Université Joseph Fourier
MAT241 Bilinear Algebra (MIN-INT)
2013-2014

Exercice 3

Question 1 Determine the nature of the solution sets of the following equations and reprensent them on the plane $\mathbb{R}^2$.

(1) $x^2 + xy + y^2 = 1$

(2) $x^2 + 4xy + y^2 = 1$

(3) $x^2 + xy + y^2 = -2$

(4) $x^2 - 2xy + y^2 = 1$

Question 2 Prove that the parametrized curve $x = \cos t$, $y = \cos t + \sin t$ ($t \in \mathbb{R}$) defines an ellipse. Determine its equation.

Question 3 Determine the nature of the solution sets of the following equations.

(1) $x^2 + 9y^2 + 4z^2 - 6xy - 12yz + 4xz = -4$

(2) $xy + xz + yz = -1$

(3) $x^2 + y^2 + z^2 - 2xy + 2xz + 3x - y + z = -1$

(4) $2x^2 + 2y^2 + z^2 + 2xz - 2yz + 4x - 2y - z = -3$

(5) $(x - y)(y - z) + (y - z)(z - x) + (z - x)(x - y) + (x - y) = 0$

Question 4 (Oral test question) Let V be a finite dimensional vector space over $\mathbb{R}$.

(1) Let $\langle\, ,\, \rangle$ be a scalar product on V and $\|.\| : V \to [0, +\infty[$ be the function defined as

$$\|x\| := \langle x, x \rangle^{1/2}.$$

Prove that, for any couple (x, y) of vectors in V, one has

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(2) Let $\phi : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V and $q : V \to \mathbb{R}$ be the quadratic form on V defined as $q(x) := \phi(x, x)$. Prove that, for any $(x, y) \in V \times V$, one has

$$q(x + y) + q(x - y) = 2(q(x) + q(y)).$$

(3) Let

$$\lambda := \sup_{0 \neq x \in V} \frac{q(x)}{\langle x, x \rangle} \quad \text{and} \quad W := \{y \in V \mid q(y) = \lambda \|y\|^2\}.$$

Prove that W is stable by multiplication by a scalar.

(4) Prove that W is stable by addition. Deduce that W is a vector subspace of V.

Question 5 (Oral test question) In this exercice, we consider $\mathbb{C}$ as a vector space of dimension 2 over $\mathbb{R}$.

(1) Prove that the map $\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, $\phi(z, w) = \mathrm{Re}(\overline{z}w)$ is a scalar product on $\mathbb{C}$.

(2) Denote by $|.| : \mathbb{C} \to \mathbb{R}$ the norm defined as $|z| := \phi(z, z)^{1/2}$. Prove the following inequality :

$$\forall\, z, w \in \mathbb{C}, \quad \big||z| - |w|\big| \leqslant |z - w|.$$

(3) Prove that, for all complex numbers z and w, one has

$$|1 - \overline{w}z|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2).$$

(4) Let D be the subset of $\mathbb{C}$ consisting of complex numbers z such that $|z| < 1$. Prove that, for all complex numbers z and w in D, the complex number

$$T_w(z) := \frac{w - z}{1 - \overline{w}z}$$

is well defined and lies in D.

(5) Let w be an element in D. Denote by $T_w : D \to D$ the map defined above. Determine $T_w^2(.) := T_w(T_w(.))$ and prove that T_w is a bijection.

Question 6 Are the following quadratic form positive ? Do they define scalar products on the corresponding vector spaces ?

(1) $q : \mathbb{R}^3 \to \mathbb{R}$, $q(x, y, z) = x^2 + y^2 + 2z(x \cos\alpha + y \sin\alpha)$, where $\alpha \in \mathbb{R}$ is a parameter.

(2) $q : \mathbb{R}^4 \to \mathbb{R}$, $q(x, y, z, t) = x^2 + 3y^2 + 4z^2 + t^2 + 2xy + xt$.

(3) $q : \mathbb{R}^n \to \mathbb{R}$,

$$q(x_1, \ldots, x_n) = a \sum_{i=1}^{n} x_i^2 + b \sum_{i \neq j} x_i x_j,$$

where a and b are two parameters.

(4) $q : \mathbb{R}[T]_n \to \mathbb{R}$,

$$q(P) = \sum_{k=0}^{n} P(k)^2.$$

Question 7 Find an orthonormal basis of $\mathbb{R}[T]_3$ with respect to the scalar product

$$\langle P, Q\rangle := \int_{-1}^{1} P(t)Q(t)\,\mathrm{d}t.$$

Question 8 We equip $\mathbb{R}^n$ with the standard scalar product. Denote by H the hyperplane

$$\{(x_1, \ldots, x_n) \,|\, x_1 + \cdots + x_n = 0\}.$$

(1) Determine an othonormal basis of H with respect to the restriction of the standard scalar product.

(2) For any point $P = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, determine the distance between P and H.

(3) Determine the matrix of the orthogonal projection of $\mathbb{R}^n$ onto H (considered as an endomorphism of $\mathbb{R}^n$) with respect to the canonical basis.

(4) Determine $H^\perp$.