

§ 3 Matrix of a bilinear form

In this section, we fix a commutative ring A and an A -module M . We assume that M admits a basis $(e_i)_{i=1}^n$.

Definition Let $\varphi: M \times M \rightarrow A$ be a bilinear form. We call the **matrix of φ** with respect to the basis $\mathbf{e} = (e_i)_{i=1}^n$ the matrix

$$B_{\varphi}^{\mathbf{e}} := \begin{pmatrix} \varphi(e_1, e_1) & \cdots & \varphi(e_1, e_n) \\ \varphi(e_2, e_1) & \cdots & \varphi(e_2, e_n) \\ \vdots & \ddots & \vdots \\ \varphi(e_n, e_1) & \cdots & \varphi(e_n, e_n) \end{pmatrix}$$

which is of size $n \times n$ and with coefficients in A .

Proposition Let $x = a_1 e_1 + \cdots + a_n e_n$ and $y = b_1 e_1 + \cdots + b_n e_n$ be two elements in M . One has

$$\varphi(x, y) = (a_1, \dots, a_n) B_{\varphi}^{\mathbf{e}} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Proof One has

$$B_{\varphi}^{\mathbf{e}} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \varphi(e_1, e_1) + \cdots + b_n \varphi(e_1, e_n) \\ b_1 \varphi(e_2, e_1) + \cdots + b_n \varphi(e_2, e_n) \\ \vdots \\ b_n \varphi(e_n, e_1) + \cdots + b_n \varphi(e_n, e_n) \end{pmatrix} = \begin{pmatrix} \varphi(e_1, y) \\ \vdots \\ \varphi(e_n, y) \end{pmatrix}$$

Therefore

$$(a_1, \dots, a_n) B_{\varphi}^{\mathbf{e}} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = (a_1, \dots, a_n) \begin{pmatrix} \varphi(e_1, y) \\ \vdots \\ \varphi(e_n, y) \end{pmatrix}$$

$$= a_1 \varphi(e_1, y) + \cdots + a_n \varphi(e_n, y) = \varphi(x, y)$$

Observation φ is symmetric $\Leftrightarrow (B_{\varphi}^{\mathbf{e}})^{\tau} = B_{\varphi}^{\mathbf{e}}$
 φ is anti-symmetric $\Leftrightarrow (B_{\varphi}^{\mathbf{e}})^{\tau} = -B_{\varphi}^{\mathbf{e}}$

$$\# \left(B_{\varphi^{\tau}}^{\mathbf{e}} = (B_{\varphi}^{\mathbf{e}})^{\tau} \right)$$

Change of basis Let $f = \{f_i\}_{i=1}^n$ be another basis of M .

We assume that $f_j = \sum_{i=1}^n P_{ij} e_i$

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and denote by P the matrix

$$\begin{pmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{pmatrix}$$

If x is an ~~vector~~ element in M which is written as

$$x = a_1 e_1 + \dots + a_n e_n = \lambda_1 f_1 + \dots + \lambda_n f_n$$

Then one has $a_i = \sum_{j=1}^n \lambda_j P_{ij}$ for any $i \in \{1, \dots, n\}$

namely $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = P \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ or $(a_1, \dots, a_n) = (\lambda_1, \dots, \lambda_n) P^T$

Similarly, if $y \in M$ which is written as

$$y = b_1 e_1 + \dots + b_n e_n = \mu_1 f_1 + \dots + \mu_n f_n$$

then $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = P \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$

Therefore the proposition shows that

$$\begin{aligned} \varphi(x, y) &= (a_1, \dots, a_n) B_\varphi^e \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \\ &= (\lambda_1, \dots, \lambda_n) P^T B_\varphi^e P \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \\ &= (\lambda_1, \dots, \lambda_n) B_\varphi^f \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \end{aligned}$$

Hence

$$B_\varphi^f = P^T B_\varphi^e P$$

§4 Universal bilinear map, duality.

In this section, we fix a commutative ring A

Definition Let M be an A -module. We denote by μ_M the bilinear map from $M^\vee \times M$ to A which sends $(\alpha, x) \in M^\vee \times M$ to $\mu_M(\alpha, x) := \alpha(x)$.

If $f: M \rightarrow N$ be an A -linear map between A -modules, we denote by $f^\vee: N^\vee \rightarrow M^\vee$ the A -linear map sending $\beta \in N^\vee$ to the A -linear form $x \mapsto \beta(f(x))$ on M .

In other words, f^\vee is characterized by the following relation

$$\forall x \in M, \quad \forall \beta \in N^\vee \quad \mu_N(\beta, f(x)) = \mu_M(f^\vee(\beta), x).$$

f^\vee is called the dual A -linear map of f .

Computation of the dual map

Proposition Let M and N be two A -modules. We assume that M admits a basis $(x_i)_{i=1}^m$ and N admits a basis $(y_j)_{j=1}^n$.

Let $(x_i^\vee)_{i=1}^m$ and $(y_j^\vee)_{j=1}^n$ be the dual bases of $(x_i)_{i=1}^m$ and

$(y_j)_{j=1}^n$ respectively. Let $f: M \rightarrow N$ be an A -linear map and

$C = (c_{ji})_{\substack{j \in \{1, \dots, n\} \\ i \in \{1, \dots, m\}}}$ be the matrix of f under the basis $(x_i)_{i=1}^m$

of M and $(y_j)_{j=1}^n$ of N

(namely $\forall i \in \{1, \dots, m\} \quad f(x_i) = c_{1i}y_1 + \dots + c_{ni}y_n$)

Then the matrix of f^\vee under the dual basis $(y_j^\vee)_{j=1}^n$ of N^\vee and

$(x_i^\vee)_{i=1}^m$ of M^\vee is equal to C^T

Proof For $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ one has

$$(f^v(y_j^v))(x_i) = y_j^v(f(x_i)) = y_j^v(c_{i1}y_1 + \dots + c_{in}y_n) = c_{ji}$$

Therefore

$$f^v(y_j^v) = c_{j1}x_1^v + \dots + c_{jm}x_m^v.$$

✱

The universal bilinear map permits to consider the elements in M as linear forms over M^v . In fact, one has

$$\begin{aligned} \text{Bil}_A(M^v \times M, A) &\cong \text{Hom}_A(M, \text{Hom}_A(M^v, A)) \\ &= \text{Hom}_A(M, (M^v)^v). \end{aligned}$$

We denote by $J_M: M \rightarrow (M^v)^v$ the A -linear map

sending $x \in M$ to the linear form $(\alpha \in M^v) \mapsto \alpha(x)$.

Definition If J_M is an A -linear isomorphism, we say that M is a reflexive A -module.

Proposition Assume that M admits a basis $(e_i)_{i=1}^n$. Then M is a reflexive A -module. In particular, any vector space of finite type over a field is reflexive.

Proof Let $(e_i^v)_{i=1}^n$ be the dual basis of $(e_i)_{i=1}^n$ and $((e_i^v)^v)_{i=1}^n$ be the dual basis of $(e_i^v)_{i=1}^n$ in $(M^v)^v$.

Then the map $J_M: M \rightarrow (M^v)^v$ sends e_i to $(e_i^v)^v$.

Thus it is an A -linear isomorphism.

✱

In this section, we fix a field K .

Let E and F be K -vector spaces of finite type, and

$\varphi: E \times F \rightarrow K$ be a K -bilinear map. Then φ induces K -linear maps

$$f_\varphi: E \rightarrow F^\vee \quad g_\varphi: F \rightarrow E^\vee$$

$$x \mapsto (y \mapsto \varphi(x, y)) \quad y \mapsto (x \mapsto \varphi(x, y))$$

Proposition The K -linear maps f_φ and g_φ have the same rank.

Recall that the rank of a K -linear map is defined as the rank of its image

Lemma Assume that both K -linear maps f_φ and g_φ are injective

Then one has $\text{rk}_K(E) = \text{rk}_K(F)$

Proof. f_φ is injective $\Rightarrow \text{rk}_K(E) \leq \text{rk}_K(F^\vee) = \text{rk}_K(F)$

g_φ is injective $\Rightarrow \text{rk}_K(F) \leq \text{rk}_K(E^\vee) = \text{rk}_K(E)$ $\#$

Proof of the proposition

Let $E_1 = \text{Ker}(f_\varphi) \subset E$, $F_1 = \text{Ker}(g_\varphi) \subset F$.

For $x \in E_1$ and $y \in F$ one has $\varphi(x, y) = 0$

For $x \in E$ and $y \in F_1$ one has $\varphi(x, y) = 0$

Therefore φ induces a K -bilinear map $\tilde{\varphi}: (E/E_1) \times (F/F_1) \rightarrow K$

such that $\tilde{\varphi}([x], [y]) = \varphi(x, y)$

Note that $\text{Ker}(f_{\tilde{\varphi}}) = \{0\}$ and $\text{Ker}(g_{\tilde{\varphi}}) = \{0\}$

By the Lemma, one has $\text{rk}_K(E/E_1) = \text{rk}_K(F/F_1)$.

Therefore $\text{rk}_K(f_\varphi) = \text{rk}_K(E) - \text{rk}_K(E_1)$

$$= \text{rk}_K(F) - \text{rk}_K(F_1) = \text{rk}_K(g_\varphi) \quad \#$$

Definition The value $\text{rk}_K(f_\varphi) = \text{rk}_K(g_\varphi)$ is called the rank of the bilinear map $\varphi: E \times F \rightarrow K$.