

Monomials in arithmetic circuits: Complete problems in the counting hierarchy

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Abstract

We consider the complexity of two questions on polynomials given by arithmetic circuits: testing whether a monomial is present and counting the number of monomials. We show that these problems are complete for subclasses of the counting hierarchy which had few or no known natural complete problems before. We also study these questions for circuits computing multilinear polynomials and for univariate multiplicatively disjoint circuits.

1 Introduction

We study the complexity of two problems on polynomials represented by arithmetic circuits. The first one is to decide whether a given monomial has zero coefficient, while the second consists in counting the number of monomials. We characterize their complexity using the counting hierarchy.

The counting hierarchy refers to the family of classes $\text{PP} \cup \text{PP}^{\text{PP}} \cup \text{PP}^{\text{PP}^{\text{PP}}} \cup \dots$. It has appeared in several recent papers. For example, Bürgisser [7] uses these classes to connect computing integers to computing polynomials, while Jansen and Santhanam [15] — building on results by Koiran and Perifel [19] — use them to derive lower bounds from derandomization. This hierarchy was originally introduced by Wagner [36] to classify the complexity of combinatorial problems. Curiously, after Wagner’s paper and another by Torán [30], this original motivation of the counting hierarchy has to the best of our knowledge not been pursued for more than twenty years. Instead, research focused on structural properties and the connection to threshold circuits [4]. As a result, there are very few natural complete problems for classes in the counting hierarchy: for instance, Kwisthout et al. give in [21] “the first problem with a practical application that is shown to be $\text{FP}^{\text{PP}^{\text{PP}}}$ -complete”. The related class $\text{C}_{\#}\text{P}$ appears to have no natural complete problems at all (see [14, p. 293]). It is however possible to define generic complete problems by starting with a $\#\text{P}$ -complete problem and considering the variant where an instance and a positive integer are provided and the question is to decide whether the number of solutions for this instance is equal to the integer. We consider these problems to be counting problems disguised as decision problems

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and thus not as natural complete problems for $C=P$, in contrast to the questions studied here. Note that the corresponding logspace counting class $C=L$ is known to have interesting complete problems from linear algebra [2].

In this paper we follow Wagner’s original idea and show that the counting hierarchy is a helpful tool to classify the complexity of several natural problems on arithmetic circuits by showing complete problems for the classes PP^{PP} , PP^{NP} and $C=P$.¹ The common setting of these problems is the use of circuits or straight-line programs to represent polynomials. Such a representation can be much more efficient than giving the list of monomials, but common operations on polynomials may become more difficult. An important example is the question of determining whether the given polynomial is identically zero. This is easy to do when the polynomial is given as a list of monomials. When the polynomial is given as a circuit however, this problem, called ACIT for *arithmetic circuit identity testing*, is not known to be in P , though it is in $coRP$. In fact, derandomizing this problem would imply circuit lower bounds, as shown in [16]. This question thus plays a crucial part in complexity and it is natural to consider other problems on polynomials represented as circuits. In this article we consider mainly two questions.

The first main problem, called ZMC for *zero monomial coefficient*, is to decide whether a given monomial in a circuit has coefficient 0 or not. This problem has already been studied by Koiran and Perifel [18]. They showed that when the formal degree of the circuit is polynomially bounded the problem is complete for $P^{\#P}$. Unfortunately this result is not fully convincing, because it is formulated with the rather obscure notion of strong non-deterministic Turing reductions. We remedy this situation by proving a completeness result for the class $C=P$ under more traditional logarithmic-space many-one reductions. This also provides a natural complete problem for this class. Koiran and Perifel also considered the general case of ZMC, where the formal degree of the circuit is not bounded. They showed that ZMC is in the counting hierarchy. We provide a better upper bound by proving that ZMC is in $coRP^{PP}$. We finally study the case of monotone circuits and show that the problem is then $coNP$ -complete.

The second main problem is to count the number of monomials in the polynomial computed by a circuit. This seems like a natural question whose solution should not be too hard, but in the general case it turns out to be PP^{PP} -complete, and the hardness holds even for weak circuits. We thus obtain another natural complete problem, in this case for the second level of the counting hierarchy. We remark that if a polynomial bound is given on the number of monomials, both the problem ZMC and the one of counting monomials become easy since an explicit description of the polynomial can be computed in polynomial time [11]. The related problem of enumerating the monomials of a given polynomial, in the black-box model, is addressed in [29].

Then we study the two above problems in the case of circuits computing multilinear polynomials. We show that our first problem becomes equivalent to the fundamental problem ACIT and that counting monomials becomes PP -complete.

Finally, we consider the case of univariate multiplicatively disjoint circuits. We show that these problems and several related ones are equivalent and complete for $LOGCFL$ in the monotone case, or close to $C=LOGCFL$ in the general case.

A preliminary version of the present paper appeared in [10].

2 Preliminaries

Arithmetic circuits An *arithmetic circuit* is a labeled directed acyclic graph (DAG) consisting of vertices or gates with indegree or fanin 0 or 2. The gates with fanin 0 are called input gates and are labeled with -1 or variables X_1, X_2, \dots, X_n . The gates with fanin 2 are called computation gates and are labeled with \times or $+$. We can also consider circuits where computation gates may receive more than two edges, in which case we say that they have *unbounded fanin*. The

¹Observe that Hemaspaandra and Ogihara [14, p. 293] state that Mundhenk et al. [27] provide natural complete problems for PP^{NP} . This appears to be a typo as Mundhenk et al. in fact present complete problems not for PP^{NP} but for the class NP^{PP} which indeed appears to have several interesting complete problems in the AI/planning literature.

polynomial computed by an arithmetic circuit is defined in the obvious way: an input gate computes the value of its label, a computation gate computes the product or the sum of its children's values, respectively. We assume that a circuit has only one sink which we call the output gate. We say that the polynomial computed by the circuit is the polynomial computed by the output gate. The *size* of an arithmetic circuit is the number of gates. The *depth* of a circuit is the length of the longest path from an input gate to the output gate in the circuit. A formula is an arithmetic circuit whose underlying graph is a tree. Finally, a circuit or formula is called *monotone* if, instead of the constant -1 , only the constant 1 is allowed. When an arithmetic circuit is the input of a problem, we consider that it is given as a graph with labels on the vertices, for instance as an adjacency list.

It is common to consider so-called *degree-bounded* arithmetic circuits, for which the degree of the computed polynomial is bounded polynomially in the number of gates of the circuit. In our opinion this kind of degree bound has two problems. One is that computing the degree of a polynomial represented by a circuit is suspected to be hard (see Section 6 and [3, 18, 17]), so problems defined with this degree bound must often be promise problems. The other problem is that the bound on the degree does not bound the size of computed constants, which by iterative squaring can have exponential bitsize. Thus even evaluating circuits on a Turing machine becomes intractable. The paper by Allender et al. [3] discusses problems that result from this. To avoid all these complications, instead of bounding the degree of the computed polynomial, we choose to bound the formal degree of the circuit or equivalently to consider multiplicatively disjoint circuits. A circuit is called *multiplicatively disjoint* if, for each \times -gate, its two input subcircuits are disjoint from one another. See [24] for a discussion of degree, formal degree and multiplicative disjointness and how they relate.

Complexity classes We assume that the reader is familiar with basic concepts of computational complexity theory (see e.g. [5]). All reductions in this paper will be logspace many-one unless stated otherwise.

We consider different counting decision classes in the counting hierarchy [36]. These classes are defined analogously to the quantifier definition of the polynomial hierarchy but, in addition to the quantifiers \exists and \forall , the quantifiers C , $C_=_$ and C_{\neq} are used.

Definition 2.1. Let \mathcal{C} be a complexity class containing P .

- $A \in C\mathcal{C}$ if and only if there is $B \in \mathcal{C}$, $f \in FP$ and a polynomial p such that

$$x \in A \Leftrightarrow \left| \left\{ y \in \{0,1\}^{p(|x|)} \mid (x,y) \in B \right\} \right| \geq f(x),$$

- $A \in C_=\mathcal{C}$ if and only if there is $B \in \mathcal{C}$, $f \in FP$ and a polynomial p such that

$$x \in A \Leftrightarrow \left| \left\{ y \in \{0,1\}^{p(|x|)} \mid (x,y) \in B \right\} \right| = f(x),$$

- $A \in C_{\neq}\mathcal{C}$ if and only if there is $B \in \mathcal{C}$, $f \in FP$ and a polynomial p such that

$$x \in A \Leftrightarrow \left| \left\{ y \in \{0,1\}^{p(|x|)} \mid (x,y) \in B \right\} \right| \neq f(x).$$

Observe that $C_{\neq}\mathcal{C} = \text{co}C_=\mathcal{C}$ with the usual definition $\text{co}\mathcal{C} = \{L^c \mid L \in \mathcal{C}\}$, where L^c is the complement of L . That is why the quantifier C_{\neq} is often also written as $\text{co}C_=_$, so $C_{\neq}P$ is sometimes called $\text{co}C_=_P$.

The counting hierarchy CH consists of the languages from all classes that we can get from P by applying the quantifiers \exists , \forall , C , $C_=_$ and C_{\neq} a constant number of times. Observe that with the definition above $PP = CP$. Torán [31] proved that this connection between PP and the counting hierarchy can be extended and that there is a characterization of CH by oracles similar to that of the polynomial hierarchy. We state some such characterizations which we will need later on, followed by other technical lemmas.

Lemma 2.2. [31] $\text{PP}^{\text{NP}} = \text{C}\exists\text{P}$.

Lemma 2.3. $\text{PP}^{\text{PP}} = \text{CC}_{\neq}\text{P}$.

Proof of Lemma 2.3. This is not stated in [31] nor is it a direct consequence, because Torán does not consider the C_{\neq} -operator. It can be shown with similar techniques and we give a proof for completeness. We show that $\text{CC}_{\neq}\text{P} = \text{CCP}$, the claim then follows, because $\text{CCP} = \text{PP}^{\text{PP}}$ by [31].

The direction from left to right is straightforward: From the definition we have $\text{CC}_{\neq}\text{P} \subseteq \text{PP}^{\#\text{P}}$. By binary search we have $\text{PP}^{\#\text{P}} = \text{PP}^{\text{PP}} = \text{CCP}$. The other direction needs a little more work. Let $L \in \text{CCP}$, there are $A \in \text{P}, f, g \in \text{FP}$ and a polynomial p such that

$$\begin{aligned}
x \in L &\Leftrightarrow \text{there are more than } f(x) \text{ values } y \in \{0, 1\}^{p(|x|)} \text{ such that} \\
&\quad \left| \left\{ z \in \{0, 1\}^{p(|x|)} \mid (x, y, z) \in A \right\} \right| \geq g(x, y) \\
&\Leftrightarrow \text{there are more than } f(x) \text{ values } y \in \{0, 1\}^{p(|x|)} \text{ such that} \\
&\quad \forall v \in \{1, \dots, 2^{p(|x|)}\}: \left| \left\{ z \in \{0, 1\}^{p(|x|)} \mid (x, y, z) \in A \right\} \right| \neq g(x, y) - v \quad (1) \\
&\Leftrightarrow \text{there are more than } 2^{p(|x|)}(2^{p(|x|)} - 1) + f(x) \text{ pairs } (x, v) \text{ with} \\
&\quad y \in \{0, 1\}^{p(|x|)} \text{ and } v \in \{1, \dots, 2^{p(|x|)}\} \text{ such that} \\
&\quad \left| \left\{ z \in \{0, 1\}^{p(|x|)} \mid (x, y, z) \in A \right\} \right| \neq g(x, y) - v. \quad (2)
\end{aligned}$$

From statement (2) we directly get $L \in \text{CC}_{\neq}\text{P}$ and thus the claim. To see the last equivalence we define $r(x, y) := \left| \left\{ z \in \{0, 1\}^{p(|x|)} \mid (x, y, z) \in A \right\} \right|$. Fix x, y , then obviously $r(x, y) \neq g(x, y) - v$ for all but at most one v . It follows that of the pairs (y, v) in the last statement $2^{p(|x|)}(2^{p(|x|)} - 1)$ always lead to inequality. So statement (2) boils down to the question how many y there are such that there is no v with $r(x, y) = g(x, y) - v$. We want these to be at least $f(x)$, so we want at least $2^{p(|x|)}(2^{p(|x|)} - 1) + f(x)$ pairs such that $r(x, y) \neq g(x, y) - v$. \square

Lemma 2.4. [12] $\exists \text{C}_{\neq}\text{P} = \text{C}_{\neq}\text{P}$.

Lemma 2.5. [28] For a large enough constant $c > 0$, it holds that for any integers n and x with $|x| \leq 2^{2^n}$ and $x \neq 0$, the number of primes p smaller than 2^{cn} such that $x \not\equiv 0 \pmod p$ is at least $2^{cn}/cn$.

Lemma 2.6. [14, p. 81] For every oracle X we have $\text{PP}^{\text{BPP}^X} = \text{PP}^X$.

3 Zero monomial coefficient

We first consider the question of deciding if a single specified monomial occurs in a polynomial. In this problem and others regarding monomials, a monomial is encoded by giving the variable powers in binary.

ZMC

Input: Arithmetic circuit C , monomial m .

Problem: Decide if m has the coefficient 0 in the polynomial computed by C .

Theorem 3.1. ZMC is $\text{C}_{=}\text{P}$ -complete for both multiplicatively disjoint circuits and formulas.

Proof. Using standard reduction techniques from the $\#\text{P}$ -completeness of the permanent (see for example [5]), one define the following generic $\text{C}_{=}\text{P}$ -complete problem, as mentioned in the introduction.

PER₌

Input: Matrix $A \in \{0, 1, -1\}^n$, $d \in \mathbb{N}$.

Problem: Decide if $\text{PER}(A) = d$.

Therefore, for the hardness of ZMC it is sufficient to show a reduction from $\text{PER}_{=}$. On input $A = (a_{ij})$ and d we compute the formula $Q := \prod_{i=1}^n \left(\sum_{j=1}^n a_{ij} Y_j \right)$. It is a classical observation by Valiant [32]² that the monomial $Y_1 Y_2 \dots Y_n$ has the coefficient $\text{PER}(A)$. Thus the coefficient of the monomial $Y_1 Y_2 \dots Y_n$ in $Q - d Y_1 Y_2 \dots Y_n$ is 0 if and only if $\text{PER}(A) = d$.

We now show that ZMC for multiplicatively disjoint circuits is in $\text{C}_{=}\text{P}$. The proof is based on the use of parse trees, which can be seen as objects tracking the formation of monomials during the computation [24] and are the algebraic analog of proof trees [34].

Define inductively the parse trees of a circuit C in the following manner:

1. the only parse tree of an input gate is the gate itself,
2. the parse trees of an addition gate α with argument gates β and γ are obtained by taking either a parse tree of β and adding the edge from β to α or by taking a parse tree of γ and adding the edge from γ to α ,
3. the parse trees of a multiplication gate α with argument gates β and γ are obtained by taking a parse tree of β and a parse tree of γ and adding the edge from β to α and the edge from γ to α , renaming vertices so that the chosen parse trees of β and γ are disjoint.

The value of a parse tree is defined as the product of the labels of each input gate in the parse tree (note that in the parse tree there may be several copies of a given input gate of the circuit, so that the corresponding label will have as power the number of copies of the gate). It is easy to see that the polynomial computed by a circuit is the sum of the values of its parse trees:

$$C(\bar{x}) = \sum_{T \text{ parse tree of } C} \text{value}(T).$$

In the case of a multiplicatively disjoint circuit, any parse tree is a subgraph of the circuit. In this case, a parse tree can be equivalently seen as a subgraph defined by a subset of T of the edges satisfying the following properties:

1. it contains the output gate,
2. for any addition gate α , if T contains an edge with origin α , then T contains exactly one edge with destination α ,
3. for any multiplication gate α , if T contains an edge with origin α , then T contains both (all) edges with destination α ,
4. for any gate α , if T contains an edge with destination α , then T contains an edge with origin α .

Consider a multiplicatively disjoint circuit C and a monomial m , where the input gates of C are labeled either by a variable or by -1 . A parse tree T contributes to the monomial m in the output polynomial if, when computing the value of the tree, we get exactly the powers in m ; this contribution has coefficient $+1$ if the number of gates labeled -1 in T is even and it has coefficient -1 if this number is odd. The coefficient of m is thus equal to 0 if and only if the number of trees contributing positively is equal to the number of trees contributing negatively.

Let us represent a parse tree by a boolean word $\bar{\epsilon}$, by indicating which edges of C appear in the parse tree (the length N of the words is therefore the number of edges in C). Some of these words will not represent a valid parse tree, but this can be tested in polynomial time. Consider the following language L composed of triples $(C, m, \epsilon_0 \bar{\epsilon})$ such that:

1. $\epsilon_0 = 0$ and $\bar{\epsilon}$ encodes a valid parse tree of C which contribute positively to m ,
2. or $\epsilon_0 = 1$ and $\bar{\epsilon}$ does not encode a valid parse tree contributing negatively to m .

²According to [35] this observation even goes back to [13].

Then the number of $\bar{\epsilon}$ such that $(C, m, 0\bar{\epsilon})$ belongs to L is the number of parse trees contributing positively to m and the number of $\bar{\epsilon}$ such that $(C, m, 1\bar{\epsilon})$ belongs to L is equal to 2^N minus the number of parse trees contributing negatively to m . Thus, the number of $\epsilon_0\bar{\epsilon}$ such that $(C, m, \epsilon_0\bar{\epsilon}) \in L$ is equal to 2^N if and only if the number of trees contributing positively is equal to the number of trees contributing negatively, if and only if the coefficient of m is equal to 0 in C . Because L is in P , ZMC for multiplicatively disjoint circuits is in $C=P$. \square

Theorem 3.2. *ZMC belongs to coRP^{PP} .*

Proof. Given a circuit C , a monomial m and a prime number p written in binary, COEFFSLP is the problem of computing modulo p the coefficient of the monomial m in the polynomial computed by C . It is shown in [17] that COEFFSLP belongs to $\text{FP}^{\#\text{P}}$.

We now describe a randomized algorithm to decide ZMC. Let c be the constant given in Lemma 2.5. Consider the following algorithm to decide ZMC given a circuit C of size n and a monomial m , using COEFFSLP as an oracle. First choose uniformly at random an integer p smaller than 2^{cn} . If p is not prime, accept. Otherwise, compute the coefficient a of the monomial m in C with the help of the oracle and accept if $a \equiv 0 \pmod{p}$. Since $|a| \leq 2^{2^n}$, Lemma 2.5 ensures that the above is a correct one-sided error probabilistic algorithm for ZMC. This yields $\text{ZMC} \in \text{coRP}^{\text{COEFFSLP}}$. Hence $\text{ZMC} \in \text{coRP}^{\text{PP}}$. \square

We now consider ZMC on monotone circuits. Notice that in the univariate case, this problem is equivalent to decide if the output of a circuit over sets of natural numbers with operations \cup and $+$ omits a given number. This was shown by McKenzie and Wagner to be coNP -complete for both formulas and circuits. This result can be easily extended to the multivariate case and gives the following.

Theorem 3.3. [25] *ZMC is coNP -complete both for monotone formulas and monotone circuits.*

We now give a result linking the ZMC problem to other questions on polynomials computed by circuits. We define the following problem.

GAPMONSLP

Input: Univariate arithmetic circuit C over X , $a, b \in \mathbb{N}$.

Problem: Decide if the polynomial computed by C contains no monomial of the form X^c for $a \leq c \leq b$.

GAPMONSLP can be seen as a generalization of the degree problem, called DEGSPLP in [3] (see also Section 6). This generalization can actually be shown to be hard as it has the same complexity as ZMC.

Proposition 3.4. *GAPMONSLP is equivalent to ZMC.*

Proof. The general case of ZMC easily reduces to ZMC for univariate circuits: we briefly explain the argument below and refer to [3] for further details. Given a circuit C over the variables X_1, \dots, X_n and a monomial $m = X_1^{d_1} \dots X_n^{d_n}$, we can compute a circuit C' over Y and a monomial $m' = Y^d$ such that the coefficient of m' in C' is zero if and only if the coefficient of m in C is zero. Indeed, define C' by substituting each variable X_i with Y^{M^i} in C for $M := 2^{|C|} + 1$ and let $d = \sum_{i=1}^n d_i M^i$. The coefficient of $m' = Y^d$ in C' is zero if and only if the coefficient of m in C is zero. Since the univariate case of ZMC is a special case of GAPMONSLP, this shows that ZMC reduces to GAPMONSLP.

For the other direction, consider (C, a, b) an instance of GAPMONSLP over the variable X . Let $C' = C \cdot (1 + XY)^{b-a}$. The circuit C' has polynomial size since $(1 + XY)^b$ can be computed with a circuit of size $O(\log b)$. The coefficient $P(Y)$ of X^b in C' is the zero polynomial if and only if (C, a, b) is a positive instance of GAPMONSLP. Now replace Y in C' with $B := 2^{2^{|C'|^2}}$ (obtained by repeated squaring from 2) to obtain a circuit C'' . Note that the polynomial P has at most $2^{|C'|}$ monomials, with coefficients bounded by $2^{2^{2^{|C'|}}}$. From the proof of [3, Prop. 2.2], it follows that $P(B) = 0$ if and only if P is the zero polynomial. That is, the coefficient of X^b in C'' is zero if and only if (C, a, b) is a positive instance of GAPMONSLP. \square

4 Counting monomials

We now turn to the problem of counting the monomials of a polynomial represented by a circuit.

COUNTMON

Input: Arithmetic circuit C , $d \in \mathbb{N}$.

Problem: Decide if the polynomial computed by C has at least d monomials.

To study the complexity of COUNTMON we will look at what we call extending polynomials. Given two monomials M and m , we say that M is m -extending if $M = mm'$ and m and m' have no common variable. We start by studying the problem of deciding the existence of an extending monomial.

EXISTEXTMON

Input: Arithmetic circuit C , monomial m .

Problem: Decide if the polynomial computed by C contains an m -extending monomial.

Proposition 4.1. EXISTEXTMON is in RP^{PP} . For multiplicatively disjoint circuits it is C_{\neq}P -complete.

Proof. We first show the first upper bound. So let (C, m) be an input for EXISTEXTMON where C is a circuit in the variables X_1, \dots, X_n . Without loss of generality, suppose that X_1, \dots, X_r are the variables appearing in m . Let $d = 2^{\lceil |C| \rceil}$: d is a bound on the degree of the polynomial computed by C . We define $C' = \prod_{i=r+1}^n (1 + Y_i X_i)^d$ for new variables Y_i . We have that C has an m -extending monomial if and only if in the product CC' the polynomial $P(Y_{r+1}, \dots, Y_n)$, which is the coefficient of $m \prod_{i=r+1}^n X_i^d$, is not identically 0. Observe that P is not given explicitly but can be evaluated modulo a random prime with an oracle for COEFFSLP. Thus it can be checked if P is identically 0 with the classical Schwartz-Zippel-DeMillo-Lipton lemma (see for example [5]). It follows that $\text{EXISTEXTMON} \in \text{RP}^{\text{PP}}$.

The upper bound in the multiplicatively disjoint setting is easier: we can guess an m -extending monomial M and then output the answer of an oracle for the complement of ZMC, to check whether M appears in the computed polynomial. This establishes containment in $\exists\text{C}_{\neq}\text{P}$ which by Lemma 2.4 is C_{\neq}P .

For hardness we reduce to EXISTEXTMON the C_{\neq}P -complete problem PER_{\neq} , i.e., the complement of the $\text{PER}_{=}$ problem introduced for the proof of Theorem 3.1. We use essentially the same reduction constructing a circuit $Q := \prod_{i=1}^n \left(\sum_{j=1}^n a_{ij} Y_j \right)$. Observe that the only potential extension of $m := Y_1 Y_2 \dots Y_n$ is m itself and has the coefficient $\text{PER}(A)$. Thus $Q - dY_1 Y_2 \dots Y_n$ has an m -extension if and only if $\text{PER}(A) \neq d$. \square

COUNTEXTMON

Input: Arithmetic circuit C , $d \in \mathbb{N}$, monomial m .

Problem: Decide if the polynomial computed by C has at least d m -extending monomials.

Proposition 4.2. COUNTEXTMON is PP^{PP} -complete.

Proof. Clearly COUNTEXTMON belongs to PP^{ZMC} and thus with Theorem 3.2 it is in $\text{PP}^{\text{coRP}^{\text{PP}}}$. Using Lemma 2.6 we get membership in PP^{PP} . To show hardness, we reduce the canonical CC_{\neq}P -complete problem $\text{CC}_{\neq}\text{3SAT}$ to COUNTEXTMON. With Lemma 2.3 the hardness for PP^{PP} follows.

$\text{CC}_{\neq}\text{3SAT}$

Input: 3SAT-formula $F(\bar{x}, \bar{y})$, $k, \ell \in \mathbb{N}$.

Problem: Decide if there are at least k assignments to \bar{x} such that there are not exactly ℓ assignments to \bar{y} such that F is satisfied.

Let $(F(\bar{x}, \bar{y}), k, \ell)$ be an instance for $\text{CC}_{\neq}3\text{SAT}$. Without loss of generality we may assume that $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$ and that no clause contains a variable in both negated and non-negated form. Let $\Gamma_1, \dots, \Gamma_c$ be the clauses of F .

For each literal u of the variables in \bar{x} and \bar{y} we define a monomial $I(u)$ in the variables $X_1, \dots, X_n, Z_1, \dots, Z_c$ in the following way:

$$\begin{aligned} I(x_i) &= X_i \prod_{\{j \mid x_i \in \Gamma_j\}} Z_j, & I(\neg x_i) &= \prod_{\{j \mid \neg x_i \in \Gamma_j\}} Z_j, \\ I(y_i) &= \prod_{\{j \mid y_i \in \Gamma_j\}} Z_j, & I(\neg y_i) &= \prod_{\{j \mid \neg y_i \in \Gamma_j\}} Z_j. \end{aligned}$$

From these monomials we compute a formula C by

$$C := \prod_{i=1}^n (I(x_i) + I(\neg x_i)) \prod_{i=1}^n (I(y_i) + I(\neg y_i)). \quad (3)$$

We fix a mapping mon from the assignments of F to the monomials computed by C : Let $\bar{\alpha}$ be an assignment to \bar{x} and $\bar{\beta}$ be an assignment to \bar{y} . We define $mon(\bar{\alpha}\bar{\beta})$ as the monomial obtained in the expansion of C by choosing the following terms. If $\alpha_i = 0$, choose $I(\neg x_i)$, otherwise choose $I(x_i)$. Similarly, if $\beta_i = 0$, choose $I(\neg y_i)$, otherwise choose $I(y_i)$.

The monomial $mon(\bar{\alpha}\bar{\beta})$ has the form $\prod_{i=1}^n X_i^{\alpha_i} \prod_{j=1}^c Z_j^{\gamma_j}$, where γ_j is the number of true literals in Γ_j under the assignment $\bar{\alpha}\bar{\beta}$. Then F is true under $\bar{\alpha}\bar{\beta}$ if and only if $mon(\bar{\alpha}\bar{\beta})$ has the factor $\prod_{j=1}^c Z_j$. Thus F is true under $\bar{\alpha}\bar{\beta}$ if and only if $mon(\bar{\alpha}\bar{\beta}) \prod_{j=1}^c (1 + Z_j + Z_j^2)$ has the factor $\prod_{i=1}^n X_i^{\alpha_i} \prod_{j=1}^c Z_j^3$. We set $C' = C \prod_{j=1}^c (1 + Z_j + Z_j^2)$.

Consider an assignment $\bar{\alpha}$ to \bar{x} . The coefficient of the monomial $\prod_{i=1}^n X_i^{\alpha_i} \prod_{j=1}^c Z_j^3$ in C' is the number of assignments $\bar{\beta}$ such that $\bar{\alpha}\bar{\beta}$ satisfies F . Thus we get

$$\begin{aligned} & (F(\bar{x}, \bar{y}), k, \ell) \in \text{CC}_{\neq}3\text{SAT} \\ \Leftrightarrow & \text{there are at least } k \text{ assignments } \bar{\alpha} \text{ to } \bar{x} \text{ such that the monomial } \prod_{i=1}^n X_i^{\alpha_i} \prod_{j=1}^c Z_j^3 \\ & \text{does not have coefficient } \ell \text{ in } C' \\ \Leftrightarrow & \text{there are at least } k \text{ assignments } \bar{\alpha} \text{ to } \bar{x} \text{ such that the monomial } \prod_{i=1}^n X_i^{\alpha_i} \prod_{j=1}^c Z_j^3 \\ & \text{occurs in } C'' := C' - \ell \prod_{i=1}^n (1 + X_i) \prod_{j=1}^c Z_j^3 \\ \Leftrightarrow & \text{there are at least } k \text{ tuples } \bar{\alpha} \text{ such that } C'' \text{ contains the monomial } \prod_{i=1}^n X_i^{\alpha_i} \prod_{j=1}^c Z_j^3 \\ \Leftrightarrow & C'' \text{ has at least } k \left(\prod_{j=1}^c Z_j^3 \right)\text{-extending monomials.} \end{aligned}$$

□

Theorem 4.3. *COUNTMON is PP^{PP} -complete. It is PP^{PP} -hard even for unbounded fan-in formulas of depth 4.*

Proof. COUNTMON can be easily reduced to COUNTEXTMON since the number of monomials of a polynomial is the number of 1-extending monomials. Therefore COUNTMON belongs to PP^{PP} .

To show hardness, it is enough to prove that instances of COUNTEXTMON constructed in Proposition 4.2 can be reduced to COUNTMON in logarithmic space. The idea of the proof is

that we make sure that the polynomial for which we count all monomials contains all monomials that are not m -extending. Thus we know how many non- m -extending monomials it contains and we can compute the number of m -extending monomials from the number of all monomials. We could use the same strategy to show in general that `COUNTTEXTMON` reduces to `COUNTMON` but by considering the instance obtained in the proof of Proposition 4.2 and analyzing the extra calculations below we get hardness for unbounded fanin formulas of depth 4.

So let (C'', k, m) be the instance of `COUNTTEXTMON` constructed in the proof of Proposition 4.2, with $m = \prod_{j=1}^c Z_j^3$. We therefore need to count the monomials computed by C'' which are of the form $f(X_1, \dots, X_n) \prod_{j=1}^c Z_j^3$. The circuit C'' is multilinear in X , and the Z_j can only appear with powers in $\{0, 1, 2, 3, 4, 5\}$. So the non- m -extending monomials computed by C'' are all products of a multilinear monomial in the X_i and a monomial in the Z_j where at least one Z_j has a power in $\{0, 1, 2, 4, 5\}$. Fix j , then all monomials that are not m -extending because of Z_j are computed by the formula

$$\tilde{C}_j := \left(\prod_{i=1}^n (X_i + 1) \right) \left(\prod_{j' \neq j} \sum_{p=0}^5 Z_{j'}^p \right) (1 + Z_j + Z_j^2 + Z_j^4 + Z_j^5). \quad (4)$$

Thus the formula $\tilde{C} := \sum_j \tilde{C}_j$ computes all non- m -extending monomials that C'' can compute. The coefficients of monomials in C'' cannot be smaller than $-\ell$ where ℓ is part of the instance of `CC \neq 3SAT` from which we constructed (C'', k, m) before. So the formula $C^* := C'' + (\ell + 1)\tilde{C}$ contains all non- m -extending monomials that C'' can compute and it contains the same extending monomials. There are $2^n 6^c$ monomials of the form that C'' can compute, only 2^n of which are m -extending, which means that there are $2^n(6^c - 1)$ monomials computed by C^* that are not m -extending. As a consequence, C'' has at least k m -extending monomials if and only if C^* has at least $2^n(6^c - 1) + k$ monomials. \square

Theorem 4.4. `COUNTMON` is PP^{NP} -complete both for monotone formulas and monotone circuits.

Proof. We first show hardness for monotone formulas. The argument is very similar to the proof of Theorem 4.3. Consider the following canonical `C \exists P`-complete problem `C \exists 3SAT`.

C \exists 3SAT
Input: 3SAT-formula $F(\bar{x}, \bar{y})$, $k \in \mathbb{N}$.
Problem: Decide if there are at least k assignments $\bar{\alpha}$ to \bar{x} such that $F(\bar{\alpha}, \bar{y})$ is satisfiable.

We reduce `C \exists 3SAT` to `COUNTMON`. With Lemma 2.2 the hardness for PP^{NP} follows. Consider a 3SAT-formula $F(\bar{x}, \bar{y})$. Let $n = |\bar{x}| = |\bar{y}|$ and let c be the number of clauses of F . Define the polynomial $C^* = C + \sum_{j=1}^c \tilde{C}_j$ where C is defined by Equation 3 and \tilde{C}_j by Equation 4. The analysis is similar to the proof of Theorem 4.3. The polynomial C^* is computed by a monotone arithmetic formula and has at least $2^n(6^c - 1) + k$ monomials if and only if (F, k) is a positive instance of `C \exists 3SAT`.

We now prove the upper bound. Recall that `COUNTMON` $\in \text{PP}^{\text{ZMC}}$. From Theorem 3.3, it follows that `COUNTMON` on monotone circuits belongs to PP^{NP} . \square

5 Multilinearity

In this section we consider the effect of multilinearity on our problems. We will not consider promise problems and therefore the multilinear variants of our problems must first check if the computed polynomial is multilinear. We start by showing that this step is not difficult, indeed, it is equivalent to the problem `ACIT`.

ACIT
Input: Arithmetic circuit C .
Problem: Decide if the polynomial computed by C is the zero polynomial.

CHECKML

Input: Arithmetic circuit C .

Problem: Decide if the polynomial computed by C is multilinear.

Proposition 5.1. CHECKML is equivalent to ACIT.

Proof. Reducing ACIT to CHECKML is easy: Simply multiply the input with X^2 for an arbitrary variable X . The resulting circuit is multilinear if and only if the original circuit was 0.

For the other direction the idea is to compute the second derivatives of the polynomial computed by the input circuit and check if they are 0.

So let C be a circuit in the variables X_1, \dots, X_n that is to be checked for multilinearity. For each i we inductively compute a circuit C_i that computes the second derivative with respect to X_i . To do so for each gate v in C the circuit C' has three gates v_i, v'_i and v''_i . The polynomial in v_i is that of v , v'_i computes the first derivative and v''_i the second. For the input gates the construction is obvious. If v is a $+$ -gate with children u and w we have $v_i = u_i + w_i, v'_i = u'_i + w'_i$ and $v''_i = u''_i + w''_i$. If v is a \times -gate with children u and w we have $v_i = u_i w_i, v'_i = u'_i w_i + u_i w'_i$ and $v''_i = u''_i w_i + 2u'_i w'_i + u_i w''_i$. It is easy to see that the constructed circuit computes indeed the second derivative with respect to X_i .

Next we compute $C' := \sum_{i=1}^n Y_i C_i$ for new variables Y_i . We have that C' is identically zero if and only if C is multilinear. Also C' can easily be constructed in logarithmic space. \square

Next we show that the problem gets much harder if, instead of asking whether *all* the monomials in the polynomial computed by a circuit are multilinear, we ask whether at least *one* of the monomials is multilinear.

MONML

Input: Arithmetic circuit C .

Problem: Decide if the polynomial computed by C contains a multilinear monomial.

The problem MONML lies at the heart of fast exact algorithms for deciding k -paths by Koutis and Williams [20, 37] (although in these papers the polynomials are in characteristic 2 which changes the problem a little). This motivated Chen and Fu [8, 9] to consider MONML, show that it is $\#P$ -hard and give algorithms for the bounded depth version. We provide further information on the complexity of this problem.

Proposition 5.2. MONML is in RP^{PP} . It is $C_{\neq}P$ -complete for multiplicatively disjoint circuits.

Proof. For the first upper bound, let C be the input in variables X_1, \dots, X_n . We set $C' = \prod_{i=1}^n (1 + X_i Y_i)$. Then C computes a multilinear monomial if and only if in the product CC' the coefficient polynomial $P(Y_1, \dots, Y_n)$ of $\prod_{i=1}^n X_i$ is not identically 0. This can be tested as in the proof of Proposition 4.1, thus establishing $MONML \in RP^{PP}$.

The $C_{\neq}P$ -completeness in the multiplicatively disjoint case can be proved in the same way as in Proposition 4.1. \square

We now turn to our first problem, namely deciding whether a monomial appears in the polynomial computed by a circuit, in the multilinear setting.

ML-ZMC

Input: Arithmetic circuit C , monomial m .

Problem: Decide if C computes a multilinear polynomial in which the monomial m has coefficient 0.

Proposition 5.3. ML-ZMC is equivalent to ACIT.

Proof. We first show that ACIT reduces to ML-ZMC. So let C be an input for ACIT. Allender et al. [3] have shown that ACIT reduces to a restricted version of ACIT in which all inputs are -1 and thus the circuit computes a constant. Let C_1 be the result of this reduction. Then C computes identically 0 if and only if the constant coefficient of C_1 is 0. This establishes the first direction.

For the other direction let (C, m) be the input, where C is an arithmetic circuit and m is a monomial. First check if m is multilinear, if not output 1 or any other nonzero polynomial. Next we construct a circuit C_1 that computes the homogeneous component of degree $\deg(m)$ of C with the classical method (see for example [6, Lemma 2.14]). Observe that if C computes a multilinear polynomial, so does C_1 . We now plug in 1 for the variables that appear in m and 0 for all other variables, call the resulting (constant) circuit C_2 . If C_1 computes a multilinear polynomial, then C_2 is zero if and only if m has coefficient 0 in C_1 . The end result of the reduction is $C^* := C_2 + ZC_3$ where Z is a new variable and C_3 is a circuit which is identically 0 iff C computes a multilinear polynomial (obtained via Proposition 5.1). C computes a multilinear polynomial and does not contain the monomial m if and only if both C_2 and ZC_3 are identically 0, which happens if and only if their sum is identically 0. \square

In the case of our second problem, counting the number of monomials, the complexity falls to PP.

ML-COUNTMON
Input: Arithmetic circuit C , $d \in \mathbb{N}$.
Problem: Decide if the polynomial computed by C is multilinear and has at least d monomials.

Proposition 5.4. ML-COUNTMON is PP-complete (for Turing reductions).

Proof. We first show $\text{ML-COUNTMON} \in \text{PP}$. To do so we use CHECKML to check that the polynomial computed by C is multilinear. Then counting monomials can be done in $\text{PP}^{\text{ML-ZMC}}$, and ML-ZMC is in coRP . By Lemma 2.6 the class PP^{coRP} is simply PP.

For hardness we reduce the computation of the $\{0, 1\}$ -permanent to ML-COUNTMON. The proposition follows, because the $\{0, 1\}$ -permanent is $\#\text{P}$ -complete for Turing reductions. So let A be a 0-1-matrix and $d \in \mathbb{N}$ and we have to decide if $\text{PER}(A) \geq d$. We get a matrix B from A by setting $b_{ij} := a_{ij}X_{ij}$. Because every entry of B is either 0 or a distinct variable, we have that, when we compute the permanent of B , every permutation that yields a non-zero summand yields a unique monomial. This means that there are no cancellations, so that $\text{PER}(A)$ is the number of monomials in $\text{PER}(B)$.

The problem is now that no small circuits for the permanent are known and thus $\text{PER}(B)$ is not a good input for ML-COUNTMON. But because there are no cancellations, we have that $\text{DET}(B)$ and $\text{PER}(B)$ have the same number of monomials. So take a small circuit for the determinant (for instance the one given in [23]) and substitute its inputs by the entries of B . The result is a circuit C which computes a polynomial whose number of monomials is $\text{PER}(A)$. Observing that the determinant, and thus the polynomial computed by C , is multilinear completes the proof. \square

6 Univariate circuits

In this section we briefly study the case of univariate circuits. One problem related to ZMC is to compute the degree of a polynomial given by an arithmetic circuit. This problem was first introduced in [3] under the name DEGSLP.

DEGSLP
Input: Arithmetic circuit C , $d \in \mathbb{N}$.
Problem: Decide if the degree of the polynomial computed by C is smaller than d .

Allender et al. [3] also introduced the problem EQUQLP, which is the problem ACIT restricted to circuits with no indeterminates (i.e., computing integers).

EQUQLP

Input: Arithmetic circuit C computing an integer.

Problem: Decide if the integer computed is 0.

In the general case, Allender et al. remark that EQUQLP and ACIT are equivalent and they are known to be in coRP . For DEQLP, the best known upper bound is coRP^{PP} [17] and it is an open problem to obtain a lower bound better than P which was obtained by Koiran and Perifel [18]. For ZMC we have given a coRP^{PP} upper bound and a $\text{C}=\text{P}$ lower bound. Finally, we have shown that COUNTMON is PP^{PP} -complete.

In contrast to these differing complexities, we first show that in the case of univariate multiplicatively disjoint circuits, all these problems have equivalent complexities (the case of COUNTMON is slightly different and treated after the others).

Proposition 6.1. *For univariate multiplicatively disjoint circuits, the problems DEQLP, ZMC, EQUQLP and ACIT are equivalent under logspace reductions. This holds in the monotone and in the general case.*

Proof. The proof we give works both in the general case and in the monotone case. Clearly EQUQLP is a special case of ACIT and it can also be decided by asking for the constant coefficient in the problem ZMC, or by asking whether the degree is smaller than 0 in DEQLP.

Conversely, we first remark that given a univariate multiplicatively disjoint circuit C of size s , we can construct a multiplicatively disjoint circuit C' of size $O(s^3)$, with $s + 1$ output gates computing the coefficients of the polynomial computed by C (the degree of C cannot be greater than s). This is done by the classical argument for computing the homogeneous components of a circuit, noting that if we start from a multiplicatively disjoint circuit we get a multiplicatively disjoint circuit. Indeed, for each gate α in C , we have in C' the gates $\alpha_0, \dots, \alpha_s$ computing the relevant coefficients of the polynomial computed by α . Then if α is an addition gate with arguments β and γ the gate α_i in C' is also an addition gate with arguments β_i and γ_i . If α is a multiplication gate with arguments β and γ the gate α_i in C' computes $\sum_{k=0}^i \beta_k \gamma_{i-k}$. Each product in this sum multiplies a “ β ” gate with a “ γ ” gate, so that multiplicatively disjointness is maintained in the construction.

It is now easy to show the converse reductions. In particular it directly gives the reduction from ZMC to EQUQLP.

To reduce ACIT to EQUQLP we apply the above construction, then square all the coefficients of the polynomial and add them up, so that the resulting integer circuit computes 0 iff the starting circuit computed the 0 polynomial.

To reduce DEQLP to EQUQLP, we just need to check whether all coefficients of degree at least d are 0, which can be done in a way similar to the reduction from ACIT to EQUQLP. \square

We now show that, for univariate multiplicatively disjoint circuits, all the problems considered above are complete for LOGCFL in the monotone case and $\text{C}=\text{LOGCFL}$ in the general case. We first recall basic facts about these classes.

LOGCFL is the class of all languages that can be reduced in logarithmic space to a context-free language. We will use the characterization of LOGCFL by logspace-uniform semi-unbounded AC^1 circuits [33], or by logspace-uniform circuits of bounded formal degree and therefore, as mentioned at the end of the paragraph on circuits, also by logspace-uniform circuits where the AND gates are disjoint [24]³.

In the non-monotone case we also need to consider the class $\text{C}=\text{LOGCFL} = \text{C}=\text{SAC}^1$. Instead of using a machine-based definition as we did in Section 2 for $\text{C}=\mathcal{C}$ when the class \mathcal{C} contains P , it is simpler to define this class by arithmetizing a circuit definition of LOGCFL. Similarly to the

³This follows from the characterization by semi-unbounded fanin circuits of logarithmic depth, and then duplicating gates to ensure that multiplications are disjoint.

class GapP defined by the difference of two $\#\text{P}$ functions, we can define the class $\text{GapLOGCFL} = \text{GapSAC}^1$ [1], either by the difference of two semi-unbounded log-depth arithmetic circuits over \mathbb{N} or just by considering such a circuit over \mathbb{Z} . Equivalently we can also consider the difference of two multiplicatively disjoint circuits over \mathbb{N} or a multiplicatively disjoint circuit over \mathbb{Z} . A language L then belongs to $\text{C}_=\text{LOGCFL}$ if there exists a function f in GapLOGCFL such that $x \in L \Leftrightarrow f(x) = 0$.

Proposition 6.2. *For monotone univariate multiplicatively disjoint circuits, the problems DEGSLP , ZMC , $\text{EQU\text{SLP}}$, ACIT and COUNTMON are LOGCFL -complete.*

Proof. For all of these problems apart from COUNTMON , it is enough to show that in the monotone case the problem $\text{EQU\text{SLP}}$ is LOGCFL -complete. Note that a monotone arithmetic circuit counts the number of satisfying parse trees of the corresponding Boolean circuit [1] (where a $+$ gate is replaced by and OR and a \times gate by an AND): this number is non-zero iff the Boolean circuit evaluates to true. The completeness follows since LOGCFL can also be characterized with circuits with disjoint AND gates [24].

For the upper bound for COUNTMON , given a circuit C and an integer d , we start from the construction given in Proposition 6.1 which yields a family of arithmetic circuits computing the coefficients of the polynomial computed by C . By the remark above, in the monotone case each of these circuits is non-zero iff the associated Boolean circuit is non-zero. We can then add a small Boolean circuit which adds up these Boolean values and compares it to d , staying in LOGCFL .

Finally, the complement of $\text{EQU\text{SLP}}$ trivially reduces to COUNTMON by asking whether the number of monomials is at least 1, so that COUNTMON is LOGCFL -hard. \square

Proposition 6.3. *For univariate multiplicatively disjoint circuits, DEGSLP , ZMC , $\text{EQU\text{SLP}}$, ACIT are $\text{C}_=\text{LOGCFL}$ -complete.*

Proof. Once again we just need to consider $\text{EQU\text{SLP}}$, and the result is then direct from the definition given above for $\text{C}_=\text{LOGCFL}$. \square

Proposition 6.4. *For univariate multiplicatively disjoint circuits, COUNTMON is $\text{C}_=\text{LOGCFL}$ -hard and is in $\text{L}^{\text{C}_=\text{LOGCFL}}$.*

Proof. The hardness follows from the argument given at the end of the proof of Proposition 6.2. The upper bound is clear using Proposition 6.3, since we only need to add up a small number of answers to ZMC and then compare to d . \square

7 Conclusion

In this paper we have strengthened the known connection between the counting hierarchy and arithmetic circuits by showing that natural questions on arithmetic circuits are complete for different classes in CH . We consider it as likely that other questions on arithmetic circuits could be shown to be connected to CH with similar techniques.

Since the preliminary version of this paper [10] was published, several problems from this paper have been considered by Mahajan, Rao and Sreenivasiah [22] for very restricted circuit classes, so-called read-once/twice formulas and branching programs. For these classes the complexity of our problems often but not always drops considerably.

Let us also remark that the techniques from this paper have found an application in a recent paper by Mittmann, Saxena and Scheiblechner [26]: The notion of degeneracy considered there, to which algebraic independence in positive characteristic can be reduced, is shown to be hard by reduction from ZMC . It would be interesting to see if a similar hardness result can be shown for algebraic independence itself.

Let us close the paper with some open questions: The $\text{C}_=\text{P}$ lower bound for ZMC does not match the upper bound of coRP^{PP} completely. Can this upper bound be derandomized to show that ZMC is in $\text{C}_=\text{P}$ also in the general case?

DEGSLP is in our opinion one of the most puzzling open questions in arithmetic circuit complexity. While it is widely believed to be hard, not even conditional hardness results are known

for it. Our contribution to the understanding of DEGSLP has been very modest, but we feel that the direction it proposes might be promising. Maybe a better understanding of tractable classes of polynomials computed by restricted classes of circuits will lead to a better understanding of the general problem. So are there any other classes of circuits for which DEGSLP is tractable? Are there any multivariate classes?

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