A Degree Bound on Decomposable Trees

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Abstract

A n-vertex graph is said to be decomposable if for any partition $(\lambda_1, \ldots, \lambda_p)$ of the integer n, there exists a sequence $(V_1, \ldots, V_p)$ of connected vertex-disjoint subgraphs with $|V_i| = \lambda_i$. In this paper, we focus on decomposable trees. We show that a decomposable tree has degree at most 4. Moreover, each degree-4 vertex of a decomposable tree is adjacent to a leaf. This leads to a polynomial time algorithm to decide if a multipode (a tree with only one vertex of degree greater than 2) is decomposable. We also exhibit two families of decomposable trees: arbitrary large trees with one vertex of degree 4, and trees with an arbitrary number of degree-3 vertices.

Key words: Tree decomposition, Integer partition, Computational complexity

1 Introduction

In this paper, we deal with a graph decomposition problem defined in [6] and [15] – see [12,7] for graphs notations and definitions. This problem can be expressed as follows. Consider a n-vertex graph $G = (V, E)$ and let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a partition of n, i.e., a set of positive integers λ_i, called parts, which sum is equal to n. A decomposition of G for λ, called a λ-decomposition, is a partition $\{V_1, \ldots, V_p\}$ of V such that:

(i) for any $1 \leqslant i \leqslant p$, we have $|V_i| = \lambda_i$,

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(ii) the subgraph of G induced by any subset V_i is connected.

In such a decomposition, we say that λ_i covers the vertices of V_i and that λ_i is placed on the subgraph induced by V_i. Given a n-vertex graph and a partition λ of n, the problem of deciding if G is λ-decomposable has been shown to be NP-complete [18] – see [4,14] for definitions about NP-completeness. In this paper, we show that this problem is still NP-complete for trees, even if restricted to trees with maximal degree 3.

The main problem we focus on here is to decide whether a given n-vertex graph G is decomposable, i.e., if it is λ-decomposable for any partition λ of n. Note that a graph G containing a hamiltonian path (a traceable graph) is decomposable, but by just knowing that G is a traceable graph, it is still a difficult problem to give a decomposition of G for any given partition λ of n. From the graph theory point of view, the problem of deciding if a graph is a perfect matching (resp. a quasi-perfect matching) [17] is equivalent to deciding if the graph is $(2, 2, \ldots, 2)$-decomposable (resp. $(2, \ldots, 2, 1)$-decomposable). Thus, any necessary condition for a graph to contain a perfect matching (or a quasi-perfect matching) is a necessary condition for a graph to be decomposable.

Graph decompositions have been intensively studied in various forms for many applications [2,3,8]. The notion of decomposition we focus on here can also be related to the management of parallel systems – parallel computers, networks of workstations, [9,16,19] – used simultaneously by different applications [1,10]. Thus, a connected subnetwork has to be efficiently assigned to each application, that is without as far as possible overlapping communications of two different applications. This is always possible if the topology of the network is decomposable.

In this paper, we focus on decomposable trees. An interest of studying trees is motivated by the fact that if a spanning tree of a graph is decomposable, then so is this graph. Decomposable trees have been studied in previous works [6,15]. In both works, the authors focus on k-podes, that is trees made of k disjoint chains connected by one extremity to a same other vertex called root. In [6] is given a polynomial time algorithm to decide if a given 3-pode is or not decomposable. In [15], the authors prove that decomposable trees have degree at most 6. We improve this bound in this paper by showing that decomposable trees have degree at most 4. In addition, we show that any degree-4 vertex of a decomposable tree is adjacent to a leaf. These results lead to a polynomial time algorithm to decide if any k-pode is decomposable or not, solving a question raised in [6]. Moreover, we exhibit an infinite family of degree-4 decomposable trees, showing that our bound on the degree of decomposable trees is tight.

Notations. The notation $\lambda \vdash n$ means that λ is a partition of the integer n. We note $\lambda = (\lambda_1^{\alpha_1}, \ldots, \lambda_k^{\alpha_k})$ the partition composed of $\sum_{i=1}^k \alpha_i$ parts with α_i parts of size λ_i. Then $\lambda \setminus \lambda_i$ denotes the partition $(\lambda_1^{\alpha_1}, \ldots, \lambda_i^{\alpha_i - 1}, \ldots, \lambda_k^{\alpha_k})$. We define $\max(\lambda) = \max\{\lambda_i, 1 \leqslant i \leqslant k\}$, the biggest part of λ. We call λ a $(q, q+1)$-partition if $\lambda = (q^\alpha, (q+1)^\beta)$ – all parts of λ have size q or $q+1$. The notation $G \rhd \lambda$ means that G is λ-decomposable, and $G \ntriangleright \lambda$ means that G is not λ-decomposable. Given a tree G with n vertices and a set Λ of partitions of n, we say that G is Λ-decomposable and we write $G \rhd \Lambda$ if G is λ-decomposable for all $\lambda \in \Lambda$. A general k-pode is called a multipode, and we note $\mathcal{P}(a_1, \ldots, a_k)$ the k-pode with branch lengths a_i – thus it has $1 + \sum_{i=1}^k a_i$ vertices. At last, (a, b) denotes the gcd of the integers a and b.

Organization of the paper. We prove the upper bound on the degree of a decomposable tree in section 2. Then we exhibit some particular families of decomposable trees in section 3. First we show that there exist decomposable trees with an arbitrary number of degree-3 vertices. Then we show that arbitrary large degree-4 decomposable trees exist. Section 4 is dedicated to the proof of the existence of a polynomial time algorithm to decide if a multipode is decomposable. In section 5, we are interested in the complexity of deciding, given a tree T and a partition λ, if T is λ-decomposable. We show that this problem is NP-complete, even if restricted to trees with maximal degree 3. Some open questions are presented in the last section.

2 Degree of decomposable trees

In this section we prove an upper bound on the degree of a decomposable tree.

Lemma 1 *[5] Let $\mathcal{P}(a_1, \ldots, a_k)$ be a decomposable multipode with $a_i \leqslant a_{i+1}$. Then for all $i \geqslant 3$, we have $a_i \geqslant \sum_{j=1}^{i-1} a_j$.*

Proof. Let $P = \mathcal{P}(a_1, \ldots, a_k)$ and $i \geqslant 2$. Let $n = 1 + \sum_{i=1}^k a_i$. We perform the euclidean division $n = q(a_i + 1) + r$, where $0 \leqslant r \leqslant a_i$. Let π be the partition of n composed of q parts of size $a_i + 1$ and one part of size r (if $r \neq 0$). Of course π must decompose P : let us consider one way for the parts of π to decompose P. First let us suppose that the part r does not cover (at all) the branch a_i. Then a_i is covered by a part of length $a_i + 1$, which also covers the root. This leaves at least two branches of size at most a_i uncovered. This is obviously impossible since there is only one part smaller than $a_i + 1$ in π. So the part r must cover the branch a_i. Therefore, the part covering the root is of size $a_i + 1$ and must also cover all branches smaller than $a_i + 1$. This gives $1 + \sum_{j=1}^{i-1} a_j \leqslant a_i + 1$. $\square$

Lemma 2 *Let $n, q \in \mathbb{N}$ such that $q^2 \leqslant n + 1$. Then there exists a $(q, q+1)$-partition of the integer n.*

Proof. We perform the euclidean division $n = \alpha q + r$ with $0 \leqslant r < q$. Thus $\alpha q + (q-1) \geqslant n \geqslant q^2 - 1$. It follows that $\alpha \geqslant q - 1$. By distributing the remainder r over the parts of size q (notice that there are enough of them), we obtain a $(q, q+1)$-partition of n. $\square$

Lemma 3 *Let $\mathcal{P}(a_1, \ldots, a_k)$ be a decomposable multipode with $a_i \leqslant a_{i+1}$. Let $n = 1 + \sum_{i=1}^{k} a_i$. Then $a_3 \geqslant \lceil \sqrt{n+2} \rceil - 1$. Moreover, if $a_1 > 1$, then $a_2 \geqslant \lceil \sqrt{n+2} \rceil - 1$.*

Proof. If an integer a is such that $(a+1)^2 > n+1$, then $(a+1)^2 \geqslant n+2$. Thus $a \geqslant \sqrt{n+2} - 1$, and as $a \in \mathbb{N}$, we obtain $a \geqslant \lceil \sqrt{n+2} \rceil - 1$.

Let $i = \max\{1 \leqslant j \leqslant k, (a_j + 1)^2 \leqslant n + 1\}$: if no such integer exists, the lemma is proved since in this case $(a_1 + 1)^2 > n + 1$, and if follows that $a_1 \geqslant \lceil \sqrt{n+2} \rceil - 1$. Thus $a_3 \geqslant a_2 \geqslant \lceil \sqrt{n+2} \rceil - 1$.

By lemma 2, there exists a $(a_i + 1, a_i + 2)$-partition of n. Our multipode must be decomposable by this partition. But the part that covers the root must also cover all a_j for $j \leqslant i$, since all parts of the partition have size at least $a_i + 1$. Thus $1 + \sum_{j \leqslant i} a_j \leqslant a_i + 2$. It follows that $\sum_{j<i} a_j \leqslant 1$. Thus $i \leqslant 2$. Then we must have $(a_3 + 1)^2 > n + 1$ and the inequality follows. Now if we suppose that $a_1 \geqslant 2$, since $\sum_{j<i} a_j \leqslant 1$, one must have $i = 1$. Thus we have $a_2 \geqslant \lceil \sqrt{n+2} \rceil - 1$. $\square$

Lemma 4 *Let $P = \mathcal{P}(a, b, c)$ be a tripode with $a \leqslant b \leqslant c$. Let $n = a+b+c+1$. If there exist $i, k \in \mathbb{N}$ such that $\lfloor \frac{n}{k} \rfloor \geqslant a + 1$ and $b + 1 \leqslant i \lfloor \frac{n}{k} \rfloor \leqslant i \lceil \frac{n}{k} \rceil \leqslant a + b$, then P is not decomposable.*

Proof. Let $\pi \vdash n$ be a $(\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil)$-partition (such a partition exists). One must use exactly i parts in order to cover the branch b and the root. Indeed, as $i \lfloor \frac{n}{k} \rfloor \geqslant b + 1$, i parts are enough. Conversely, $(i-1) \lceil \frac{n}{k} \rceil \leqslant a + b - \lceil \frac{n}{k} \rceil \leqslant a + b - (a + 1) < b$, which shows that $i - 1$ parts are not enough. Now, as all parts have size at least $a + 1$, the i-th part that covers the root must also cover the branch a, but this is impossible since $i \lceil \frac{n}{k} \rceil < a + b + 1$. The tripode P is therefore not π-decomposable. $\square$

Lemma 5 *There does not exist a decomposable tripode with n vertices that has its smaller branch greater or equal to $\lceil \sqrt{n+2} \rceil + 1$.*

Proof. Let $T = \mathcal{P}(a, b, c)$ be a tripode with n vertices satisfying $c \geqslant b \geqslant a \geqslant \sqrt{n+2}+1$. If $c < a+b$, the tripode is not decomposable by lemma 1. Now let us suppose that $a+b \leqslant c$. As $a+b+c+1 = n$, we have $a+b < n/2$.

Let $N = \lfloor n/(a+1) \rfloor$. Notice that $n/(N+1) \leqslant n/(\lfloor n/(a+1) \rfloor + 1) \leqslant n/(n/(a+1)) \leqslant a+1$.

Let $\theta = \frac{1}{n}(b + \frac{a}{2})$. Of course $0 < \theta < 1$. By Dirichlet's theorem of diophantine approximation (for example see [11]), there exist $0 \leqslant i \leqslant k \leqslant N$ such that $\left|\theta - \frac{i}{k}\right| \leqslant \frac{1}{k(N+1)}$. Of course one can also suppose that $(i, k) = 1$. If $i = 0$, we would obtain $b + a/2 < a+1$ which is impossible because that would give $a \leqslant b < a/2+1$, but $a \geqslant 3$. In the same way, $i = k = 1$ would give $n - b - a/2 < a+1$ and thus $c < a/2$, a contradiction. Hence $k \geqslant 2$. Now if $i/k \geqslant 1/2$, then $n/2 - b - a/2 < (a+1)/2$. That would lead to $n/2 < b+a+1/2$, then $n/2 \leqslant a+b$, which is once again a contradiction. Thus $i/k < 1/2$ and $k \geqslant 3$.

As $i > 0$, we have both $|in/k - i\lfloor n/k \rfloor| < i$ and $|in/k - i\lceil n/k \rceil| < i$. Thus $|b + a/2 - i\lfloor n/k \rfloor| < n/(k(N+1)) + i$ and $|b + a/2 - i\lceil n/k \rceil| < n/(k(N+1)) + i$. It is clear from lemma 4 that the tripode T is not decomposable as soon as $n/(k(N+1)) + i \leqslant a/2$. As $n/(N+1) \leqslant a+1$, the tripode T is not decomposable as soon as $(a+1)/k + i \leqslant a/2$.

Let us notice that $a(a-2) \geqslant (\sqrt{n+2}+1)(\sqrt{n+2}-1) = n+1$. Thus $k \leqslant N \leqslant n/a \leqslant a-2$. As we know that $i/k < 1/2$, we have $i \leqslant k/2 - 1/2$. Let $M(k) = (a+1)/k + k/2 - 1/2$. Thus $(a+1)/k + i \leqslant M(k)$. Let us now suppose that $a \geqslant 8$. One can check that $M(3) \leqslant a/2$. Moreover $M(a-2) = (a+1)/(a-2) + (a-2)/2 - 1/2 \leqslant a/2$. As $\{x, M(x) \leqslant a/2\}$ is an interval (the graph of M is a convex parabola), it follows that $M(j) \leqslant a/2$ for all $j \in \{3, \ldots, a-2\}$. Thus $(a+1)/k + i \leqslant M(k) \leqslant a/2$ since $k \in \{3, \ldots, a-2\}$. The tripode T is therefore not decomposable.

Let us now suppose that $a \leqslant 7$. As a consequence, $\lceil \sqrt{n+2} \rceil + 1 \leqslant 7$. Thus $n \leqslant 34$. The polynomial time algorithm for tripodes allows us to check that there does not exist any decomposable tripode $\mathcal{P}(a, b, c)$ $(a \leqslant b \leqslant c)$ with n vertices satisfying both $a \geqslant \sqrt{n+2}+1$ and $n \leqslant 34$. □

Lemma 6 *Merging two branches of a decomposable multipode gives a decomposable multipode. Stated differently, if $\mathcal{P}(a_1, \ldots, a_k)$ is decomposable, then for all $i \neq j$, $\mathcal{P}(\{a_1, \ldots, a_k\} \setminus \{a_i, a_j\} \cup \{a_i + a_j\})$ is decomposable too.*

Proof. Let us consider a partition of $1 + \sum a_i$. Let us suppose that we can decompose $\mathcal{P}(a_1, \ldots, a_k)$ following this partition. Then it should be obvious that $\mathcal{P}(\{a_1, \ldots, a_k\} \setminus \{a_i, a_j\} \cup \{a_i + a_j\})$ can also be decomposed following this partition: the branch of length $a_i + a_j$ is now covered by the parts that covered either a_i or a_j (including some pieces of the central part if it covered some vertices of a_i or a_j). $\square$

Theorem 1 *There is no decomposable k-pode for $k \geqslant 5$. All decomposable 4-podes have (at least) a branch of length 1.*

Proof. Let $P = \mathcal{P}(a_1, \ldots, a_k)$ be a decomposable k-pode of size n, with $a_i \leqslant a_{i+1}$. Let $b = \sum_{i=1}^{k-2} a_i$. The tripode $T = \mathcal{P}(b, a_{k-1}, a_k)$ is decomposable by lemma 6. By lemma 1, we have $b \leqslant a_{k-1}$. Now let us suppose that $k \geqslant 5$. By lemma 3, we have $b \geqslant a_1 + a_2 + a_3 \geqslant \lceil \sqrt{n+2} \rceil + 1$. This is in contradiction with lemma 5. Let us now suppose that $k = 4$ and $a_1 \geqslant 2$. We can handle this case in the same way as before with the tripode $T = \mathcal{P}(a_1 + a_2, a_3, a_4)$, noticing that $a_1 + a_2 \geqslant 2 + a_2 \geqslant \lceil \sqrt{n+2} \rceil + 1$. $\square$

Corollary 1 *Decomposable trees have degree a most 4. Moreover, every degree-4 vertex of a decomposable tree is adjacent to a leaf.*

Proof. Let T be a decomposable tree and v a vertex of T with degree $d \geqslant 3$. Let $a_1, a_2, \ldots a_d$ be the sizes of the connected components of $T \setminus \{v\}$. As T is decomposable, the multipode $\mathcal{P}(a_1, \ldots, a_d)$ is obviously decomposable too. Then theorem 1 ends the proof. $\square$

3 Some families of decomposable trees

A chain is of course decomposable. In this section, we are interested in decomposable trees with at least one vertex of degree greater than 2.

It is shown in [6] that there exist infinitely many decomposable tripodes. Indeed, a tripode $\mathcal{P}(1, a, b)$ is decomposable if and only if $(1+a, 2+a+b) = 1$. Hence, there exist decomposable tripodes of size n for $n \in A = \{5\} \cup \{7, \ldots, \infty[$ – one can also check that there does not exist a decomposable tripode (or even a decomposable tree different from a chain) of size $n \notin A$. A decomposable tree with two degree-3 vertices is also exhibited in [6]. We generalize this result in the following proposition.

Proposition 1 *There exist decomposable trees with an arbitrary number of degree-3 vertices.*

Proof. Let us define a comb as a chain $(v_1, \ldots, v_m)$ on which, for each $1 < i < m$, we possibly connect a new vertex to v_i. An oriented comb is a comb whose one of the extremities is marked. Let us denote $\mathcal{C}(N, a_1, \ldots, a_n)$, with $1 < a_1$, $a_i + 1 < a_{i+1}$ and $a_n < N - 1$, the oriented comb defined in the following way. We take a chain $(v_1, v_2, \ldots, v_{N-n})$ composed of $N - n$ vertices and we connect a new vertex to each $v_{a_i-(i-1)}$. At last we give an orientation to it by marking the vertex v_1. Of course each oriented comb can be uniquely written $\mathcal{C}(M, b_1, \ldots, b_l)$. For such an oriented comb P, we define its size M and the set of its break points $R(P) = \{b_1, \ldots, b_l\} \subset \mathbb{N}$.

Let us consider a sequence of prime numbers $(p_i)_{i>0}$ with $p_1 = 3$ satisfying $p_i > \prod_{j<i} p_j + 1$ for all i. For $i > 0$, let $\lambda_i = \prod_{j \leqslant i} p_j$. Let $n \geqslant 1$. We shall build a comb G possessing exactly n vertices of degree 3. Let $k > (\lambda_n + 1)^2$. Let $N = \lambda_n k + 1$. We now define $G = \mathcal{C}(N, \lambda_1, \ldots, \lambda_n)$.

Let us show that G is decomposable. Let A be the part of the graph G composed of the λ_n vertices the closest to the marked vertex plus the tooth in position λ_n. Let π be a partition of N. Our aim is to show that $G \rhd \pi$. Obviously, it is enough to find some parts of π that cover (at least) the part A of G: indeed, the rest of the graph is a chain. Let $a = |A| = \lambda_n + 1$. If a part π_i of π satisfies $\pi_i \geqslant a$, then of course $G \rhd \pi$. In the following, we shall assume that $\pi_i < a$ for all i.

We shall say that a sequence of integers $(\alpha_1, \ldots, \alpha_d)$ covers A in this order if we can cover A (and possibly some additional vertices) by placing α_1 on the marked vertex, then α_2 next to it, and so on. It should be obvious that $(\alpha_1, \ldots, \alpha_d)$ covers A in this order if and only if $\sum_{i=1}^{d} \alpha_i \geqslant a$ and for all $i \leqslant d$ one has $\sum_{j \leqslant i} \alpha_j \notin R(G)$.

Remember that we have supposed that all parts have size at most $a - 1 = \lambda_n$. Thus, the number of parts is at least $N/\lambda_n \geqslant (\lambda_n k + 1)/\lambda_n \geqslant k \geqslant (\lambda_n + 1)^2 \geqslant a\lambda_n$. Thus we have at least a parts of the same size α. Moreover, we can also assume that $\alpha \neq 1$ – otherwise we cover A with these parts and we are done. Let us decompose $\alpha = Q \prod_{i=1}^{e} p_i^{s_i}$ with $p_i \nmid Q$ for all $i \geqslant 1$. If $Q \neq 1$, then a long enough sequence $(\alpha, \ldots, \alpha)$ covers A in this order. Indeed, for all m, we have $m\alpha \notin R(G)$. For the same reason, if there exists i such that $s_i \geqslant 2$, then $m\alpha \notin R(G)$ for all $m \in \mathbb{N}$ and it follows that a long enough sequence $(\alpha, \ldots, \alpha)$ covers A in this order. It remains to deal with the case where $\alpha = \prod_{j=1}^{d} p_{i_j}$ with $i_j < i_{j+1}$.

First case: $i_d > d$. As $p_{i_1} \nmid N$, there exists a part $\beta \in \pi$ such that $p_{i_1} \nmid \beta$. Let us show that a long enough sequence $(\alpha, \beta, \alpha, \alpha, \ldots, \alpha)$ covers A in this order. First $\alpha \notin R(G) = \{\lambda_1, \ldots, \lambda_n\}$. Moreover, for all $j \geqslant 1$, $j\alpha + \beta \notin R(G)$. Indeed, let $j \geqslant 1$ and $z = j\alpha + \beta$. One has $z \geqslant p_{i_1}$ and $z \not\equiv 0 \bmod p_{i_1}$. But $R(G) \cap [p_{i_1}, +\infty[\subset (\prod_{i \leqslant i_1} p_i)\mathbb{N} \subset p_{i_1}\mathbb{N}$ since $p_{i_1} > \lambda_{i_1 - 1} + 1$.

Second case: $i_d = d$. This means that $\alpha = \prod_{i=1}^{d} p_i$. As $p_1 \nmid N$, there is a part $\beta \in \pi$ such that $p_1 \nmid \beta$. Let us show that a long enough sequence $(\beta, \alpha, \alpha, \ldots, \alpha)$ covers A in this order. First $\beta \notin R(G) \subset p_1\mathbb{N}$. Moreover, for all $j \geqslant 1$, one has $j\alpha + \beta \not\equiv 0 \bmod p_1$. It follows that $j\alpha + \beta \notin R(G)$. □

Theorem 1 gives a necessary condition for a degree-4 tree to be decomposable. However, one can wonder if such trees do exist. We show in the next proposition that there exist infinitly many of them.

Lemma 7 *[5] A 4-pode $\mathcal{P}(a, b, 1, 1)$ is decomposable if and only if $\mathcal{P}(a, b, 2)$ is decomposable, a and b are both even, and $a \not\equiv 1 \bmod 3$ or $b \not\equiv 1 \bmod 3$.*

Proof. Let us assume that $\mathcal{P}(a, b, 1, 1)$ is decomposable. Then $\mathcal{P}(a, b, 2)$ is decomposable by lemma 6, and from [6] we know that $(a, b, 1, 1)$ contains at most two odd integers. Thus, a and b are even. Finally, if $a \equiv 1 \bmod 3$ and $b \equiv 1 \bmod 3$, then $\mathcal{P}(a, b, 1, 1)$ is not $(3, \ldots, 3, 2)$-decomposable (it is enough to consider the only possible part covering three vertices including the root in a decomposition). Thus, $a \not\equiv 1 \bmod 3$ or $b \not\equiv 1 \bmod 3$.

Conversely, let $P = \mathcal{P}(a, b, 2)$ be a decomposable tripode, with a and b even, and either $a \not\equiv 1 \bmod 3$ or $b \not\equiv 1 \bmod 3$. Let us assume that P is π-decomposable with $\pi = (\pi_1, \ldots, \pi_k)$, $\pi_1 \geqslant \ldots \geqslant \pi_k$. If $\pi_1 \geqslant 4$, then it is easy to find a π-decomposition of P such that the π_1 part covers all the vertices of the branch of length 2. Thus, $\mathcal{P}(a, b, 1, 1)$ is π-decomposable. We now assume that $\pi_1 \leqslant 3$. If $\pi_1 \leqslant 2$, this decomposition can be obtained from any λ-decomposition with $\lambda = (2, \ldots, 2, 1)$ by splitting some parts. Since a and b are even, $\mathcal{P}(a, b, 1, 1)$ is λ-decomposable so that both the branchs of length a and b are covered by parts equal to 2. So, if $\pi_1 \leqslant 2$, then $\mathcal{P}(a, b, 1, 1)$ is π-decomposable.

It remains to deal with the case where $\pi_1 = 3$. If there exists a π-decomposition of P such that the root is covered by the maximal part of π and such that this part does not cover any vertex of either the branch of length a or the branch of length b, then it is easy to see that in all cases, $\mathcal{P}(a, b, 1, 1)$ is π-decomposable. Hence, let us assume that π_1 covers $V_1 = \{r, v_1, v_2\}$ where r is the root of P and v_1 (resp. v_2) is a vertex on the branch of length a (resp. b). We consider two cases.

First case: $\pi_k = 1$. If π_k covers the vertex on the branch of length 1, then $\mathcal{P}(a, b, 1, 1)$ is clearly π-decomposable. Thus, consider without loss of generality that the vertex covered by π_k is on the branch of length a of P. Then $\mathcal{P}(a, b, 1, 1)$ is π-decomposable in the following way: cover the root and the two branches of length 1 of $\mathcal{P}(a, b, 1, 1)$ with π_1, place the part(s) that covered the branch of length 2 of P on the branch a of $\mathcal{P}(a, b, 1, 1)$, and place π_k on the branch of length b of $\mathcal{P}(a, b, 1, 1)$ – the other parts remain on the branch a or b according to where they were placed on P. This shows that $\mathcal{P}(a, b, 1, 1)$ is π-decomposable.

Second case: $\pi_k = 2$. By hypothesis, without loss of generality, $a \not\equiv 1 \bmod 3$. Then, there exists $\pi_i = 2$ such that the vertex-set V_i covered by π_i is on the branch of length a and there exists $\pi_j = 2$ covering the branch of length 2. Moreover, since b is even, there exists $\pi_m = 3$ such that V_m is on the branch of length b. As in the first case, $\mathcal{P}(a,b,1,1)$ can be shown to be π-decomposable: cover the root and the two branches of length 1 of $\mathcal{P}(a,b,1,1)$ with π_1, place π_i and π_j on the branch b, and place π_m on the branch a.

Thus $\mathcal{P}(a,b,1,1)$ is decomposable. □

For $a \leqslant b \leqslant c$, let $\Lambda(a,b,c)$ be the set of partitions of $a+b+c+1$ of the form $(q^\alpha,(q+1)^\beta)$ with $q \leqslant a+b$ and $\beta > 0$ or of the form $(q^\alpha,(q+1)^\beta,r)$ with $q \leqslant a-2$, $\beta > 0$ and $0 < r \leqslant q-1$. Let us recall the following result, allowing to design a polynomial time algorithm to decide if a tripode is decomposable.

Fact 1 *[6] A tripode $\mathcal{P}(a,b,c)$ with $a \leqslant b \leqslant c$ is decomposable if and only if it is $\Lambda(a,b,c)$-decomposable.*

We can now show that our bound on the degree of decomposable trees given in theorem 1 is tight.

Proposition 2 *The number of decomposable 4-podes is infinite.*

Proof. Let $t \geqslant 3$ such that $t \not\equiv 2 \bmod 3$ and $P_t = \mathcal{P}(2,2t-2,2t)$. Since one of the two integers $2t$ and $2t-2$ is not equivalent to 1 modulo 3, then by lemma 7, the 4-pode $\mathcal{P}(1,1,2t-2,2t)$ is decomposable if and only if the tripode P_t is decomposable. Thus, to prove the proposition, it is enough to show that P_t is decomposable.

By fact 1, it is enough to show that $(q,q+1)$-partitions (without rest r) decompose P_t. If a partition π contains a part equal to 2, then we clearly have $P_t \rhd \pi$. Since $3 \nmid 4t+1$, then $(2,3)$-partitions of n without part equal to 2 do not exist. Thus, we have now to deal with the $(q,q+1)$-partitions where $q \geqslant 3$.

Let π be a $(q,q+1)$-partition made of λ parts equal to q and μ parts equal to $q+1$. If $\lambda+\mu$ is even, then $\lambda \equiv \mu \equiv 1 \bmod 2$ since $n = 4t+1$ is odd. Assume $\lambda = 2\alpha+1$ and $\mu = 2\beta+1$. We have $\alpha q + \beta(q+1) + q = 2t$ since $2(\alpha q + \beta(q+1) + q) = n-1 = 4t$. These parts of size q can cover exactly the branch of length $2t$ of P_t. Thus, $P_t \rhd \pi$. We consider now that $\lambda+\mu$ is odd.

First case: $\lambda = 2\alpha+1$ and $\mu = 2\beta$. Let $x = \alpha q + \beta(q+1)$. Thus, $P_t \rhd \pi$ as soon as $x \leqslant 2t-2$. Indeed, it is enough to place α parts of size q and β parts of size $q+1$ on each branch of length $2t-2$ and $2t$, and it remains to place the last part on the root. Since $4t+1 = n = 2x+q$ then $P_t \rhd \pi$ if $q \geqslant 5$.

We have now to deal with $(3,4)$-decompositions and $(4,5)$-decompositions. We consider first the case $q=4$. We have $n = 4\lambda + 5\cdot 2\beta$, with $n = 4t+1$ odd, a contradiction. Consider now $q=3$. Since $3 \nmid n$, then $\mu > 0$ and thus $\beta \geqslant 1$. So $\alpha\cdot 3 + (\beta-1)\cdot 4 = 2t-5$ since $2(\alpha\cdot 3+(\beta-1)\cdot 4)+3+2\cdot 4 = 4t+1$. Thus,

$\alpha + 1$ parts of size 3 and $\beta - 1$ parts of size 4 can be placed such that the branch of length $2t - 2$ is exactly covered. The remaining part of the 4-pode to be covered is a chain, and thus, $P_t \rhd \pi$.

Second case: $\lambda = 2\alpha$ and $\mu = 2\beta + 1$. Let $x = \alpha q + \beta(q + 1)$. As in the first case, $P_t \rhd \pi$ as soon as $x \leqslant 2t - 2$. Since $4t + 1 = n = 2x + q + 1$ then this is true when $q \geqslant 4$. Thus, we only have to deal with $(3, 4)$-decompositions. But we have $n = 3 \cdot 2\alpha + 4\mu$, with $n = 4t + 1$ odd, a contradiction. $\square$

4 A polynomial time algorithm for multipodes

We present here a polynomial time algorithm to decide whether a multipode is decomposable or not. As in the previous works, the multipode is given as a tree: if it has n vertices, its size is therefore $\Theta(n)$. From the complexity point of view, it would be equivalent to give the branches lengths in unary. Hence, a polynomial time algorithm is an algorithm with time bounded by a polynomial in the number of vertices. We first recall the following lemma.

Lemma 8 *[6] Consider a multipode P and a partition π such that $P \rhd \pi$. Then, there exists a π-decomposition of P such that a part* $\max(\pi)$ *contains the root of P.*

Recall the definition of $\Lambda(a, b, c)$ from the previous section. The next lemma is an adaption of the proof of the main result of [6].

Lemma 9 *If $P = \mathcal{P}(1, a, b, c)$, where $a \leqslant b \leqslant c$, is not π-decomposable with* $\max(\pi) \neq a$ *and* $1 \notin \pi$*, then there exists $\lambda \in \Lambda(a, b, c + 1)$ such that P is not λ-decomposable.*

Proof. We shall only sketch the proof. It consists in building a not decomposing P partition $\lambda \in \Lambda(a, b, c + 1)$ from a not decomposing P partition π by using consecutive reductions. We give here the main steps of such a construction. Note that for a partition π decomposing P with $1 \notin \pi$, the part containing the root also contains the branch of size 1. Thus, we basically have to deal with P being a tripode.

Lemma 2 of the original proof [6] can be expressed here as follows. Let $n = a + b + c + 2$ and $\pi = (\pi_1, \ldots, \pi_k) \vdash n$ with $\pi_k \geqslant a + 1$ (and thus $1 \notin \pi$). Then, $P \rhd \pi$ if and only if there exists a sum s of some parts in $\pi \setminus \pi_1$ such that $2 + a + b - \pi_1 \leqslant s \leqslant b$. This way of expressing this lemma 2 does not change the reduction of π into a $\lambda \in \Lambda(a, b, c + 1)$.

Let $P \not\rhd \pi$ such that $1 \notin \pi$ and $\max(\pi) \neq a$. If $\max(\pi) \leqslant a - 1$, then the reduction given in [6] to deal with the tripode $\mathcal{P}(a, b, c)$ does not change. Thus, we obtain from π a partition $\lambda \in \Lambda(a, b, c+1)$ such that $P \not\triangleright \lambda$. Note that all along this reduction, the greatest part of π never increases, and the smaller part never decreases. Thus, for any partition μ along the reduction from π to λ, $\max(\mu) \leqslant a-1$ et $1 \notin \mu$. This guarantees a ad hoc transformation for the 4-pode P.

If $\max(\pi) \geqslant a+1$ then as in the original proof, we can assume that $\pi_k \geqslant a+1$. The transformation of π to obtain $\lambda \in \Lambda(a, b, c+1)$ such that $P \not\triangleright \lambda$ still works. Indeed, it is enough to note that the smallest part never decreases in this transformation and thus no part equal to a (and so equal to 1) appears in the transformation. $\square$

Lemma 10 *The 4-pode $\mathcal{P}(1, a, b, c)$, with $a \leqslant b \leqslant c$, is decomposable if and only if $\mathcal{P}(1, a, b, c) \triangleright \Lambda(a, b, c+1)$ and $\mathcal{P}(a, b, c)$ is decomposable.*

Proof. Let $n = a+b+c+2$ and suppose that $P = \mathcal{P}(1, a, b, c)$ is decomposable. Of course $\mathcal{P}(1, a, b, c) \triangleright \Lambda(a, b, c+1)$. Let $\pi \vdash n-1$. Thus, $P \triangleright (\pi, 1)$ where $(\pi, 1)$ is the partition of n obtained from π by adding a part of size 1. Moreover, there is a $(\pi, 1)$-decomposition of $\mathcal{P}(1, a, b, c)$ where a part equal to 1 is placed on the branch of length 1. Thus, $\mathcal{P}(a, b, c) \triangleright \pi$. This shows that $\mathcal{P}(a, b, c)$ is decomposable.

Conversely, let us suppose that P is not decomposable. Then, there exists $\pi \vdash n$ such that $P \not\triangleright \pi$. If $1 \in \pi$, then $\mathcal{P}(a, b, c) \not\triangleright \pi \setminus 1$. Let us now assume that $1 \notin \pi$. If $\max(\pi) \geqslant a+1$ or $\max(\pi) \leqslant a-1$, then by lemma 9 there exists $\lambda \in \Lambda(a, b, c+1)$ such that $P \not\triangleright \lambda$. Assume now that $\max(\pi) = a$ and $1 \notin \pi$. If π contains only one part equal to a, then the partition $\tilde{\pi}$, obtained by replacing a by $a-1$ in π, does not decompose $\mathcal{P}(a, b, c)$. Otherwise, we would have $\mathcal{P}(a, 8, c) \triangleright \tilde{\pi}$ with the maximal part (equal to $a-1$) covering the root by lemma 8, and then $P \triangleright \pi$, a contradiction. Hence we can additionaly suppose that π contains at least two parts equal to a.

Thus, $\mathcal{P}(1, b, c) \not\triangleright \pi \setminus a$ with $\max(\pi \setminus a) = a$. Note that if π contains at least one part different from a then $\mathcal{P}(1, b, c) \triangleright \pi \setminus a$. Indeed, let us place the parts in increasing order on the branch of length b on $\mathcal{P}(1, b, c)$. If we obtain a subsum equal to $b+1$ (the only possibility for this decomposition not to decompose $\mathcal{P}(1, b, c)$) we can replace a part less than a (that we can suppose to be the last placed) by a part equal to a (there is still one remaining such part since $c \geqslant a$). This part covers the branch of length 1 and thus $\mathcal{P}(1, b, c) \triangleright \pi \setminus a$, a contradiction. This shows that $\pi = (a^\alpha)$. Thus $\pi \in \Lambda(a, b, c+1)$ with $\mathcal{P}(1, a, b, c) \not\triangleright \pi$. $\square$

Corollary 2 *There is a polynomial time algorithm deciding whether a given multipode is decomposable or not.*

Proof. In [6] is given a polynomial time algorithm dealing with tripodes. From theorem 1, to prove the corollary, it is enough to give a polynomial time algorithm for 4-podes with at least a branch of size 1. This can be easily obtained from lemma 10. □

5 Decomposition following a given partition

What is the complexity of the problem of deciding whether a given tree is decomposable or not? The upper bound we can give is a direct consequence of the definition of a decomposable tree, i.e., this problem is in Π_2. One way to improve this bound would be to show that, given a tree T and a partition λ, deciding if $T \rhd \lambda$ can be performed in polynomial time.

We thus investigate the complexity of the following problem TP. Given a tree T and an integer partition λ, is the tree T λ-decomposable? Problem TP is clearly in NP. We show here that TP_3, i.e., problem TP restricted to trees with maximal degree 3, is NP-complete. As a consquence, problem TP is NP-complete too.

Proposition 3 *Problem* TP_3 *is* NP-*complete.*

Proof. In this proof, a full comb Q_m of length m is the $2m$-vertex tree built as follows. Take a chain $(q_1, \ldots, q_m)$ and link each vertex q_i, with $1 \leqslant i \leqslant m$, to a new vertex q_{i+m}. In the following, "linking a full comb Q_m to a vertex v" means that we link v to q_1 – called the first vertex of Q_m.

Clearly, $\mathrm{TP}_3 \in \mathrm{NP}$. We shall make a polynomial time many-one reduction from X3C (exact cover by 3-sets) to TP_3. Consider an instance $(U_1, \ldots, U_p)$ of $X3C$ with $U_i \subset U = \{1, \ldots, 3n\}$ and $|U_i| = 3$. Remind that such an instance is in $X3C$ if and only if there exist $i_1, \ldots, i_n$ such that $\bigcup_t U_{i_t} = U$. Whitout loss of generality, we assume that $p > n$. Let us now give the construction of a tree T. First, we consider a chain $(c_1, d_1, c_2, d_2, \ldots, c_{p-1}, d_{p-1}, c_p)$. We link each vertex c_j with the extremity of a new chain $(k_j, a_j^1, a_j^2, a_j^3)$, with k_j the vertex linked to c_j. We call this chain (together with the 4 full combs we shall connect to it) the branch of T associated to the set $U_j = \{u_j^1, u_j^2, u_j^3\}$. Let $q = 9n + p$. We now link each vertex k_j to a full comb K_j of length q. At last, we similarly link each vertex a_j^i to a full comb A_j^i of length u_j^i.

The partition we consider is made of $N = 3n+n+1$ parts λ_i defined as follows. For $1 \leqslant i \leqslant 3n$, let $\lambda_i = 2i + 1$. For $3n + 1 \leqslant i \leqslant 3n + n$, let $\lambda_i = 2q + 1$. Finally, the last part is the remaining number of vertices in the tree T, i.e., $\lambda_N = (2p - 1) + (p - n)(2q + 1) + \sum_{j=1}^{p}(2u_j^1 + 2u_j^2 + 2u_j^3 + 3) - \sum_{i=1}^{3n}(2i + 1)$.

If $(U_1, \ldots, U_p) \in$ X3C, then T is λ-decomposable. Indeed, $U = \bigcup_t U_{i_t}$ and thus the n parts equal to $2q+1$ can be placed on the vertices $K_{i_t} \cup \{k_{i_t}\}$, the parts λ_i for $1 \leqslant i \leqslant 3n$ on the vertices $A^m_{i_t} \cup \{a^m_{i_t}\}$, and the remaining part on the remaining vertex set – which induces a connected subtree.

Conversely, assume that $T \rhd \lambda$. Note that all the parts are odd, and that there is no part equal to 1. Thus, the part placed on vertex k_j should also cover each vertex of K_j. Otherwise there would be an isolated vertex with no part placed on it (impossible to be covered) or a not covered full comb; but a full comb can't be covered by one or several odd parts greater or equal to 3. Of course this also holds for any vertex a^i_j: the part placed on a^i_j must also cover all vertices of A^i_j. Consider now a part equal to $2q+1$. Because of its size, it must cover a vertex k_j. By the previous remark, it covers exactly $K_j \cup \{k_j\}$. Thus, the parts equal to $2q+1$ cover the vertices of $K_{i_t} \cup \{k_{i_t}\}$ for all $1 \leqslant t \leqslant n$. Then the part λ_N, because of its size, has to cover all the vertices c_i, d_i and the $p-n$ branches still not covered. We have now to decide where the remaining parts λ_i, for $1 \leqslant i \leqslant 3n$, can be placed. As previously, each still not covered branch can't be covered by more that three parts. Thus each such branch is covered by exactly three parts. Then, each part λ_i covers exactly one $A^m_{i_t} \cup \{a^m_{i_t}\}$. This shows that $U = \bigcup_t U_{i_t}$ and thus that $(U_1, \ldots, U_p) \in$ X3C. $\square$

6 Questions

Many questions remain to investigate regarding properties of decomposable trees. Given a tree, let us define its structure. It is obtained by deleting every degree-2 vertex, and each time, pasting both parts – otherwise stated, pieces of chains are replaced with one edge only. One can wonder whether every reasonable degree-4 structure is the structure of a decomposable tree – by reasonable, we mean that at least one of the four neigbours of a degree-4 vertex is a leaf. If this is not the case, can we give a characterization of the structures that correspond to decomposable trees? Another interesting question would be to give a tight lower bound on the number of vertices of a decomposable tree containing a least k degree-3 vertices.

From an algorithmic point of view, the main question adressing the polynomial time decidability of decomposable trees is still open. If it is not polynomial time decidable, another interesting question would be to know if this problem is fixed parameter tractable [13], with as parameter the number of vertices of degree at least 3.

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