Université Joseph Fourier
MAT241 Bilinear Algebra (MIN-INT)
2013-2014

Exercice 1

Question 1 Let $V = \mathbb{R}^3$ and $(e_i)_{i=1}^3$ be the canonical basis of V. Denote by $(e_i^\vee)_{i=1}^3$ the dual basis of $(e_i)_{i=1}^3$. Write the following linear form as a linear combination of vectors in the dual basis.

(1) $\alpha(x, y, z) = 3x + 4y + 5z$.

(2) The differential at $(0, 0, 0)$ of the function $f(x, y, z) = \exp(x + y + z)$.

(3)
$$\beta(x, y, z) = (1, 1, 1)A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Question 2 Let V be the vector space of all polynomials in $\mathbb{R}[T]$ of degree $\leqslant 2$. Let $e_1 = 1$, $e_2 = 1 + T$ and $e_3 = 1 + T + T^2$.

(1) Prove that $(e_i)_{i=1}^3$ is a basis of the vector space V.

(2) Prove that $I : V \to \mathbb{R}$, $I(f) = \int_0^1 f(t)\,\mathrm{d}t$ is a linear form on V.

(3) Write I as a linear combination of the vectors in the dual basis of $(e_i)_{i=1}^3$.

Question 3 Let V be a finite dimensional vector space over $K = (\mathbb{R}$ or $\mathbb{C})$. Prove that the map from V to $V^{\vee\vee}$, which sends $x \in V$ to the linear form $\alpha \mapsto \alpha(x)$ on $V^\vee$, is a K-linear isomorphism.

Question 4 Are the following functions bilinear forms, symmetric bilinear forms ?

(1) $\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x, y) = x_1x_2 + y_1y_2$.

(2) $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\varphi(x, y) = x_1y_2 + x_2y_1$.

(3) $\varphi : \mathbb{R}[T] \times \mathbb{R}[T] \to \mathbb{R}$, $\varphi(P, Q) = P'(1)Q(0) + P'(0)Q(1)$.

(4) $\varphi : C^0([0, 1]) \times C^0([0, 1]) \to \mathbb{R}$, $\varphi(f, g) = \int_0^1 f(t)g(1 - t)\,\mathrm{d}t$.

(5) $\varphi : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$, $\varphi(A, B) = \mathrm{tr}(AB)$.

Question 5 (Oral test question) Let V be a vector space over $K = (\mathbb{R}$ or $\mathbb{C})$ and W be a vector subspace of V.

(1) Denote by $\sim$ the binary relation on V defined as

$$x \sim y \Longleftrightarrow x - y \in W.$$

Prove that it is an equivalence relation.

(2) For any element $x \in V$, denote by $[x]$ the equivlence class of x with respect to the equivalence relation $\sim$. Let V/W be the quotient set $V/\sim$. Prove that the maps

$$\begin{array}{ccc} (V/W) \times (V/W) & \longrightarrow & V/W \\ ([x],[y]) & \longmapsto & [x+y] \end{array}$$

and

$$\begin{array}{ccc} K \times (V/W) & \longrightarrow & V/W \\ (a,[x]) & \longmapsto & [ax] \end{array}$$

are well defined.

(3) Prove that the set V/W equipped with the above maps forms a vector space over K and that the projection $\pi : V \to V/W$, $\pi(x) = [x]$ is a K-linear map.

(4) Determine the kernel of π.

(5) Prove that $\pi^\vee : (V/W)^\vee \to V^\vee$ is an injective map.

(6) Let $\eta : W \to V$ be the inclusion map of W into V. Prove that $\eta^\vee$ is a surjective map.

(7) Let $\Delta : V \times V^\vee \to \mathbb{R}$ be the bilinear map defined as $\Delta(x,\alpha) = \alpha(x)$. Denote by $W_\Delta^\perp$ the set of all $\alpha \in V^\vee$ such that $\alpha(x) = 0$ for any $x \in W$. Prove that $W_\Delta^\perp$ identifies with the image of $\pi^\vee$ and the kernel of $\eta^\vee$.

(8) Assume that V is finite dimensional. Prove that $\dim(W_\Delta^\perp) = \dim(V) - \dim(W)$.

Question 6 Determine the matrix of the following bilinear forms with respect to the basis assigned.

(1) $\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x,y) = x_1y_2 + x_2y_1 + x_1y_1 - x_2y_2$, canonical basis.

(2) $\varphi : \mathbb{R}[T]_2 \times \mathbb{R}[T]_2 \to \mathbb{R}$, $\varphi(P,Q) = \int_0^1 P(t)Q(t)dt$, the basis $\{1, T, T^2\}$.

Question 7 Let $\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the bilinear form defined as

$$\varphi(x,y) = x_1y_1 - \frac{1}{2}x_2y_1 + \frac{3}{2}x_1y_2 - x_2y_2.$$

Is this bilinear form symmetric ? If it is not the case, please decompose φ into the sum of a symmetric bilinear form and a anti-symmetric bilinear form.

Question 8 Let $E = M_n(\mathbb{R})$ be the vector space of square matrices of size $n \times n$ and $\varphi : E \times E \to \mathbb{R}$ defined as $\varphi(A,B) = \mathrm{tr}(AB)$.

(1) Is the bilinear form φ symmetric ?

(2) Determine the kernel and the dimension of the bilinear form φ.

(3) Let I be the identity matrix. Determine the vector space $\{I\}^\perp$.