Université Joseph Fourier
L2 MAT241 MIN-INT
2013-2014

Examination

All documents are allowed.

Answers to be returned before March 13th, 2014 at 10h00

First part

Let E be a finite dimensional vector space over $\mathbb{R}$.

1. Let $q : E \to \mathbb{R}$ be a positive quadratic form. By using the symmetric bilinear form $\phi : E \times E \to \mathbb{R}$ associated to q, prove the following formula

$$\forall\, x, y \in E, \quad q(x+y) + q(x-y) = 2(q(x) + q(y)).$$

2. By using the result obtained in the previous question, prove that the isotopic cone

$$H = \{x \in E \,|\, q(x) = 0\}$$

is a vector subspace of E.

3. Prove that, for any $x \in H$ and any $y \in E$ one has $\phi(x, y) = 0$. One can use the Cauchy-Schwarz inequality for example.

4. Let $\langle\, , \,\rangle$ be a scalar product on E and $u : E \to E$ be an auto-adjoint endomorphism such that $q(x) = \langle x, u(x)\rangle$. Prove that for all x and y in E one has

$$\phi(x, y) = \langle x, u(y)\rangle.$$

5. Deduce that H identifies with the kernel of u.

Second part

Let $E = \mathbb{R}^3$ be equipped with the usual scalar product $\langle\, , \,\rangle$ such that

$$\langle x, y\rangle := x_1x_2 + y_1y_2 + z_1z_2.$$

Consider the quadratic form $q : E \to \mathbb{R}$ such that

$$q(x) := \frac{1}{3}(x_2^2 - x_3^2 + 4x_1x_2 - 4x_1x_3).$$

6. Determine the symmetric bilinear form ϕ associated to the quadratic form q and write its matrix A with respect to the canonical basis.

7. Determine the kernel of the bilinear form ϕ. Is the quadratic form q non-degenerated (namely the kernel of ϕ is $\{0\}$) ?

8. Determine the rank and the signature of the quadratic form q.

9. Write the quadratic form q into the sum of squares of linearly independent linear forms. Find an orthogonal basis of the bilinear form ϕ.

10. Let $B = A^2$. Prove that B is the matrix of a projection, namely the endomorphism $p : E \to E$ associated to B verifies the relation $p^2 = p$. Prove that p is an orthogonal projection.

11. Find the image and the kernel of the endomorphism $p : E \to E$ associated to B and determine an orthonormal basis of E such that B is diagonalized under this basis.

TSVP

Third part

In this exercice, we consider a matrix of size $n \times n$ with coefficients in $\mathbb{R}$. Let H be the matrix $A^T A$. We equip $\mathbb{R}^n$ with the usual scalar product which sends $(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n$ to $X^T Y$.

12. By using a result of the course, prove that the vector space $\mathbb{R}^n$ has an orthonormal basis which consisits of the eigenvectors of H.

13. Let $X \in \mathbb{R}^n$ be a vector. Prove that $X^T H X \geqslant 0$. Deduce that all eigenvalues of H are non-negative.

14. In and only in this question, we assume that

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -1 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$

Compute H, and then find the eigenspaces of H. Determiner a basis of $\mathbb{R}^3$ consisting of eigenvectors of H.

15. Verify that the trace of H equals to the sum of all eigenvalues of H (where we count the multiplicity).

16. Deduce that the function $\langle \, , \, \rangle : M_n(\mathbb{R}) \times M_n(\mathbb{R})$, which sends (M, N) to $\mathrm{Tr}(M^T N)$, is a scalar product, where $M_n(\mathbb{R})$ denotes the vector space of all matrix of size $n \times n$ with coefficients in $\mathbb{R}$.

17. We say that a matrix M is symmetric (resp. antisymmetric) if $M^T = M$ (resp. $M^T = -M$). Prove that, if M is symmetric and if N is antisymmetric, then $\langle M, N \rangle = 0$.

18. Let $S_n(\mathbb{R})$ the set of all symmetric matrices in $M_n(\mathbb{R})$. Prove that $S_n(\mathbb{R})$ is a vector subspace of $M_n(\mathbb{R})$ and that $S_n(\mathbb{R})^\perp$ is the set of all antisymmetric matrices.

19. Prove that the map $\pi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, which sends M to $\frac{1}{2}(M + M^T)$, is the orthogonal projection onto $S_n(\mathbb{R})$.

20. For the matrix A in Question **14**, compute the distance of the matrix A and the vector subspace $S_n(\mathbb{R})$.