On the shape of decomposable trees

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Abstract

A $n$-vertex graph is said to be decomposable if for any partition $(\lambda_1, \ldots, \lambda_p)$ of the integer $n$, there exists a sequence $(V_1, \ldots, V_p)$ of connected vertex-disjoint subgraphs with $|V_i| = \lambda_i$. The aim of the paper is to study the homeomorphism classes of decomposable trees. More precisely, we show that homeomorphism classes containing decomposable trees with an arbitrarily large minimal distance between all pairs of distinct vertices of degree different from 2 is exactly the set of combs.

Keywords: Tree partitions; Arbitrarily partitionable trees; Homeomorphism class; Integer partitions

1 Introduction

A partition of the integer $n$ is a non-increasing sequence of positive integers that sums up to $n$. We are interested here in the following notion. Given a graph $G = (V, E)$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of the integer $|V|$, the graph $G$ is said to be $\lambda$-decomposable if there exists a partition $(V_1, \ldots, V_p)$ of $V$ such that the set $V_i$ induces a connected subgraph of $G$, and $|V_i| = \lambda_i$ (such a partition is called a $\lambda$-decomposition of $G$). A graph is said to be decomposable if it is $\lambda$-decomposable for all partition $\lambda$ of the integer $|V|$. These graphs are sometimes called arbitrarily vertex decomposable graphs. The term $k$-partitionable is also used in the literature for graphs which are $\lambda$-decomposable for all partitions $\lambda$ with at most $k$ parts.

Questions related to this notion of graph decomposition are old. In 1976 and 1977 respectively, Győri [7] and Lovász [10] independently proved that $k$-connected graphs are $k$-partitionable. Algorithmic versions of $k$-partitioning have been studied. Is was shown by Suzuki et al. [13] that, given a 2-connected graph $G$ and a partition $\lambda$ with 2 parts, a $\lambda$-decomposition of $G$ can be computed in linear time. A polynomial time algorithm for the similar problem with $k = 3$ was found by Miyano et al. [11]. The case $k = 4$ is treated in Nakano et al. [12] for the special case of planar graphs.

Some special classes of graphs have also been studied such as the one of plane triangulations: it was proved by Diwan and Kurhekar [4] that these graphs are 6-partitionable. At last, let us mention that some on-line version of decomposability has been studied by Hornák et al. [8].

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In this paper we focus on the case of decomposable trees. A motivation is that trees are the smallest graph candidates to be decomposable, and the property of being decomposable is monotone (adding edges to a decomposable graph yields a decomposable graph). Some works have been carried out both on the structure and the algorithmic aspects of decomposable trees. From the algorithmic point of view, it was shown that it can be decided in polynomial time whether a tree homeomorphic to $K_{1,3}$ (i.e. a tree with one degree 3 vertex and three paths attached to it) is decomposable or not [2]. In the general case, the complexity of deciding if a tree is decomposable is largely unknown. (However, given a tree $T$ and an integer partition $\lambda$, deciding if $T$ is $\lambda$-decomposable is known to be NP-complete [3].)

Concerning the structure of decomposable trees, it was first shown by Horňák and Woźniak [9] that decomposable trees have degree at most 6; it was later proved in [3] that a decomposable tree is of degree at most 4, and each degree 4 vertex in such a tree is adjacent to a leaf (and this bound is tight). In the same paper, a family of decomposable trees with an arbitrary number of degree 3 vertices is presented. However, these trees are combs made of a long paths with vertices attached to it (all at distance 1 from the path). Thus, it can be said that these trees look like a path. The question we address in this paper is to characterize the possible shapes (i.e. homeomorphism classes) of decomposable trees that can be realized by trees in which the minimal distance between all pairs of distinct vertices of degree different from 2 (later called the length) is arbitrarily large. We show that these shapes are exactly combs.

The paper is organized as follows. We give some definitions in Section 2. Then we show that a decomposable tree with length at least 2 must be a comb in Section 3. A construction of families of decomposable combs of arbitrary shape and arbitrarily large length is given in Section 4. At last, the main result is given in Section 5, where some open problems are presented.

## 2 Definitions

We recall that a partition of the integer $n$ is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_p)$ such that $\lambda_1 + \ldots + \lambda_p = n$. A tree $T = (V, E)$ is said to be $\lambda$-decomposable if there exists a partition $(V_1, \ldots, V_p)$ of $V$ such that $|V_i| = \lambda_i$ for all $i$, each $V_i$ inducing a subtree on $T$. Such a partition of $V$ will be called a $\lambda$-partition of $T$. An $n$-vertex tree $T$ is said to be decomposable if it is $\lambda$-decomposable for all partitions $\lambda$ of the integer $n$.

Given a tree $T$, we define the shape of $T$ to be the smallest tree homeomorphic to $T$; that is, this is the tree obtained from $T$ by contracting in turn all edges adjacent to a degree 2 vertex. Hence the set of shapes is exactly the set of trees without degree 2 vertices. Given a tree $T = (V, E)$, we also define its length

$$\ell(T) = \min\{d(u, v) \mid u, v \in V, u \neq v, \deg(u) \neq 2, \deg(v) \neq 2\}$$

where $d(u, v)$ denotes the distance between $u$ and $v$, and $\deg(u)$ the degree of $u$. That is, $\ell(T)$ is the length of the shortest path $(v_0, \ldots, v_k)$ with $\deg(v_0) \neq 2, \deg(v_k) \neq 2$, and whose internal vertices $v_1, \ldots, v_{k-1}$ all have degree 2. To put it differently, every tree $T$ of shape $s = (V_s, E_s)$ is obtained in a unique way by replacing each edge $e \in E_s$ with a path of length $w(e)$: then $\ell(T) = \min\{w(e) \mid e \in E_s\}$.

We now define $D_m$ to be the set of shapes $s$ such that there exists a decomposable tree $T$ of shape $s$ with $\ell(T) \geq m$. At last we define $D = \bigcap_{m \geq 1} D_m$. Thus $D$ is the set of all shapes
realized by decomposable trees with an arbitrarily large length.

We recall that a comb is a tree of maximum degree 3 with the property that all vertices of degree 3 belong to a single path of the tree. We shall denote as $\mathcal{C} = \{c_k \mid k \in \mathbb{N}\}$ the set of shapes of combs, where $c_k$ is the shape of combs with $k$ vertices of degree 3 – it thus includes the shape $c_0$ of paths.

### 3 Forbidden structure in decomposable trees

The aim of this work is to characterize $\mathcal{D}$. The following bound on the degree of decomposable trees gives a strong restriction on $\mathcal{D}$.

**Lemma 1** [3] Decomposable trees have degree at most 4. Moreover, every degree 4 vertex of a decomposable tree is adjacent to a leaf.

Let us denote by $S(a, b, c)$ the $(1 + a + b + c)$-vertex tree made of a single vertex of degree 3 with three paths of lengths $a$, $b$ and $c$ attached to it. We need to recall two other facts on decomposable trees.

**Lemma 2** [1, 3] Let $T$ be a decomposable tree and $v$ a vertex of $T$ of degree 3. Let $a, b, c$ the cardinals of the connected components of $T - v$. Then $S(a, b, c)$ is a decomposable tree.

**Lemma 3** [3] If $S(a, b, c)$ is a decomposable tree, then

$$\min\{a, b, c\} \leq \left\lfloor \sqrt{n + 2} \right\rfloor$$

where $n = 1 + a + b + c$.

Notice that an immediate corollary of Lemma 1 is that $\mathcal{D}_2$ (and thus $\mathcal{D}$) is made of trees of degree at most 3. We now strengthen this by showing that $\mathcal{D}_2$ only contains combs.

**Lemma 4** A decomposable tree of length at least 2 is a comb, i.e. $\mathcal{D}_2 \subseteq \mathcal{C}$.

**Proof:** Let $T$ a decomposable tree of length $\ell(T) \geq 2$ and shape $s$. By Lemma 1 we know that $s$ is of degree at most 3. Our aim is to show that $s$ is a comb. For this, all we have to do is to show that no vertex in $s$ is adjacent to three vertices of degree 3. Indeed, if no vertex is adjacent to three vertices of degree 3 in $s$, each degree 3 vertex has at most two neighbours of degree 3 in $s$; hence there exists a path containing all vertices of degree 3 in $s$, i.e. this shape is a comb.

Suppose by contradiction that $T$ is a decomposable tree of length at least 2 which is not a comb. In $T$, there is a vertex $v_0$ connected by edge-disjoint paths to three vertices $v_1, v_2, v_3$ of degree 3 – see the left part of Figure 1. We denote by $T'_1$ and $T''_1$ the two connected components of $T - v_1$ not containing $v_0$.

Let $a_1$ be the order of the connected component of $T - v_0$ containing the vertex $v_1$. By Lemma 2, the tree $S(a_1, a_2, a_3)$ is decomposable – see the right part of Figure 1. Without loss of generality, we can assume $a_1 = \min\{a_1, a_2, a_3\}$. Now by Lemma 3, we have $a_1 \leq \sqrt{n + 2} + 1$. Without loss of generality we can assume $|T'_1| \leq |T''_1|$; let $b = |T''_1|$. Since $\ell(T) \geq 2$, we have $b + 4 \leq a_1$. It follows that $b + 3 \leq \sqrt{n + 2}$, which yields $(b + 1)^2 \leq n + 1$. Then it is easy to see there exists a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of the integer $n$ with $\lambda_i \in \{b + 1, b + 2\}$ for all $i -$
Figure 1: A tree $T$ whose shape is not a comb and the tree $S(a_1, a_2, a_3)$ associated to the vertex $v_0$ of $T$.

see e.g. [3, Lemma 2]. Now it can be checked that $T$ is not $\lambda$-decomposable. Indeed, the part covering $v_1$ should cover both $T'_1$ and $T''_1$ because no part in $\lambda$ is smaller than $b + 1$. However, $|T''_1| \geq 2$ because $\ell(T) \geq 2$. Thus $|T'_1| + |T''_1| + 1 \geq b + 3$, but there is no part that large in the partition $\lambda$. Hence $T$ is not decomposable. \[\square\]

4 Construction of decomposable combs

Our aim is to show the existence of decomposable combs of arbitrary shape and arbitrary large length. Roughly speaking, our method is based on an inductive construction building a decomposable comb of shape $c_{k+1}$ from a decomposable comb of shape $c_k$ – of course all paths (i.e. combs of shape $c_0$) are decomposable which gives a starting point for the construction.

Given an integer sequence of odd length $C = (a_0, b_1, a_1, b_2, \ldots, a_{k-1}, b_k, a_k)$ we define the associated comb which is a path $(u[0], u[1], \ldots, u[a_0 + \ldots + a_k + k])$, with a path of length $b_i$ attached to $u[a_0 + \ldots + a_{i-1} + (i - 1)]$ for all $i \in \{1, \ldots, k\}$ – see Figure 2. We shall denote by $B_i$ the set of vertices corresponding to the path of length $b_i$. We define $v_0 = u[0]$ and $v_i = u[a_0 + \ldots + a_{i-1} + (i - 1)]$ for all $i \in \{1, \ldots, k+1\}$. The set of vertices strictly between $v_i$ and $v_{i+1}$ is denoted by $A_i$ – again see Figure 2. The number of vertices of this comb, which is equal to $a_0 + \ldots + a_k + b_1 + \ldots + b_k + k$, will be denoted by $n$. In the following we shall freely speak about the comb $C$ instead of the comb associated to the integer sequence $C$.

Figure 2: Comb associated to a tuple.

We say that the comb $C$ of order $n$ has property (E) if both the following conditions hold:
• $C$ is $\lambda$-decomposable for every partition $\lambda$ of $n$ of the form $\lambda = (q, q, \ldots, q)$;
• for any partition of $n$ of the form $(q, q, \ldots, q, r)$ with $0 < r < q$, the comb $C$ can be decomposed following this partition in such a way that one of its extremities $v_0$ or $v_{k+1}$ is covered by the smallest part (i.e. the part of size $r$).

Notice that property (E) is not comparable with decomposability: a comb with this property may not be decomposable by all partitions, but at the same time an additional property is required for partitions of a certain type.

The extension of a comb $C$, denoted as $C'$, is defined in the following way. Let $C = (a_0, b_1, a_1, b_2, \ldots, b_k, a_k)$. Recall that its size is denoted by $n$. Let $N = (4n)!$. The extension of $C$ is defined as $C' = (a'_0, b'_1, a'_1, b'_2, \ldots, a'_k, b'_{k+1}, a'_{k+1})$ where:

- $a'_0 = N \cdot (4n! + 1) - n + a_0$;
- $b'_i = b_i$ and $a'_i = a_i$ for all $i \in \{1, \ldots, k\}$;
- $b'_{k+1} = n!$;
- $a'_{k+1} = N \cdot (4n! + 1) + 2n! - n - 1$.

We shall use the same notations $A'_i, B'_i, v'_i$ for $C'$ as previously defined for $C$ – see Figure 3. It can be seen there is a canonical inclusion of $C$ into $C'$; we shall omit this inclusion mapping and denote by $A_0$ the set of vertices of $C'$ corresponding to the image of $A_0$ by this inclusion mapping. In the same way, we denote by $V$ the vertices of $C'$ corresponding to the image of the vertices of $C$.

![Figure 3: The extension $C'$ of a comb $C$.](image)

We shall make constant use of the following simple remark.

**Remark 5** Consider a $n$-vertex tree $T = (V, E)$ and an integer partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of $n$. If there exist $I \subseteq \{1, \ldots, p\}$ and some $\{V_i \mid i \in I\}$, each $V_i$ inducing a subtree, such that $|V_i| = \lambda_i$, and such that $V \setminus \left( \bigcup_{i \in I} V_i \right)$ induces a path on $T$, then the tree $T$ is $\lambda$-decomposable.

We now present the main lemma which allows to obtain decomposable combs of arbitrary shape.

**Lemma 6** If $C$ is a comb of size $n \geq 3$ with property (E), then its extension $C'$ is decomposable and has property (E).
Proof: Let $C$ be a comb of size $n \geq 3$ with property (E). Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition of $n'$ (the size of the extended comb $C'$) with $\lambda_1 \geq \ldots \geq \lambda_p$. We will prove that $C'$ can be decomposed with respect to $\lambda$. Moreover, we will show that this can be achieved with the smallest part of $\lambda$ placed on one of the extremities of $C'$ — this of course implies property (E). We consider two cases according to how large the biggest part of $\lambda$ is.

**Case 1 — one part of $\lambda$ is large:** $\lambda_1 > 4n!$. Recall that $V$ is the set of vertices of $C'$ corresponding to the vertices of $C$ (through the canonical inclusion defined before). Let

$$W = (V \setminus A_0) \cup \{v'_{k+1}\} \cup B'_{k+1}.$$ 

Remark that $|W| = n! + n + 1 - a_0$. Our strategy to decompose $C'$ following $\lambda$ is this one. The comb $C'$ is made of one long path with a complex part $W$ of size approximately $n!$ in the middle. Keeping the largest part $\lambda_1$ aside, we start placing parts of $\lambda$ starting from the rightmost vertex of $C'$, as long as it does not cover any vertex of $W$. Then we try to cover $W$ all at once with the largest part $\lambda_1 > 4n!$. Of course this may fail: but in this case, using this strategy starting from the leftmost extremity of $C'$ will succeed because $a'_{k+1} - a_0 \approx 2n!$.

Let us check this carefully. Let $\sigma_0 = 0$ and $\sigma_i = \lambda_2 + \ldots + \lambda_{i+1}$ for all $i \in \{1, \ldots, p-1\}$. Let

$$t = \max\{i \in \{0, \ldots, p-1\} \mid \sigma_i \leq a'_{k+1}\}.$$

If $a'_{k+1} + |W| \leq \sigma_i + \lambda_1$, let us show that $C'$ is decomposable following $\lambda$. Indeed, we can put parts of size $\lambda_2, \ldots, \lambda_{t+1}$ on $A'_{k+1}$, then a part of size $\lambda_1$ covering the uncovered part of $A'_{k+1}$, the whole $W$, and some vertices from $A_0'$. The remaining vertices induce a path and it follows by Remark 5 that $C'$ is $\lambda$-decomposable.

Now suppose that $\sigma_t + \lambda_1 < a'_{k+1} + |W|$. In this case, it holds that

$$a'_{k+1} < \sigma_t + \lambda_1 < a'_{k+1} + |W|$$

where the left inequality comes from the definition of $t$ and the fact that $\lambda_1$ is the largest part. Since $a'_{k+1} - a'_{k+1} = -2n! + a_0 + 1$ we obtain (by adding this quantity to each term in the previous inequality):

$$a_0 < \sigma_t + \lambda_1 - 2n! + a_0 + 1 < a_0 + |W|.$$  

(1)

Recall that $|W| = n! + n - a_0 + 1$ and we supposed $\lambda_1 > 4n!$. Thus the right inequality in Equation (1) gives $\sigma_t \leq a_0 - n! + n - 2a_0 \leq \sigma_t$. In the same way, adding $|W|$ to each term of the left inequality in Equation (1), we obtain $a_0 + |W| \leq \sigma_t + \lambda_1 - (-n! + n + 1) \leq \sigma_t + \lambda_1$ as soon as $n \geq 3$ (as we supposed). Thus we have obtained:

$$\sigma_t \leq a_0 < a'_0 + |W| \leq \sigma_t + \lambda_1.$$ 

It follows that $C'$ is $\lambda$-decomposable. Indeed, we can put parts of size $\lambda_2, \ldots, \lambda_{t+1}$ along $A'_0$, then a part of size $\lambda_1$ covering the uncovered part of $A'_0$, the whole $W$, and some vertices from $A'_{k+1}$. The remaining vertices inducing a path, we obtain that $C'$ is $\lambda$-decomposable using Remark 5.

In both cases above, one of the smallest part of $\lambda$ has been placed on a portion of path either to the left or to the right of $W$. It is easy to check that this smallest part can be shifted so as to cover one of the extremities of $C'$. 


**Case 2 – all parts of \( \lambda \) are small:** \( \lambda_i \leq 4n! \). In this case \( \lambda_i \in \{1, \ldots, 4n!\} \) for all \( i \). Let \( n_i \) be the number of parts of size \( i \) in \( \lambda \). For all \( 1 \leq i \leq 4n! \), let \( \sigma_i = i \cdot n_i \) – that is, \( \sigma_i \) is the sum of all parts of size \( i \) in \( \lambda \). For all \( i \), we also define \( q_i \) and \( r_i \) to be the quotient and the remainder of the Euclidean division of \( \sigma_i \) by \( N \); that is \( \sigma_i = q_i \cdot N + r_i \), with \( q_i \in \mathbb{N} \) and \( 0 \leq r_i < N \).

Notice that \( 2N \cdot (4n! + 1) < n' = \sum_{i=1}^{4n!} (q_i \cdot N + r_i) \). Since \( \sum_{i=1}^{4n!} r_i < N \cdot 4n! \) we get \( N(8n! + 2 - 4n!) < N \sum_{i=1}^{4n!} q_i \). Thus it holds that:

\[
4n! + 2 < q_1 + \ldots + q_{4n!}.
\]

Moreover, notice that \( i | N \) (\( i \) is a divisor of \( N \)) for all \( i \leq 4n! \). Using some parts of \( \lambda \), we can create the partition

\[
\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_{4n!+2}^{m_{4n!+2}})
\]

with \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{4n!+2}, \) and \( m_i \cdot \mu_i = N \) for all \( i \) – the notation used here means that \( \mu \) is made of \( m_i \) parts of size \( \mu_i \) for all \( 1 \leq i \leq 4n! + 2 \). Let us call \( \nu \) the partition of the integer \( n' - N \cdot (4n! + 2) \) made of the parts from \( \lambda \) not used in \( \mu \). Of course the choice of \( \mu \) above may not be unique: we choose the largest possible \( \mu \). This choice ensures that the partition \( \nu \) always contains (one of) the smallest part of \( \lambda \). We now consider two subcases.

**Case 2a:** \( \mu_1 \geq n \). Notice that \( |A'_0 \cup V| = N(4n! + 1) \). In this case we can cover \( A'_0 \cup V \) by using parts \( (\mu_1^{m_1}, \ldots, \mu_{4n!+1}^{m_{4n!+1}}) \) – using one part of size \( \mu_1 \) in order to cover \( V \). The remaining part of the comb to be covered \( B'_{k+1} \cup \{v_{k+1}\} \cup A'_{k+1} \) induces a path; hence it can be covered by the remaining parts of the partition (by Remark 5). Moreover, at least one part of minimal size remains to cover \( B'_{k+1} \cup \{v_{k+1}\} \cup A'_{k+1} \), since there is one such part in \( \nu \): such a part can be placed on \( v'_{k+2} \), at the rightmost extremity of \( C' \).

**Case 2b:** \( \mu_1 < n \). In this case it holds for all \( i \) that \( \mu_i < n \), and thus \( \mu_i | n! \). Moreover, recall that \( m_i \cdot \mu_i = N \geq 3n! \). We can use \( n!/\mu_{4n!+2} \) parts of size \( \mu_{4n!+2} \) to cover \( B'_{k+1} \) (which is of size \( n! \)) and \( 2n!/\mu_{4n!+2} \) parts of size \( \mu_{4n!+2} \) to cover the \( 2n! \) rightmost vertices \( A'_{k+1,r} \) of \( A'_{k+1} \) – this corresponds to hachured parts on Figure 4.

![Figure 4: The combs \( C_l \) and \( C_r \).](image)

Let \( \tilde{A}'_0 = A'_0 \setminus V \) and \( \tilde{A}'_{k+1} = A'_{k+1} \setminus A'_{k+1,r} \). Let \( C_l \) be the comb induced by vertices \( \tilde{A}'_0 \cup V \) and \( C_r \) be the comb induced by vertices \( V \cup \{v'_{k+1}\} \cup \tilde{A}'_{k+1} \) – these two combs are
picted in Figure 4 (note that $C_l$ and $C_r$ do not contain any of the hachured parts). The number of vertices of these two combs is $|C_l| = |C_r| = N(4n! + 1)$. We are going to show that (at least) one of $C_l$ or $C_r$ can be covered with $\mu' = (\mu_1^{m_1}, \ldots, \mu_{4n!+1}^{m_{4n!+1}})$. Indeed, recall that comb $C$ has property (E); thus it can be covered with $(\mu_1, \mu_1, \ldots, \mu_1)$ and a part of size $\mu'_1 = |C|$ mod $\mu_1$ (if not zero) with the part of size $\mu'_1$ covering one of the extremities of $C$. In case $(\mu_1, \ldots, \mu_1, \mu'_1)$ decompose $C$ with the part of size $\mu'_1$ placed on the left extremity of $C$, then we can use $\mu'$ to cover $C_l$; otherwise, $C$ can be decomposed following $(\mu_1, \ldots, \mu_1, \mu'_1)$ with $\mu'_1$ placed on the right extremity of $C$ and this implies that $C_r$ can be decomposed by $\mu'$. In both cases, the part of $C'$ that remains to be covered induces a path (either $A'_0$ or $A'_{k+1}$). Thus $C'$ is $\lambda$-decomposable by Remark 5.

In both cases above, remaining parts (including some parts of size $\mu_{4n!+2}$ and all parts of $\nu$) are placed either on $A'_0$ or on $A'_{k+1}$. In the first case the smallest part of $\lambda$ (which is present in $\nu$) can be placed so as to cover $v'_0$. In the second case where it is placed along $A'_{k+1}$, it can be shifted to the right by permuting it with parts used to cover $A'_{k+1}$ in order to cover $v'_{k+2}$. In both cases, one of the smallest part of $\lambda$ has been placed on one of the extremities of $C'$ as required. □

5 Main result and final remarks

We are now ready to state the main result:

**Theorem 7** The set of shapes that can be realized by decomposable trees with an arbitrarily large length is exactly the set of combs, i.e. $D = C$.

**Proof:** By Lemma 4 we know that $D \subseteq D_2 \subseteq C$. Let us show the converse inclusion. Starting with $C_m^0$, a path of length $m \geq 3$, we define $C_m^{k+1} = (C_m^k)'$ for all $k \in \mathbb{N}$. It is easy to check that for any comb $C$, it holds that $\ell(C') \geq \ell(C)$; hence $\ell(C_m^k) \geq m$ for all $k$ and $m$. Moreover, using Lemma 6, we get that all $C_m^k$ are decomposable. It follows that $c_k \in D_m$ for all $k$ and $m$, and thus $C \subseteq D$. □

The construction of decomposable combs proposed here yields huge trees – there is a doubly exponential blow-up in size when building a decomposable comb of shape $c_{k+1}$ from a comb of shape $c_k$. It would be nice to obtain smaller decomposable trees of a given shape and length. More generally, it would be interesting to study lower bound on the size of a decomposable comb of prescribed shape and length.

From the algorithmic point of view, one may wonder about the complexity of deciding if a comb is decomposable. A question of interest is the complexity of this problem parameterized – as defined in [6, 5] – by the number of degree 3 nodes. In particular, it was shown in [2] that it can be decided in polynomial time whether a tree homeomorphic to $K_{1,3}$ is decomposable or not. In fact, it is shown that such a tree is decomposable if and only if it is $\lambda$-decomposable for all partitions $\lambda$ having at most three different sizes of parts – i.e. partitions $\lambda$ of the form $(\lambda_1, \lambda_2, \lambda_3)$. This raises the question whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that a comb of shape $c_k$ is decomposable if and only if it is decomposable by partitions involving at most $f(k)$ different sizes of parts.
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