies that the triangle inequality is satisfied.

We have just proved that (M, d) is a metric space. In addition, the matrix P records the Gromov products of (M, d) at base r , because for any $x_i, x_j \in M \setminus \{r\}$, we have:

$$2(x_i | x_j)_r = d(x_i, r) + d(x_j, r) - d(x_i, x_j) = P_{ii} + P_{jj} - (P_{ii} + P_{jj} - 2P_{ij}) = 2P_{ij}.$$

We now argue that for any $a, b, c \in M$ such that $\min\{(a | c)_r, (c | b)_r\} - (a | b)_r > 0$, we must have $(a, b, c) = (x_i, x_j, x_k)$ for distinct $i, j, k \in \{1, \dots, n\}$. For any $y \in M$, we have $(y | r)_r = 0$, and thus $r \notin \{a, b, c\}$. So we must have $(a, b, c) = (x_i, x_j, x_k)$ for some $i, j, k \in \{1, \dots, n\}$. Thus, we only need to argue that a, b and c are distinct. Indeed, if $a = c$, then

$$\min\{(a | c)_r, (c | b)_r\} - (a | b)_r = \min\{(a | a)_r, (a | b)_r\} - (a | b)_r \leq 0$$

and if $a = b$,

$$\begin{aligned} 2(\min\{(a | c)_r, (b | c)_r\} - (a | b)_r) &= 2(a | c)_r - 2(a | a)_r \\ &= 2(a | c)_r - 2d(a, r) \\ &= d(c, r) - d(c, a) - d(a, r) \leq 0. \end{aligned}$$

Thus, the hyperbolicity δ_r at base r of (M, d) is larger than α if and only if there are distinct $i, j, k \in \{1, \dots, n\}$ such that $\min\{(x_i | x_k)_r, (x_k | x_j)_r\} - (x_i | x_j)_r > \alpha$. So we showed that $\delta_r > \alpha$ if and only if there are distinct i, j, k such that $\min\{P_{ik}, P_{kj}\} - P_{ij} > \alpha$, or equivalently, $\min\{W_{ik}, W_{kj}\} - W_{ij} > \alpha$, which means that G has a positive triangle. \square

Theorem 3 *If fixed-base hyperbolicity can be decided in time $O(n^\nu)$, with $\nu \geq 2$, then the (max,min)-product of two matrices of size $n \times n$ can be done in time $O(n^{2+\nu/3} \log n)$.*

Proof: Let us assume fixed-base hyperbolicity can be decided in time $O(n^\nu)$. By Lemma 2, the tripartite negative triangle problem can also be solved in time $O(n^\nu)$. It remains to show that (max,min)-product can be done in time $O(n^{2+\nu/3} \log n)$. By duality, the complexity of computing (max,min) products and (min,max) products are the same. Moreover, we can assume without loss of generality that the two matrices of which we want to compute the product have integer inputs in the range $\{0, \dots, 2n^2\}$. (Indeed, one can sort the inputs of the two matrices, replace input values with their ranks, perform the product, and replace back the ranks with the initial values in the product.) Applying Theorem 1 gives complexity $O(n^{2+\nu/3} \log n)$ to compute this type of products. \square

The above theorem immediately implies that any $O(n^\nu)$ -time algorithm with $\nu < 3(\omega - 1)/2$ to solve fixed-base hyperbolicity would give an $O(n^\tau)$ -time algorithm with $\tau < (3 + \omega)/2$ for the (max,min)-product of two matrices of size $n \times n$. Since the state of the art is $3(\omega - 1)/2 > 2.05$, any $O(n^{2.05})$ -time algorithm for fixed-base Gromov hyperbolicity would yield an improvement on the complexity of (max, min)-product.

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