Université Joseph Fourier
L2 MAT241 MIN-INT
2013-2014

Examination of May 5th, 2014

Documents and electronic equipments are not allowed
Duration : 2 hours

First part

Let E be an Euclidean space.

1. State the Cauchy-Schwarz inequality for the Euclidean product of E and precise the condition under which the inequality becomes an equality.

2. Recall the definition of auto-adjoint endomorphisms of E. Prove that, if $u : E \to E$ is an auto-adjoint endomorphism and if x and y are eigenvectors associated to different eigenvalues, then x and y are orthogonal.

3. Let $u : E \to E$ be an endomorphism and u^* be its adjoint. Prove that there exists an orthonomal basis of E consisiting of eigenvectors of $u^* \circ u$.

4. Let $u : E \to E$ be an endomorphism of E. We assume that $u^* \circ u = u \circ u^*$. Prove that any eigenspace of $u^* \circ u$ is stable by the action of u.

Second part

We equip $\mathbb{R}^2$ with the usual scalar product. Recall that an endomorphism $u : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be *orthogonal* if u^* is the inverse u. In the following, we fix an orthogonal endomorphism u of $\mathbb{R}^2$ and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be its matrix under the canonical basis of $\mathbb{R}^2$. We denote by A^T the transposition of the matrix A and by $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the identity matrix of size 2×2.

5. Prove that $A^T A = AA^T = I$.

6. Deduce that there exists $\theta \in \mathbb{R}$ such that $a = \cos(\theta)$ and $b = \sin(\theta)$.

7. Prove that the determinant of A is either 1 or -1.

8. Assume that $\text{dét}\,(A) = 1$. Determine the values of c and d and represent the matrix A as a function of θ, which we denote by A_θ in what follows. What is the nature of u in this case ?

9. Assume that $\text{dét}\,(A) = -1$. Prove that the matrix A can be written in the form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_\theta.$$

What is the nature of u in this case ?

10. Let $v : \mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism such that $v^* \circ v = \lambda \, \mathrm{Id}_{\mathbb{R}^2}$, where λ is a non-negative number. Determine the matrix of v under the canonical basis in terms of λ and A_θ (for some $\theta \in \mathbb{R}$).

Third part

We consider the following quadratic form on $\mathbb{R}^3$:

$$q(x, y, z) = x^2 + xy + xz - yz.$$

11. Determine the polar form ϕ of q.

12. Write q into a linear combination of squqres of linearly independent linear forms. Determine its rank and signature.

13. Determine the nature of the quadratic surface

$$\{(x, y, z) \in \mathbb{R}^3 \,|\, q(x, y, z) = 1\}.$$

14. Determine a basis of $\mathbb{R}^3$ with is orthogonal with respect to the bilinear form ϕ.

Fourth part

Let E and F be two finite dimensional vector spaces over $\mathbb{R}$, equipped with scalar products $\langle\, ,\, \rangle_E$ and $\langle\, ,\, \rangle_F$ respectively. Denote by $\|.\|_E$ and $\|.\|_F$ the corresponding Euclidean norms on E and on F respectively.

Let $f : E \to F$ be a linear map. Recall that the adjoint of f is defined as the linear map $f^* : F \to E$ such that

$$\forall\, x \in E,\, \xi \in F, \quad \langle f^*(\xi), x\rangle_E = \langle \xi, f(x)\rangle_F.$$

15. Prove that $\mathrm{Ker}(f) \subset \mathrm{Ker}(f^* \circ f)$.

16. Show that if $f^*(f(x)) = 0$ then $\|f(x)\|_F = 0$. Deduce that $\mathrm{Ker}(f) = \mathrm{Ker}(f^* \circ f)$

17. Prove the equality $\mathrm{Im}(f^*) = \mathrm{Im}(f^* \circ f)$

18. Prove that the eigenvalues of $f^* \circ f$ are positive real numbers.

19. Show that there exists an orthonormal basis of E which consisits of eigenvectors of $f^* \circ f$.

20. Assume that $\dim(F) = 1$. Determine a polynomial P of degree 2 which satisfies $P(f^* \circ f) = 1$. Determine the eigenspaces of $f^* \circ f$.