

§1 Linear forms

We fix a commutative ring A

Definition Let M and N be two A -modules. We denote by $\text{Hom}_A(M, N)$ the set of all A -linear maps from M to N . The set $\text{Hom}_A(M, A)$ is also denoted by M^* , whose elements are called A -linear forms over M .

Observation If $f: M \rightarrow N$ and $g: M \rightarrow N$ are two A -linear maps, then

$f+g: M \rightarrow N$ $(f+g)(x) := f(x) + g(x)$ is also an A -linear map

- If $f: M \rightarrow N$ is an A -linear map, and $a \in A$. then

$af: M \rightarrow N$ $(af)(x) = a f(x)$ is also an A -linear map

- The set $\text{Hom}_A(M, N)$ equipped with the above composition laws forms an A -module.

Proposition Let M be an A -module. Assume that M admits a basis

$(e_i)_{i=1}^n$ of finite cardinal. For any $j \in \{1, \dots, n\}$, let $e_j^*: M \rightarrow A$ be the A -linear map defined as

$$e_j^*(a_1 e_1 + \dots + a_n e_n) := a_j$$

Then $(e_j^*)_{j=1}^n$ forms a basis of M^* . (called the dual basis of $(e_i)_{i=1}^n$)

Proof: The maps e_j^* are well defined since any element $x \in M$ can be written in a unique way as A -linear combination of e_1, \dots, e_n .

By definition, e_j^* are A -linear maps.

If $\alpha: M \rightarrow A$ is an A -linear form over M , we claim that

$$\alpha = \sum_{j=1}^n \alpha(e_j) \cdot e_j^*$$

In fact, for any $x = a_1 e_1 + \dots + a_n e_n \in M$. one has

$$\sum_{j=1}^n \alpha(e_j) e_j^*(x) = \sum_{j=1}^n \alpha(e_j) a_j = \alpha(a_1 e_1 + \dots + a_n e_n) = \alpha(x).$$

Moreover, if b_1, \dots, b_n are elements in A such that $\beta := b_1 e_1^* + \dots + b_n e_n^* = 0$ then one has

$$\forall i \in \{1, \dots, n\} \quad 0 = \beta(e_i) = \sum_{j=1}^n b_j e_j^*(e_i) = b_i. \quad \text{So } (e_j^*)_{j=1}^n \text{ is a basis of } M^*.$$

Corollary Let K be a field and E be a vector space over K . Page 2

Suppose that E is of finite type. Then $\text{rg}_K(E) = \text{rg}_K(E^\vee)$

Proposition Let M and N be two A -modules. We assume that M has a basis $(x_i)_{i=1}^n$, and N has a basis $(y_j)_{j=1}^m$. Then, for any $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$, the map $x_i^\vee \otimes y_j$ sending $\sum_{k=1}^n a_k x_k$ ($a_1, \dots, a_n \in A$) to $a_i y_j$ is an A -linear map from M to N .

Moreover, $(x_i^\vee \otimes y_j)_{\substack{i \in \{1, \dots, n\} \\ j \in \{1, \dots, m\}}}$ is a basis of $\text{Hom}_A(M, N)$.

Proof One has $(x_i^\vee \otimes y_j) \left(\sum_{k=1}^n a_k x_k + \sum_{k=1}^n b_k x_k \right) = (a_i + b_i) y_j$

$$= a_i y_j + b_i y_j$$

$$(x_i^\vee \otimes y_j) \left(\lambda \sum_{k=1}^n a_k x_k \right) = (\lambda a_i) y_j = \lambda (a_i y_j) \quad (\lambda \in A)$$

Therefore $x_i^\vee \otimes y_j : M \rightarrow N$ is an A -linear map.

If $(c_{ij})_{\substack{i \in \{1, \dots, n\} \\ j \in \{1, \dots, m\}}}$ are elements in A such that $\sum_{i=1}^n \sum_{j=1}^m c_{ij} (x_i^\vee \otimes y_j) = 0$

Then for any $k \in \{1, \dots, n\}$

$$0 = \sum_{i=1}^n \sum_{j=1}^m c_{ij} (x_i^\vee \otimes y_j)(x_k) = \sum_{j=1}^m c_{kj} y_j$$

Since $(y_j)_{j=1}^m$ is a basis of N , one has $c_{kj} = 0$ for any j .

Therefore $c_{ij} = 0$ for any $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$.

Finally, for any $f \in \text{Hom}_A(M, N)$ one can write each $f(x_i)$ into the form

$$f(x_i) = \lambda_{i1} y_1 + \dots + \lambda_{im} y_m$$

Then one has $f = \sum_{i=1}^n \sum_{j=1}^m \lambda_{ij} (x_i^\vee \otimes y_j)$ since for $x = \sum_{k=1}^n a_k x_k$

in M , one has

$$\sum_{i=1}^n \sum_{j=1}^m \lambda_{ij} (x_i^\vee \otimes y_j)(x) = \sum_{i=1}^n \sum_{j=1}^m \lambda_{ij} a_i y_j = \sum_{i=1}^n a_i f(x_i) = f(x)$$

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Corollary Let E and F be two vector spaces over a field

K , which are of finite type. Then

$$\text{rk}_K(\text{Hom}_K(E, F)) = \text{rk}_K(E) \cdot \text{rk}_K(F)$$

Example $(A^n)^V = \{(a_1, \dots, a_n) \mid a_i \in A\}$ is the dual A -module of A^n . In fact, one can consider (a_1, \dots, a_n) as an A -linear form on A^n which sends a column vector $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ to

$$(a_1, \dots, a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \dots + a_n b_n$$

§2 Bilinear maps

Definition Let M , N and P be three A -modules. We call A -bilinear map from $M \times N$ to P any map $f: M \times N \rightarrow P$ which is A -linear in each of the coordinates, namely

$$\forall a, b \in A, \quad x, x' \in M, \quad y, y' \in N,$$

$$f(ax + bx', y) = a f(x, y) + b f(x', y)$$

$$f(x, ay + by') = a f(x, y) + b f(x, y')$$

Observation ① If $f: M \times N \rightarrow P$ and $g: M \times N \rightarrow P$ are A -bilinear maps, then $f+g: M \times N \rightarrow P$

$$(f+g)(x, y) := f(x, y) + g(x, y)$$

is an A -bilinear map

② If $f: M \times N \rightarrow P$ is an A -bilinear map, and $a \in A$. Then $af: M \times N \rightarrow P$

$$(af)(x, y) := a f(x, y)$$

is an A -bilinear map.

(3) The set $\text{Bil}_A(M \times N, P)$ of all A -bilinear maps forms an A -module with respect to the above composition laws

(4) For each $f \in \text{Bil}_A(M \times N, P)$, one can construct an A -linear map from M to $\text{Hom}_A(N, P)$ which sends $x \in M$ to the A -linear map $f(x, \cdot) : N \rightarrow P$
 $y \mapsto f(x, y)$.

Thus we obtain an A -linear map

$$\text{Bil}_A(M \times N, P) \longrightarrow \text{Hom}_A(M, \text{Hom}_A(N, P))$$

This is actually an A -linear isomorphism. The inverse of this map is given by the map from $\text{Hom}_A(M, \text{Hom}_A(N, P))$ to $\text{Bil}_A(M \times N, P)$ sending $\underline{\Phi} \in \text{Hom}_A(M, \text{Hom}_A(N, P))$ to the A -bilinear map
 $((x, y) \in M \times N) \longmapsto (\underline{\Phi}(x))(y)$.

In the case where M, N and P are three vector spaces of finite type over a field K , one has

$$\begin{aligned} \text{rk}_K(\text{Bil}_K(M \times N, P)) &= \text{rk}_K(\text{Hom}_K(M, \text{Hom}_K(N, P))) \\ &= \text{rk}_K(M) \cdot \text{rk}_K(\text{Hom}_K(N, P)) = \text{rk}_K(M) \cdot \text{rk}_K(N) \cdot \text{rk}_K(P). \end{aligned}$$

Definition Let M be an A -module. We call A -bilinear form over M any element in $\text{Bil}_A(M \times M, A)$.

If $\varphi : M \times M \rightarrow A$ is an A -bilinear form, we denote by φ^τ the A -bilinear form on M sending $(x, y) \in M \times M$ to $\varphi(y, x)$.

If $\varphi = \varphi^\tau$, we say that the A -bilinear form is symmetric.

If $\varphi = -\varphi^\tau$, we say that the A -bilinear form is anti-symmetric.

By definition, one has $(\varphi^\tau)^\tau = \varphi$.

The symmetric/anti-symmetric forms consist of sub- A -modules of $\text{Bil}_A(M \times M, A)$

Proposition Assume that $2=1+1$ is invertible in A .

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Then any A -bilinear form φ on an A -module M can be decomposed in a unique way into the sum of two A -bilinear forms φ^s and φ^a where φ^s and φ^a are respectively symmetric and anti-symmetric A -bilinear forms over M .

Proof Existence Let $\varphi^s = \frac{1}{2}(\varphi + \varphi^\tau)$, $\varphi^a = \frac{1}{2}(\varphi - \varphi^\tau)$

As $(\varphi^\tau)^\tau = \varphi$, one has

$$(\varphi^s)^\tau = \frac{1}{2}(\varphi + \varphi^\tau)^\tau = \frac{1}{2}(\varphi^\tau + \varphi) = \varphi^s$$

$$(\varphi^a)^\tau = \frac{1}{2}(\varphi - \varphi^\tau)^\tau = \frac{1}{2}(\varphi^\tau - \varphi) = -\varphi^a$$

Therefore φ^s is symmetric and φ^a is anti-symmetric

Moreover, one has $\varphi = \varphi^s + \varphi^a$.

Uniqueness Assume that φ has another decomposition $\varphi = \psi + \eta$ with ψ symmetric and η anti-symmetric. Then one has

$$\psi - \varphi^s = -\eta + \varphi^a \text{ is symmetric and anti-symmetric}$$

Lemma Assume that $2=1+1$ is invertible in A . If ϕ is an A -bilinear form on M which is symmetric and anti-symmetric, then $\phi = 0$

Proof For $(x, y) \in M \times M$ one has

$$\phi(y, x) = \phi(x, y) \quad \text{and} \quad \phi(y, x) = -\phi(x, y)$$

Therefore $2\phi(x, y) = 0$. Since 2 is invertible, $\phi(x, y) = 0$

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