Errata of "Arakelov geometry over adelic curves"
— July 6th, 2021—

• Proof of Theorem 1.1.7: It is not adequate to apply Corollary 1.1.6 (2) to show that, given a finite-dimensional semi-normed vector space $(V_2, \|\cdot\|_2)$ over a trivially valued field $(k, |\cdot|)$, if the function $\|\cdot\|_2$ is not identically zero, then it is bounded from above and from below by positive real numbers outside of its null sub-space. In fact, here in Theorem 1.1.7 we do not assume that the semi-norm $\|\cdot\|_2$ is ultrametric, while this condition is included in the assumption of Corollary 1.1.6. Following is an errata for the proof.

Let $(e_i)_{i=1}^r$ be a basis of V_2. For any $(a_1, \ldots, a_r) \in k^r$, one has

$$\|a_1 e_1 + \cdots + a_r e_r\|_2 \leqslant \sum_{i=1}^r |a_i| \cdot \|e_i\|_2 \leqslant \sum_{i=1}^r \|e_i\|_2,$$

which shows the boundedness from above. We now show the boundedness from below by contradiction. Suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $V_2 \setminus N_{\|\cdot\|_2}$ such that

$$\lim_{n\to+\infty} \|x_n\| = 0.$$

Since V_2 is finite-dimensional, there exists $N \in \mathbb{N}$ and a vector subspace F of V_2 such that, for any integer $n \geqslant N$, the equality

$$F = \mathrm{Vect}_k(\{x_\ell \,:\, \ell \in \mathbb{N},\ \ell \geqslant n\})$$

holds. Then, by the proof of boundedness from above, we obtain that

$$\sup_{y\in F} \|y\| \leqslant \inf_{n\in\mathbb{N},\, n\geqslant N} \Big(r \sup_{\ell\in\mathbb{N},\, \ell\geqslant n} \|x_\ell\| \Big) = r \limsup_{n\to+\infty} \|x_n\| = 0.$$

However, by definition F is not contained in the null sub-space of $\|\cdot\|_2$, which leads to a contradiction.

• Proof of Theorem 1.2.54: The arguments for showing that Θ^+ is convex and the function $\log\det(\cdot)$ is strictly concave on the convex open set Θ^+ of positive definite self-adjoint operators are not correct. They should be replaced by the arguments as follows.

Let V be an n-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with an inner product $\langle , \rangle'$. For any self-adjoint operator $u : V \to V$. By diagonalizing the operator u one can show that there exists a positive definite operator, that one denotes by $u^{\frac{1}{2}}$, such that $u = u^{\frac{1}{2}} \circ u^{\frac{1}{2}}$.

Let u and v be two positive definite self-adjoint operators. For $x \in V$,

$$\langle x, (tu + (1-t)v)(x)\rangle' = t\langle x, u(x)\rangle' + (1-t)\langle x, v(x)\rangle' \geqslant 0,$$

and the equality holds if and only if $x = 0$. Thus Θ^+ is convex.

Since the determinant function is multiplicative,

$$\begin{aligned}\det(tu + (1-t)v) &= \det(u)\det(tI + (1-t)u^{-\frac{1}{2}} \circ v \circ u^{-\frac{1}{2}}) \\ &= \det(u)\prod_{i=1}^n (t + (1-t)\lambda_i),\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $u^{-\frac{1}{2}} \circ v \circ u^{-\frac{1}{2}}$ (counting multiplicity), and I denotes the identity operator. Note that $\lambda_i > 0$ for all i. By the concavity of the function log, we obtain

$$
\begin{aligned}
\log\det(tu+(1-t)v) &\geqslant \log\det(u)+(1-t)\sum_{i=1}^{n}\log(\lambda_i)\\
&\geqslant \log\det(u)+(1-t)\log\det(u^{-\frac{1}{2}}\circ v\circ u^{-\frac{1}{2}})\\
&= t\log\det(u)+(1-t)\log\det(v),
\end{aligned}
$$

which shows the concavity of $\log\det(\cdot)$.

• Proposition 2.3.12

PROPOSITION 2.3.12. *Let L be an invertible $\mathcal{O}_X$-module which is generated by global sections. Let $\varphi_{\mathscr{L}}$ be the metric induced by a model $(\mathscr{X},\mathscr{L})$ of (X,L). Let φ be a continuous metric of L and $\mathscr{H} := \{s \in H^0(\mathscr{X},\mathscr{L}) \,:\, \|s\|_{\varphi} \leqslant 1\}$. Moreover, let $\mathcal{E}$ be an $\mathfrak{o}_k$-submodule of $H^0(\mathscr{X},\mathscr{L})$ such that $\mathcal{E}' := \mathcal{E}/\mathcal{E}_{\mathrm{tor}}$ yields a lattice of $H^0(X,L)$. Then one has the following:*

(1) *If $\mathscr{H}\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}} \to \mathscr{L}$ is surjective, then $\varphi \leqslant \varphi_{\mathscr{L}}$.*
(2) *If φ is the quotient metric on L induced by $\|\cdot\|_{\mathcal{E}'}$ (see Definition 1.1.27 for the norm induced by a lattice), then $\varphi \geqslant \varphi_{\mathscr{L}}$.*
(3) *If φ is the quotient metric on L induced by $\|\cdot\|_{\mathcal{E}'}$, and the natural homomorphism $\mathcal{E}\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}} \to \mathscr{L}$ is surjective, then $\varphi = \varphi_{\mathscr{L}}$.*

PROOF. If $\mathscr{X}$ is flat over $\mathfrak{o}_k$, then the proof of the book works well. Note that in this case, $H^0(\mathscr{X},\mathscr{L})$ is torsion free.

In general, let $\mathscr{X}'$ and $\mathscr{L}'$ be the same one as the beginning of Subsection 2.3.2. Then, $(\mathscr{X}',\mathscr{L}')$ is a flat model of (X,L), and if we set

$$
\mathscr{H}' := \{s \in H^0(\mathscr{X}',\mathscr{L}') \,:\, \|s\|_{\varphi} \leqslant 1\},
$$

then $\mathscr{H}/\mathscr{H}_{\mathrm{tor}} \subseteq \mathscr{H}'$. Moreover, note that $\varphi_{\mathscr{L}} = \varphi_{\mathscr{L}'}$ and $\mathcal{E}' \subseteq H^0(\mathscr{X}',\mathscr{L}')$. Observing the following diagrams:

$$
\begin{array}{ccc}
\mathscr{H}\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}} & \longrightarrow & \mathscr{L}\\
\downarrow & & \downarrow\\
\mathscr{H}'\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}'} & \longrightarrow & \mathscr{L}',
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{E}\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}} & \longrightarrow & \mathscr{L}\\
\downarrow & & \downarrow\\
\mathcal{E}'\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}'} & \longrightarrow & \mathscr{L}',
\end{array}
$$

one can see the assertions. □

• Proposition 2.3.17

The proof of Proposition 2.3.17 in the case where $\mathscr{L}$ is ample can be done in the following way.

There is a positive number n such that $H^0(\mathscr{X},\mathscr{L}^{\otimes n})\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}} \to \mathscr{L}$ is surjective. Since $\mathscr{X}$ is quasi-compact, there is a finitely generated $\mathfrak{o}_k$-sub-module $\mathcal{E}$ of $H^0(\mathscr{X},\mathscr{L}^{\otimes n})$ such that $\mathcal{E}\otimes_{\mathfrak{o}_k}\mathcal{O}_{\mathscr{X}} \to \mathscr{L}^{\otimes n}$ is surjective and $\mathcal{E}\otimes_{\mathfrak{o}_k} k = H^0(X,L^{\otimes n})$. Thus $\mathcal{E}' := \mathcal{E}/\mathcal{E}_{\mathrm{tor}}$ yields a lattice of $H^0(X,L^{\otimes n})$, so that, by Proposition 2.3.12, $\varphi_{\mathcal{L}^{\otimes n}}$ is the quotient metric induced by $\|\cdot\|_{\mathcal{E}'}$. Therefore, $\varphi_{\mathcal{L}^{\otimes n}}$ is semipostive. Moreover, as $\varphi_{\mathcal{L}^{\otimes n}} = n\varphi_{\mathcal{L}}$ by Proposition 2.3.15, $\varphi_{\mathcal{L}}$ is also semipositive by Proposition 2.3.2.

• Proposition 6.4.20

The original proof works under the assumption $\mathrm{vol}(\Omega_\infty) > 0$. For the general case we need a supplementary condition that there exists an integrable function ψ on Ω such that

$$
\int_{\Omega}\psi\,\nu(\mathrm{d}\omega) > 0.
$$

Then we replace the function φ in the original proof by

$$\varphi(\omega) := (1/a)(\ln \|f\|_{ag_\omega} + \psi(\omega)), \quad \omega \in \Omega.$$

By using this new φ, one can see that the original proof works well.