# Geometry of arithmetic varieties

Lecture Hall, Jia Yi Bing Building, 82# Jing Chun Yuan

BICMR, Peking University

# Schedule

## Monday September 2nd

09:00 – 10:15   Reception
10:15 – 11:15   **Marc Hindry** (Université de Paris)
       *On the size of generators of the Mordell-Weil group of an abelian variety*
11:15 – 11:30   Tea break
11:30 – 12:30   **Ke Chen** (Nanjing University)
       *On CM points away from the Torelli locus*
12:30 – 14:30   Lunch
14:30 – 15:30   **Tong Zhang** (East China Normal University)
       *Linear system on arithmetic varieties fibered by curves*
15:30 – 15:45   Tea break
15:45 – 16:45   **Hideaki Ikoma** (International Buddhist University)
       *Differentiability of the arithmetic volume function along the base conditions*

## Tuesday September 3rd

09:00 – 10:00   **Anna Cadoret** (Sorbonne Université & Kyoto University)
       *Tannakian Cebotarev density theorem*
10:00 – 10:15   Tea break
10:15 – 11:15   **Yongqiang Zhao** (Westlake University)
       *A Galois theoretic perspective of scrollar syzygy theory*
11:15 – 11:30   Tea break
11:30 – 12:30   **Yang Cao** (Leibniz Universität, Hannover)
       *Arithmetic purity of strong approximation for linear algebraic groups*
12:30 – 14:30   Lunch
14:30 – 15:30   **Yongqi Liang** (University of Science and Technology of China)
       *Non-invariance of strong and weak approximation properties under extension of the ground field*
15:30 – 15:45   Tea break
15:45 – 16:45   **Fei Xu** (Capital Normal University)
       *Strong approximation with Brauer-Manin obstruction for a family of norm varieties*

## Wednesday September 4th

09:00 – 10:00   **Huilin Zhu** (Xianmen University)
       *On pure ternary exponential Diophantine equations*
10:00 – 10:15   Tea break
10:15 – 11:15   **Mounir Hajli** (Shanghai Jiaotong University)
       *A new formula for the Mahler measure*
11:15 – 11:30   Tea break
11:30 – 12:30   **Tamás Szamuely** (Universita degli studi di Pisa)
       *A generalization of Beilinson's geometric height pairing*
12:30 – 14:30   Lunch

# Schedule

**Thursday September 5th**

09 : 00 – 10 : 00   **Zhizhong Huang** (Leibniz Universität, Hannover)
*Distribution of Mordell-Weil ranks in elliptic fibrations*

10 : 00 – 10 : 15   Tea break

10 : 15 – 11 : 15   **Fabien Pazuki** (University of Copenhagen)
*Regulators of number fields and abelian varieties*

11 : 15 – 11 : 30   Tea break

11 : 30 – 12 : 30   **Per Salberger** (Chalmers University of Technology)
*Counting rational points with the determinant method*

12 : 30 – 14 : 30   Lunch

14 : 30 – 15 : 30   **Yu Yasufuku** (Nihon University)
*The smallness of the GCD in Higher Dimensions*

15 : 30 – 15 : 45   Tea break

15 : 45 – 16 : 45   **Yong Hu** (Southern University of Science and Technology)
*Rost kernels of division algebras over complete discrete valuation fields*

**Friday September 6th**

09 : 00 – 10 : 00   **Yanbo Fang** (Université de Paris)
*Metric extension for sections on semipositive line bundle over non-Archimedean fields*

10 : 00 – 10 : 15   Tea break

10 : 15 – 11 : 15   **Klaus Künnemann** (Universität Regensburg)
*Positive forms and currents on toric varieties*

11 : 15 – 11 : 30   Tea break

11 : 30 – 12 : 30   **Jose Ignacio Burgos Gil** (Instituto de Ciencias Matemáticas)
*Convex analysis on polyhedral spaces*

12 : 30 – 14 : 30   Lunch

14 : 30 – 15 : 30   **Victoria Cantoral Farfán** (Katholieke Universiteit Leuven)
*The Mumford-Tate conjecture implies the algebraic Sato-Tate conjecture*

15 : 30 – 15 : 45   Tea break

15 : 45 – 16 : 45   **Atsushi Moriwaki** (Kyoto University)
*Arithmetic dynamical system and Dirichlet's property*

# Abstracts

**Jose Ignacio Burgos Gil** (Instituto de Ciencias Matemáticas)
*Convex analysis on polyhedral spaces*

Convex functions have many good properties. For instance, if a sequence of convex functions converge pointwise in a dense subset of an open subset $U$, then they converge in the whole $U$ and the convergence is uniform on each compact subset of U. In this talk we will propose various notions of convexity on balanced quasi-embedded polyhedral spaces (to be defined during the talk) and show that they share the strong continuity properties of classical convex functions. We will also discuss the existence of Monge Ampere type measures for such functions.

As an application of the theory we show that any semipositive toroidal $b$-divisor on an algebraic variety is integrable.

This is join work in progress with A. Botero and M. Sombra.

**Anna Cadoret** (Sorbonne Université & Kyoto University)
*Tannakian Cebotarev density theorem*

Let $\mathcal{C}_0$ be either a lisse Weil $\overline{\mathbb{Q}}_\ell$-sheaf, an almost $\mathfrak{u}$-tame sheaf or an overconvergent $\overline{\mathbb{Q}}_p$-$F$-isocrystal on a smooth, separated, geometrically connected scheme $X_0$ over a finite field. We prove the following Tannakian Cebotarev density theorem : let $S \subset |X_0|$ be a subset of closed points of upper Dirichlet density $> 0$. Then the Zariski-closure of the union of the corresponding conjugacy classes of Frobenii contains at least one connected component of the Tannakian group of $\mathcal{C}_0$. The key point is to use the weight filtration to reduce to the case of semisimple $\mathbb{Q}$-coefficients and then apply the theory of companions to reduce to the case of $\overline{\mathbb{Q}}_\ell$-coefficients, where the assertion is an easy consequence of the (classical) Cebotarev density theorem.

**Victoria Cantoral Farfán** (Katholieke Universiteit Leuven)
*The Mumford-Tate conjecture implies the algebraic Sato-Tate conjecture*

The famous Mumford—Tate conjecture asserts that, for every prime number $\ell$, Hodge cycles are $\mathbb{Q}_\ell$ linear combinations of Tate cycles, through Artin's caparisons theorems between Betti and étale cohomology. The algebraic Sato-Tate conjecture, introduced by Serre and developed by Banaszak and Kedlaya, is a powerful tool in order to prove new instances of the generalized Sato-Tate conjecture. This previous conjecture is related with the equidistribution of Frobenius traces. Our main goal is to prove that the Mumford-Tate conjecture for an abelian variety $A$ implies the algebraic Sato-Tate conjecture for $A$. The relevance of this result lies mainly in the fact that the list of known cases of the Mumford-Tate conjecture was up to now a lot longer than the list of known cases of the algebraic Sato-Tate conjecture.

This is a joint work with Johan Commelin.

**Yang Cao** (Leibniz Universität, Hannover)
*Arithmetic purity of strong approximation for linear algebraic groups*

Strong approximation with Brauer-Manin obstruction is defined by Colliot-Thélène and Xu to study the local-global for the integral points. For a semi-simple, simply connected $k$-simple linear algebraic group $G$, it is conjectured that $G$ satisfies the propriety of purity : the complement of any

codimension $\geqslant 2$ closed subset satisfies strong approximation. We prove this conjecture for $k$-isotropic groups by an analogue of fibration method and for Spin groups by using the density of rational points with almost prime polynomial values. This is joint work with Zhizhong Huang.

**Ke Chen** (Nanjing University)
*On CM points away from the Torelli locus*

Using properties of the stable Faltings height and the Sato-Tate equidistribution for CM abelian varieties, we show that rational points in the Hecke orbits of certain type of CM points in the Siegel modular variety do not lie in the open Torelli locus when the genus is large enough. In other words certain CM structure cannot be realized on Jacobians of curves of sufficiently high genus. This is a joint work with X. Lu and K. Zuo.

**Yanbo Fang** (Université de Paris)
*Metric extension for sections on semipositive line bundle over non-Archimedean fields*

For a projective variety $X$ defined over a non-Archimedean complete (non-trivially) valued field, and an ample line bundle $L$ equipped with a semipositive metric, we consider the problem of metric extension for sections of $L^{\otimes n}$ from a closed subvariety $Y$ to $X$. Namely, for any given restricted section on $Y$, find an extended section on $X$ with smallest possible distorsion of supremum norm. This problem is equivalent to the comparison of a quotient norm and a supremum norm on the algebra of restricted sections. We improve the result of Chen and Moriwaki in obtaining a uniform asymptotic bound.

We form normed algebras of sections and study their Berkovich spectra, relating them to the dual unit disc bundle of L with respect to the envelop metric. There are two methods for this norm comparison : one exploits the holomorphic convexity, another relies on special properties of affinoid algebras. Work based on `arXiv: 1904.03696`

**Mounir Hajli** (Shanghai Jiaotong University)
*A new formula for the Mahler measure*

Let $f$ be a polynomial with coefficients in a number field. In this talk, I shall show that the Mahler measure of $f$ can be a approximated by a sequence of real numbers given in terms of the coefficients of $f$.

**Marc Hindry** (Université de Paris)
*On the size of generators of the Mordell-Weil group of an abelian variety*

Given an abelian variety over a global field, theorems of Mordell, Néron and Weil assert than the group of rational points is finitely generated and further is equipped with a canonical bilinear form (Néron-Tate height). A natural question is the search of an upper bound for the size (height) of a minimal set of generators. I will discuss known results and problems, explaining the link with the L-function of the abelian variety.

**Yong Hu** (Southern University of Science and Technology)
*Rost kernels of division algebras over complete discrete valuation fields*

Let $F$ be a field, $D$ be a central division $F$-algebra of prime power degree. By the Rost kernel of $D$ we mean the subgroup of $F^*$ consisting of elements $\lambda$ such that the cohomology class $(D)\cup(\lambda) \in H^3(F)$ vanishes. In general, this subgroup contains the Suslin kernel, which we define to be the group generated

by $m$-th powers of reduced norms from $D^{\otimes m}$, for all $m \geq 1$. In 1985, Suslin conjectured that the Rost and the Suslin kernels always coincide. In this talk we will discuss some new cases of his conjecture, for complete discrete valuation fields. This is based on a joint work with Zhengyao Wu.

**Zhizhong Huang** (Leibniz Universität, Hannover)
*Distribution of Mordell-Weil ranks in elliptic fibrations*

In this talk we report some quantitative results on the distribution of Mordell-Weil ranks and the related Hilbert Property in several isotrivial elliptic surfaces, inspired by the recent works of Corvaja-Zannier, Demeio, Loughran-Salgado, Colliot-Thélène. We show that for a quadratic family of the congruent number elliptic curve, as a rational elliptic surface, fibres of rank 0 and 1 both have positive density, providing evidence to a conjecture of Silverman. We also discuss a quartic family (as a K3 surface) and show that the set of fibres of rank strictly larger than the generic one is not thin.

**Hideaki Ikoma** (International Buddhist University)
*Differentiability of the arithmetic volume function along the base conditions*

The arithmetic volume of a pair of an adelic $\mathbb{R}$-Cartier divisor and an $\mathbb{R}$-Cartier divisor is an invariant measuring the asymptotic behavior of the numbers of the strictly small sections of the high multiples of the pair. In this paper, we establish that the arithmetic volume function defined on an open cone of the space of pairs is Gâteaux differentiable along the directions of $\mathbb{R}$-Cartier divisors and that the derivatives are given by arithmetic restricted positive intersection numbers.

**Klaus Künnemann** (Universität Regensburg)
*Positive forms and currents on toric varieties*

We report on joint work in progress with José Burgos, Walter Gubler and Philipp Jell. We compare Lagerberg forms and currents on the partial compactification induced by a smooth fan with invariant forms and currents on the corresponding complex toric variety.

**Yongqi Liang** (University of Science and Technology of China)
*Non-invariance of strong and weak approximation properties under extension of the ground field*

We consider algebraic varieties defined over a number field. We will define strong and weak approximation properties (with or without Brauer-Manin obstruction) for rational points on varieties. We will talk about some results on the non-invariance of such properties under the extension of the ground field.

**Atsushi Moriwaki** (Kyoto University)
*Arithmetic dynamical system and Dirichlet's property*

Dirichlet's property comes from the essential part of the proof of Dirichlet's unit theorem, that is, the property that a pseudo-effective arithmetic $\mathbb{R}$-Cartier is R-linearly equivalent to an effective arithmetic $\mathbb{R}$-Cartier. In the case of arithmetic dynamical system, Dirichlet's property is a subtle problem. In the talk, we present several examples of this property.

**Fabien Pazuki** (University of Copenhagen)
*Regulators of number fields and abelian varieties*

In the general study of regulators, we present three inequalities, some corollaries about the Northcott property and a counting problem for rational points on elliptic curves. We first bound from below the regulators of number fields, following previous works of Silverman and Friedman. We then bound from below the regulators of Mordell-Weil groups of abelian varieties defined over a number field, assuming a conjecture of Lang and Silverman. Finally we explain how to prove an unconditional statement for elliptic curves of rank at least 4. This third inequality is joint work with Pascal Autissier and Marc Hindry.

**Per Salberger** (Chalmers University of Technology)
*Counting rational points with the determinant method*

I will present new uniform estimates for the number of rational points of bounded height on non-singular threefolds in $\mathbb{P}^4$. These are obtained by means of a global version of Heath-Brown's $p$-adic determinant method and recent results on covering gonality. As an application we obtain new estimates for equal sums of three powers, which are much stronger than the previous estimates of Hua, Browning/Heath-Brown and myself.

**Tamás Szamuely** (Universita degli studi di Pisa)
*A generalization of Beilinson's geometric height pairing*

In the first section of his seminal paper on height pairings, Beilinson constructed an $\ell$-adic height pairing for rational Chow groups of homologically trivial cycles of complementary codimension on smooth projective varieties over a global function field. In joint work with Damian Rössler we extend his construction to arbitrary finitely generated base fields of positive characteristic and speculate about its geometric origin.

**Fei Xu** (Capital Normal University)
*Strong approximation with Brauer-Manin obstruction for a family of norm varieties*

Weak approximation of smooth varieties $N_{L/k}(x) = \prod_{i=1}^{n} p_i(t)$ has been studied extensively by various methods, where $L/k$ is a finite extension of number fields and $p_i(t)$'s are distinct irreducible polynomials. For strong approximation of this equation, the first non-trivial example was given by Derenthal and Wei where $[L : k] = 4$ and $n = 1$ and $p_1(t)$ has a root in $K$ with $\deg(p_1(t)) = 2$. In this talk, I will show this equation satisfies strong approximation with Brauer-Manin obstruction when $L$ is contained in $k[t]/(p_i(t))$ for all $1 \leqslant i \leqslant n$. Under Schinzel's hypothesis, we can remove the restriction that $L$ is contained in $k[t]/(p_i(t))$ for all $1 \leqslant i \leqslant n$ when $L/k$ is cyclic. This is a joint work with Cao and Wei.

**Yu Yasufuku** (Nihon University)
*The smallness of the GCD in Higher Dimensions*

Recently, Ru–Vojta introduced a birational invariant based on Schmidt subspace theorem, and derived Diophantine approximation results. We compute this invariant in the case of some blowups, and show smallness of gcd's. When we restrict to integral points, this recovers the gcd result of Levin on $\mathbb{G}_m^n$. We will also compare our results with what would be implied by Vojta's conjecture. This is a joint work with Julie Tzu-Yueh Wang (Academia Sinica).

**Tong Zhang** (East China Normal University)
*Linear system on arithmetic varieties fibered by curves*

We consider a hermitian line bundle on an arithmetic variety fibered by curves. For such a line bundle, we give an upper bound on the number of its effective sections by its volume together with other explicit terms.

**Yongqiang Zhao** (Westlake University)
*A Galois theoretic perspective of scrollar syzygy theory*

The theory of scrollar syzygy resolution of algebraic curves was introduced by Schreyer in his work for Green's conjecture. In this talk, we will give a Galois theoretic perspective of this theory. We will explain the number theory motivations for this project and will discuss, if time permits, its number fields analogues. This is a joint project with Wouter Castryck.

**Huilin Zhu** (Xiamen University)
*On pure ternary exponential Diophantine equations*

In this talk, we give a survey about pure ternary exponential Diophantine equations and some developments in results as well as in tools. These tools include Baker's method, the Bilu-Hanrot-Voutier Theorem about the existence of primitive divisors of Lucas numbers and Lehmer numbers, algeraic number theory methods, and others. We generalize the conjectures of Jesmanowicz, Terai-Cao-Le and Yuan-Han and also present some results that we proved and plan to further improve in this area.

# Geometry of arithmetic varieties

Lecture Hall, Jia Yi Bing Building, 82# Jing Chun Yuan

BICMR, Peking University

## Registered participants

| | |
|---|---|
| François Ballaÿ | BICMR, Peking University |
| Jose Ignacio Burgos Gil | Instituto de Ciencias Matemáticas (CSIC) |
| Anna Cadoret | IMJ-PRG, Sorbonne Université & Kyoto University |
| Victoria Cantoral Farfán | Katholieke Universiteit Leuven |
| Yang Cao | Leibniz Universität Hannover |
| Huayi Chen | IMJ-PRG, Université de Paris |
| Ke Chen | Nanjing University |
| Sheng Chen | Capital Normal University |
| Yanbo Fang | IMJ-PRG, Université de Paris |
| Mounir Hajli | Shanghai Jiao Tong University |
| Marc Hindry | IMJ-PRG, Université de Paris |
| Guang Hu | University of Science and Technology of China |
| Yong Hu | Southern University of Science and Technology |
| Zhizhong Huang | Leibniz Universität Hannover |
| Chun Yin Hui | Tsinghua University |
| Hideaki Ikoma | International Buddhist University |
| Zhongyu Jin | Peking University |
| Rongji Kang | Academy of Mathematics and Systems Science |
| Klaus Künnemann | Universität Regensburg |
| Yao Li | Peking University |
| Yongqi Liang | University of Science and Technology of China |
| Jing Liu | Capital Normal University |
| Atsushi Moriwaki | Kyoto University |
| Fabien Pazuki | University of Copenhagen |
| Léo Poyeton | BICMR, Peking University |
| Matthieu Rambaud | Telecom ParisTech |
| Per Salberger | Chalmers University of Technology |
| Antoine Sedillot | ENS Paris-Saclay & BICMR, Peking University |
| Christophe Soulé | CNRS, Institut des Hautes Études Scientifiques |
| Heng Song | Capital Normal University |
| Jie Sun | Hubei University of Arts and Science |
| Tamás Szamuely | Università degli studi di Pisa |
| Longke Tang | Peking University |
| Yupeng Wang | BICMR, Peking University |
| Han Wu | University of Science and Technology of China |
| Fei Xu | Capital Normal University |
| Quan Xu | China Jiliang University |
| Kazuhiko Yamaki | Kyoto University |
| Yu Yasufuku | Nihon University |
| Tong Zhang | East China Normal University |
| Yongqiang Zhao | Westlake University |
| Peng Zheng | IMJ-PRG, Sorbonne Université |
| Huilin Zhu | Xiamen University |

# Geometry of arithmetic varieties

Lecture Hall, Jia Yi Bing Building, 82# Jing Chun Yuan

BICMR, Peking University

## Organizing and scientific committees

| | |
|---|---|
| François Ballaÿ | BICMR, Peking University |
| Huayi Chen | IMJ-PRG, Université de Paris |
| Noriko Hirata-Kohno | Nihon University |
| Philippe Gille | CNRS, Institut Camille Jordan |
| Atsushi Moriwaki | Kyoto University |
| Marusia Rebolledo | Université Clerment-Auvergne |

## Secretary

He Liu   BICMR, Peking University   `liuhe@math.pku.edu.cn`

## Sponsors

Beijing International Center for Mathematical Research

GDR Structuration de la théorie des nombres

Institut de Mathématiques de Jussieu - Paris Rive Gauche

Université Paris Diderot - Paris 7