**Université Joseph Fourier**
**MAT241 Bilinear Algebra (MIN-INT)**
**2013-2014**

# Exercice 2

**Question 1** Transform the following quadratic forms into the sum of squares of independent linear forms. Determiner an orthogonal basis and the signature of each quadratic form.

(1) $q : \mathbb{R}^2 \to \mathbb{R}$, $q(x) = 5x_1^2 + 2x_1x_2 + 5x_2^2$.

(2) $q : \mathbb{R}^3 \to \mathbb{R}$, $q(x) = (x_1 + x_2 + x_3)^2 + (x_1 + 2x_2 + x_3)^2 + (x_1 + x_3)^2$.

(3) $q : \mathbb{R}^3 \to \mathbb{R}$, $q(x) = 7x_1x_2 + 8x_1x_3 + 4x_2x_3$.

(4) $q : \mathbb{R}[T]_2 \to \mathbb{R}$, $q(P) = P(2)P(1) + P(1)P(0)$.

**Question 2** Let $E = \mathbb{R}[T]_n$. We consider the map $q : E \to \mathbb{R}$ defined as

$$q(P) = \int_0^1 P(t)P'(t)\,\mathrm{d}t$$

(1) Prove that $q$ is a quadratic form.

(2) Write $q$ as the difference of squares of two linearly independent linear forms.

(3) Compute the kernel and the rank of the polar form of $q$ (namely the symmetric bilinear form associated to $q$).

(4) Determine an orthogonal basis for the polar form of $q$.

(5) Determine the isotopic cone of $q$.

**Question 3** Let $V = \mathbb{R}^n$ and let $\ell_1$ and $\ell_2$ be two non-zero linear forms on $V$. Let $q : V \to \mathbb{R}$ be the map defined as

$$q(x) = \ell_1(x)\ell_2(x).$$

(1) Prove that $q$ is a quadratic form on $V$.

(2) Prove that the kernel of $q$ equals $\mathrm{Ker}(\ell_1) \cap \mathrm{Ker}(\ell_2)$.

(3) Determine all possible values of the signatures of the quadratic form $q$.

**Question 4** Let $q$ be a quadratic form on a finite dimensional vector space $V$ over $\mathbb{R}$, $f_1, \ldots, f_p$ be linear forms on $V$ and $\alpha_1, \ldots, \alpha_p$ be real numbers. Assume that $q$ can be written as

$$q = \alpha_1 f_1^2 + \cdots + \alpha_p f_p^2.$$

(1) Prove that $\mathrm{rk}(f_1, \ldots, f_p) \geqslant \mathrm{rk}(q)$.

(2) Prove that, if $f_1, \ldots, f_p$ are linearly independent and if $\alpha_1, \ldots, \alpha_p$ are non-zero, then the rank of the quadratic form $q$ is equal to $p$.

**Question 5 (Oral test question)** Let $n$ and $p$ be two integers in $\mathbb{Z}_{\geqslant 1}$. Let $A$ be a matrix in $M_{n,p}(\mathbb{R})$ and $B$ be the matrix $AA^T$.

(1) Prove that the matrix $B$ is symmetric.

(2) Let $q : \mathbb{R}^n \to \mathbb{R}$ be the quadratic form defined as

$$q(x) = (x_1, \cdots, x_n)B\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Prove that $q$ is positive.

(3) Determine the kernel of the quadratic form $q$.

(4) Determine the signature of $q$ in terms of the rank of $A$.

(5) Detremine the signature of the quadratic form

$$q_1 : \mathbb{R}^n \to \mathbb{R}, \quad q_1(x) = \sum_{1 \leqslant i,j \leqslant n} x_i x_j.$$

(6) Determine the signature of the quadratic form

$$q_2 : \mathbb{R}^n \to \mathbb{R}, \quad q_2(x) = \sum_{1 \leqslant i,j \leqslant n} ij x_i x_j.$$

**Question 6** Let $V = \mathbb{R}[T]_n$ be the vector space of all polynomials of degree $\leqslant n$. For $P$ and $Q$ in $V$, we define

$$\varphi(P, Q) = \int_0^1 P'(t)Q'(t)\,\mathrm{d}t$$

(1) Prove that $\varphi$ is a symmetric bilinear form on $V$.

(2) Let $q$ be the quadratic form associated to $\varphi$. Prove that $q$ is positive.

(3) Determine the isotropic cone of $q$.

(4) Prove that the kernel of $\varphi$ is equal to the isotropic cone of $q$.

(5) Determine the rank of $\varphi$.

(6) Let $f : V \to \mathbb{R}$ be the map defined as

$$f(P) := \int_0^1 P'(t)\,\mathrm{d}t.$$

Prove that $f$ is a linear form on $V$.

(7) Let $q_1$ be the quadratic form on $V$ defined as

$$q_1(P) := q(P) - f(P)^2.$$

Prove that

$$q_1(P) = \int_0^t (P'(t) - f(P))^2\,\mathrm{d}t.$$

(8) Determine the signature of $q_1$.