On the shape of decomposable trees

Dominique Barth* Hervé Fournier* Romain Ravaux*

December 8, 2008

Abstract

A n-vertex graph is said to be decomposable if for any partition $(\lambda_1, \ldots, \lambda_p)$ of the integer n, there exists a sequence $(V_1, \ldots, V_p)$ of connected vertex-disjoint subgraphs with $|V_i| = \lambda_i$. The aim of the paper is to study the homeomorphism classes of decomposable trees. More precisely, we show that homeomorphism classes containing decomposable trees with an arbitrarily large minimal distance between all pairs of distinct vertices of degree different from 2 is exactly the set of combs.

Keywords: Tree partitions; Arbitrarily partitionable trees; Homeomorphism class; Integer partitions

1 Introduction

A partition of the integer n is a non-increasing sequence of positive integers that sums up to n. We are interested here in the following notion. Given a graph $G = (V, E)$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of the integer $|V|$, the graph G is said to be λ-decomposable if there exists a partition $(V_1, \ldots, V_p)$ of V such that the set V_i induces a connected subgraph of G, and $|V_i| = \lambda_i$ (such a partition is called a λ-decomposition of G). A graph is said to be decomposable if it is λ-decomposable for all partition λ of the integer $|V|$. These graphs are sometimes called arbitrarily vertex decomposable graphs. The term k-partitionable is also used in the literature for graphs which are λ-decomposable for all partitions λ with at most k parts.

Questions related to this notion of graph decomposition are old. In 1976 and 1977 respectively, Györi [7] and Lovász [10] independently proved that k-connected graphs are k-partitionable. Algorithmic versions of k-partitioning have been studied. Is was shown by Suzuki et al. [13] that, given a 2-connected graph G and a partition λ with 2 parts, a λ-decomposition of G can be computed in linear time. A polynomial time algorithm for the similar problem with $k = 3$ was found by Miyano et al. [11]. The case $k = 4$ is treated in Nakano et al. [12] for the special case of planar graphs.

Some special classes of graphs have also been studied such as the one of plane triangulations: it was proved by Diwan and Kurhekar [4] that these graphs are 6-partitionable. At last, let us mention that some on-line version of decomposability has been studied by Horňák et al. [8].

*Laboratoire PRiSM, CNRS UMR 8144 and Université de Versailles Saint-Quentin-en-Yvelines, 45 avenue des États-Unis, 78035 Versailles Cedex, France. Email: {dominique.barth, herve.fournier, romain.ravaux}@prism.uvsq.fr.

In this paper we focus on the case of decomposable trees. A motivation is that trees are the smallest graph candidates to be decomposable, and the property of being decomposable is monotone (adding edges to a decomposable graph yields a decomposable graph). Some works have been carried out both on the structure and the algorithmic aspects of decomposable trees. From the algorithmic point of view, it was shown that it can be decided in polynomial time whether a tree homeomorphic to $K_{1,3}$ (i.e. a tree with one degree 3 vertex and three paths attached to it) is decomposable or not [2]. In the general case, the complexity of deciding if a tree is decomposable is largely unknown. (However, given a tree T and an integer partition λ, deciding if T is λ-decomposable is known to be NP-complete [3].)

Concerning the structure of decomposable trees, it was first shown by Horňák and Woźniak [9] that decomposable trees have degree at most 6; it was later proved in [3] that a decomposable tree is of degree at most 4, and each degree 4 vertex in such a tree is adjacent to a leaf (and this bound is tight). In the same paper, a family of decomposable trees with an arbitrary number of degree 3 vertices is presented. However, these trees are combs made of a long paths with vertices attached to it (all at distance 1 from the path). Thus, it can be said that these trees look like a path. The question we address in this paper is to characterize the possible shapes (i.e. homeomorphism classes) of decomposable trees that can be realized by trees in which the minimal distance between all pairs of distinct vertices of degree different from 2 (later called the *length*) is arbitrarily large. We show that these shapes are exactly combs.

The paper is organized as follows. We give some definitions in Section 2. Then we show that a decomposable tree with length at least 2 must be a comb in Section 3. A construction of families of decomposable combs of arbitrary shape and arbitrarily large length is given in Section 4. At last, the main result is given in Section 5, where some open problems are presented.

2 Definitions

We recall that a partition of the integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_p)$ such that $\lambda_1 + \dots + \lambda_p = n$. A tree $T = (V, E)$ is said to be λ-*decomposable* if there exists a partition $(V_1, \dots, V_p)$ of V such that $|V_i| = \lambda_i$ for all i, each V_i inducing a subtree on T. Such a partition of V will be called a λ-*partition* of T. An n-vertex tree T is said to be *decomposable* if it is λ-decomposable for all partitions λ of the integer n.

Given a tree T, we define the *shape* of T to be the smallest tree homeomorphic to T; that is, this is the tree obtained from T by contracting in turn all edges adjacent to a degree 2 vertex. Hence the set of shapes is exactly the set of trees without degree 2 vertices. Given a tree $T = (V, E)$, we also define its *length*

$$\ell(T) = \min\{\mathrm{d}(u, v) \mid u, v \in V,\ u \neq v,\ \deg(u) \neq 2,\ \deg(v) \neq 2\}$$

where $\mathrm{d}(u, v)$ denotes the distance between u and v, and $\deg(u)$ the degree of u. That is, $\ell(T)$ is the length of the shortest path $(v_0, \dots, v_k)$ with $\deg(v_0) \neq 2$, $\deg(v_k) \neq 2$, and whose internal vertices $v_1, \dots, v_{k-1}$ all have degree 2. To put it differently, every tree T of shape $\mathbf{s} = (V_{\mathbf{s}}, E_{\mathbf{s}})$ is obtained in a unique way by replacing each edge $e \in E_{\mathbf{s}}$ with a path of length $w(e)$: then $\ell(T) = \min\{w(e) \mid e \in E_{\mathbf{s}}\}$.

We now define $\mathcal{D}_m$ to be the set of shapes $\mathbf{s}$ such that there exists a decomposable tree T of shape $\mathbf{s}$ with $\ell(T) \geqslant m$. At last we define $\mathcal{D} = \bigcap_{m \geqslant 1} \mathcal{D}_m$. Thus $\mathcal{D}$ is the set of all shapes realized by decomposable trees with an arbitrarily large length.

We recall that a *comb* is a tree of maximum degree 3 with the property that all vertices of degree 3 belong to a single path of the tree. We shall denote as $\mathcal{C} = \{\mathbf{c}_k \mid k \in \mathbb{N}\}$ the set of shapes of combs, where $\mathbf{c}_k$ is the shape of combs with k vertices of degree 3 – it thus includes the shape $\mathbf{c}_0$ of paths.

3 Forbidden structure in decomposable trees

The aim of this work is to characterize $\mathcal{D}$. The following bound on the degree of decomposable trees gives a strong restriction on $\mathcal{D}$.

Lemma 1 *[3] Decomposable trees have degree at most 4. Moreover, every degree 4 vertex of a decomposable tree is adjacent to a leaf.*

Let us denote by $S(a,b,c)$ the $(1+a+b+c)$-vertex tree made of a single vertex of degree 3 with three paths of lengths a, b and c attached to it. We need to recall two other facts on decomposable trees.

Lemma 2 *[1, 3] Let T be a decomposable tree and v a vertex of T of degree 3. Let a, b, c the cardinals of the connected components of $T - v$. Then $S(a,b,c)$ is a decomposable tree.*

Lemma 3 *[3] If $S(a,b,c)$ is a decomposable tree, then*

$$\min\{a,b,c\} \leqslant \left\lceil \sqrt{n+2} \right\rceil$$

where $n = 1 + a + b + c$.

Notice that an immediate corollary of Lemma 1 is that $\mathcal{D}_2$ (and thus $\mathcal{D}$) is made of trees of degree at most 3. We now strengthen this by showing that $\mathcal{D}_2$ only contains combs.

Lemma 4 *A decomposable tree of length at least 2 is a comb, i.e. $\mathcal{D}_2 \subseteq \mathcal{C}$.*

Proof: Let T a decomposable tree of length $\ell(T) \geqslant 2$ and shape $\mathbf{s}$. By Lemma 1 we know that $\mathbf{s}$ is of degree at most 3. Our aim is to show that $\mathbf{s}$ is a comb. For this, all we have to do is to show that no vertex in $\mathbf{s}$ is adjacent to three vertices of degree 3. Indeed, if no vertex is adjacent to three vertices of degree 3 in $\mathbf{s}$, each degree 3 vertex has at most two neighbours of degree 3 in $\mathbf{s}$; hence there exists a path containing all vertices of degree 3 in $\mathbf{s}$, i.e. this shape is a comb.

Suppose by contradiction that T is a decomposable tree of length at least 2 which is not a comb. In T, there is a vertex v_0 connected by edge-disjoint paths to three vertices v_1, v_2, v_3 of degree 3 – see the left part of Figure 1. We denote by T_i' and T_i'' the two connected components of $T - v_i$ not containing v_0.

Let a_i be the order of the connected component of $T - v_0$ containing the vertex v_i. By Lemma 2, the tree $S(a_1, a_2, a_3)$ is decomposable – see the right part of Figure 1. Without loss of generality, we can assume $a_1 = \min\{a_1, a_2, a_3\}$. Now by Lemma 3, we have $a_1 \leqslant \sqrt{n+2}+1$. Without loss of generality we can assume $|T_1''| \leqslant |T_1'|$; let $b = |T_1'|$. Since $\ell(T) \geqslant 2$, we have $b + 4 \leqslant a_1$. It follows that $b + 3 \leqslant \sqrt{n+2}$, which yields $(b+1)^2 \leqslant n+1$. Then it is easy to see there exists a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of the integer n with $\lambda_i \in \{b+1, b+2\}$ for all i –

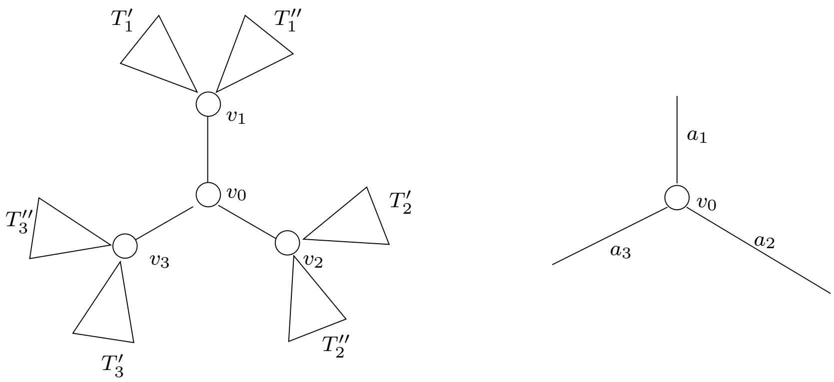

Figure 1: A tree T whose shape is not a comb and the tree $S(a_1, a_2, a_3)$ associated to the vertex v_0 of T.

see e.g. [3, Lemma 2]. Now it can be checked that T is not λ-decomposable. Indeed, the part covering v_1 should cover both T_1' and T_1'' because no part in λ is smaller than $b+1$. However, $|T_1''| \geqslant 2$ because $\ell(T) \geqslant 2$. Thus $|T_1'| + |T_1''| + 1 \geqslant b+3$, but there is no part that large in the partition λ. Hence T is not decomposable. $\square$

4 Construction of decomposable combs

Our aim is to show the existence of decomposable combs of arbitrary shape and arbitrary large length. Roughly speaking, our method is based on an inductive construction building a decomposable comb of shape $\mathbf{c}_{k+1}$ from a decomposable comb of shape $\mathbf{c}_k$ – of course all paths (i.e. combs of shape $\mathbf{c}_0$) are decomposable which gives a starting point for the construction.

Given an integer sequence of odd length $C = (a_0, b_1, a_1, b_2, \ldots, a_{k-1}, b_k, a_k)$ we define the associated comb which is a path $(u[0], u[1], \ldots, u[a_0 + \ldots + a_k + k])$, with a path of length b_i attached to $u[a_0 + \ldots + a_{i-1} + (i-1)]$ for all $i \in \{1, \ldots, k\}$ – see Figure 2. We shall denote by B_i the set of vertices corresponding to the path of length b_i. We define $v_0 = u[0]$ and $v_i = u[a_0 + \ldots + a_{i-1} + (i-1)]$ for all $i \in \{1, \ldots, k+1\}$. The set of vertices strictly between v_i and v_{i+1} is denoted by A_i – again see Figure 2. The number of vertices of this comb, which is equal to $a_0 + \ldots + a_k + b_1 + \ldots + b_k + k$, will be denoted by n. In the following we shall freely speak about the comb C instead of the comb associated to the integer sequence C.

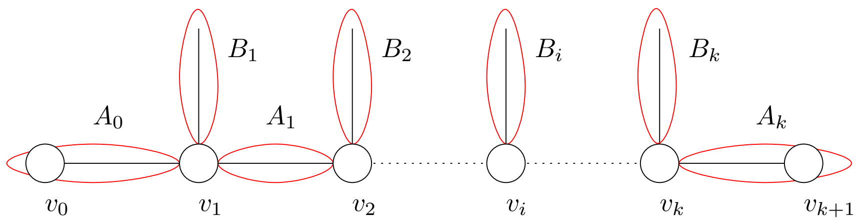

Figure 2: Comb associated to a tuple.

We say that the comb C of order n has *property (E)* if both the following conditions hold:

- C is λ-decomposable for every partition λ of n of the form $\lambda = (q, q, \dots, q)$;
- for any partition of n of the form $(q, q, \dots, q, r)$ with $0 < r < q$, the comb C can be decomposed following this partition in such a way that one of its extremities v_0 or v_{k+1} is covered by the smallest part (i.e. the part of size r).

Notice that property (E) is not comparable with decomposability: a comb with this property may not be decomposable by all partitions, but at the same time an additional property is required for partitions of a certain type.

The *extension of a comb* C, denoted as C', is defined in the following way. Let $C = (a_0, b_1, a_1, b_2, \dots, b_k, a_k)$. Recall that its size is denoted by n. Let $N = (4n!)!$. The extension of C is defined as $C' = (a'_0, b'_1, a'_1, b'_2, \dots, a'_k, b'_{k+1}, a'_{k+1})$ where:

- $a'_0 = N \cdot (4n! + 1) - n + a_0$;
- $b'_i = b_i$ and $a'_i = a_i$ for all $i \in \{1, \dots, k\}$;
- $b'_{k+1} = n!$;
- $a'_{k+1} = N \cdot (4n! + 1) + 2n! - n - 1$.

We shall use the same notations A'_i, B'_i, v'_i for C' as previously defined for C – see Figure 3. It can be seen there is a canonical inclusion of C into C'; we shall omit this inclusion mapping and denote by A_0 the set of vertices of C' corresponding to the image of A_0 by this inclusion mapping. In the same way, we denote by V the vertices of C' corresponding to the image of the vertices of C.

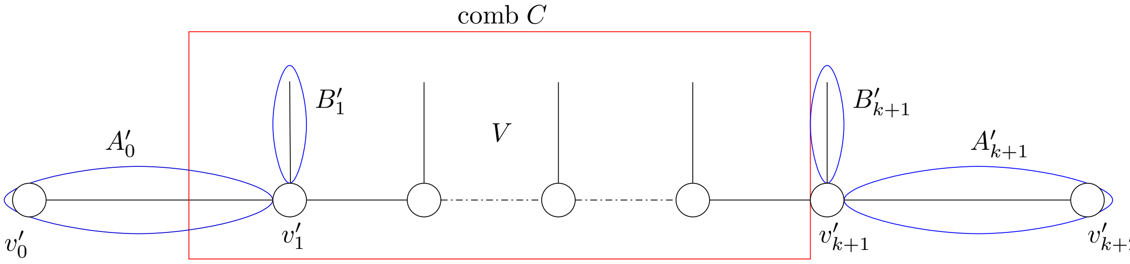

Figure 3: The extension C' of a comb C.

We shall make constant use of the following simple remark.

Remark 5 *Consider a n-vertex tree $T = (V, E)$ and an integer partition $\lambda = (\lambda_1, \dots, \lambda_p)$ of n. If there exist $I \subseteq \{1, \dots, p\}$ and some $\{V_i \mid i \in I\}$, each V_i inducing a subtree, such that $|V_i| = \lambda_i$, and such that $V \setminus \left(\bigcup_{i \in I} V_i\right)$ induces a path on T, then the tree T is λ-decomposable.*

We now present the main lemma which allows to obtain decomposable combs of arbitrary shape.

Lemma 6 *If C is a comb of size $n \geqslant 3$ with property (E), then its extension C' is decomposable and has property (E).*

Proof: Let C be a comb of size $n \geqslant 3$ with property (E). Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition of n' (the size of the extended comb C') with $\lambda_1 \geqslant \ldots \geqslant \lambda_p$. We will prove that C' can be decomposed with respect to λ. Moreover, we will show that this can be achieved with the smallest part of λ placed on one of the extremities of C' – this of course implies property (E). We consider two cases according to how large the biggest part of λ is.

Case 1 – one part of λ is large: $\lambda_1 > 4n!$. Recall that V is the set of vertices of C' corresponding to the vertices of C (through the canonical inclusion defined before). Let

$$W = (V \setminus A_0) \cup \{v'_{k+1}\} \cup B'_{k+1}.$$

Remark that $|W| = n! + n + 1 - a_0$. Our strategy to decompose C' following λ is this one. The comb C' is made of one long path with a complex part W of size approximately $n!$ in the middle. Keeping the largest part λ_1 aside, we start placing parts of λ starting from the rightmost vertex of C', as long as it does not cover any vertex of W. Then we try to cover W all at once with the largest part $\lambda_1 > 4n!$. Of course this may fail: but in this case, using this strategy starting from the leftmost extremity of C' will succeed because $a'_{k+1} - a'_0 \approx 2n!$.

Let us check this carefully. Let $\sigma_0 = 0$ and $\sigma_i = \lambda_2 + \ldots + \lambda_{i+1}$ for all $i \in \{1, \ldots, p-1\}$. Let

$$t = \max\{i \in \{0, \ldots, p-1\} \mid \sigma_i \leqslant a'_{k+1}\}.$$

If $a'_{k+1} + |W| \leqslant \sigma_t + \lambda_1$, let us show that C' is decomposable following λ. Indeed, we can put parts of size $\lambda_2, \ldots, \lambda_{t+1}$ on A'_{k+1}, then a part of size λ_1 covering the uncovered part of A'_{k+1}, the whole W, and some vertices from A'_0. The remaining vertices induce a path and it follows by Remark 5 that C' is λ-decomposable.

Now suppose that $\sigma_t + \lambda_1 < a'_{k+1} + |W|$. In this case, it holds that

$$a'_{k+1} < \sigma_t + \lambda_1 < a'_{k+1} + |W|$$

where the left inequality comes from the definition of t and the fact that λ_1 is the largest part. Since $a'_0 - a'_{k+1} = -2n! + a_0 + 1$ we obtain (by adding this quantity to each term in the previous inequality):

$$a'_0 < \sigma_t + \lambda_1 - 2n! + a_0 + 1 < a'_0 + |W|. \tag{1}$$

Recall that $|W| = n! + n - a_0 + 1$ and we supposed $\lambda_1 > 4n!$. Thus the right inequality in Equation (1) gives $\sigma_t \leqslant a'_0 - n! + n - 2a_0 \leqslant a'_0$. In the same way, adding $|W|$ to each term of the left inequality in Equation (1), we obtain $a'_0 + |W| \leqslant \sigma_t + \lambda_1 + (-n! + n + 1) \leqslant \sigma_t + \lambda_1$ as soon as $n \geqslant 3$ (as we supposed). Thus we have obtained:

$$\sigma_t \leqslant a'_0 < a'_0 + |W| \leqslant \sigma_t + \lambda_1.$$

It follows that C' is λ-decomposable. Indeed, we can put parts of size $\lambda_2, \ldots, \lambda_{t+1}$ along A'_0, then a part of size λ_1 covering the uncovered part of A'_0, the whole W, and some vertices from A'_{k+1}. The remaining vertices inducing a path, we obtain that C' is λ-decomposable using Remark 5.

In both cases above, one of the smallest part of λ has been placed on a portion of path either to the left or to the right of W. It is easy to check that this smallest part can be shifted so as to cover one of the extremities of C'.

Case 2 – all parts of λ are small: $\lambda_1 \leqslant 4n!$. In this case $\lambda_i \in \{1, \ldots, 4n!\}$ for all i. Let n_i be the number of parts of size i in λ. For all $1 \leqslant i \leqslant 4n!$, let $\sigma_i = i \cdot n_i$ – that is, σ_i is the sum of all parts of size i in λ. For all i, we also define q_i and r_i to be the quotient and the remainder of the Euclidean division of σ_i by N; that is $\sigma_i = q_i N + r_i$, with $q_i \in \mathbb{N}$ and $0 \leqslant r_i < N$.

Notice that $2N \cdot (4n! + 1) < n' = \sum_{i=1}^{4n!}(q_i N + r_i)$. Since $\sum_{i=1}^{4n!} r_i < N \cdot 4n!$ we get $N(8n! + 2 - 4n!) < N \sum_{i=1}^{4n!} q_i$. Thus it holds that:

$$4n! + 2 < q_1 + \ldots + q_{4n!}.$$

Moreover, notice that $i|N$ (i is a divisor of N) for all $i \leqslant 4n!$. Using some parts of λ, we can create the partition

$$\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_{4n!+2}^{m_{4n!+2}})$$

with $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{4n!+2}$, and $m_i \cdot \mu_i = N$ for all i – the notation used here means that μ is made of m_i parts of size μ_i for all $1 \leqslant i \leqslant 4n! + 2$. Let us call ν the partition of the integer $n' - N \cdot (4n! + 2)$ made of the parts from λ not used in μ. Of course the choice of μ above may not be unique: we choose the largest possible μ_i. This choice ensures that the partition ν always contains (one of) the smallest part of λ. We now consider two subcases.

Case 2a: $\mu_1 \geqslant n$. Notice that $|A'_0 \cup V| = N(4n! + 1)$. In this case we can cover $A'_0 \cup V$ by using parts $(\mu_1^{m_1}, \ldots, \mu_{4n!+1}^{m_{4n!+1}})$ – using one part of size μ_1 in order to cover V. The remaining part of the comb to be covered $B'_{k+1} \cup \{v_{k+1}\} \cup A'_{k+1}$ induces a path; hence it can be covered by the remaining parts of the partition (by Remark 5). Moreover, at least one part of minimal size remains to cover $B'_{k+1} \cup \{v_{k+1}\} \cup A'_{k+1}$, since there is one such part in ν: such a part can be placed on v'_{k+2}, at the rightmost extremity of C'.

Case 2b: $\mu_1 < n$. In this case it holds for all i that $\mu_i < n$, and thus $\mu_i | n!$. Moreover, recall that $m_i \mu_i = N \geqslant 3n!$. We can use $n!/\mu_{4n!+2}$ parts of size $\mu_{4n!+2}$ to cover B'_{k+1} (which is of size $n!$) and $2n!/\mu_{4n!+2}$ parts of size $\mu_{4n!+2}$ to cover the $2n!$ rightmost vertices $A'_{k+1,r}$ of A'_{k+1} – this corresponds to hachured parts on Figure 4.

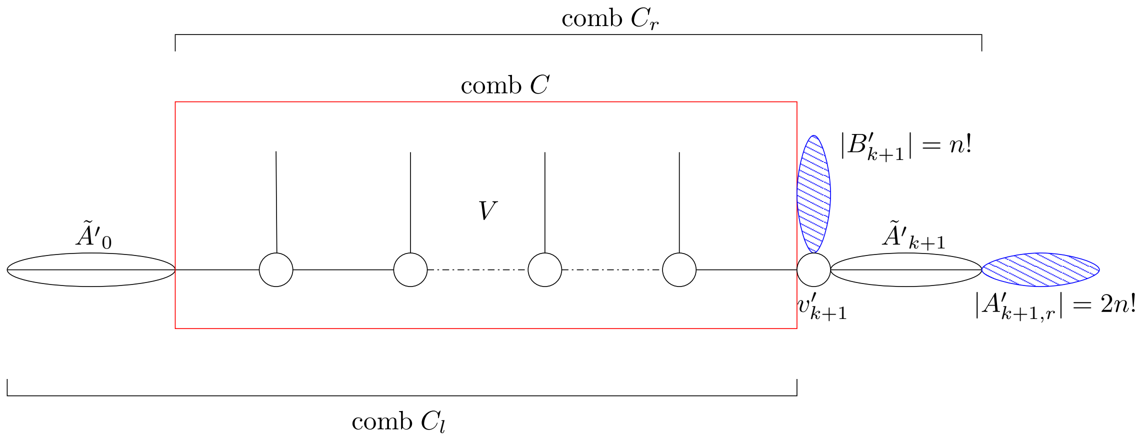

Figure 4: The combs C_l and C_r.

Let $\tilde{A}'_0 = A'_0 \setminus V$ and $\tilde{A}'_{k+1} = A'_{k+1} \setminus A'_{k+1,r}$. Let C_l be the comb induced by vertices $\tilde{A}'_0 \cup V$ and C_r be the comb induced by vertices $V \cup \{v'_{k+1}\} \cup \tilde{A}'_{k+1}$ – these two combs are pictured in Figure 4 (note that C_l and C_r do not contain any of the hachured parts). The number of vertices of these two combs is $|C_l| = |C_r| = N(4n! + 1)$. We are going to show that (at least) one of C_l or C_r can be covered with $\mu' = (\mu_1^{m_1}, \dots, \mu_{4n!+1}^{m_{4n!+1}})$. Indeed, recall that comb C has property (E); thus it can be covered with $(\mu_1, \mu_1, \dots, \mu_1)$ and a part of size $\mu'_1 = |C| \mod \mu_1$ (if not zero) with the part of size μ'_1 covering one of the extremities of C. In case $(\mu_1, \dots, \mu_1, \mu'_1)$ decompose C with the part of size μ'_1 placed on the left extremity of C, then we can use μ' to cover C_l; otherwise, C can be decomposed following $(\mu_1, \dots, \mu_1, \mu'_1)$ with μ'_1 placed on the right extremity of C and this implies that C_r can be decomposed by μ'. In both cases, the part of C' that remains to be covered induces a path (either $\tilde{A}'_0$ or $\tilde{A}'_{k+1}$). Thus C' is λ-decomposable by Remark 5.

In both cases above, remaining parts (including some parts of size $\mu_{4n!+2}$ and all parts of ν) are placed either on $\tilde{A}'_0$ or on $\tilde{A}'_{k+1}$. In the first case the smallest part of λ (which is present in ν) can be placed so as to cover v'_0. In the second case where it is placed along $\tilde{A}'_{k+1}$, it can be shifted to the right by permuting it with parts used to cover $A'_{k+1,r}$ in order to cover v'_{k+2}. In both cases, one of the smallest part of λ has been placed on one of the extremities of C' as required. □

5 Main result and final remarks

We are now ready to state the main result:

Theorem 7 *The set of shapes that can be realized by decomposable trees with an arbitrarily large length is exactly the set of combs, i.e.* $\mathcal{D} = \mathcal{C}$.

Proof: By Lemma 4 we know that $\mathcal{D} \subseteq \mathcal{D}_2 \subseteq \mathcal{C}$. Let us show the converse inclusion. Starting with C_m^0 a path of length $m \geqslant 3$, we define $C_m^{k+1} = (C_m^k)'$ for all $k \in \mathbb{N}$. It is easy to check that for any comb C, it holds that $\ell(C') \geqslant \ell(C)$; hence $\ell(C_m^k) \geqslant m$ for all k and m. Moreover, using Lemma 6, we get that all C_m^k are decomposable. It follows that $\mathbf{c}_k \in \mathcal{D}_m$ for all k and m, and thus $\mathcal{C} \subseteq \mathcal{D}$. □

The construction of decomposable combs proposed here yields huge trees – there is a doubly exponential blow-up in size when building a decomposable comb of shape $\mathbf{c}_{k+1}$ from a comb of shape $\mathbf{c}_k$. It would be nice to obtain smaller decomposable trees of a given shape and length. More generally, it would be interesting to study lower bound on the size of a decomposable comb of prescribed shape and length.

From the algorithmic point of view, one may wonder about the complexity of deciding if a comb is decomposable. A question of interest is the complexity of this problem parameterized – as defined in [6, 5] – by the number of degree 3 nodes. In particular, it was shown in [2] that it can be decided in polynomial time whether a tree homeomorphic to $K_{1,3}$ is decomposable or not. In fact, it is shown that such a tree is decomposable if and only if it is λ-decomposable for all partitions λ having at most three different sizes of parts – i.e. partitions λ of the form $(\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3})$. This raises the question whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that a comb of shape $\mathbf{c}_k$ is decomposable if and only if it is decomposable by partitions involving at most $f(k)$ different sizes of parts.

Acknowledgment

We thank the anonymous referees for their helpful comments.

References

[1] D. Barth, O. Baudon, and J. Puech. Networks sharing: a polynomial algorithm for tripodes. Technical Report 1164, LRI, Université Paris Sud, 1998.

[2] D. Barth, O. Baudon, and J. Puech. Decomposable trees: a polynomial algorithm for tripodes. *Discrete applied mathematics*, 119:205–216, 2002.

[3] D. Barth and H. Fournier. A degree bound on decomposable trees. *Discrete mathematics*, 306(5):469–477, 2006.

[4] A. Diwan and M. Kurhekar. Plane triangulations are 6-partitionable. *Discrete Mathematics*, 256(1-2):91–103, 2002.

[5] R. G. Downey and M. R. Fellows. *Parameterized Complexity.* Monographs in Computer Science. Springer, 1999.

[6] J. Flum and M. Grohe. *Parameterized complexity theory.* Springer, 2006.

[7] E. Györi. On division of graphs to connected subgraphs. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976) vol I*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 485–494, 1978.

[8] M. Horňák, Z. Tuza, and M. Woźniak. On-line arbitrarily vertex decomposable trees. *Discrete Applied Mathematics*, 155(11):1420–1429, 2007.

[9] M. Horňák and M. Woźniak. Arbitrarily vertex decomposable trees are of degree at most 6. *Opuscula Mathematica*, 23:49–62, 2003.

[10] L. Lovász. A homology theory for spanning trees of a graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, 30(3-4):241–251, 1977.

[11] H. Miyano, T. Nishizeki, N. Takahashi, and S. Uneo. An algorithm for tripartitioning 3-connected graphs. *Journal of Information Processing Society of Japan*, 31:584–592, 1990.

[12] S. Nakano, S. Rahman, and T. Nishizeki. A linear-time algorithm for four-partitioning four-connected planar graphs. *Information Processing Letters*, 62(6):315–322, 1997.

[13] H. Suzuki, N. Takahashi, and T. Nishizeki. A linear time algorithm for bipartition of biconnected graphs. *Information Processing Letters*, 33(5):227–231, 1990.