Bilinear algebra M241

Licence 2, DLST, Université Joseph Fourier, 2014

Documents **not** allowed

Final examination, 2 hours

A particular attention will be paied to the lucidity of writing the answer.

One should specify the hypotheses of statement used in the answer. The results of the course are allowed in the answer. However, when these results are used, one should write down clearly their hypotheses and conclusions.

Exercice 1

Let E be an Euclidean space. An endomorphism $u : E \to E$ is said to be *normal* if it commutes with its adjoint, namely if $u \circ u^* = u^* \circ u$.

1. Let u be an endomorphism of E. Write (without proof) the matrix of u^* in an orthonormal basis of E in terms of the matrix of u in the same basis.
2. (a) Prove that any orthogonal endomorphism is normal.
   (b) Prove that any auto-adjoint endomorphism is normal.
3. In this question, we consider the vector space $\mathbb{R}^2$ equipped with the canonical Euclidean space structure. Prove that the endomorphism of $\mathbb{R}^2$, whose matrix in the canonical basis is given by $M = \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$, is normal but neither orthogonal nor auto-adjoint.
4. We suppose in this question that the dimension of E is 2. Prove that, if u is a normal endomorphism and if the characteristic polynomial of u has a real root, then there exists an orthonormal basis of E consisting of eigenvectors of u. (*Indication : given an eigenvector of u, try to prove that there exists an orthonormal basis in which the matrix of u is upper triangular.*)
5. Let v be an arbitrary endomorphisme of E.
   (a) Prove that $v^* \circ v$ is diagonalisable in an orthonormal basis and that any of its eigenvalue is non-negative.

(b) Prove that if v is an invertible endomorphism, then the eigenvalues of $v^* \circ v$ are positve.

6. Suppose that E is of dimension 2. Let u be an endomorphism. Suppose that u is normal and that the characteristic polynomial of u does not have any real root.

   (a) Prove that, for certain orthonormal basis $\mathcal{B}$ of E, the matrix $\mathrm{Mat}_{\mathcal{B}}(u)$ of u in the basis $\mathcal{B}$ is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b = -c$ and $a = d$.

   (b) Deduce that u is the composition of a rotation with a dilatation $h = \lambda \mathrm{Id}$.

Exercice 2

We consider the following quadratic form on $\mathbb{R}^3$:

$$q(x, y, z) = x^2 + 2y^2 - 2xy - 2yz.$$

1. Let ϕ_q be the polar form of q. Recall the nature of this object, give its formula in the coordinates and write its matrix M in the canonical basis.
2. Decompose q into a linear combination of linearly independent linear forms (decomposition of Gauss). Precis each linear form in the decomposition.
3. Deduce a basis in $\mathbb{R}^3$ in which the quadatic form q becomes diagonal. Is it a basis of diagonalization of the matrix M ?
4. Determine the signature of q. Is q degenerate ? Does it have a non-trivial isotropic cone ? Justify your answer.
5. Determine the type of the quadratic surface

$$\mathcal{Q} = \{(x, y, z) \in \mathbb{R}^3 \,|\, q(x, y, z) = 1\}.$$

   Draw a brief picture of it (indicate the axes if possible).
6. Prove that

$$\mathcal{Q} = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x - y + z)(x - y - z) = (1 - y + z)(1 + y - z)\}.$$

7. Choose $\lambda \neq 0$. What is the geometric nature of the set $\mathcal{D}_\lambda$ of points (x, y, z) satisfying simultaneously $(x - y + z) = \lambda(1 - y - z)$ and $(x - y - z) = \frac{1}{\lambda}(1 + y + z)$?
8. Prove that $\mathcal{D}_\lambda \subset \mathcal{Q}$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. Deduce that $\mathcal{Q}$ is the union of a family of disjoint lines.