EXTENSION PROPERTY OF SEMIPOSITIVE INVERTIBLE SHEAVES OVER A NON-ARCHIMEDEAN FIELD

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Abstract. In this article, we prove an extension property of semipositively metrized ample invertible sheaves on a projective scheme over a complete non-archimedean valued field.

Introduction

Let k be a field and X be a projective scheme over $\operatorname{Spec} k$, equipped with an ample invertible $\mathcal{O}_X$-module L. If Y is a closed subscheme of X, then for sufficiently positive integer n, any section ℓ of $L|_Y^{\otimes n}$ on Y extends to a global section of $L^{\otimes n}$ on X. In other words, the restriction map $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$ is surjective. A simple proof of this result relies on Serre's vanishing theorem, which ensures that $H^1(X, \mathcal{I}_Y \otimes L^{\otimes n}) = 0$ for sufficiently positive integer n, where $\mathcal{I}_Y$ is the ideal sheaf of Y.

The metrized version (with $k = \mathbb{C}$) of this result has been widely studied in the literature and has divers applications in complex analytic geometry and in arithmetic geometry. We assume that the ample invertible sheaf L is equipped with a continuous (with respect to the analytic topology) metric $|\cdot|_h$, which induces a continuous metric $|\cdot|_{h^n}$ on each tensor power sheaf $L^{\otimes n}$, where $n \in \mathbb{N}$, $n \geqslant 1$. The metric $|\cdot|_{h^n}$ leads to a supremum norm $\|\cdot\|_{h^n}$ on the global section space $H^0(X, L)$ such that

$$\forall\, s \in H^0(X, L),\ \|s\|_{h^n} = \sup_{x \in X(\mathbb{C})} |s|_{h^n}(x).$$

Similarly, it induces a supremum norm $\|\cdot\|_{Y,h^n}$ on the space $H^0(Y, L|_Y^{\otimes n})$ with $\|s\|_{Y,h^n} = \sup_{y \in Y(\mathbb{C})} |s|_{h^n}(y)$. Note that for any section $s \in H^0(X, L^{\otimes n})$ one has $\|s|_Y\|_{Y,h^n} \leqslant \|s\|_{h^n}$. The metric extension problem consists of studying the extension of global sections of $L|_Y$ to those of L with an estimation on the supremum norms. Note that a positivity condition on the metric h is in general necessary to obtain interesting upper bounds. This problem has been studied by using Hörmander's L^2 estimates (see [3] for example), under smoothness conditions on the metric. More recently, it has proved (without any regularity condition) that, if the metric $|\cdot|_h$ is semi-positive, then for any $\epsilon > 0$ and any section $l \in H^0(Y, L|_Y)$ there exists an integer $n \geqslant 1$ and $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^{\otimes n}$ and that $\|s\|_{h^n} \leqslant e^{\epsilon n} \|s|_Y\|_{Y,h^n}$. We refer the readers to [10, 9] for more details.

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The purpose of this article is to study the non-archimedean counterpart of the above problem. We will establish the following result (see Theorem 4.2 and Corollary 1.2).

Theorem 0.1. *Let k be a field equipped with a complete and non-archimedean absolute value $|\cdot|$ (which could be trivial). Let X be a projective scheme over* Spec k *and L be an ample invertible sheaf on X, equipped with a continuous and* semi-positive *metric $|\cdot|_h$. Let Y be a closed subscheme of X and $l \in H^0(Y, L|_Y)$. For any $\epsilon > 0$ there exists an integer $n_0 \geq 1$ such that, for any integer $n \geq n_0$, the section $l^{\otimes n}$ extends to a section $s \in H^0(X, L^{\otimes n})$ verifying $\|s\|_h \leq e^{\epsilon n}\|l\|^n_{Y,h}$.*

The semi-positivity condition of the metric means that the metric $|\cdot|_h$ can be written as a uniform limit of Fubini-Study metrics. We will show that, if the absolute value $|\cdot|$ is non-trivial, then this condition is equivalent to the classical semi-positivity condition (namely uniform limit of nef model metrics, see Proposition 3.17) of Zhang [12], see also [4, 8], and compare with the complex analytic case [11]. The advantage of the new definition is that it also works in the trivial valuation case, where the model metrics are too restrictive. We use an argument of extension of scalars to the ring of formal Laurent series to obtain the result of the above theorem in the trivial valuation case.

The article is organized as follows. In the first section we introduce the notation of the article and prove some preliminary results, most of which concern finite dimensional normed vector spaces over a non-archimedean field. In the second section, we discuss some property of model metrics. In the third section, we study various properties of continuous metrics on an invertible sheaf, where an emphasis is made on the positivity of such metrics. Finally, in the fourth section, we prove the extension theorem.

1. Notation and preliminaries

1.1. **Notation.** Throughout this paper, we fix the following notation.

1.1.1. Fix a field k with a complete and non-archimedean absolute value $|\cdot|$. The valuation ring of k and the maximal ideal of the valuation ring are denoted by $\mathfrak{o}_k$ and $\mathfrak{m}_k$, respectively, that is,

$$\mathfrak{o}_k := \{a \in k \mid |a| \leq 1\} \quad \text{and} \quad \mathfrak{m}_k := \{x \in k \mid |x| < 1\}.$$

In the case where $|\cdot|$ is discrete, we fix a uniformizing parameter ϖ of $\mathfrak{m}_k$, that is, $\mathfrak{m}_k = \varpi\mathfrak{o}_k$.

1.1.2. A norm $\|\cdot\|$ of a finite-dimensional vector space V over k is always assumed to be ultrametric, that is, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$. A pair $(V, \|\cdot\|)$ is called a normed finite-dimensional vector space over k.

1.1.3. Fix an algebraic scheme X over Spec k, that is, X is a scheme of finite type over Spec(k). Let X^{an} be the analytification of X in the sense of Berkovich [1]. For $x \in X^{\mathrm{an}}$, the residue field of the associated scheme point of x is denoted by $\kappa(x)$. Note that the seminorm $|\cdot|_x$ at x yields an absolute value of $\kappa(x)$. By abuse of notation, it is denoted by $|\cdot|_x$. Let $\hat{\kappa}(x)$ be the completion of $\kappa(x)$ with respect to $|\cdot|_x$. The extension of $|\cdot|_x$ to $\hat{\kappa}(x)$ is also denoted by the same symbol $|\cdot|_x$. The valuation ring of $\hat{\kappa}(x)$ and the maximal ideal of the valuation ring are denoted by $\mathfrak{o}_x$ and $\mathfrak{m}_x$, respectively. Let L be an invertible sheaf on X. For $x \in X^{\mathrm{an}}$, $L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$ is denoted by $L(x)$.

1.1.4. By *continuous metric* on L, we refer to a family $h = \{|\cdot|_h(x)\}_{x\in X^{\mathrm{an}}}$, where $|\cdot|_h(x)$ is a norm on $L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$ over $\hat{\kappa}(x)$ for each $x \in X^{\mathrm{an}}$, such that for any local basis ω of L over a Zariski open subset U, $|\omega|_h(\cdot)$ is a continuous function on U^{an}. We assume that X is projective. Given a continuous metric h on L, we define a norm $\|\cdot\|_h$ on $H^0(X, L)$ such that

$$\forall\, s \in H^0(X, L), \quad \|s\|_h := \sup_{x\in X^{\mathrm{an}}} |s|_h(x).$$

Similarly, if Y is a closed subscheme of X, we define a norm $\|\cdot\|_{Y,h}$ on $H^0(Y, L)$ such that

$$\forall\, l \in H^0(Y, L), \quad \|l\|_{Y,h} := \sup_{y\in Y^{\mathrm{an}}} |l|_h(y).$$

Clearly one has

$$\|s\|_h \geqslant \|\, s|_Y\|_{Y,h} \tag{1}$$

for any $s \in H^0(X, L)$.

• In the following 1.1.5, 1.1.6 and 1.1.7, X is always assumed to be projective.

1.1.5. Given a continuous metric h on L, the metric induces for each integer $n \geqslant 1$ a continuous metric on $L^{\otimes n}$ which we denote by h^n: for any point $x \in X^{\mathrm{an}}$ and any local basis ω of L over a Zariski open neighborhood of x one has

$$|\omega^{\otimes n}|_{h^n}(x) = |\omega|_h(x)^n.$$

Note that for any section $s \in H^0(X, L)$ one has $\|s^{\otimes n}\|_{h^n} = \|s\|_h^n$. By convention, h^0 denotes the trivial metric on $L^{\otimes 0} = \mathscr{O}_X$, namely $|\mathbf{1}|_{h^0}(x) = 1$ for any $x \in X^{\mathrm{an}}$, where $\mathbf{1}$ denotes the section of unity of $\mathscr{O}_X$.

Conversely, given a continuous metric $g = \{|\cdot|_g(x)\}_{x\in X^{\mathrm{an}}}$ on $L^{\otimes n}$, there is a unique continuous metric h on L such that $h^n = g$. We denote by $g^{1/n}$ this metric. This observation allows to define continuous metrics on an element in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ as follows. Given $M \in \mathrm{Pic}(X) \otimes \mathbb{Q}$, we denote by $\Gamma(M)$ the subsemigroup of $\mathbb{N}_{\geq 1}$ of all positive integers n such that $M^{\otimes n} \in \mathrm{Pic}(X)$. We call *continuous metric* on M any family $g = (g_n)_{n\in\Gamma(M)}$ with g_n being a continuous metric on $M^{\otimes n}$, such that $g_n^m = g_{mn}$ for any $n \in \Gamma(M)$ and any $m \in \mathbb{N}_{\geq 1}$. Note that the family $g = (g_n)_{n\in\Gamma(M)}$ is uniquely determined by any of its elements. In fact, given an element $n \in \Gamma(M)$, one has $g_m = g_{mn}^{1/n} = (g_n^m)^{1/n}$ for any $m \in \Gamma(M)$. In particular, for any positive rational number p/q, the family $g^{p/q} = (g_{Nnp}^{1/Nq})_{n\in\Gamma(M^{\otimes(p/q)})}$ is a continuous metric on $M^{\otimes(p/q)}$, where N is a positive integer such that $M^{\otimes N} \in \mathrm{Pic}(X)$, and the metric $g^{p/q}$ does not depend on the choice of the positive integer N.

Let M be an element in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ equipped with a continuous metric $g = (g_n)_{n \in \Gamma(M)}$. By abuse of notation, for $n \in \Gamma(M)$ we also use the expression g^n to denote the continuous metric g_n on $M^{\otimes n}$.

1.1.6. We call *model* of X any projective and flat $\mathfrak{o}_k$-scheme $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ such that the generic fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ is X. We denote by $\mathscr{X}_\circ := \mathscr{X} \otimes_{\mathfrak{o}_k} (\mathfrak{o}_k/\mathfrak{m}_k)$ the central fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$. By the valuative criterion of properness, for any point $x \in X^{\mathrm{an}}$, the canonical k-morphism $\mathrm{Spec}\,\hat{\kappa}(x) \to X$ extends in a unique way to an $\mathfrak{o}_k$-morphism of schemes $\mathscr{P}_x : \mathrm{Spec}\,\mathfrak{o}_x \to \mathscr{X}$. We denote by $r_{\mathscr{X}}(x)$ the image of $\mathfrak{m}_x \in \mathrm{Spec}\,\mathfrak{o}_x$ by the map $\mathscr{P}_x$. Thus we obtain a map $r_{\mathscr{X}}$ from X^{an} to $\mathscr{X}_\circ$, called the *reduction map* of $\mathscr{X}$.

Let $\mathscr{L}$ be an element of $\mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$ such that $\mathscr{L}|_X = L$ in $\mathrm{Pic}(X) \otimes \mathbb{Q}$. The $\mathbb{Q}$-invertible sheaf $\mathscr{L}$ yields a continuous metric $|\cdot|_{\mathscr{L}}$ as follows.

First we assume that $\mathscr{L} \in \mathrm{Pic}(\mathscr{X})$ and $\mathscr{L}|_X = L$ in $\mathrm{Pic}(X)$. For any $x \in X^{\mathrm{an}}$, let ω_x be a local basis of $\mathscr{L}$ around $r_{\mathscr{X}}(x)$ and $\bar{\omega}_x$ the class of ω_x in $L(x) := L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$. For $l \in L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$, if we set $l = a_x \bar{\omega}_x$ ($a_x \in \hat{\kappa}(x)$), then $|l|_{\mathscr{L}}(x) := |a_x|_x$. Here we set $h := \{|\cdot|_{\mathscr{L}}(x)\}_{x \in X^{\mathrm{an}}}$. Note that h is continuous because, for a local basis ω of $\mathscr{L}$ over an open set $\mathscr{U}$ of $\mathscr{X}$, $|\omega|_{\mathscr{L}}(x) = 1$ for all $x \in r_{\mathscr{X}}^{-1}(\mathscr{U}_\circ)$. Moreover,

$$|\cdot|_{h^n}(x) = |\cdot|_{\mathscr{L}^n}(x) \tag{2}$$

for all $n \geq 0$ and $x \in X^{\mathrm{an}}$. Indeed, if we set $l = a_x \bar{\omega}_x$ for $l \in L(x)$, then $l^{\otimes n} = a_x^n \bar{\omega}_x^{\otimes n}$. Thus

$$|l^{\otimes n}|_{h^n}(x) = (|l|_h(x))^n = |a_x|_x^n = |l^{\otimes n}|_{\mathscr{L}^n}(x).$$

In general, there are $\mathscr{M} \in \mathrm{Pic}(\mathscr{X})$ and a positive integer m such that $\mathscr{L}^{\otimes m} = \mathscr{M}$ in $\mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$ and $\mathscr{M}|_X = L^{\otimes m}$ in $\mathrm{Pic}(X)$. Then

$$|\cdot|_{\mathscr{L}}(x) := (|\cdot|_{\mathscr{M}}(x))^{1/m}.$$

Note that the above definition does not depend on the choice of $\mathscr{M}$ and m. Indeed, let $\mathscr{M}'$ and m' be another choice. As $\mathscr{M}^{\otimes m'} = \mathscr{M}'^{\otimes m}$ in $\mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$, there is a positive integer N such that $\mathscr{M}^{\otimes Nm'} = \mathscr{M}'^{\otimes Nm}$ in $\mathrm{Pic}(\mathscr{X})$, so that, by using (2),

$$(|\cdot|_{\mathscr{M}}(x))^{Nm'} = |\cdot|_{\mathscr{M}^{\otimes Nm'}}(x) = |\cdot|_{\mathscr{M}'^{\otimes Nm}}(x) = (|\cdot|_{\mathscr{M}'}(x))^{Nm},$$

as desired.

1.1.7. Let $\mathscr{X}$ be a model of X. As $\mathscr{X}$ is flat over $\mathfrak{o}_k$, the natural homomorphism $\mathscr{O}_{\mathscr{X}} \to \mathscr{O}_X$ is injective. Let Y be a closed subscheme of X and $I_Y \subseteq \mathscr{O}_X$ the defining ideal sheaf of Y. Let $\mathscr{I}_{\mathscr{Y}}$ be the kernel of $\mathscr{O}_{\mathscr{X}} \to \mathscr{O}_X/I_Y$, that is, $\mathscr{I}_{\mathscr{Y}} := \mathscr{I}_Y \cap \mathscr{O}_{\mathscr{X}}$. Obviously $\mathscr{I}_{\mathscr{Y}} \otimes_{\mathfrak{o}_k} k = I_Y$, so that if we set $\mathscr{Y} = \mathrm{Spec}(\mathscr{O}_{\mathscr{X}}/\mathscr{I}_{\mathscr{Y}})$, then $\mathscr{Y} \times_{\mathrm{Spec}(\mathfrak{o}_k)} \mathrm{Spec}(k) = Y$. Moreover, $\mathscr{Y}$ is flat over $\mathfrak{o}_k$ because $\mathscr{O}_{\mathscr{Y}} \to \mathscr{O}_Y$ is injective. Therefore, $\mathscr{Y}$ is a model of Y. We say that $\mathscr{Y}$ is the *Zariski closure of Y in $\mathscr{X}$*.

1.2. **Extension obstruction index.** In this subsection, we introduce an invariant to describe the obstruction to the extension property. Let X be a projective scheme over $\operatorname{Spec} k$, L be an invertible sheaf on X equipped with a continuous metric h, and Y be a closed subscheme of X. For any non-zero element l of $H^0(Y, L|_Y)$, we denote by $\lambda_h(l)$ the following number (if there does not exist any section $s \in H^0(X, L^{\otimes n})$ extending $l^{\otimes n}$, then the infimum in the formula is defined to be $+\infty$ by convention)

$$\lambda_h(l) = \limsup_{n \to +\infty} \inf_{\substack{s \in H^0(X, L^{\otimes n}) \\ s|_Y = l^{\otimes n}}} \left(\frac{\log \|s\|_{h^n}}{n} - \log \|l\|_{Y,h} \right) \in [0, +\infty]. \tag{3}$$

This invariant allows to describe in a numerically way the obstruction to the metric extendability of the section l. In fact, the following assertions are equivalent:

(a) $\lambda_h(l) = 0$,
(b) for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}_{\geq 1}$ such that, for any integer $n \geq n_0$, the element $l^{\otimes n}$ extends to a section $s \in H^0(X, L^{\otimes n})$ such that $\|s\|_h \leq e^{\epsilon n} \|l\|_{Y,h}^n$.

The following proposition shows that, if $l^{\otimes n}$ extends to a global section of $L^{\otimes n}$ for sufficiently positive n (it is the case notably when the line bundle L is ample), then the limsup defining $\lambda_h(l)$ is actually a limit.

Proposition 1.1. *For any integer $n \geqslant 1$, let*

$$a_n = \inf_{\substack{s \in H^0(X, L^{\otimes n}) \\ s|_Y = l^{\otimes n}}} \Big(\log \|s\|_{h^n} - n \log \|l\|_{Y,h} \Big).$$

Then the sequence $(a_n)_{n \geq 1}$ is sub-additive, namely one has $a_{m+n} \leq a_m + a_n$ for any $(m, n) \in \mathbb{N}_{\geq 1}$. In particular, if for sufficiently positive integer n, the section l^n lies in the image of the restriction map $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$, then "$\limsup$" in (3) is actually "$\lim$".

Proof. By (1), one has $a_n \geq 0$ for any integer $n \geq 1$. Moreover, $a_n < +\infty$ if and only if l^n lies in the image of the restriction map $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$. To verify the inequality $a_{m+n} \leq a_m + a_n$, it suffices to consider the case where both a_m and a_n are finite. Let s_m and s_n be respectively sections in $H^0(X, L^{\otimes m})$ and $H^0(X, L^{\otimes n})$ such that $s_m|_Y = l^{\otimes m}$ and $s_n|_Y = l^{\otimes n}$, then the section $s = s_m \otimes s_n \in H^0(X, L^{\otimes (m+n)})$ verifies the relation $s|_Y = l^{\otimes (n+m)}$. Moreover, one has

$$\|s\|_h = \sup_{x \in X^{\mathrm{an}}} |s|_h(x) = \sup_{x \in X^{\mathrm{an}}} |s_m|_h(x) \cdot |s_n|_h(x) \leqslant \|s_m\|_h \cdot \|s_n\|_h.$$

Since s_m and s_n are arbitrary, one has $a_{m+n} \leq a_m + a_n$. Finally, by Fekete's lemma, if $a_n < +\infty$ for sufficiently positive integer n, then the sequence $(a_n/n)_{n \geq 1}$ actually converges in $\mathbb{R}_+$. The proposition is thus proved. □

Corollary 1.2. *Assume that the invertible sheaf L is ample, then the following conditions are equivalent.*

(a) $\lambda_h(l) = 0$,

(b) for any $\epsilon > 0$, there exists $n \in \mathbb{N}_{\geq 1}$ and a section $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^n$ and that $\|s\|_h \leq e^{\epsilon n}\|l\|_{Y,h}$.

Proof. We keep the notation of the previous proposition. By definition the second condition is equivalent to

$$\liminf_{n\to+\infty} \frac{a_n}{n} = 0. \tag{4}$$

Since L is ample, Proposition 1.1 leads to the convergence of the sequence $(a_n/n)_{n\geq 1}$ in $\mathbb{R}_+$. Hence the condition (4) is equivalent to $\lambda_h(l) = 0$. □

1.3. **Normed vector space over a non-archimedean field.** In this subsection, we recall several facts on (ultrametric) norms over a non-archimedean field. Throughout this paper, a norm is always assumed to be ultrametric. Let V be a finite-dimensional vector space over k and $\|\cdot\|$ a norm of V over $(k, |\cdot|)$.

1.3.1. *Orthogonality of norms.* For $\alpha \in (0, 1]$, a basis $(e_1, \ldots, e_r)$ of V is called an *α-orthogonal basis of V with respect to $\|\cdot\|$* if

$$\alpha \max\{|a_1|\|e_1\|, \ldots, |a_r|\|e_r\|\} \leq \|a_1e_1 + \cdots + a_re_r\| \quad (\forall a_1, \ldots, a_r \in k).$$

If $\alpha = 1$ (resp. $\alpha = 1$ and $\|e_1\| = \cdots = \|e_r\| = 1$), then the above basis is called an *orthogonal basis of V* (resp. an *orthonormal basis of V*). Let $(e'_1, \ldots, e'_r)$ be another basis of V. We say that $(e_1, \ldots, e_r)$ *is compatible with* $(e'_1, \ldots, e'_r)$ if $ke_1 + \cdots + ke_i = ke'_1 + \cdots + ke'_i$ for $i = 1, \ldots, r$.

Proposition 1.3. *Fix a basis $(e'_1, \ldots, e'_r)$ of V. For any $\alpha \in (0, 1)$, there exists an α-orthogonal basis $(e_1, \ldots, e_r)$ of V with respect to $\|\cdot\|$ such that $(e_1, \ldots, e_r)$ is compatible with $(e'_1, \ldots, e'_r)$. Moreover, if the absolute value $|\cdot|$ is discrete, then there exists an orthogonal basis $(e_1, \ldots, e_r)$ of V compatible with $(e'_1, \ldots, e'_r)$.*

Proof. We prove it by induction on $\dim_k V$. If $\dim_k V = 1$, then the assertion is obvious. By the hypothesis of induction, there is a $\sqrt{\alpha}$-orthogonal basis $(e_1, \ldots, e_{r-1})$ of $V' := ke'_1 + \cdots + ke'_{r-1}$ with respect to $\|\cdot\|$ such that

$$ke_1 + \cdots + ke_i = ke'_1 + \cdots + ke'_i$$

for $i = 1, \ldots, r-1$. Choose $v \in V \setminus V'$. As

$$\mathrm{dist}(v, V') := \inf\{\|v - x\| : x \in V'\} > 0,$$

there is $y \in V'$ such that $\|v - y\| \leq (\sqrt{\alpha})^{-1}\mathrm{dist}(v, V')$. We set $e_r = v - y$. Clearly $(e_1, \ldots, e_{r-1}, e_r)$ forms a basis of V. It is sufficient to see that

$$\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\| \geq \alpha \max\{|a_1|\|e_1\|, \ldots, |a_{r-1}|\|e_{r-1}\|, \|e_r\|\}$$

for all $a_1, \ldots, a_{r-1} \in k$. Indeed, as $\|e_r\| \leq (\sqrt{\alpha})^{-1}\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\|$, we have

$$\alpha\|e_r\| \leq \sqrt{\alpha}\|e_r\| \leq \|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\|.$$

If $\|a_1e_1 + \cdots + a_{r-1}e_{r-1}\| \leq \|e_r\|$, then

$$\begin{aligned}\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\| &\geq \sqrt{\alpha}\|e_r\| \geq \sqrt{\alpha}\|a_1e_1 + \cdots + a_{r-1}e_{r-1}\| \\ &\geq \sqrt{\alpha}\left(\sqrt{\alpha}\max\{|a_1|\|e_1\|, \ldots, |a_{r-1}|\|e_{r-1}\|\}\right) \\ &= \alpha\max\{|a_1|\|e_1\|, \ldots, |a_{r-1}|\|e_{r-1}\|\}.\end{aligned}$$

Otherwise,

$$\begin{aligned}\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\| &= \|a_1e_1 + \cdots + a_{r-1}e_{r-1}\| \\ &\geq \sqrt{\alpha}\max\{|a_1|\|e_1\|, \ldots, |a_{r-1}|\|e_{r-1}\|\} \\ &\geq \alpha\max\{|a_1|\|e_1\|, \ldots, |a_{r-1}|\|e_{r-1}\|\},\end{aligned}$$

as required.

For the second assertion, it is sufficient to show the following lemma because it implies that the set $\{\|v - x\| \mid x \in V'\}$ has the minimal value. □

Lemma 1.4. *If $|\cdot|$ is discrete, then the set $\{\|v\| \mid v \in V \setminus \{0\}\}$ is discrete in $\mathbb{R}_{>0}$.*

Proof. Let us consider a map $\beta : V \setminus \{0\} \to \mathbb{R}_{>0}/|k^\times|$ given by

$$\beta(v) = \text{the class of } \|v\| \text{ in } \mathbb{R}_{>0}/|k^\times|.$$

It is sufficient to see that $\beta(V \setminus \{0\})$ is finite. Let $\beta_1, \ldots, \beta_l$ be distinct elements of $\beta(V \setminus \{0\})$. We choose $v_1, \ldots, v_l \in V \setminus \{0\}$ with $\beta(v_i) = \beta_i$ for $i = 1, \ldots, l$. If $i \neq j$, then $\|a_iv_i\| \neq \|a_jv_j\|$ for all $a_i, a_j \in k^\times$. Therefore, we obtain

$$\|a_1v_1 + \cdots + a_lv_l\| = \max\{\|a_1v_1\|, \ldots, \|a_1v_l\|\}$$

for all $a_1, \ldots, a_l \in k$. In particular, $v_1, \ldots, v_l$ are linearly independent. Therefore, we have $\#(\beta(V \setminus \{0\})) \leq \dim_k V$. □

1.3.2. *Scalar extension of norms.* Let V' be a vector space over k and $\|\cdot\|'$ a norm of V'.

Lemma 1.5. *For $\phi \in \operatorname{Hom}_k(V, V')$, the set $\left\{ \frac{\|\phi(v)\|'}{\|v\|} \,\middle|\, v \in V \setminus \{0\} \right\}$ is bounded from above.*

Proof. Fix $\alpha \in (0, 1)$. Let $(e_1, \ldots, e_r)$ be an α-orthogonal basis of V (cf. Proposition 1.3). We set

$$C_1 = \max\{\|\phi(e_1)\|', \ldots, \|\phi(e_r)\|'\} \quad \text{and} \quad C_2 = \min\{\|e_1\|, \ldots, \|e_r\|\}.$$

Then, for $v = a_1e_1 + \cdots + a_re_r \in V \setminus \{0\}$,

$$\begin{aligned}\frac{\|\phi(v)\|'}{\|v\|} &\leq \frac{\max\{|a_1|\|\phi(e_1)\|', \ldots, |a_r|\|\phi(e_r)\|'\}}{\alpha\max\{|a_1|\|e_1\|, \ldots, |a_r|\|e_r\|\}} \\ &\leq \frac{\max\{|a_1|C_1, \ldots, |a_r|C_1\}}{\alpha\max\{|a_1|C_2, \ldots, |a_r|C_2\}} = \frac{C_1}{\alpha C_2},\end{aligned}$$

as desired. □

By the above lemma, we define $\|\phi\|_{\mathrm{Hom}_k(V,V')}$ to be

$$\|\phi\|_{\mathrm{Hom}_k(V,V')} := \sup\left\{ \frac{\|\phi(v)\|'}{\|v\|} \mid v \in V \setminus \{0\} \right\}.$$

Note that $\|\cdot\|_{\mathrm{Hom}_k(V,V')}$ yields a norm on $\mathrm{Hom}_k(V,V')$. We denote $\|\cdot\|_{\mathrm{Hom}_k(V,k)}$ by $\|\cdot\|^\vee$ (i.e. the case where $V' = k$ and $\|\cdot\|' = |\cdot|$).

Lemma 1.6. *Let W be a subspace of V and $\psi \in W^\vee := \mathrm{Hom}_k(W,k)$. For any $\alpha \in (0,1)$, there is $\varphi \in V^\vee := \mathrm{Hom}_k(V,k)$ such that $\varphi|_W = \psi$ and*

$$\|\psi\|^\vee \leq \|\varphi\|^\vee \leq \alpha^{-1}\|\psi\|^\vee.$$

Proof. Let $(e_1,\ldots,e_r)$ be an α-orthogonal basis of V such that $W = ke_1 + \cdots + ke_l$ (cf. Proposition 1.3). We define $\varphi \in V^\vee$ to be

$$\varphi(a_1e_1 + \cdots + a_re_r) := \psi(a_1e_1 + \cdots + a_le_l)$$

for $a_1,\ldots,a_r \in k$. Then $\varphi|_W = \psi$. Moreover, note that

$$\begin{aligned}\alpha\|a_1e_1 + \cdots + a_le_l\| &\leq \alpha\max\{|a_1|\|e_1\|,\ldots,|a_l|\|e_l\|\} \\ &\leq \alpha\max\{|a_1|\|e_1\|,\ldots,|a_r|\|e_r\|\} \leq \|a_1e_1+\cdots+a_re_r\|,\end{aligned}$$

so that

$$\frac{|\varphi(a_1e_1+\cdots+a_re_r)|}{\|a_1e_1+\cdots+a_re_r\|} \leq \alpha^{-1}\frac{|\psi(a_1e_1+\cdots+a_le_l)|}{\|a_1e_1+\cdots+a_le_l\|} \leq \alpha^{-1}\|\psi\|^\vee$$

for all $a_1,\ldots,a_r \in k$ with $(a_1,\ldots,a_l) \neq (0,\ldots,0)$. Thus the assertion follows. □

Corollary 1.7. *The natural homomorphism $V \to (V^\vee)^\vee$ is an isometry.*

Proof. We denote the norm of $(V^\vee)^\vee$ by $\|\cdot\|'$, that is,

$$\|v\|' = \sup\left\{ \frac{|\phi(v)|}{\|\phi\|^\vee} \mid \phi \in V^\vee \setminus \{0\} \right\}.$$

Note that $|\phi(v)| \leq \|v\|\|\phi\|^\vee$ for all $v \in V$ and $\phi \in V^\vee$. In particular, $\|v\|' \leq \|v\|$. For $v \in V \setminus \{0\}$, we set $W := kv$ and choose $\psi \in W^\vee$ with $\psi(v) = 1$. Then $\|\psi\|^\vee = 1/\|v\|$. For any $\alpha \in (0,1)$, by Lemma 1.6, there is $\varphi \in V^\vee$ such that $\varphi|_W = \psi$ and $\|\varphi\|^\vee \leq \alpha^{-1}\|\psi\|^\vee$. As $|\varphi(v)|/\|\varphi\|^\vee \leq \|v\|'$, we have $\alpha\|v\| \leq \|v\|'$. Thus we obtain $\|v\| \leq \|v\|'$ by taking $\alpha \to 1$. □

Definition 1.8. Let k' be an extension field of k, and let $|\cdot|'$ be a complete absolute value of k' which is an extension of $|\cdot|$. We set $V_{k'} := V \otimes_k k'$. Identifying $V_{k'}$ with

$$\mathrm{Hom}_k(\mathrm{Hom}_k(V,k),k'),$$

we can give a norm $\|\cdot\|_{k'}$ of $V_{k'}$, that is,

$$\|v'\|_{k'} = \sup\left\{ \frac{|(\phi\otimes 1)(v')|'}{\|\phi\|^\vee} \,\middle|\, \phi \in V^\vee \right\}.$$

The norm $\|\cdot\|_{k'}$ is called the *scalar extension of* $\|\cdot\|$. Note that $\|v\otimes 1\|_{k'} = \|v\|$ for $v \in V$. Indeed, by Corollary 1.7,

$$\|v\otimes 1\|_{k'} = \sup\left\{ \frac{|\phi(v)|}{\|\phi\|^\vee} \,\middle|\, \phi \in V^\vee \right\} = \|v\|.$$

Proposition 1.9. *For $\alpha \in (0,1]$, let $(e_1,\ldots,e_r)$ be an α-orthogonal basis of V with respect to $\|\cdot\|$. Then $(e_1 \otimes 1,\ldots,e_r \otimes 1)$ also yields an α-orthogonal basis of $V_{k'}$ with respect to $\|\cdot\|_{k'}$.*

Proof. Let $(e_1^\vee,\ldots,e_r^\vee)$ be the dual basis of $(e_1,\ldots,e_r)$. For $a_1,\ldots,a_r \in k$ with $a_i \neq 0$,

$$\frac{|(e_i^\vee)(a_1e_1+\cdots+a_re_r)|}{\|a_1e_1+\cdots+a_re_r\|} \leq \frac{|a_i|}{\alpha\max\{|a_1|\|e_1\|,\ldots,|a_r|\|e_r\|\}} \leq \frac{|a_i|}{\alpha|a_i|\|e_i\|} = \frac{1}{\alpha\|e_i\|},$$

and hence $\|e_i^\vee\|^\vee \leq (\alpha\|e_i\|)^{-1}$. Therefore, for $a'_1,\ldots,a'_r \in k'$,

$$\begin{aligned}\|a'_1e_1+\cdots+a'_re_r\| &\geq \frac{|(e_i^\vee\otimes 1)(a'_1e_1+\cdots+a'_re_r)|'}{\|e_i^\vee\|^\vee}\\ &= \frac{|a'_i|'}{\|e_i^\vee\|^\vee} \geq \frac{|a'_i|'}{(\alpha\|e_i\|)^{-1}} = \alpha|a'_i|'\|e_i\|.\end{aligned}$$

Thus we have the assertion. □

Lemma 1.10. *Let k'' be an extension field of k', and let $|\cdot|''$ be a complete absolute value of k'' as an extension of $|\cdot|'$. We set $V_{k''} := V \otimes_k k''$. Note that $V_{k''} = V_{k'} \otimes_{k'} k''$. Let $\|\cdot\|_{k''}$ (resp. $\|\cdot\|_{k',k''}$) be a norm of $V_{k''}$ obtained by the scalar extension of $\|\cdot\|$ on V (resp. the scalar extension of $\|\cdot\|_{k'}$ on $V_{k'}$). Then $\|\cdot\|_{k''} = \|\cdot\|_{k',k''}$.*

Proof. For $\epsilon > 0$, let $(e_1,\ldots,e_r)$ be an $e^{-\epsilon}$-orthogonal basis of V with respect to $\|\cdot\|$. Then, by Proposition 1.9, $(e_1,\ldots,e_r)$ forms an $e^{-\epsilon}$-orthogonal basis of $V_{k'}$ and $V_{k''}$ with respect to $\|\cdot\|_{k'}$ and $\|\cdot\|_{k''}$, respectively, so that $(e_1,\ldots,e_r)$ is also an $e^{-\epsilon}$-orthogonal basis of $V_{k''}$ with respect to $\|\cdot\|_{k',k''}$. Note that $\|e_i\| = \|e_i\|_{k''} = \|e_i\|_{k',k''}$ for all $i = 1,\ldots,r$. Thus, for $a''_1,\ldots,a''_r \in k''$,

$$\begin{aligned}\|a''_1e_1+\ldots+a''_re_r\|_{k',k''} &\leq \max\{|a''_1|''\|e_1\|,\ldots,|a''_r|''\|e_r\|\}\\ &\leq e^{\epsilon}\|a''_1e_1+\ldots+a''_re_r\|_{k''}\end{aligned}$$

and

$$\begin{aligned}\|a''_1e_1+\ldots+a''_re_r\|_{k''} &\leq \max\{|a''_1|''\|e_1\|,\ldots,|a''_r|''\|e_r\|\}\\ &\leq e^{\epsilon}\|a''_1e_1+\ldots+a''_re_r\|_{k',k''}.\end{aligned}$$

Thus, we have the assertion by taking $\epsilon \to 0$. □

Lemma 1.11. *Let $f : V \to W$ be a surjective homomorphism of finite-dimensional vector spaces over k. Let $\|\cdot\|_V$ and $\|\cdot\|_W$ be norms of V and W, respectively. We assume that $\dim_k W = 1$ and $\|\cdot\|_W$ is the quotient norm of $\|\cdot\|_V$ in terms of the surjection $f : V \to W$. We set $V_{k'} := V \otimes_k k'$ and $W_{k'} := W \otimes_k k'$. Let $\|\cdot\|_{V,k'}$ and $\|\cdot\|_{W,k'}$ be the norms of $V_{k'}$ and $W_{k'}$ obtained by the scalar extensions of $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Then $\|\cdot\|_{W,k'}$ is the quotient norm of $\|\cdot\|_{V,k'}$ in terms of the surjection $f_{k'} := f \otimes \mathrm{id}_{k'} : V_{k'} \to W_{k'}$.*

Proof. Let $\|\cdot\|'_{W_{k'}}$ be the quotient norm of $\|\cdot\|_{V,k'}$ with respect to the surjection $f_{k'} : V_{k'} \to W_{k'}$. Let e be an non-zero element of W. As $\|e\|_{W,k'} = \|e\|_W$, it is sufficient to show that $\|e\|'_{W_{k'}} = \|e\|_W$. Note that

$$\{v \in V \mid f(v) = e\} \subseteq \{v' \in V_{k'} \mid f_{k'}(v') = e\},$$

so that we have $\|e\|_W \geq \|e\|'_{W_{k'}}$. Let us consider an inequality $\|e\|_W \leq \|e\|'_{W_{k'}}$. For $\epsilon > 0$, let $(e_1, \ldots, e_r)$ be an $e^{-\epsilon}$-orthogonal basis of V such that $(e_2, \ldots, e_r)$ forms a basis of $\mathrm{Ker}(f)$. Clearly we may assume that $f(e_1) = e$. Then

$$\begin{aligned}\|e\|'_{W_{k'}} &= \inf\{\|e_1 + a'_2 e_2 + \cdots + a'_r e_r\|_{V,k'} \mid a'_2, \ldots, a'_r \in k'\} \\ &\geq \inf\{e^{-\epsilon} \max\{\|e_1\|, |a'_2|'\|e_2\|_V, \ldots, |a'_r|'\|e_r\|_V\} \mid a'_2, \ldots, a'_r \in k'\} \\ &\geq e^{-\epsilon}\|e_1\| \geq e^{-\epsilon}\|e\|_W.\end{aligned}$$

Therefore, we have $\|e\|'_{W_{k'}} \geq \|e\|_W$ by taking $\epsilon \to 0$. □

Lemma 1.12. *We assume that the absolute value $|\cdot|$ of k is trivial. Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space over $(k, |\cdot|)$. Then we have the following:*

(1) The set $\{\|v\| \mid v \in V\}$ is a finite set.

(2) Let k' be a field and $|\cdot|'$ a complete and non-trivial absolute value of k' such that $k \subseteq k'$ and $|\cdot|'$ is an extension of $|\cdot|$. Let $\mathfrak{o}_{k'}$ be the valuation ring of $(k', |\cdot|')$ and $\mathfrak{m}_{k'}$ the maximal ideal of $\mathfrak{o}_{k'}$. We assume the following:

(i) The natural map $k \to \mathfrak{o}_{k'}$ induces an isomorphism $k \xrightarrow{\sim} \mathfrak{o}_{k'}/\mathfrak{m}_{k'}$.

(ii) If an equation $|a'|' = \|v\|/\|v'\|$ holds for some $a' \in k'^{\times}$ and $v, v' \in V \setminus \{0\}$, then $\|v\| = \|v'\|$.

Let $\|\cdot\|'$ be a norm of $V_{k'} := V \otimes_k k'$ over $(k', |\cdot|')$ such that $\|v\| = \|v \otimes 1\|'$ for all $v \in V$. If $(e_1, \ldots, e_r)$ is an orthogonal basis of $(V, \|\cdot\|)$, then $(e_1, \ldots, e_r)$ forms an orthogonal basis of $(V_{k'}, \|\cdot\|')$. In particular, $\|\cdot\|' = \|\cdot\|_{k'}$.

Proof. (1) Let $(e_1, \ldots, e_r)$ be an orthogonal basis of $(V, \|\cdot\|)$ (cf. Proposition 1.3). Then

$$\|a_1 e_1 + \cdots + a_r e_r\| = \max\{|a_1|\|e_1\|, \ldots, |a_r|\|e_r\|\}$$

for all $a_1, \ldots, a_r \in k$, so that

$$\|a_1 e_1 + \cdots + a_r e_r\| \in \{0, \|e_1\|, \ldots, \|e_r\|\}.$$

(2) First we assume that

$$\|e_1\| = \cdots = \|e_r\| = c.$$

Then, for any $v \in V$,

$$\|v\| = \begin{cases} c & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Let us see that

$$\|a'_1 e_1 + \cdots + a'_r e_r\|' = c \max\{|a'_1|', \ldots, |a'_r|'\}$$

for $a'_1, \ldots, a'_r \in k'$. Clearly we may assume that

$$(a'_1, \ldots, a'_r) \neq (0, \ldots, 0).$$

We set $\gamma := \max\{|a'_1|', \ldots, |a'_r|'\}$. We fix $\omega \in k'$ with $|\omega|' = \gamma$. By the assumption (i), for each $j = 1, \ldots, r$, we can find $a_j \in k$ and $b'_j \in k'$ such that

$$a'_j = a_j \omega + b'_j \quad \text{and} \quad |b'_j|' < \gamma.$$

Note that

$$a'_1 e_1 + \cdots + a'_r e_r = \omega \left(\textstyle\sum_{j=1}^r a_j e_j \right) + b'_1 e_1 + \cdots + b'_r e_r.$$

Moreover, as $\sum_{j=1}^r a_j e_j \neq 0$, we have

$$\left\| \omega \left(\textstyle\sum_{j=1}^r a_j e_j \right) \right\|' = \gamma \left\| \textstyle\sum_{j=1}^r a_j e_j \right\| = c\gamma$$

and

$$\|b'_1 e_1 + \cdots + b'_r e_r\|' \leq c \max\{|b'_1|', \ldots, |b'_r|'\} < c\gamma.$$

Therefore,

$$\|a'_1 e_1 + \cdots + a'_r e_r\|' = c\gamma = c \max\{|a'_1|', \ldots, |a'_r|'\}.$$

In general, we take positive numbers $c_1 < \cdots < c_b$ and non-empty subsets $I_1, \ldots, I_b$ of $\{1, \ldots, r\}$ such that $\{\|e_l\| \mid l \in I_s\} = \{c_s\}$ for $s = 1, \ldots, b$ and $I_1 \cup \cdots \cup I_b = \{1, \ldots, r\}$. Note that $I_s \cap I_{s'} = \emptyset$ for $s \neq s'$. Let us consider

$$x = a'_1 e_1 + \cdots + a'_r e_r = \sum_{s=1}^b x_s \in V_{k'} \quad (a'_1, \ldots, a'_r \in k'),$$

where $x_s = \sum_{l \in I_s} a'_l e_l$. Note that $(e_l)_{l \in I_s}$ forms an orthogonal basis of $\bigoplus_{l \in I_s} k e_l$ and $\|e_l\| = c_s$ for all $l \in I_s$. Therefore, by the above observation,

$$\|x_s\|' = c_s \max_{l \in I_s}\{|a'_l|'\} = \max_{l \in I_s}\{\|a'_l e_l\|'\},$$

so that it is sufficient to see that

$$\|x\|' = \max_{s=1,\ldots,b} \left\{ \|x_s\|' \right\}.$$

Clearly we may assume that $x \neq 0$. We set

$$\Sigma := \{s \in \{1, \ldots, b\} \mid x_s \neq 0\}.$$

For $s, s' \in \Sigma$ with $s \neq s'$, we have $\|x_s\|' \neq \|x_{s'}\|'$. Indeed, we choose $l_s \in I_s$ and $l_{s'} \in I_{s'}$ with $\|x_s\|' = \|a'_{l_s} e_{l_s}\|'$ and $\|x_{s'}\|' = \|a'_{l_{s'}} e_{l_{s'}}\|'$. If $\|x_s\|' = \|x_{s'}\|'$, then

$$\left| a'_{l_s} / a'_{l_{s'}} \right|' = \|e_{l_{s'}}\| / \|e_{l_s}\|,$$

so that, by the assumption (ii), $\|e_{l_{s'}}\| = \|e_{l_s}\|$, which is a contradiction. Therefore,

$$\|x\|' = \left\| \textstyle\sum_{s \in \Sigma} x_s \right\|' = \max_{s \in \Sigma}\{\|x_s\|'\} = \max_{s=1,\ldots,b}\{\|x_s\|'\},$$

as required. □

Remark 1.13. We assume that $|\cdot|'$ is discrete and

$$|a'|' = \exp(-\alpha\,\mathrm{ord}_{\mathfrak{o}_{k'}}(a')) \qquad (a' \in k')$$

for $\alpha \in \mathbb{R}_{>0}$. If

$$\alpha \notin \bigcup_{v,v' \in V\setminus\{0\}} \mathbb{Q}(\log\|v\| - \log\|v'\|),$$

then the assumption (ii) holds. Indeed, we suppose that $|a'|' = \|v\|/\|v'\|$ for some $a' \in {k'}^{\times}$ and $v, v' \in V \setminus \{0\}$. Then

$$-\alpha\,\mathrm{ord}_{\mathfrak{o}_{k'}}(a') = \log\|v\| - \log\|v'\|,$$

so that $\mathrm{ord}_{\mathfrak{o}_{k'}}(a') = 0$, and hence $\|v\| = \|v'\|$, as required.

1.3.3. *Lattices and norms.* From now on and until the end of the subsection , we assume that $|\cdot|$ is non-trivial. Let $\mathscr{V}$ be an $\mathfrak{o}_k$-submodule of V. We say that $\mathscr{V}$ is a *lattice of* V if $\mathscr{V} \otimes_{\mathfrak{o}_k} k = V$ and

$$\sup\{\|v\|_0 \mid v \in \mathscr{V}\} < \infty$$

for some norm $\|\cdot\|_0$ of V. Note that the condition $\sup\{\|v\|_0 \mid v \in \mathscr{V}\} < \infty$ does not depend on the choice of the norm $\|\cdot\|_0$ since all norms on V are equivalent. For a lattice $\mathscr{V}$ of V, we define $\|\cdot\|_{\mathscr{V}}$ to be

$$\|v\|_{\mathscr{V}} := \inf\{|a|^{-1} \mid a \in k^{\times} \text{ and } av \in \mathscr{V}\}.$$

Note that $\|\cdot\|_{\mathscr{V}}$ forms a norm of V. Moreover, for a norm $\|\cdot\|$ of V,

$$(V, \|\cdot\|)_{\leq 1} := \{v \in V \mid \|v\| \leq 1\}$$

is a lattice of V.

Proposition 1.14. *Let $\mathscr{V}$ be a lattice of V. We assume that, as an $\mathfrak{o}_k$-module, $\mathscr{V}$ admits a free basis $(e_1, \ldots, e_r)$. Then $(e_1, \ldots, e_r)$ is an orthonormal basis of V with respect to $\|\cdot\|_{\mathscr{V}}$.*

Proof. For $v = a_1e_1 + \cdots + a_re_r \in V$ and $a \in k^{\times}$,

$$\begin{aligned} av \in \mathscr{V} &\iff aa_i \in \mathfrak{o}_k \text{ for all } i = 1, \ldots, r \\ &\iff |a_i| \leq |a|^{-1} \text{ for all } i = 1, \ldots, r \\ &\iff \max\{|a_1|, \ldots, |a_r|\} \leq |a|^{-1}, \end{aligned}$$

so that $\|v\|_{\mathscr{V}} = \max\{|a_1|, \ldots, |a_r|\}$. □

Let us consider the following lemmas.

Lemma 1.15. *A subgroup G of $(\mathbb{R}, +)$ is either discrete or dense in $\mathbb{R}$.*

Proof. Clearly we may assume that $G \neq \{0\}$, so that $G \cap \mathbb{R}_{>0} \neq \emptyset$. We set $\delta = \inf(G \cap \mathbb{R}_{>0})$. If $\delta \in G \cap \mathbb{R}_{>0}$, then $G = \mathbb{Z}\delta$. Indeed, for $g \in G$, let n be an integer such that $n \leq g/\delta < n+1$. Thus $0 \leq g - n\delta < \delta$, and hence $g = n\delta$. Therefore, G is discrete.

Next we assume that $\delta \notin G \cap \mathbb{R}_{>0}$. Then there is a sequence $\{\delta_n\}_{n=1}^{\infty}$ in $G \cap \mathbb{R}_{>0}$ such that $\delta_n > \delta_{n+1}$ for all n and $\lim_{n\to\infty} \delta_n = \delta$. If we set $a_n = \delta_n - \delta_{n+1}$, then $a_n \in G \cap \mathbb{R}_{>0}$ and $\lim_{n\to\infty} a_n = 0$. For an open interval (α, β) of $\mathbb{R}$ ($\alpha < \beta$), we choose a_n and an integer m such that $a_n < \beta - \alpha$ and $m < \beta/a_n \leq m+1$. Then we have $ma_n < \beta$ and

$$\alpha < \beta - a_n \leq (m+1)a_n - a_n = ma_n,$$

so that $ma_n \in (\alpha, \beta) \cap G$. Thus G is dense. □

Lemma 1.16. *Let $\|\cdot\|$ be a norm of V and $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$. Then*

$$\|v\|_{\mathscr{V}} = \inf\{|b| \mid b \in k^\times \text{ and } \|v\| \leq |b|\}.$$

Moreover, $\|\cdot\| \leq \|\cdot\|_{\mathscr{V}}$ and $\|\cdot\|_{\mathscr{V}} \leq |\alpha|\|\cdot\|$ for all $\alpha \in k^\times$ with $|\alpha| > 1$.

Proof. The first assertion is obvious because, for $a \in k^\times$, $av \in \mathscr{V}$ if and only if $\|v\| \leq |a|^{-1}$.

For $v \in V$, let $a \in k^\times$ with $av \in \mathscr{V}$. Then $\|av\| \leq 1$, that is, $\|v\| \leq |a|^{-1}$, and hence $\|v\| \leq \|v\|_{\mathscr{V}}$.

Finally we consider the second inequality, that is, $\|v\|_{\mathscr{V}} \leq |\alpha|\|v\|$ for $v \in V$. Clearly we may assume that $v \neq 0$. As $|\alpha|^{-1} < 1$, there is $\epsilon > 0$ with $|\alpha|^{-1}e^{\epsilon} < 1$. By the first assertion, we can choose $b \in k^\times$ such that $\|v\| \leq |b| \leq e^{\epsilon}\|v\|_{\mathscr{V}}$. If $\|v\| < |b\alpha^{-1}|$, then

$$\|v\|_{\mathscr{V}} \leq |b||\alpha|^{-1} \leq e^{\epsilon}\|v\|_{\mathscr{V}}|\alpha|^{-1}.$$

Thus $1 \leq e^{\epsilon}|\alpha|^{-1}$. This is a contradiction, so that $\|v\| \geq |b\alpha^{-1}|$. Therefore,

$$\|v\|_{\mathscr{V}} \leq |b| \leq |\alpha|\|v\|,$$

as required. □

Proposition 1.17. *We assume that $|\cdot|$ is discrete. Then we have the following:*

1. *Every lattice $\mathscr{V}$ of V is a finitely generated $\mathfrak{o}_k$-module.*
2. *If we set $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$ for a norm of $\|\cdot\|$ of V, then $\|\cdot\| \leq \|\cdot\|_{\mathscr{V}} \leq |\varpi|^{-1}\|\cdot\|$.*

Proof. (1) Let $(e'_1, \ldots, e'_r)$ be an orthogonal basis of V with respect to $\|\cdot\|_{\mathscr{V}}$ (cf. Proposition 1.3). As $|\cdot|$ is discrete, there is $\lambda_i \in k^\times$ with $|\lambda_i| = \|e'_i\|_{\mathscr{V}}$. If we set $e_i = \lambda_i^{-1}e'_i$ for $i = 1, \ldots, r$, then $(e_1, \ldots, e_r)$ forms an orthonormal basis of V with respect to $\|\cdot\|_{\mathscr{V}}$. Therefore,

$$\mathscr{V} \subseteq (V, \|\cdot\|_{\mathscr{V}})_{\leq 1} = \mathfrak{o}_k e_1 + \cdots + \mathfrak{o}_k e_r.$$

Thus we have (1) because $\mathfrak{o}_k$ is noetherian.

(2) follows from Lemma 1.16. □

Proposition 1.18. *We assume that $|\cdot|$ is not discrete. If we set $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$ for a norm of $\|\cdot\|$ of V, then $\|\cdot\| = \|\cdot\|_{\mathscr{V}}$.*

Proof. By Lemma 1.15, we can find a sequence $\{\beta_n\}_{n=1}^{\infty}$ such that $|\beta_n| > 1$ and $\lim_{n\to\infty} |\beta_n| = 1$. On the other hand, by Lemma 1.16,

$$\|\cdot\| \leq \|\cdot\|_{\mathscr{V}} \leq |\beta_n|\|\cdot\|.$$

Therefore the assertion follows. □

Proposition 1.19. *We assume that the absolute value $|\cdot|$ is not discrete. Let $\|\cdot\|$ be a norm of V and $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$. For any $\epsilon > 0$, there is a sub-lattice $\mathscr{V}'$ of $\mathscr{V}$ such that $\mathscr{V}'$ is finitely generated over $\mathfrak{o}_k$ and $\|\cdot\| \leq \|\cdot\|_{\mathscr{V}'} \leq e^{\epsilon}\|\cdot\|$.*

Proof. Let $(e_1, \ldots, e_r)$ be an $e^{-\epsilon/2}$-orthogonal basis of V with respect to $\|\cdot\|$ (cf. Proposition 1.3). As $\|\cdot\| = \|\cdot\|_{\mathscr{V}}$ by Proposition 1.18, we can find $\lambda_i \in k^{\times}$ such that $\|e_i\| \le |\lambda_i| \le e^{\epsilon/2}\|e_i\|$ for each i. We set $\omega_i := \lambda_i^{-1} e_i$ $(i = 1, \ldots, r)$ and $\mathscr{V}' := \mathfrak{o}_k\omega_1 + \cdots + \mathfrak{o}_k\omega_r$. Note that $\omega_i \in \mathscr{V}$ for all i, that is, $\mathscr{V}'$ is a sub-lattice of $\mathscr{V}$ and $\mathscr{V}'$ is finitely generated over $\mathfrak{o}_k$. For $c_1, \ldots, c_r \in k$, by Proposition 1.14,

$$\begin{aligned}\|c_1e_1 + \cdots + c_re_r\|_{\mathscr{V}'} &= \|c_1\lambda_1\omega_1 + \cdots + c_r\lambda_r\omega_r\|_{\mathscr{V}'} = \max\{|c_1\lambda_1|, \ldots, |c_r\lambda_r|\} \\ &\le e^{\epsilon/2}\{|c_1|\|e_1\|, \ldots, |c_r|\|e_r\|\} \le e^{\epsilon}\|c_1e_1 + \cdots + c_re_r\|,\end{aligned}$$

so that we have $\|\cdot\|_{\mathscr{V}'} \le e^{\epsilon}\|\cdot\|$. □

2. Seminorm and integral extension

Let $\mathscr{A}$ be a finitely generated $\mathfrak{o}_k$-algebra, which contains $\mathfrak{o}_k$ as a subring. We set $A := \mathscr{A} \otimes_{\mathfrak{o}_k} k$. Note that A coincides with the localization of $\mathscr{A}$ with respect to $S := \mathfrak{o}_k \setminus \{0\}$. Let $\mathrm{Spec}(A)^{\mathrm{an}}$ be the analytification of $\mathrm{Spec}(A)$, that is, the set of all seminorms of A over the absolute value of k. For $x \in \mathrm{Spec}(A)^{\mathrm{an}}$, let $\mathfrak{o}_x$ and $\mathfrak{m}_x$ be the valuation ring of $(\hat{\kappa}(x), |\cdot|_x)$ and the maximal ideal of $\mathfrak{o}_x$, respectively (see §1.1.3 for the definition of $\hat{\kappa}(x)$). We denote the natural homomorphism $A \to \hat{\kappa}(x)$ by φ_x. It is easy to see that the following are equivalent:

(1) $\mathrm{Spec}(\hat{\kappa}(x)) \to \mathrm{Spec}(A)$ extends to $\mathrm{Spec}(\mathfrak{o}_x) \to \mathrm{Spec}(\mathscr{A})$, that is, there is a ring homomorphism $\tilde{\varphi}_x : \mathscr{A} \to \mathfrak{o}_x$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\tilde{\varphi}_x} & \mathfrak{o}_x \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow{\varphi_x} & \hat{\kappa}(x) \end{array}$$

(2) $|a|_x \le 1$ for all $a \in \mathscr{A}$.

Moreover, under the above conditions, the image of $\mathfrak{m}_x$ of $\mathrm{Spec}(\mathfrak{o}_x)$ is given by $\tilde{\varphi}_x^{-1}(\mathfrak{m}_x) = (\mathscr{A}, |\cdot|_x)_{<1}$, and $(\mathscr{A}, |\cdot|_x)_{<1} \in \mathrm{Spec}(\mathscr{A})_{\circ}$, where

$$\begin{cases} (\mathscr{A}, |\cdot|_x)_{<1} := \{a \in \mathscr{A} \mid |a|_x < 1\}, \\ \mathrm{Spec}(\mathscr{A})_{\circ} := \{P \in \mathrm{Spec}(\mathscr{A}) \mid P \cap \mathfrak{o}_k = \mathfrak{m}_k\}. \end{cases}$$

Let $\mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$ be the set of all $x \in \mathrm{Spec}(A)^{\mathrm{an}}$ such that the above condition (2) is satisfied. The map $r_{\mathscr{A}} : \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}} \to \mathrm{Spec}(\mathscr{A})_{\circ}$ given by

$$x \mapsto (\mathscr{A}, |\cdot|_x)_{<1}$$

is called the reduction map (cf. §1.1.6). Note that the reduction map is surjective (cf. [1, Proposition 2.4.4] or [5, 4.13 and Proposition 4.14]).

Theorem 2.1. *If we set $\mathscr{B} := \{\alpha \in A \mid \alpha$ is integral over $\mathscr{A}\}$, then*

$$\mathscr{B} = \bigcap_{x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}} (A, |\cdot|_x)_{\le 1},$$

where $(A, |\cdot|_x)_{\le 1} := \{\alpha \in A \mid |\alpha|_x \le 1\}$.

Proof. First let us see that $\mathscr{B} \subseteq (A, |\cdot|_x)_{\leq 1}$ for all $x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$. If $a \in \mathscr{B}$, then there are $a_1, \ldots, a_n \in \mathscr{A}$ such that $a^n + a_1 a^{n-1} + \cdots + a_n = 0$. We assume that $|a|_x > 1$. Then

$$\begin{aligned} |a|_x^n = |a^n|_x = |a_1 a^{n-1} + \cdots + a_n|_x &\leq \max_{i=1,\ldots,n} \{|a_i|_x |a|_x^{n-i}\} \\ &\leq \max_{i=1,\ldots,n} \{|a|_x^{n-i}\} = |a|_x^{n-1}, \end{aligned}$$

so that $|a|_x \leq 1$, which is a contradiction.

Let $a \in A$ such that a is not integral over $\mathscr{A}$. We show that there exists a prime ideal $\mathfrak{q}$ of $\mathscr{A}$ such that the canonical image of a in $A/S^{-1}\mathfrak{q}$ is not integral over $\mathscr{A}/\mathfrak{q}$. In fact, since A is a k-algebra of finite type, it is a noetherian ring. In particular, it admits only finitely many minimal prime ideals $S^{-1}\mathfrak{p}_1, \ldots, S^{-1}\mathfrak{p}_n$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals of $\mathscr{A}$ which do not intersect $S = \mathfrak{o}_k \setminus \{0\}$. Assume that, for any $i \in \{1, \ldots, n\}$, f_i is a monic polynomial in $(\mathscr{A}/\mathfrak{p}_i)[T]$ such that $f_i(\lambda_i) = 0$, where λ_i is the class of a in $A/S^{-1}(\mathfrak{p}_i)$. Let F_i be a monic polynomial in $\mathscr{A}[T]$ whose reduction modulo $\mathfrak{p}_i[T]$ identifies with f_i. One has $F_i(a) \in S^{-1}\mathfrak{p}_i$ for any $i \in \{1, \ldots, n\}$. Let F be the product of the polynomials $F_1, \ldots, F_n$. Then $F(a)$ belongs to the intersection $\bigcap_{i=1}^n S^{-1}\mathfrak{p}_i$, hence is nilpotent, which implies that a is integral over $\mathscr{A}$. To show that there exists $x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$ such that $|a|_x > 1$ we may replace $\mathscr{A}$ (resp. A) by $\mathscr{A}/\mathfrak{q}$ (resp. $A/S^{-1}\mathfrak{q}$) and hence assume that $\mathscr{A}$ is an integral domain without loss of generality.

We set $b = a^{-1}$. Let us see that

$$b\mathscr{A}[b] \cap \mathfrak{o}_k \neq \{0\} \quad \text{and} \quad 1 \notin b\mathscr{A}[b].$$

We set $a = a'/s$ for some $a' \in \mathscr{A}$ and $s \in S$. Then $s = ba' \in b\mathscr{A}[b] \cap \mathfrak{o}_k$, so that $b\mathscr{A}[b] \cap \mathfrak{o}_k \neq \{0\}$. Next we assume that $1 \in b\mathscr{A}[b]$. Then

$$1 = a'_1 b + a'_2 b^2 + \cdots + a'_{n'} b^{n'}$$

for some $a'_1, \ldots, a'_{n'} \in \mathscr{A}$, so that $a^{n'} = a'_1 a^{n'-1} + \cdots + a'_{n'}$, which is a contradiction.

Let $\mathfrak{p}$ be the maximal ideal of $\mathscr{A}[b]$ such that $b\mathscr{A}[b] \subseteq \mathfrak{p}$. As $\mathfrak{p} \cap \mathfrak{o}_k \neq \{0\}$ and $\mathfrak{p} \cap \mathfrak{o}_k \subseteq \mathfrak{m}_k$, we have $\mathfrak{p} \cap \mathfrak{o}_k = \mathfrak{m}_k$, and hence $\mathfrak{p} \in \mathrm{Spec}(\mathscr{A}[b])_\circ$. Note that $\mathscr{A}[b]$ is finitely generated over $\mathfrak{o}_k$ and $\mathscr{A}[b] \otimes_{\mathfrak{o}_k} k = A[b]$. Thus, since the reduction map

$$r_{\mathscr{A}[b]} : \mathrm{Spec}(A[b])^{\mathrm{an}}_{\mathscr{A}[b]} \to \mathrm{Spec}(\mathscr{A}[b])_\circ$$

is surjective, there is $x \in \mathrm{Spec}(A[b])^{\mathrm{an}}_{\mathscr{A}[b]}$ such that $r_{\mathscr{A}[b]}(x) = \mathfrak{p}$. Clearly $x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$. As $b \in \mathfrak{p}$, we have $|b|_x < 1$, so that $|a|_x > 1$ because $ab = 1$. Therefore,

$$a \notin \bigcap_{x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}} (A, |\cdot|_x)_{\leq 1},$$

as required. □

We assume that X is projective. Let $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ be a flat and projective scheme over $\mathrm{Spec}\,\mathfrak{o}_k$ such that the generic fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ is X. Let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$ such that $\mathscr{L}|_X = L$. We set $h := \{|\cdot|_{\mathscr{L}}(x)\}_{x \in X^{\mathrm{an}}}$. For the definition of the metric $|\cdot|_{\mathscr{L}}(x)$ at x, see §1.1.6.

Corollary 2.2. *Fix $l \in H^0(X, L)$. If $|l|_{\mathscr{L}}(x) \leq 1$ for all $x \in X^{\mathrm{an}}$, then there is $s \in \mathfrak{o}_k \setminus \{0\}$ such that $sl^{\otimes n} \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$ for all $n \geq 0$.*

Proof. Let $\mathscr{X} = \bigcup_{i=1}^N \mathrm{Spec}(\mathscr{A}_i)$ be an affine open covering of $\mathscr{X}$ with the following properties:

(1) $\mathscr{A}_i$ is a finitely generated over $\mathfrak{o}_k$ for every i.
(2) $\mathrm{Spec}(\mathscr{A}_i)_\circ \neq \emptyset$ for all i.
(3) There is a basis ω_i of $\mathscr{L}$ over $\mathrm{Spec}(\mathscr{A}_i)$ for every i.

We set $l = a_i\omega_i$ for some $a_i \in A_i := \mathscr{A}_i \otimes_{\mathfrak{o}_k} k$. By our assumption, $|a_i|_x \leq 1$ for all $x \in \mathrm{Spec}(A_i)^{\mathrm{an}}_{\mathscr{A}_i}$. Therefore, by Theorem 2.1, a_i is integral over $\mathscr{A}_i$, so that, by the following Lemma 2.3, we can find $s_i \in S$ such that $s_i a_i^n \in \mathscr{A}_i$ for all $n \geq 0$. We set $s = s_1 \cdots s_N$. Then, as $sa_i^n \in \mathscr{A}_i$ for all $n \geq 0$ and $i = 1, \ldots, N$, we have the assertion. □

Lemma 2.3. *Let A be a commutative ring and S a multiplicatively closed subset of A, which consists of regular elements of A. If $t \in S^{-1}A$ and t is integral over A, then there is $s \in S$ such that $st^n \in A$ for all $n \geq 0$.*

Proof. As t is integral over A, there are $a_1, \ldots, a_{r-1} \in A$ such that

$$t^r = a_1 t^{r-1} + \cdots + a_{r-1}t + a_r.$$

We choose $s \in S$ such that $st^i \in A$ for $i = 0, \ldots, r-1$. By induction on n, we prove that $st^n \in A$ for all $n \geq 0$. Note that

$$t^n = a_1 t^{n-1} + \cdots + a_{r-1}t^{n-r+1} + a_r t^{n-r}.$$

Thus, if $st^i \in A$ for $i = 0, \ldots, n-1$, then $st^n \in A$ because

$$st^n = a_1(st^{n-1}) + \cdots + a_{r-1}(st^{n-r+1}) + a_r(st^{n-r}).$$

□

3. Continuous metrics of invertible sheaves

In this section, we consider several properties of continuous metrics of invertible sheaves. Let $h = \{|\cdot|_h(x)\}_{x \in X^{\mathrm{an}}}$ and $h' = \{|\cdot|_{h'}(x)\}_{x \in X^{\mathrm{an}}}$ be continuous metrics of L^{an} (cf. §1.1.4). As $L(x) := L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$ is a 1-dimensional vector space over $\hat{\kappa}(x)$, $h + h' := \{|\cdot|_h(x) + |\cdot|_{h'}(x)\}_{x \in X^{\mathrm{an}}}$ forms a continuous metric of L^{an}. Indeed, we can find a continuous positive function φ on X^{an} such that $|\cdot|_{h'}(x) = \varphi(x)|\cdot|_h(x)$ for any $x \in X^{\mathrm{an}}$. Thus

$$h + h' = \{(1 + \varphi(x))|\cdot|_h(x)\}_{x \in X^{\mathrm{an}}}$$

is a continuous metric of L^{an}.

Lemma 3.1. *There is a continuous metric of L^{an}.*

Proof. Let us choose an affine open covering $X = \bigcup_{i=1}^N U_i$ together with a local basis ω_i of L on each U_i. Let h_i be a metric of L^{an} over U_i^{an} given by $|\omega_i|_{h_i}(x) = 1$ for $x \in U_i^{\mathrm{an}}$. As X^{an} is paracompact (locally compact and σ-compact), we can find a partition of unity $\{\rho_i\}_{i=1,\ldots,N}$ of continuous functions on X^{an} such that $\mathrm{supp}(\rho_i) \subseteq U_i^{\mathrm{an}}$ for all i. If we set $|\cdot|_h(x) = \sum_{i=1}^N \rho_i(x)|\cdot|_{h_i}(x)$, then $h = \{|\cdot|_h(x)\}_{x \in X^{\mathrm{an}}}$ yields a continuous metric of L^{an}. □

3.1. **Extension theorem for a metric arising from a model.** We assume that X is projective. Let $\mathscr{X} \to \operatorname{Spec} \mathfrak{o}_k$ be a model of X. We let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$ such that $\mathscr{L}|_X = L$. We have seen in §1.1.6 that $\mathscr{L}$ induces a continuous metric $h = \{|\cdot|_{\mathscr{L}}(x)\}_{x \in X^{\mathrm{an}}}$ of L^{an}.

Theorem 3.2. *We assume that $|\cdot|$ is non-trivial and $\mathscr{L}$ is an ample invertible sheaf. Fix a closed subscheme Y of X, $l \in H^0(Y, L|_Y)$ and a positive number ϵ. Then there are a positive integer n and $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^{\otimes n}$ and*

$$\|s\|_{h^n} \leq e^{n\epsilon} \left(\|l\|_{Y,h}\right)^n.$$

Proof. Clearly, we may assume that $l \neq 0$. Let $\mathscr{Y}$ be the Zariski closure of Y in $\mathscr{X}$ (cf. §1.1.7).

Claim 3.2.1. *There are a positive integer a and $\alpha \in k^\times$ such that*

$$e^{-a\epsilon/2} \leq \|\alpha l^{\otimes a}\|_{Y,h^a} \leq 1.$$

Proof. First we assume that $|\cdot|$ is discrete. We take a positive integer a such that $e^{-\epsilon a/2} \leq |\varpi|$. We also choose $\alpha \in k^\times$ such that

$$|\alpha^{-1}| = \min\{|\gamma| \mid \gamma \in k^\times \text{ and } \|l^{\otimes a}\|_{Y,h^a} \leq |\gamma|\}.$$

Then, as $\|l^{\otimes a}\|_{Y,h^a} \leq |\alpha^{-1}| \leq |\varpi|^{-1}\|l^{\otimes a}\|_{Y,h^a}$, we have

$$e^{-a\epsilon/2} \leq |\varpi| \leq \|\alpha l^{\otimes a}\|_{Y,h^a} \leq 1.$$

Next we assume that $|\cdot|$ is not discrete. In this case, $|k^\times|$ is dense in $\mathbb{R}_{>0}$ by Lemma 1.15, so that we can choose $\beta \in k^\times$ such that

$$e^{-\epsilon/2} \leq \|l\|_{Y,h}/|\beta| \leq 1.$$

Thus if we set $\alpha = \beta^{-1}$ and $a = 1$, we have the assertion. □

By Corollary 2.2, there is $\beta \in \mathfrak{o}_K \setminus \{0\}$ such that

$$\beta(\alpha l^{\otimes a})^{\otimes m} \in H^0(\mathscr{Y}, \mathscr{L}^{\otimes am}|_{\mathscr{Y}})$$

for all $m \geq 0$. We choose a positive integer m such that $|\beta|^{-1} \leq e^{am\epsilon/2}$ and

$$H^0(\mathscr{X}, \mathscr{L}^{\otimes am}) \to H^0(\mathscr{Y}, \mathscr{L}^{\otimes am}|_{\mathscr{Y}})$$

is surjective, so that we can find $l_m \in H^0(\mathscr{X}, \mathscr{L}^{\otimes am})$ such that $l_m|_{\mathscr{Y}} = \beta(\alpha l^{\otimes a})^{\otimes m}$. Note that $\|l_m\|_{h^{am}} \leq 1$. Thus, if we set $s = \beta^{-1}\alpha^{-m} l_m$, then $s|_{\mathscr{Y}} = l^{\otimes am}$ and

$$\begin{aligned} \|s\|_{h^{am}} &= |\beta|^{-1}|\alpha|^{-m}\|l_m\|_{h^{am}} \leq e^{am\epsilon/2}|\alpha|^{-m} \\ &\leq e^{am\epsilon/2}|\alpha|^{-m}\left(e^{a\epsilon/2}\|\alpha l^{\otimes a}\|_{Y,h^a}\right)^m = e^{am\epsilon}\left(\|l\|_{Y,h}\right)^{am}, \end{aligned}$$

as required. □

3.2. **Quotient metric.** Let V be a finite-dimensional vector space over k. We assume that there is a surjective homomorphism

$$\pi : V \otimes_k \mathscr{O}_X \to L.$$

For each $e \in V$, $\pi(e \otimes 1)$ yields a global section of L, that is, $\pi(e \otimes 1) \in H^0(X, L)$. We denote it by $\tilde{e}$. Let $\|\cdot\|$ be a norm of V and $\overline{V} := (V, \|\cdot\|)$. Let $\|\cdot\|_{\hat{\kappa}(x)}$ be a norm of $V \otimes_k \hat{\kappa}(x)$ obtained by the scalar extension of $\|\cdot\|$ (cf. Definition 1.8). Let $|\cdot|^{\mathrm{quot}}_{\overline{V}}(x)$ be the quotient norm of $L(x) := L \otimes \hat{\kappa}(x)$ induced by $\|\cdot\|_{\hat{\kappa}(x)}$ and the surjective homomorphism $V \otimes_k \hat{\kappa}(x) \to L(x)$.

Lemma 3.3. *Let h be a continuous metric of L^{an} (cf. Lemma 3.1). Let $(e_0, \ldots, e_r)$ be an orthogonal basis of V with respect to $\|\cdot\|$. Then, for $s \in H^0(X, L)$,*

$$|s|^{\mathrm{quot}}_{\overline{V}}(x) = \frac{|s|_h(x)}{\displaystyle\max_{i=0,\ldots,r} \left\{ \frac{|\tilde{e}_i|_h(x)}{\|e_i\|} \right\}}$$

on X^{an}.

Proof. We set $I := \{i \mid \tilde{e}_i \neq 0 \text{ in } H^0(X, L)\}$ and $U_i := \{p \in X \mid \tilde{e}_i \neq 0 \text{ at } p\}$ for $i \in I$.

Claim 3.3.1. *For a fixed $j \in I$, if we set $\tilde{e}_i = a_{ij}\tilde{e}_j$ on U_j ($a_{ij} \in \mathscr{O}_{U_j}$), then*

$$|\tilde{e}_j|^{\mathrm{quot}}_{\overline{V}}(x) = \frac{1}{\displaystyle\max_{i=0,\ldots,r} \left\{ \frac{|a_{ij}|_x}{\|e_i\|} \right\}}$$

on U_j^{an}.

Proof. We set $c_i = \|e_i\|$ for $i = 0, \ldots, r$. Without loss of generality, we may assume that $j = 0$, that is, we need to show that

$$|\tilde{e}_0|^{\mathrm{quot}}_{\overline{V}}(x) = \frac{1}{\max\{1/c_0, |a_{10}|_x/c_1, \ldots, |a_{r0}|_x/c_r\}}.$$

Since

$$\ker(\pi_x : V \otimes_k \hat{\kappa}(x) \to L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)) = \langle e_1 - a_{10}(x)e_0, \ldots, e_r - a_{r0}(x)e_0 \rangle$$

for $x \in U_0^{\mathrm{an}}$, we have

$$|\tilde{e}_0|^{\mathrm{quot}}_{\overline{V}}(x) = \inf \left\{ f(\lambda_1, \ldots, \lambda_r) \mid (\lambda_1, \ldots, \lambda_r) \in \hat{\kappa}(x)^r \right\},$$

where $f(\lambda_1, \ldots, \lambda_r) := \left\| e_0 + \sum_{i=1}^r \lambda_i(e_i - a_{i0}(x)e_0) \right\|_{\hat{\kappa}(x)}$. Note that

$$f(\lambda_1, \ldots, \lambda_r) = \max\left\{ c_0 \left| 1 - \textstyle\sum_{i=1}^r \lambda_i a_{i0}(x) \right|_x, c_1|\lambda_1|_x, \ldots, c_r|\lambda_r|_x \right\}.$$

As

$$\max\{\alpha_0, \ldots, \alpha_r\} \max\{\beta_0, \ldots, \beta_r\} \geq \max\{\alpha_0\beta_0, \ldots, \alpha_r\beta_r\}$$

for $\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_r \in \mathbb{R}_{\geq 0}$, we have

$$f(\lambda_1, \ldots, \lambda_r) \cdot \max\{1/c_0, |a_{10}(x)|_x/c_1, \ldots, |a_{r0}(x)|_x/c_r\}$$
$$\geq \max\left\{\left|1 - \sum_{i=1}^r \lambda_i a_{i0}(x)\right|_x, |\lambda_1 a_{10}(x)|_x, \ldots, |\lambda_r a_{r0}(x)|_x\right\}$$
$$\geq \left|1 - \sum_{i=1}^r \lambda_i a_{i0}(x) + \sum_{i=1}^r \lambda_i a_{i0}(x)\right|_x = 1.$$

Therefore, we obtain

$$\inf\{f(\lambda_1, \ldots, \lambda_r) \mid (\lambda_1, \ldots, \lambda_r) \in \hat{\kappa}(x)^n\}$$
$$\geq \frac{1}{\max\{1/c_0, |a_{10}(x)|_x/c_1, \ldots, |a_{r0}(x)|_x/c_r\}}.$$

We need to see that

$$f(\eta_1, \ldots, \eta_r) = \frac{1}{\max\{1/c_0, |a_{10}(x)|_x/c_1, \ldots, |a_{r0}(x)|_x/c_r\}}.$$

for some $\eta_1, \ldots, \eta_r \in \hat{\kappa}(x)$. As $f(0, \ldots, 0) = c_0$, the assertion holds if

$$\max\{1/c_0, |a_{10}(x)|_x/c_1, \ldots, |a_{r0}(x)|_x/c_r\} = 1/c_0.$$

Next we assume that

$$\max\{1/c_0, |a_{10}(x)|_x/c_1, \ldots, |a_{r0}(x)|_x/c_r\} = |a_{i0}(x)|_x/c_i$$

for some i. Clearly $a_{i0}(x) \neq 0$. If we set

$$\eta_j = \begin{cases} 0 & \text{if } j \neq i, \\ 1/a_{i0}(x) & \text{if } j = i, \end{cases}$$

then $f(\eta_1, \ldots, \eta_n) = c_i/|a_{i0}(x)|_x$, as required. □

If we set $s = f\tilde{e}_j$ on U_j ($f \in \mathscr{O}_{U_j}$), then $|s|^{\mathrm{quot}}_{\overline{V}}(x) = |f|_x |\tilde{e}_j|^{\mathrm{quot}}_{\overline{V}}(x)$ on U_j^{an}, so that, by Claim 3.3.1,

$$|s|^{\mathrm{quot}}_{\overline{V}}(x) = \frac{|f|_x}{\max\limits_{i=0,\ldots,r}\left\{\dfrac{|a_{ij}|_x}{\|e_i\|}\right\}}.$$

On the other hand, $|s|_h(x) = |f|_x |\tilde{e}_j|_h(x)$ and $|\tilde{e}_i|_h(x) = |a_{ij}|_x |\tilde{e}_j|_h(x)$ for $i = 0, \ldots, r$. Thus

$$|s|^{\mathrm{quot}}_{\overline{V}}(x) = \frac{|s|_h(x)}{\max\limits_{i=0,\ldots,n}\left\{\dfrac{|\tilde{e}_i|_h(x)}{\|e_i\|}\right\}}$$

on U_j^{an}. Therefore, the assertion follows because $X = \bigcup_{j \in I} U_j$. □

Corollary 3.4. *$\left\{|\cdot|^{\mathrm{quot}}_{\overline{V}}(x)\right\}_{x \in X^{\mathrm{an}}}$ yields a continuous metric of L^{an}.*

Proof. If V has an orthogonal basis with respect to $\|\cdot\|$, then the assertion follows from Lemma 3.3.

In general, by Proposition 1.3, for each $n \in \mathbb{Z}_{>0}$, we choose a basis

$$(e_{n,0}, e_{n,1}, \ldots, e_{n,r})$$

of V such that

$$(1-1/n)\max\{|c_0|\|e_{n,0}\|, \ldots, |c_r|\|e_{n,r}\|\} \leq \|c_0 e_{n,0} + \cdots + c_r e_{n,r}\|$$

for all $c_0, \ldots, c_r \in k$. If we set

$$\|c_0 e_{n,0} + \cdots + c_r e_{n,r}\|_n := \max\{|c_0|\|e_{n,0}\|, \ldots, |c_r|\|e_{n,r}\|\}$$

for $c_0, \ldots, c_r \in k$. Then $(1-1/n)\|\cdot\|_n \leq \|\cdot\| \leq \|\cdot\|_n$, so that

$$(1-1/n)|\cdot|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}(x) \leq |\cdot|^{\mathrm{quot}}_{(V,\|\cdot\|)}(x) \leq |\cdot|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}(x)$$

for all $x \in X^{\mathrm{an}}$. Let ω be a local basis of L over an open set U. Then the above inequalities imply that

$$\log(1-1/n) \leq \log\left(|\omega|^{\mathrm{quot}}_{(V,\|\cdot\|)}(x)\right) - \log\left(|\omega|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}(x)\right) \leq 0$$

for all $x \in U^{\mathrm{an}}$, which shows that the sequence $\left\{\log\left(|\omega|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}(x)\right)\right\}_{n=1}^{\infty}$ converges to $\log\left(|\omega|^{\mathrm{quot}}_{(V,\|\cdot\|)}(x)\right)$ uniformly on U^{an}. Thus, by the previous observation, $\log\left(|\omega|^{\mathrm{quot}}_{(V,\|\cdot\|)}(x)\right)$ is continuous on U^{an}. □

From now on and until the end of the subsection, we assume that X is projective and L is generated by global sections. Let $h = \{|\cdot|_h(x)\}_{x \in X^{\mathrm{an}}}$ be a continuous metric of L^{an}. As $H^0(X, L) \otimes_k \mathscr{O}_X \to L$ is surjective, by Corollary 3.4,

$$h^{\mathrm{quot}} = \left\{|\cdot|^{\mathrm{quot}}_{(H^0(X,L),\|\cdot\|_h)}(x)\right\}_{x \in X^{\mathrm{an}}}$$

yields a continuous metric of L^{an}. For simplicity, we denote $|\cdot|^{\mathrm{quot}}_{(H^0(X,L),\|\cdot\|_h)}(x)$ by $|\cdot|^{\mathrm{quot}}_h(x)$. Moreover, the supreme norm of $H^0(X, L)$ arising from h^{quot} is denoted by $\|\cdot\|^{\mathrm{quot}}_h$, that is, $\|\cdot\|^{\mathrm{quot}}_h := \|\cdot\|_{h^{\mathrm{quot}}}$.

Lemma 3.5. *(1) $|\cdot|_h(x) \leq |\cdot|^{\mathrm{quot}}_h(x)$ for all $x \in X^{\mathrm{an}}$.*

(2) $\|\cdot\|_h = \|\cdot\|^{\mathrm{quot}}_h$.

(3) Let (L', h') be a pair of an invertible sheaf L' on X and a continuous metric $h' = \{|\cdot|_{h'}(x)\}_{x \in X^{\mathrm{an}}}$ of L'^{an} such that L' is generated by global sections. Then

$$|l \cdot l'|^{\mathrm{quot}}_{h \otimes h'}(x) \leq |l|^{\mathrm{quot}}_h(x)|l'|^{\mathrm{quot}}_{h'}(x)$$

for $l \in L(x)$ and $l' \in L'(x)$.

Proof. (1) Fix $l \in L(x) \setminus \{0\}$. For $\epsilon > 0$, let $(e_1, \ldots, e_n)$ be an $e^{-\epsilon}$-orthogonal basis of $H^0(X, L)$ with respect to $\|\cdot\|_h$. There is $s \in H^0(X, L) \otimes_k \hat{\kappa}(x)$ such that $s(x) = l$

and $\|s\|_{h,\hat{\kappa}(x)} \leq e^{\epsilon}|l|_h^{\mathrm{quot}}(x)$. We set $s = a_1e_1 + \cdots + a_ne_n$ $(a_1, \ldots, a_n \in \hat{\kappa}(x))$. Then, by Proposition 1.9,

$$\begin{aligned}\|s\|_{h,\hat{\kappa}(x)} &\geq e^{-\epsilon}\max\{|a_1|_x\|e_1\|_h, \ldots, |a_n|_x\|e_n\|_h\} \\ &\geq e^{-\epsilon}\max\{|a_1|_x|e_1|_h(x), \ldots, |a_n|_x|e_n|_h(x)\} \geq e^{-\epsilon}|l|_h(x),\end{aligned}$$

so that $|l|_h(x) \leq e^{2\epsilon}|l|_h^{\mathrm{quot}}(x)$, and hence the assertion follows because ϵ is an arbitrary positive number.

(2) By (1), we have $\|\cdot\|_h \leq \|\cdot\|_h^{\mathrm{quot}}$. On the other hand, as $|s|_h^{\mathrm{quot}}(x) \leq \|s\|_h$ for $s \in H^0(X, L)$, we have $\|s\|_h^{\mathrm{quot}} \leq \|s\|_h$.

(3) For $\epsilon > 0$, there are $s \in H^0(X, L) \otimes_k \hat{\kappa}(x)$ and $s' \in H^0(X, L') \otimes_k \hat{\kappa}(x)$ such that

$$s(x) = l,\ s'(x) = l',\ \|s\|_{h,\hat{\kappa}(x)} \leq e^{\epsilon}|l|_h^{\mathrm{quot}}(x) \text{ and } \|s'\|_{h',\hat{\kappa}(x)} \leq e^{\epsilon}|l'|_{h'}^{\mathrm{quot}}(x).$$

Here let us see that $\|s \cdot s'\|_{h\otimes h',\hat{\kappa}(x)} \leq e^{2\epsilon}\|s\|_{h,\hat{\kappa}(x)}\|s'\|_{h',\hat{\kappa}(x)}$. Let $(s_1, \ldots, s_m)$ and $(s'_1, \ldots, s'_{m'})$ be $e^{-\epsilon}$-orthogonal bases of $H^0(X, L)$ and $H^0(X, L')$, respectively. If we set $s = t_1s_1 + \cdots + t_ms_m$ and $s' = t'_1s'_1 + \cdots + t'_{m'}s'_{m'}$ $(t_1, \ldots, t_m, t'_1, \ldots, t'_{m'} \in \hat{\kappa}(x))$, then

$$s \cdot s' = \sum_{i,j} t_it'_js_i \cdot s'_j.$$

Thus,

$$\begin{aligned}\|s \cdot s'\|_{h\otimes h',\hat{\kappa}(x)} &\leq \max_{i,j}\left\{|t_i|_x|t'_j|_x\|s_i \cdot s'_j\|_{h\otimes h'}\right\} \leq \max_{i,j}\left\{|t_i|_x|t'_j|_x\|s_i\|_h\|s'_j\|_{h'}\right\} \\ &\leq \max_i\{|t_i|_x\|s_i\|_h\}\max_j\left\{|t'_j|_x\|s'_j\|_{h'}\right\} \\ &\leq e^{2\epsilon}\|s\|_{h,\hat{\kappa}(x)}\|s'\|_{h',\hat{\kappa}(x)}.\end{aligned}$$

Therefore, we have $(s \cdot s')(x) = l \cdot l'$ and

$$|l \cdot l'|_{h\otimes h'}^{\mathrm{quot}}(x) \leq \|s \cdot s'\|_{h\otimes h',\hat{\kappa}(x)} \leq e^{2\epsilon}\|s\|_{h,\hat{\kappa}(x)}\|s'\|_{h',\hat{\kappa}(x)} \leq e^{4\epsilon}|l|_h^{\mathrm{quot}}(x)|l'|_{h'}^{\mathrm{quot}}(x),$$

as required. □

Proposition 3.6. *If there are a normed finite-dimensional vector space $(V, \|\cdot\|)$ and a surjective homomorphism $V \otimes_k \mathscr{O}_X \to L$ such that h is given by $\left\{|\cdot|_{(V,\|\cdot\|)}^{\mathrm{quot}}(x)\right\}_{x\in X^{\mathrm{an}}}$, then $|\cdot|_{h^n}(x) = |\cdot|_{h^n}^{\mathrm{quot}}(x)$ for all $n \geq 1$.*

Proof. First we consider the case $n = 1$. Fix $l \in L(x) \setminus \{0\}$. For $\epsilon > 0$, there is $s \in V \otimes_k \hat{\kappa}(x)$ such that $\tilde{s}(x) = l$ and $\|s\|_{\hat{\kappa}(x)} \leq e^{\epsilon}|l|_h(x)$.

Note that $\|\tilde{e}\|_h \leq \|e\|$ for all $e \in V$. Let $(e_1, \ldots, e_r)$ be an $e^{-\epsilon}$-orthogonal basis of V with respect to $\|\cdot\|$. If we set $s = a_1e_1 + \cdots + a_re_r$ $(a_1, \ldots, a_r \in \hat{\kappa}(x))$, then, by Proposition 1.9,

$$\begin{aligned}\|\tilde{s}\|_{h,\hat{\kappa}(x)} &\leq \max\{|a_1|_x\|\tilde{e}_1\|_h, \ldots, |a_r|_x\|\tilde{e}_r\|_h\} \\ &\leq \max\{|a_1|_x\|e_1\|, \ldots, |a_r|_x\|e_r\|\} \\ &\leq e^{\epsilon}\|s\|_{\hat{\kappa}(x)},\end{aligned}$$

so that

$$|l|_h^{\mathrm{quot}}(x) \le \|\tilde{s}\|_{h,\hat{\kappa}(x)} \le e^{\epsilon}\|s\|_{\hat{\kappa}(x)} \le e^{2\epsilon}|l|_h(x),$$

and hence $|l|_h^{\mathrm{quot}}(x) \le |l|_h(x)$ by taking $\epsilon \to 0$. Thus the assertion for $n = 1$ follows from (1) in Lemma 3.5.

In general, by using (3) in Lemma 3.5,

$$|l^n|_{h^n}(x) = (|l|_h(x))^n = \left(|l|_h^{\mathrm{quot}}(x)\right)^n \ge |l^n|_{h^n}^{\mathrm{quot}}(x),$$

and hence we have the assertion by (1) in Lemma 3.5. □

Lemma 3.7. *We assume that there are a normed finite-dimensional vector space $(V, \|\cdot\|)$ and a surjective homomorphism $V \otimes_k \mathscr{O}_X \to L$ such that h is given by $\left\{|\cdot|_{(V,\|\cdot\|)}^{\mathrm{quot}}(x)\right\}_{x \in X^{\mathrm{an}}}$. Let k' be an extension field of k, and let $|\cdot|'$ be a complete absolute value of k' as an extension of $|\cdot|$. We set*

$$X' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k'), \quad L = L \otimes_k k' \quad \textit{and} \quad V' := V \otimes_k k'.$$

Let $\|\cdot\|'$ be a norm of V' obtained by the scalar extension of $\|\cdot\|$. Moreover, let h' be a continuous metric of L'^{an} given by the scalar extension of h. Then h' coincides with $\left\{|\cdot|_{(V',\|\cdot\|')}^{\mathrm{quot}}(x')\right\}_{x' \in X'^{\mathrm{an}}}$.

Proof. Let $f : X' \to X$ be the projection. For $x' \in X'^{\mathrm{an}}$, we set $x = f^{\mathrm{an}}(x')$. Then $\hat{\kappa}(x) \subseteq \hat{\kappa}(x')$ and $(L \otimes_k \hat{\kappa}(x)) \otimes_{\hat{\kappa}(x)} \hat{\kappa}(x') = L' \otimes_{k'} \hat{\kappa}(x')$, that is, $L(x) \otimes_{\hat{\kappa}(x)} \hat{\kappa}(x') = L'(x')$. Moreover, $V' \otimes_{k'} \hat{\kappa}(x') = (V \otimes_k \hat{\kappa}(x)) \otimes_{\hat{\kappa}(x)} \hat{\kappa}(x')$, and by Lemma 1.10, $\|\cdot\|'_{\hat{\kappa}(x')} = \|\cdot\|_{\hat{\kappa}(x')} = \|\cdot\|_{\hat{\kappa}(x),\hat{\kappa}(x')}$. Thus the assertion follows from Lemma 1.11. □

Proposition 3.8. *We assume that there is a subspace H of $H^0(X, L)$ such that $H \otimes_k \mathscr{O}_X \to L$ is surjective and the morphism $\phi_H : X \to \mathbb{P}(H)$ induced by H is a closed embedding. We identify X with $\phi_H(X)$, so that $L = \mathscr{O}_{\mathbb{P}(H)}(1)\Big|_X$. Let $\|\cdot\|$ be a norm of H such that H has an orthonormal basis $(e_1, \ldots, e_r)$ with respect to $\|\cdot\|$. We set*

$$h := \left\{|\cdot|_{(H,\|\cdot\|)}^{\mathrm{quot}}(x)\right\}_{x \in X^{\mathrm{an}}} \quad \textit{and} \quad \mathscr{H} := \mathfrak{o}_k e_1 + \cdots + \mathfrak{o}_k e_r = (H, \|\cdot\|)_{\le 1}.$$

Let $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}(\mathscr{H})$ (cf. §1.1.7) and $\mathscr{L} := \mathscr{O}_{\mathbb{P}(\mathscr{H})}(1)\Big|_{\mathscr{X}}$. Then $|\cdot|_h(x) = |\cdot|_{\mathscr{L}}(x)$ for all $x \in X^{\mathrm{an}}$.

Proof. First let us see that $|s|_h(x) \le |s|_{\mathscr{L}}(x)$ for $s \in H$. Let ω_ξ be a local basis of $\mathscr{L}$ at $\xi = r_{\mathscr{X}}(x)$. If we set $s = s_\xi \omega_\xi$, then

$$|s|_{\mathscr{L}}(x) = |s_\xi|_x.$$

As $s_\xi^{-1} s \in \mathscr{L}_\xi$ and $\mathscr{H} \otimes_{\mathfrak{o}_k} \mathscr{O}_{\mathscr{X},\xi} \to \mathscr{L}_\xi$ is surjective, there are $l_1, \ldots, l_r \in \mathscr{H}$ and $a_1, \ldots, a_r \in \mathscr{O}_{\mathscr{X},\xi}$ such that $s_\xi^{-1} s = a_1 l_1 + \cdots + a_r l_r$. Therefore,

$$\begin{aligned} \left|s_\xi^{-1} s\right|_h(x) &\le \max\{|a_1 l_1|_h(x), \ldots, |a_r l_r|_h(x)\} \\ &= \max\{|a_1|_x |l_1|_h(x), \ldots, |a_r|_x |l_r|_h(x)\} \le 1, \end{aligned}$$

so that $|s|_h(x) \leq |s_\xi|_x = |s|_{\mathscr{L}}(x)$, as required.

Next let us see that $|l|_{\mathscr{L}}(x) \leq \|l\|_{\hat{\kappa}(x)}$ for all $l \in H \otimes \hat{\kappa}(x)$. By Proposition 1.9, $(e_1, \ldots, e_r)$ is an orthonormal basis of $H \otimes \hat{\kappa}(x)$ with respect to $\|\cdot\|_{\hat{\kappa}(x)}$. Thus, if we set $l = a_1 e_1 + \cdots + a_r e_r$ $(a_1, \ldots, a_r \in \hat{\kappa}(x))$, then

$$\begin{aligned} |l|_{\mathscr{L}}(x) &\leq \max\{|a_1|_x |e_1|_{\mathscr{L}}(x), \ldots, |a_r|_x |e_r|_{\mathscr{L}}(x)\} \\ &\leq \max\{|a_1|_x, \ldots, |a_r|_x\} = \|l\|_{\hat{\kappa}(x)}. \end{aligned}$$

Finally let us see that $|s|_{\mathscr{L}}(x) \leq |s|_h(x)$ for $s \in H$. For $\epsilon > 0$, we choose $l \in H \otimes \hat{\kappa}(x)$ such that $l(x) = s(x)$ and $\|l\|_{\hat{\kappa}(x)} \leq e^{\epsilon}|s|_h(x)$. Then, by the previous observation,

$$|s|_{\mathscr{L}}(x) = |l|_{\mathscr{L}}(x) \leq \|l\|_{\hat{\kappa}(x)} \leq e^{\epsilon}|s|_h(x).$$

Thus the assertion follows. □

Remark 3.9. We assume that $|\cdot|$ is non-trivial and $\|\cdot\| = \|\cdot\|_{\mathscr{H}}$ for some finitely generated lattice $\mathscr{H}$ of H. Then a free basis $(e_1, \ldots, e_r)$ of $\mathscr{H}$ yields an orthonormal basis of H with respect to $\|\cdot\|$ (cf. Proposition 1.14). Moreover, $\mathscr{H} = (H, \|\cdot\|)_{\leq 1}$.

3.3. **Semipositive metric.** We assume that L is semiample, namely certain tensor power of L is generated by global sections. We say that a continuous metric $h = \{|\cdot|_h(x)\}_{x \in X^{\mathrm{an}}}$ is *semipositive* if there are a sequence $\{e_n\}$ of positive integers and a sequence $\{(V_n, \|\cdot\|_n)\}$ of normed finite-dimensional vector spaces over k such that there is a surjective homomorphism $V_n \otimes_k \mathscr{O}_X \to L^{\otimes e_n}$ for every n, and that the sequence

$$\left\{ \frac{1}{e_n} \log \frac{|\cdot|^{\mathrm{quot}}_{(V_n, \|\cdot\|_n)}(x)}{|\cdot|_{h^{e_n}}(x)} \right\}_{n=1}^{\infty}$$

converges to 0 uniformly on X^{an}.

Proposition 3.10. *If X is projective, L is generated by global sections, and h is semipositive, then the sequence*

$$\left\{ \frac{1}{m} \log \frac{|\cdot|^{\mathrm{quot}}_{h^m}(x)}{|\cdot|_{h^m}(x)} \right\}_{m=1}^{\infty}$$

converges to 0 *uniformly on* X^{an}.

Proof. We set

$$a_m = \max_{x \in X^{\mathrm{an}}} \left\{ \log \frac{|\cdot|^{\mathrm{quot}}_{h^m}(x)}{|\cdot|_{h^m}(x)} \right\}.$$

Then $a_{m+m'} \leq a_m + a_{m'}$ by (3) in Lemma 3.5, and hence $\lim_{m\to\infty} a_m/m = \inf\{a_m/m\}$ by Fekete's lemma. For $\epsilon > 0$, there is e_n such that

$$e^{-e_n\epsilon}|\cdot|_{h^{e_n}}(x) \leq |\cdot|_{h_n}(x) \leq e^{e_n\epsilon}|\cdot|_{h^{e_n}}(x)$$

for all $x \in X^{\mathrm{an}}$, where $h_n = \left\{|\cdot|^{\mathrm{quot}}_{(V_n, \|\cdot\|_n)}(x)\right\}_{x \in X^{\mathrm{an}}}$. Thus

$$e^{-e_n\epsilon}\|\cdot\|_{h^{e_n}} \leq \|\cdot\|_{h_n} \leq e^{e_n\epsilon}\|\cdot\|_{h^{e_n}},$$

so that $e^{-e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x) \leq |\cdot|^{\mathrm{quot}}_{h_n}(x) \leq e^{e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x)$. Thus, by Proposition 3.6,

$$e^{-e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x) \leq |\cdot|_{h_n}(x) \leq e^{e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x).$$

Therefore,

$$1 \leq \frac{|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x)}{|\cdot|_{h^{e_n}}(x)} = \frac{|\cdot|_{h_n}(x)}{|\cdot|_{h^{e_n}}(x)} \frac{|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x)}{|\cdot|_{h_n}(x)} \leq e^{2e_n\epsilon},$$

that is, $0 \leq a_{e_n}/e_n \leq 2\epsilon$, and hence $0 \leq \lim_{m\to\infty} a_m/m \leq 2\epsilon$, as required. □

Corollary 3.11. *A continuous metric h is semipositive if and only if, for any $\epsilon > 0$, there is a positive integer n such that, for all $x \in X^{\mathrm{an}}$, we can find $s \in H^0(X, L^{\otimes n})_{\hat{\kappa}(x)} \setminus \{0\}$ with $\|s\|_{h^n,\hat{\kappa}(x)} \leq e^{n\epsilon}|s|_{h^n}(x)$.*

Proof. First we assume that h is semipositive. By using Proposition 3.10, we can find a positive integer n such that $L^{\otimes n}$ is generated by global sections and

$$|\cdot|_{h^n}(x) \leq |\cdot|^{\mathrm{quot}}_{h^n}(x) \leq e^{n\epsilon/2}|\cdot|_{h^n}(x)$$

for all $x \in X^{\mathrm{an}}$. On the other hand, there is $s \in H^0(X, L^{\otimes n})_{\hat{\kappa}(x)} \setminus \{0\}$ such that $\|s\|_{h^n,\hat{\kappa}(x)} \leq e^{n\epsilon/2}|s|^{\mathrm{quot}}_{h^n}(x)$. Thus,

$$\|s\|_{h^n,\hat{\kappa}(x)} \leq e^{n\epsilon/2}|s|^{\mathrm{quot}}_{h^n}(x) \leq e^{n\epsilon}|s|_{h^n}(x).$$

Next we consider the converse. For a positive integer m, there is a positive integer e_m such that, for any $x \in X^{\mathrm{an}}$, we can find $s \in H^0(X, L^{\otimes e_m})_{\hat{\kappa}(x)} \setminus \{0\}$ with $\|s\|_{h^{e_m},\hat{\kappa}(x)} \leq e^{e_m/m}|s|_{h^{e_m}}(x)$. Clearly $L^{\otimes e_m}$ is generated by global sections. Moreover,

$$|s|_{h^{e_m}}(x) \leq |s|^{\mathrm{quot}}_{(H^0(X,L^{\otimes e_m}),\|\cdot\|_{h^{e_m}})}(x) \leq e^{e_m/m}|s|_{h^{e_m}}(x),$$

that is,

$$0 \leq \frac{1}{e_m}\log\left(\frac{|\cdot|^{\mathrm{quot}}_{(H^0(X,L^{\otimes e_m}),\|\cdot\|_{h^{e_m}})}(x)}{|\cdot|_{h^{e_m}}(x)}\right) \leq \frac{1}{m}.$$

Thus h is semipositive. □

Corollary 3.12. *Let h be a continuous metric of L^{an}. If there are a sequence $\{e_n\}$ of positive integers and a sequence $\{h_n\}$ of metrics such that h_n is a semipositive metric of $(L^{\otimes e_n})^{\mathrm{an}}$ for each n and*

$$\frac{1}{e_n}\log\frac{|\cdot|_{h_n}(x)}{|\cdot|_{h^{e_n}}(x)}$$

converges to 0 uniformly as $n \to \infty$, then h is semipositive.

Proof. For a positive number $\epsilon > 0$, choose a positive integer n such that

$$e^{-\epsilon e_n/3}h^{e_n} \leq h_n \leq e^{\epsilon e_n/3}h^{e_n}.$$

As h_n is semipositive, by Corollary 3.11, there is a positive integer m such that, for all $x \in X^{\mathrm{an}}$, we can find $s \in H^0(X, L^{\otimes me_n})_{\hat{\kappa}(x)} \setminus \{0\}$ with $\|s\|_{h_n^m,\hat{\kappa}(x)} \leq e^{me_n\epsilon/3}|s|_{h_n^m}(x)$, so that

$$\|s\|_{h^{me_n},\hat{\kappa}(x)} \leq e^{\epsilon me_n/3}\|s\|_{h_n^m,\hat{\kappa}(x)} \leq e^{2me_n\epsilon/3}|s|_{h_n^m}(x) \leq e^{me_n\epsilon}|s|_{h^{me_n}}(x).$$

Therefore, the assertion follows from Corollary 3.11. □

3.4. **The functions σ and μ on X^{an}.** Throughout this subsection, we assume that X is projective. Let $\widehat{\mathrm{Pic}}_{C^0}(X)$ denote the group of isomorphism classes of pairs (L, h) consisting of an invertible sheaf L on X and a continuous metric h of L^{an}. Fix $\overline{L} = (L, h) \in \widehat{\mathrm{Pic}}_{C^0}(X)$. We assume that L is generated by global sections. We define $\sigma_{\overline{L}}(x)$ to be

$$\sigma_{\overline{L}}(x) := \log\left(\frac{|\cdot|_h^{\mathrm{quot}}(x)}{|\cdot|_h(x)}\right).$$

Lemma 3.13. *For $\overline{L}$ and $\overline{L}' \in \widehat{\mathrm{Pic}}_{C^0}(X)$ such that both L and L' are generated by global sections, we have the following:*

(1) $\sigma_{\overline{L}} \geq 0$ on X^{an}.
(2) $\sigma_{\overline{L}\otimes\overline{L}'}(x) \leq \sigma_{\overline{L}}(x) + \sigma_{\overline{L}'}(x)$ for $x \in X^{\mathrm{an}}$.
(3) If $\overline{L} \simeq \overline{L}'$, then $\sigma_{\overline{L}} = \sigma_{\overline{L}'}$ on X^{an}.

Proof. (1) and (3) are obvious. (2) follows from (3) in Lemma 3.5. □

We assume that L is semiample. We set

$$\mathbb{N}(L) := \left\{ n \in \mathbb{Z}_{\geq 1} \mid L^{\otimes n} \text{ is generated by global sections} \right\}.$$

Note that $\mathbb{N}(L) \neq \emptyset$ and $\mathbb{N}(L)$ forms a subsemigroup of $\mathbb{Z}_{\geq 1}$ with respect to the addition of $\mathbb{Z}_{\geq 1}$. For $x \in X^{\mathrm{an}}$, we define $\mu_{\overline{L}}(x)$ to be

$$\mu_{\overline{L}}(x) := \inf\left\{ \frac{\sigma_{\overline{L}^{\otimes n}}(x)}{n} \;\middle|\; n \in \mathbb{N}(L) \right\}.$$

Note that $\mu_{\overline{L}}$ is upper-semicontinuous on X^{an} because $\sigma_{\overline{L}^{\otimes n}}$ is continuous for all $n \in \mathbb{N}(L)$. We set

$$\widehat{\mathrm{Pic}}^{+}_{C^0}(X) := \{(L, h) \in \widehat{\mathrm{Pic}}_{C^0}(X) \mid L \text{ is semiample}\}.$$

Note that $\widehat{\mathrm{Pic}}^{+}_{C^0}(X)$ forms a semigroup with respect to $\otimes$.

Lemma 3.14. *Let $\overline{L} = (L, h)$ and $\overline{L}' = (L', h')$ be elements of $\widehat{\mathrm{Pic}}^{+}_{C^0}(X)$. Then we have the following:*

(1) $\mu_{\overline{L}} \geq 0$ on X^{an}.
(2) $\mu_{\overline{L}}(x) = \lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}(L)}} \dfrac{\sigma_{\overline{L}^{\otimes n}}(x)}{n}$ for $x \in X^{\mathrm{an}}$.
(3) $\mu_{\overline{L}\otimes\overline{L}'}(x) \leq \mu_{\overline{L}}(x) + \mu_{\overline{L}'}(x)$ for $x \in X^{\mathrm{an}}$.
(4) If $\overline{L} \simeq \overline{L}'$, then $\mu_{\overline{L}} = \mu_{\overline{L}'}$ on X^{an}.
(5) For $n \geq 0$, $\mu_{\overline{L}^{\otimes n}} = n\mu_{\overline{L}}$ on X^{an}.

Proof. (1) follows from (1) in Lemma 3.13.

(2) Since $\sigma_{\overline{L}^{\otimes(n+n')}}(x) \leq \sigma_{\overline{L}^{\otimes n}}(x) + \sigma_{\overline{L}^{\otimes n'}}(x)$ for $n, n' \in \mathbb{N}(L)$ by (2) in Lemma 3.13, the assertion follows from Fekete's lemma.

(3) and (4) follow from (2) and (3) in Lemma 3.13 together with (2), respectively.

(5) If $n = 0$, then the assertion is obvious, so that we may assume that $n \geq 1$. We fix $n_0 \in \mathbb{N}(L)$. Then $n_0 \in \mathbb{N}(L^{\otimes n})$. Thus, by (2),

$$\mu_{\overline{L}^{\otimes n}}(x) = \lim_{m\to\infty} \frac{\sigma_{L^{\otimes mn_0 n}}(x)}{mn_0} = n \lim_{m\to\infty} \frac{\sigma_{L^{\otimes mn_0 n}}(x)}{mn_0 n} = n\mu_{\overline{L}}(x).$$

□

We set $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}} := \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and

$$\widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}} := \{(L,h) \in \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}} \mid L \text{ is semiample}\}.$$

Let $\iota : \widehat{\mathrm{Pic}}_{C^0}(X) \to \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}}$ be the canonical homomorphism. For $\overline{L} \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$, we choose a positive integer n and $\overline{L}_n \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)$ with $\iota(\overline{L}_n) = \overline{L}^{\otimes n}$. Then $\mu_{\overline{L}_n}(x)/n$ does not depend on the choice of n and $\overline{L}_n$. Indeed, let us choose another $n' \in \mathbb{Z}_{\geq 1}$ and $\overline{L}_{n'} \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)$ with $\iota(\overline{L}_{n'}) = \overline{L}^{\otimes n'}$. As $\iota(\overline{L}_n^{\otimes n'}) = \iota(\overline{L}_{n'}^{\otimes n}) = \overline{L}^{\otimes nn'}$, there is a positive integer m such that $\overline{L}_n^{\otimes mn'} = \overline{L}_{n'}^{\otimes mn}$. By (5) in Lemma 3.14,

$$mn'\mu_{\overline{L}_n}(x) = \mu_{\overline{L}_n^{\otimes mn'}}(x) = \mu_{\overline{L}_{n'}^{\otimes mn}}(x) = mn\mu_{\overline{L}_{n'}}(x),$$

that is, $\mu_{\overline{L}_n}(x)/n = \mu_{\overline{L}_{n'}}(x)/n'$, as required. By abuse of notation, it is also denoted by $\mu_{\overline{L}}(x)$.

Lemma 3.15. *For $\overline{L}, \overline{L}' \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$, we have the following:*

(1) $\mu_{\overline{L}\otimes\overline{L}'}(x) \leq \mu_{\overline{L}}(x) + \mu_{\overline{L}'}(x)$ for $x \in X^{\mathrm{an}}$.
(2) For $a \in \mathbb{Q}_{\geq 0}$, $\mu_{\overline{L}^{\otimes a}} = a\mu_{\overline{L}}$ on X^{an}.
(3) Let $\overline{L}_1, \ldots, \overline{L}_r$ be elements of $\widehat{\mathrm{Pic}}_{C_0}(X)_{\mathbb{Q}}$. We assume that there are open intervals $I_1, \ldots, I_r$ of $\mathbb{R}$ such that

$$\overline{L} \otimes \overline{L}_1^{\otimes t_1} \otimes \cdots \otimes \overline{L}_r^{\otimes t_r} \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$$

for all $(t_1, \ldots, t_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$. Then, for a fixed $x \in X^{\mathrm{an}}$, there is a continuous function $f : I_1 \times \cdots \times I_r \to \mathbb{R}$ such that

$$f(t_1, \ldots, t_r) = \mu_{\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r}}(x)$$

for all $(t_1, \ldots, t_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$.

Proof. (1) and (2) are consequences of (3) and (5) in Lemma 3.14, respectively.

(3) We set

$$f_0(t_1, \ldots, t_r) := \mu_{\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r}}(x)$$

for $(t_1,\dots,t_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$. By (1) and (2), for $\lambda \in [0,1] \cap \mathbb{Q}$ and $(t_1,\dots,t_r), (t'_1,\dots,t'_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$, we have

$$
\begin{aligned}
f_0(\lambda(t_1,\dots,t_r) &+ (1-\lambda)(t'_1,\dots,t'_r)) \\
&= \mu^{\mathrm{a}}_{(\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r})^{\otimes\lambda}\otimes(\overline{L}\otimes\overline{L}_1^{\otimes t'_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t'_r})^{\otimes(1-\lambda)}}(x) \\
&\leq \lambda\mu_{\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r}}(x) + (1-\lambda)\mu_{\overline{L}\otimes\overline{L}_1^{\otimes t'_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t'_r}}(x) \\
&= \lambda f_0(t_1,\dots,t_r) + (1-\lambda) f_0(t'_1,\dots,t'_r),
\end{aligned}
$$

that is, f_0 is concave on $(I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$. Therefore, the assertion (3) follows from [7, Corollary 1.3.2]. □

Let (L,h) be an element of $\widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$. We say that h is semipositive if there is a positive integer n such that $L^{\otimes n} \in \mathrm{Pic}(X)$ and h^n is semipositive. The following characterization of the semipositivity of h is a consequence of Proposition 3.10.

Proposition 3.16. *For $\overline{L} = (L,h) \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$, h is semipositive if and only if $\mu_{\overline{L}} = 0$ on X^{an}.*

We assume that $|\cdot|$ is non-trivial. Let $\mathscr{X}$ be a model of X over $\mathrm{Spec}(\mathfrak{o}_k)$. Let $L \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ and $\mathscr{L} \in \mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$ with $\mathscr{L}|_X = L$. Let m be a positive integer such that $L^{\otimes m} \in \mathrm{Pic}(X)$. Then we define $\overline{L} = (L,h)$ to be

$$(L,h) := \left(L^{\otimes m}, \{|\cdot|_{\mathscr{L}^{\otimes m}}(x)\}_{x \in X^{\mathrm{an}}}\right)^{\otimes 1/m}.$$

Proposition 3.17. *If L is ample and $\mathscr{L}$ is nef, then h is semipositive.*

Proof. First we assume that $\mathscr{L}$ is ample. We choose a positive integer n such that $\mathscr{L}^{\otimes n} \in \mathrm{Pic}(\mathscr{X})$ and $\mathscr{L}^{\otimes n}$ is very ample. Then we have an embedding $\iota : \mathscr{X} \to \mathbb{P}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}))$ and $\mathscr{L}^{\otimes n} = \iota^*(\mathscr{O}_{\mathbb{P}(H^0(\mathscr{X},\mathscr{L}^{\otimes n}))}(1))$. Let $(e_1,\dots,e_r)$ be a free basis of $H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$. We define a norm $\|\cdot\|$ of $H^0(X, L^{\otimes n})$ to be

$$\|a_1e_1 + \cdots + a_re_r\| := \max\{|a_1|,\dots,|a_r|\}.$$

Note that $(H^0(X,L^{\otimes n}), \|\cdot\|)_{\leq 1} = H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$, so that, by Proposition 3.8, we have $|\cdot|^{\mathrm{quot}}_{(H,\|\cdot\|)}(x) = |\cdot|_{\mathscr{L}^{\otimes n}}(x)$ for $x \in X^{\mathrm{an}}$. Thus h is semipositive.

In general, let $\mathscr{A}$ be an ample invertible sheaf on $\mathscr{X}$ and $A := \mathscr{A}|_X$. We choose $\delta \in \mathbb{Q}_{>0}$ such that $L \otimes A^{\otimes a}$ is ample for all $a \in (-\delta,\delta) \cap \mathbb{Q}$. Note that $\overline{L} \otimes (A, |\cdot|_{\mathscr{A}})^{\otimes\epsilon} = (L \otimes A^{\otimes\epsilon}, |\cdot|_{\mathscr{L}\otimes\mathscr{A}^{\otimes\epsilon}})$, so that $\mu_{\overline{L}\otimes(A,|\cdot|_{\mathscr{A}})^{\otimes\epsilon}} = 0$ for $\epsilon \in (0,\delta) \cap \mathbb{Q}$ by the previous observation together with Proposition 3.16. On the other hand, by (3) in Lemma 3.15,

$$\mu_{\overline{L}}(x) = \lim_{\substack{\epsilon\downarrow 0\\ \epsilon\in\mathbb{Q}}} \mu_{\overline{L}\otimes(A,|\cdot|_{\mathscr{A}})^{\otimes\epsilon}}(x).$$

Therefore, $\mu_{\overline{L}} = 0$, and hence h is semipositive by Proposition 3.16. □

Remark 3.18. Assume that the absolute value $|\cdot|$ is non-trivial. Let L be an ample invertible sheaf on X, equipped with a semipositive continuous metric h. Then there exists a sequence $\{(\mathscr{X}_n, \mathscr{L}_n)\}_{n\geqslant 1}$, where $\mathscr{X}_n$ is a model of X and $\mathscr{L}_n$ is a nef invertible sheaf on $\mathscr{X}_n$ such that $\mathscr{L}_n|_X = L^{\otimes n}$ and that $h_n = (|\cdot|_{\mathscr{L}_n}(x)^{1/n})_{x\in X^{\mathrm{an}}}$ converges uniformly to h. This follows from Proposition 3.10 and the comparison between quotient metrics and model metrics (via the embedding into the projective spaces of lattices). Combining with Proposition 3.17 and Corollary 3.11, we obtain that, in the non-trivial valuation case, our semipositivity coincides with that of Zhang [12] and Moriwaki [8]. We refer the readers to [6, §6] and to [2, §6.8] for the descriptions of the semipositivity in terms of plurisubharmonic currents. Note that their semipositivity is also equivalent to our semipositivity.

4. Extension theorem

Throughout this section, we assume that X is projective. Let us begin with a special case of the extension theorem. The general extension theorem is a consequence of the special case.

Theorem 4.1. *We assume that L is very ample. Let $\|\cdot\|$ be a norm of $H^0(X,L)$ and h a continuous metric of L^{an} given by $\{|\cdot|^{\mathrm{quot}}_{(H^0(X,L),\|\cdot\|)}(x)\}_{x\in X^{\mathrm{an}}}$. Let Y be a closed subschme of X and $l \in H^0(Y, L|_Y)$. Then, for any $\epsilon > 0$, there are a positive integer n and $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^{\otimes n}$ and $\|s\|_{h^{\otimes n}} \leq e^{n\epsilon}(\|l\|_{Y,h})^n$.*

Proof. First we assume that $|\cdot|$ is non-trivial. Let us begin with the following:

Claim 4.1.1. *There are a positive integer a and a finitely generated lattice $\mathscr{H}$ of $H^0(X, L^{\otimes a})$ such that*

$$\|\cdot\|_{h^a} \leq \|\cdot\|_{\mathscr{H}} \leq e^{a\epsilon/2}\|\cdot\|_{h^a}.$$

Proof. First we assume that $|\cdot|$ is discrete. We choose a positive integer a such that $|\varpi|^{-1} \leq e^{a\epsilon/2}$. We set $\mathscr{H} := \{s \in H^0(X, L^{\otimes a}) \mid \|s\|_{h^a} \leq 1\}$. Note that $\mathscr{H}$ is a finitely generated lattice of $H^0(X, L^{\otimes a})$ by Proposition 1.17. As $\|\cdot\|_{h^a} \leq \|\cdot\|_{\mathscr{H}} \leq |\varpi|^{-1}\|\cdot\|_{h^a}$ by Proposition 1.17, we have the assertion.

Next we assume that $|\cdot|$ is not discrete. By Proposition 1.18, there is a lattice $\mathscr{V}$ of $H^0(X,L)$ such that $\|\cdot\|_h = \|\cdot\|_{\mathscr{V}}$. By Proposition 1.19, there is a finitely generated lattice $\mathscr{H}$ of $H^0(X,L)$ such that $\mathscr{H} \subseteq \mathscr{V}$ and $\|\cdot\|_h \leq \|\cdot\|_{\mathscr{H}} \leq e^{\epsilon/2}\|\cdot\|_h$, as desired. □

Let $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}(\mathscr{H})$ (cf. §1.1.7) and $\mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{H})}(1)\Big|_{\mathscr{X}}$. Moreover, let h' be a continuous metric of $(L^{\otimes a})^{\mathrm{an}}$ given by

$$\{|\cdot|^{\mathrm{quot}}_{(H,\|\cdot\|_{\mathscr{H}})}(x)\}_{x\in X^{\mathrm{an}}}.$$

Then, by Proposition 3.8 and Remark 3.9, $|\cdot|_{h'} = |\cdot|_{\mathscr{L}}$. Therefore, by virtue of Theorem 3.2, there are a positive integer m and $s \in H^0(X, L^{\otimes am})$ such that $s|_Y = l^{\otimes am}$ and

$$\|s\|_{h'^m} \leq e^{am\epsilon/2}(\|l^{\otimes a}\|_{Y,h'})^m. \tag{5}$$

As $\|\cdot\|_{h^a} \leq \|\cdot\|_{\mathscr{H}} \leq e^{a\epsilon/2}\|\cdot\|_{h^a}$, we have

$$|\cdot|^{\mathrm{quot}}_{h^a}(x) \leq |\cdot|_{h'}(x) \leq e^{a\epsilon/2}|\cdot|^{\mathrm{quot}}_{h^a}(x)$$

for all $x \in X^{\mathrm{an}}$. Therefore, by Proposition 3.6,

$$|\cdot|_{h^a}(x) \leq |\cdot|_{h'}(x) \leq e^{a\epsilon/2}|\cdot|_{h^a}(x) \tag{6}$$

for all $x \in X^{\mathrm{an}}$. In particular, $|\cdot|_{h^{am}}(x) \leq |\cdot|_{h'^m}(x)$. Therefore,

$$\|s\|_{h^{am}} \leq \|s\|_{h'^m}. \tag{7}$$

On the other hand, by using (6),

$$\|l^{\otimes a}\|_{Y,h'} \leq e^{a\epsilon/2} \sup\{|l^{\otimes a}|_{h^a}(y) \mid y \in Y^{\mathrm{an}}\} \leq e^{a\epsilon/2}(\|l\|_{Y,h})^a. \tag{8}$$

Thus the assertion follows from (5), (7) and (8).

Next we assume that $|\cdot|$ is trivial. Clearly we may assume that $l \neq 0$. Let k' be the field $k((T))$ of formal Laurent power series over k, that is, the quotient field of the ring $k[\![T]\!]$ of formal power series over k. We set

$$\Sigma := \bigcup_{i=0}^{\infty} \left(\bigcup_{s,s' \in H^0(X,L^{\otimes i}) \setminus \{0\}} \mathbb{Q}\left(\log \|s\|_{h^i} - \log \|s'\|_{h^i}\right) \right).$$

As $\left\{\|s\|_{h^i} \mid s \in H^0(X,L^{\otimes i}) \setminus \{0\}\right\}$ is a finite set by (1) in Lemma 1.12, we have $\#(\Sigma) \leq \aleph_0$. Therefore, we can find $\alpha \in \mathbb{R}_{>0} \setminus \Sigma$. Here we consider an absolute value $|\cdot|'$ of k' given by

$$|\phi(T)|' := \exp(-\alpha \operatorname{ord}(\phi(T))) \quad (\phi(T) \in k').$$

We set

$$X' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k'), \quad Y' := Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') \quad \text{and} \quad L' = L \otimes_k k'.$$

Note that $H^0(X', L') = H^0(X,L) \otimes_k k'$. Let h' be a continuous metric of L'^{an} given by the scalar extension of h. Then, by Lemma 3.7, h' is given by

$$\{|\cdot|^{\mathrm{quot}}_{(H^0(X',L'),\|\cdot\|_{k'})}(x')\}_{x' \in X'^{\mathrm{an}}},$$

where $\|\cdot\|_{k'}$ is the scalar extension of $\|\cdot\|$. Moreover, for $s \in H^0(X,L)$, $|s|_{h'}(x') = |s|_h(p^{\mathrm{an}}(x'))$ for $x' \in X'^{\mathrm{an}}$, where $p : X' \to X$ is the projection. Note that $p^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$ is surjective. Therefore, $\|s\|_{h'} = \|s\|_h$ for all $s \in H^0(X,L)$.

By the previous observation, there are a positive integer n and $s' \in H^0(X', L'^{\otimes n})$ such that

$$s'\big|_{Y'} = l^{\otimes n} \quad \text{and} \quad \|s'\|_{h'^n} \leq e^{n\epsilon}(\|l\|_{Y',h'})^n = e^{n\epsilon}(\|l\|_{Y,h})^n.$$

Note that, for a positive integer d,

$$s'^{\otimes d} \in H^0(X', L'^{\otimes dn}), \quad s'^{\otimes d}\Big|_{Y'} = l^{\otimes dn} \quad \text{and} \quad \|s'^{\otimes d}\|_{h'^{dn}} \leq e^{dn\epsilon}(\|l\|_{Y,h})^{dn}.$$

Thus we may assume that $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$ is surjective. Let $(e_1, \ldots, e_r)$ be an orthogonal basis of $H^0(X, L^{\otimes n})$ with respect to $\|\cdot\|_{h^n}$ such that $(e_{t+1}, \ldots, e_r)$ forms a basis of $\mathrm{Ker}(H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n}))$ (cf. Proposition 1.3). We set

$$s' = a_1(T)e_1 + \cdots + a_t(T)e_t + a_{t+1}(T)e_{t+1} + \cdots + a_r(T)e_r$$

for some $a_1(T), \ldots, a_r(T) \in k' = k((T))$. As $s'|_{Y'} = l^{\otimes n} \in H^0(Y, L|_Y^{\otimes n})$ and $(e_1|_Y, \ldots, e_t|_Y)$ forms a basis of $H^0(Y, L|_Y^{\otimes n})$, we have $a_1(T), \ldots, a_t(T) \in k$. Note that

$$\alpha \notin \bigcup_{s,s' \in H^0(X, L^{\otimes n}) \setminus \{0\}} \mathbb{Q}\left(\log \|s\|_{h^n} - \log \|s'\|_{h^n}\right),$$

so that, by (2) in Lemma 1.12 and Remark 1.13, $(e_1, \ldots, e_r)$ forms an orthogonal basis of $H^0(X', L'^{\otimes n})$ with respect to $\|\cdot\|_{h'^n}$. Therefore, if we set $s = a_1e_1 + \cdots + a_te_t$, then $s \in H^0(X, L^{\otimes n})$, $s|_Y = l^{\otimes n}$ and

$$\begin{aligned} \|s\|_{h^n} &= \max\{|a_1|\|e_1\|_{h^n}, \ldots, |a_t|\|e_t\|_{h^n}\} \\ &\leq \max\left\{|a_1|\|e_1\|_{h^n}, \ldots, |a_t|\|e_t\|_{h^n}, |a_{t+1}(T)|'\|e_{t+1}\|_{h^n}, \ldots, |a_r(T)|'\|e_r\|_{h^n}\right\} \\ &= \|s'\|_{h'^n} \leq e^{n\epsilon}(\|l\|_{Y,h})^n, \end{aligned}$$

as required. □

Theorem 4.2. *We assume that L is ample and h is a semipositive continuous metric of L^{an}. Fix a closed subscheme Y, $l \in H^0(Y, L|_Y)$ and $\epsilon \in \mathbb{R}_{>0}$. Then there is a positive integer n_0 such that, for all $n \geq n_0$, we can find $s \in H^0(X, L^{\otimes n})$ with*

$$s|_Y = l^{\otimes n} \quad \text{and} \quad \|s\|_{h^n} \leq e^{n\epsilon}(\|l\|_{Y,h})^n.$$

Proof. Clearly we may assume that $l \neq 0$. Let us begin with the following claim:

Claim 4.2.1. *For any $\epsilon' > 0$, there are a positive integer N and $s_N \in H^0(X, L^{\otimes N})$ such that*

$$s_N|_Y = l^{\otimes N} \quad \text{and} \quad \|s_N\|_{h^N} \leq e^{N\epsilon'}(\|l\|_{Y,h})^N.$$

Proof. By using Proposition 3.10, we can find a positive integer a such that $L^{\otimes a}$ is very ample and

$$|\cdot|_{h^a}(x) \leq |\cdot|_{h^a}^{\mathrm{quot}}(x) \leq e^{a\epsilon'/2}|\cdot|_{h^a}(x)$$

for all $x \in X^{\mathrm{an}}$. We set $h' = \{|\cdot|_{h^a}^{\mathrm{quot}}(x)\}$. Then, the above inequalities means that

$$|\cdot|_{h^a}(x) \leq |\cdot|_{h'}(x) \leq e^{a\epsilon'/2}|\cdot|_{h^a}(x) \tag{9}$$

for all $x \in X^{\mathrm{an}}$. Further, by Theorem 4.1, there are a positive integer b and $s_{ab} \in H^0(X, L^{\otimes ab})$ such that $s_{ab}|_Y = l^{\otimes ab}$ and

$$\|s_{ab}\|_{h'^b} \leq e^{ab\epsilon'/2}(\|l^{\otimes a}\|_{Y,h'})^b.$$

By (9),

$$\|l^{\otimes a}\|_{Y,h'} \leq e^{a\epsilon'/2}\|l^{\otimes a}\|_{Y,h^a} = e^{a\epsilon'/2}(\|l\|_{Y,h})^a.$$

Moreover, as $|\cdot|_{h^{ab}}(x) \leq |\cdot|_{h'^b}(x)$ by (9), we have $\|s_{ab}\|_{h^{ab}} \leq \|s_{ab}\|_{h'^b}$, so that

$$\begin{aligned} \|s_{ab}\|_{h^{ab}} &\leq \|s_{ab}\|_{h'^b} \leq e^{ab\epsilon'/2}(\|l^{\otimes a}\|_{Y,h'})^b \\ &\leq e^{ab\epsilon'/2}(e^{a\epsilon'/2}(\|l\|_{Y,h})^a)^b \leq e^{ab\epsilon'}(\|l\|_{Y,h})^{ab}. \end{aligned}$$

Therefore, if we set $N = ab$, then we have the assertion of the claim. □

Since L is ample, by Corollary 1.2, the above claim is actually equivalent to the assertion of the theorem. Thus the theorem is proved. □

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