Documents, calculators and cell phones are not allowed.
A particular attention will be paied to the concision and the lucidity of writing the answer.
All answers should be justified. The mark standards are indicative.

Questions about the course (8 pts)

1) State and prove the Cauchy-Schwarz inequality.

2) Let E be a vector space over $\mathbb{R}$, $q : E \to \mathbb{R}$ be a quadratic form on E and φ be the symmetric bilinear form associated to q. Prove that two vectors u and v of E are φ-orthogonal if and only if $q(u+v) = q(u) + q(v)$.

3) Reminder : Let $(E, \langle\cdot,\cdot\rangle)$ be an Euclidean vector space and F be a vector subspace of E. Then one has a decomposition of E into an orthogonal direct sum $E = F \oplus F^{\perp}$. Hence for any vector $v \in E$ there exists a unique vector $p(v) \in F$ such that $v - p(v) \in F^{\perp}$. The map $p : v \in E \mapsto p(v) \in F$ is $\mathbb{R}$-linear and is called the orthogonal projection from E to F.

a) Prove that for any $u \in F$ one has $\|v - p(v)\|^2 + \|p(v) - u\|^2 = \|v - u\|^2$, where $\|.\|$ denotes the Euclidean norm on E.

b) Deduce that $\mathrm{dist}(v, F) = \|v - p(v)\|$, where $\mathrm{dist}(v, F) = \inf_{u \in F} \|v - u\|$.

Exercice 1 (2 pts)

Determine the type of the quadratic surface Q in $\mathbb{R}^3$ of equation

$$x^2 + 2xy + 2xz + 2y^2 + 2z^2 = 1.$$

Exercice 2 (12 pts) Let $E = \mathbb{R}^3$ and φ be the symmetric bilinear form on $E \times E$ whose matrix in the canonical basis $\mathcal{E} = (e_i)_{i=1,2,3}$ of E is as follows :

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1) Express φ in coordinates and determine the associated quadratic form q.

2) Prove that φ defines a scalar product.

In what follows, we denote by $\langle\cdot,\cdot\rangle$ this scalar product and consider the Euclidean space $(E, \langle\cdot,\cdot\rangle)$.

Reminder : The angle $\alpha \in [0, \pi]$ between two non-zero vectors $u, v \in E$ is defined by $\cos\alpha = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$.

3) Find the angle between the vectors e_1 and e_2 of the basis $\mathcal{E}$ in $(E, \langle\cdot,\cdot\rangle)$.

4) Let $f : E \to \mathbb{R}$ be the linear form defined as $f(x, y, z) = 2x - y + 3z$. Find a vector $v \in E$ such that for any $u \in E$ one has $f(u) = \langle u, v\rangle$ (*Riesz representation* of f).

5) Determine the vector subspace $\mathrm{Ker}\, f$ and justify the relation $E = \mathrm{Ker}\, f \oplus \mathrm{Vect}(v)$.

6) Determine a vector v_0 of norm 1 which generates $(\mathrm{Ker}\, f)^{\perp}$. Deduce that $v = f(v_0)v_0$.

7) Determine a basis of $\mathrm{Ker}\, f$. Find an orthonormal basis $\mathcal{C} = (v_0, v_1, v_2)$ of $(E, \langle\cdot,\cdot\rangle)$ such that $\mathrm{Ker}\, f = \mathrm{Vect}(v_1, v_2)$ by using the Gram-Schmidt method.

8) Express the orthogonal projection p of E on $\mathrm{Ker}\, f$ and write the matrix of p in the basis $\mathcal{E}$.
Find the image $p(e_1)$ and the distance $\mathrm{dist}(e_1, \mathrm{Ker}\, f)$.

9) Express the orthogonal symmetry s of E writh respect to $\mathrm{Ker}\, f$ and write the matrix of s in the basis $\mathcal{E}$.

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MAT241 2013-2014 Short answer to the examination of March 18th, 2014

About the course :

1) Se the course.

2) By definition, one has $\varphi(u,v) = \frac{1}{2}(q(u+v)-q(u)-q(v))$. Hence $\varphi(u,v)=0$ if and only if $q(u+v)-q(u)-q(v)=0$.

3a) One has $v-u = (v-p(v)) + (p(v)-u)$, where $v-p(v) \in F^{\perp}$ and $p(v)-u \in F$ are orthogonal. It remains to apply the theorem of Pythagore.

3b) By the result of the question a), for any $u \in F$ one has $\|v-u\| \leq \|v-p(v)\|$, and the equality holds when $u = p(v)$, which lead to the assertion.

Exercice 2 : The Gauss reduction formula gives

$$x^2 + 2xy + 2xz + 2y^2 + 2z^2 = (x+y+z)^2 + y^2 + z^2\,,$$

hence Q is an ellipsoid.

Exercice 3 :

1) For two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ one has

$$\varphi(x,y) = xAy^t = 2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 + x_3y_3\,,$$

and hence $q(x) = \varphi(x,x) = 2x_1^2 - 2x_1x_2 + 2x_2^2 + x_3^2$.

2) The Gauss reduction of q gives $q(x) = 2(x_1 - \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 + x_3^2$. Hence the signature of φ is $(3,0)$, φ is positively defined, hence defines a scalar product.

3) One has : $\langle e_1, e_2\rangle = \varphi(e_1,e_2) = -1$, $\langle e_1, e_1\rangle = q(e_1) = 2$, et $\langle e_2, e_2\rangle = q(e_2) = 2$. Hence the angle α between e_1 and e_2 verfies $\cos\alpha = \dfrac{\langle u,v\rangle}{\|u\|\cdot\|v\|} = -\dfrac{1}{2}$, namely $\alpha = \dfrac{2\pi}{3}$.

4) If $u = (x,y,z)$ and $v = (a,b,c)$ then by the formula in the question 1), the identity $f(u) = \langle u, v\rangle$ can be written in coordinates as $2x - y + 3z = 2ax - bx - ay + 2by + cz$. Thus $2a - b = 2$, $2b - a = -1$, and $c = 3$. Hence $v = (1,0,3)$.

5) By the definition of the kernel of a linear map and by 4) we obtain :

$$\mathrm{Ker}f = \{u = (x,y,z) \in \mathbb{R}^3 \mid 2x - y + 3z = 0\} = \{u \in \mathbb{R}^3 \mid \langle u, v\rangle = 0\} = (\mathrm{Vect}(v))^{\perp}\,.$$

Moreover, for any vector subspace F of E, one has $F^{\perp\perp} = F$ and $E = F \oplus F^{\perp}$. For $F = \mathrm{Ker}f$ one obtains $(\mathrm{Ker}f)^{\perp} = \mathrm{Vect}(v)$ and $E = \mathrm{Ker}\,f \oplus \mathrm{Vect}(v)$.

6) By the previous question, one can choose $v_0 = v/||v||$, where $||v|| = \sqrt{q(v)} = \sqrt{11}$. By using the equality $f(u) = \langle u, v\rangle$ we obtain $f(v_0)v_0 = \dfrac{1}{11}f(v)v = \dfrac{1}{11}\langle v,v\rangle v = v$.

7) We can choose non-proportional vectors $u_1 = (1,2,0)$ and $u_2 = (0,3,1)$ in $\mathrm{Ker}f$ to form a basis of $\mathrm{Ker}f$. On uses the Gram-Schmidt method to find an orthonormal basis :

$$v_1 = u_1/\|u_1\| = u_1/\sqrt{6} \quad \text{and} \quad v_2 = \frac{u_2 - \langle u_2, v_1\rangle v_1}{\|u_2 - \langle u_2, v_1\rangle v_1\|} = \frac{1}{\sqrt{22}}v_2'\,,$$

where $v_2' = (3,0,-2)$. Hence $\mathcal{C} = (v_0, v_1, v_2)$ is an orthonormal basis of E since $v_0 \perp \mathrm{Ker}f$ and $\|v_0\| = 1$.

8) By the course, for $u \in E$ one has$p(u) = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2$. In particular,

$$p(e_1) = \frac{3}{11}(3,0,-2), \quad p(e_2) = \frac{1}{11}(1,11,3), \quad p(e_3) = \frac{-1}{11}(3,0,-2)\,.$$

Hence

$$P = \mathrm{Mat}_{\mathcal{E}}(p) = \frac{1}{11}\begin{pmatrix} 9 & 1 & -3 \\ 0 & 11 & 0 \\ -6 & 3 & 2 \end{pmatrix}$$

(one can verify that $P^2 = P$).

One has : $\mathrm{dist}(e_1, \mathrm{Ker}f) = \|e_1 - p(e_1)\| = \sqrt{q(e_1 - p(e_1))} = \frac{2}{\sqrt{11}}$ où $e_1 - p(e_1) = \frac{1}{11}(2,0,6)$.

9) By the course, the orthogonal symmetry s of E with respect to $\mathrm{Ker}\,f$ can be written as $s = 2p - \mathrm{id}_E$. Hence

$$S = \mathrm{Mat}_{\mathcal{E}}(s) = 2P - I_3 = \frac{1}{11}\begin{pmatrix} 7 & 2 & -6 \\ 0 & 11 & 0 \\ -12 & 6 & -7 \end{pmatrix}$$

(we can verify that $S^2 = I_3$ is the identity matrix).

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