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Explicit uniform estimation of rational points I. Estimation of heights

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Abstract. By using the slope method in Arakelov geometry, we study the complexity of the singular locus of an arithmetic projective variety and explicit estimations of the arithmetic Hilbert–Samuel function.

1. Introduction

Let K be a number field and X be a projective variety defined over K. The complexity of the rational points of X is measured by the height function. Northcott's theorem asserts that there are only finitely many rational points of X with bounded height. Namely, for any real number $B > 0$, the set

$$S(X;B) := \{P \in X(K) \mid H(P) \leqq B\}$$

is finite, where $H(P)$ is the height of P. Let $N(X;B) = \#S(X;B)$ be the *counting function* of X. The asymptotic behavior of $N(X;B)$ when B goes to infinity describes the density of $X(K)$. For example, $N(X;B) = O(1)$ if and only if $X(K)$ is finite.

Among the estimates of the counting function $N(X;B)$, the work of Heath-Brown [31] is of uniform nature, where the word "uniform" concerns all closed subvarieties with given degree and dimension in a projective space. His idea is to use a determinant argument (inspired by Bombieri and Pila [2], and Pila [42]), which can be summarized as follows. The monomials of a certain degree evaluated on a family of rational points in $S(X;B)$ having the same reduction modulo some prime number form a matrix whose determinant is zero by a local estimation. Hence there exists a hypersurface of X containing all rational points in the family. The set $S(X;B)$ is then covered by several hypersurfaces of bounded degree. The upper bound of $N(X;B)$ is thus obtained by estimating the number of these auxiliary hypersurfaces.

The results in [31] are obtained for $K = \mathbb{Q}$. Further research in this direction includes works of Broberg, Browning, Heath-Brown, Helfgott, Venkatesh, Salberger etc. (see [9], [12], [13], [14], [15], [24], [33], [32], [44], [45]). In particular, Broberg has generalized [31] to the number field case. It should be pointed out that the determinant argument plays an important role in most of the works cited above.

Based on an observation of Bost that the determinant argument mentioned above is quite similar to his slope method [3], or to the interpolation matrix method of Laurent [37], we revisit this problem in the context of Arakelov geometry by using the slope method. The aim of this approach is twofold. On one hand, it avoids using Siegel's lemma and hence treats the problem for all number fields in a uniform way without supplementary difficulties. On the other hand, the geometrical interpretation permits us to establish explicit estimates.

Recall the main result of [9] which generalizes Theorem 14 of [31]. Let X be an integral closed sub-variety of $\mathbb{P}^n_K$, $n \in \mathbb{N}\setminus\{0\}$. Let d and δ be respectively the dimension and the degree of X, and $\varepsilon > 0$ be a positive number. Assume that the ideal $I \subset K[T_0, \ldots, T_n]$ of X is generated by homogeneous polynomials of degrees at most τ, where $\tau \in \mathbb{N}\setminus\{0\}$. Broberg has proved that there exists an integer a depending only on n, τ and ε, an integer k satisfying[1)]

$$k \ll_{n,\tau,\varepsilon} B^{(d+1)/\sqrt[d]{\delta}+\varepsilon}$$

and a family $(F_i)_{i=0}^k$ of homogeneous polynomials of degree $\leqq a$ which are not identically zero on X and such that

$$S(X;B) \subset \bigcup_{i=0}^{k} \{x \in X(K) \mid F_i(x) = 0\},$$

where $S(X;B)$ denotes the set of all rational points of X with height $\leqq B$.

Note that Heath-Brown had considered the case where X is a hypersurface (that is, the ideal I is principal) defined over $\mathbb{Q}$ and Broberg has investigated the general case.

The aim of this article and the companion one [19] is to remove the supplementary assumption[2)] on the degrees of polynomials generating I and to calculate explicitly the constants a and k figuring in the above theorem. We shall actually establish the following result.

Theorem A. *Let $\varepsilon > 0$ and D be an integer such that*

$$D > \max\left\{(\varepsilon^{-1}+1)\left(2\delta^{-\frac{1}{d}}(d+1)+\delta-2\right), 2(n-d)(\delta-1)+d+2\right\}.$$

There is an explicitly computable constant $C = C(\varepsilon, \delta, n, d, K)$ such that, for any $B \geqq e^{\varepsilon}$, the set $S(X;B)$ of rational points of X with exponential height $\leqq B$ is covered by no more than $CB^{(1+\varepsilon)\delta^{-\frac{1}{d}}(d+1)} + 1$ hypersurfaces of degree $\leqq D$ not containing X.

[1)] Here the Vinogradov symbol $\ll_{n,\tau,\varepsilon}$ signifies that there exists a constant $C(n,\tau,\varepsilon)$ only depending on n, τ and ε such that $k \leqq C(n,\tau,\varepsilon)B^{(d+1)/\sqrt[d]{\delta}+\varepsilon}$.

[2)] From an inexplicit point of view, the assumption on the degrees of polynomials generating I can be removed by using a result of Kleiman on Hilbert polynomials for geometrically reduced subschemes of a projective space. We refer to [44], p. 126, for a detailed discussion on this point.

In the case where X is a plane curve, we prove a refinement of Theorem A and use it to establish the following estimation.

Theorem B. *Assume that X is an integral plane curve of degree δ. Then, for any $\varepsilon > 0$, one has*

$$\#S(X;\delta) \ll_K \delta^{2+\varepsilon}.$$

This answers a question of Heath-Brown.

For removing the assumption on the generating system of the variety (in an explicit way), we shall use the theory of Cayley–Chow form to construct a system of generators of the variety X. This theory, in its classical form, is due to Chow and van der Waerden [21]. Their original objective was to describe a projective variety X by one single homogeneous equation, called the *Chow form* of X. Later this theory has been applied in transcendental number theory by Gelfond [28], Nesterenko [38], Brownawell and Waldschmidt [10], and Philippon [39]. By [38] (see also [11]), one can explicitly construct a system of generators of a projective variety from its Chow form in controlling the degrees and the heights of these polynomials. However, the degrees of the polynomials thus constructed are in general much larger than the degree of the variety, which leads to an extra error term in the estimate. To overcome this difficulty, we shall use a variant called the Cayley form. This approach is inspired by a work of Catanese [16] (see also [27]). The Cayley form permits to construct a system of generators of lower degree. By using this theory, we are able to remove the supplementary condition on the degree of homogeneous polynomials in the generating system of X.

The explicit computation of the constants a and k requires highly non-trivial effective minoration of the arithmetic Hilbert–Samuel function developed in the article [22] of David and Philippon where higher Chow forms involve, and also several new estimates in the slope theory. It is known since the article [29] of Gillet and Soulé that the coefficient of the leading term of the arithmetic Hilbert–Samuel function is equal to the normalized auto-intersection number in the sense of the arithmetic intersection theory. In order to obtain explicit lower bounds of the arithmetic Hilbert–Samuel function, we shall reformulate the result of David and Philippon in the language of slope method. Note that the lower bound thus obtained is not asymptotically optimal. The coefficient of its leading term is smaller than the normalized auto-intersection number. This result will be used to prove that all rational points of small height are contained in a single hypersurface of low degree.

We also study effective upper bounds of the arithmetic Hilbert–Samuel function (or more generally, the maximal slope variant of it) and obtain an explicit upper bound in terms of the essential minimum of the variety. The proof is based on the slope inequality applied to the evaluation map on points of small height.

It turns out that these results have their own interest in Arakelov geometry, and deserve to be written independently. In the forthcoming article [19], we shall prove Theorems A and B.

This article is organized as follows. In the second section, we establish several slope inequalities and discuss their arithmetic consequences. The third section is devoted to the construction of Cayley forms and its application in the estimation of the complexity of the singular locus of an arithmetic projective variety. Finally in the fourth section, we discuss the estimation of the geometric and arithmetic Hilbert–Samuel functions in the framework of Arakelov geometry. The computation of norms of several tensor operators on Hermitian spaces are left in the Appendix.

We present the notation and the terminologies that we shall use in the current article and in [19].

Notation. 1. Denote by K a number field and by $\mathcal{O}_K$ its integer ring. If E is a projective $\mathcal{O}_K$-module of finite rank and if V is a vector subspace of E_K, the *saturation* of V in E is by definition the largest sub-$\mathcal{O}_K$-module F of E such that $F_K = V$. Note that E/F is then a torsion-free (hence projective) $\mathcal{O}_K$-module.

2. Any maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$ corresponds to a discrete valuation $v_\mathfrak{p}$ on K. Denote by $\mathbb{F}_\mathfrak{p}$ its residue field, by $N_\mathfrak{p}$ the cardinality of $\mathbb{F}_\mathfrak{p}$ and by $|\cdot|_\mathfrak{p}$ the absolute value on K such that $|a|_\mathfrak{p} = N_\mathfrak{p}^{-v_\mathfrak{p}(a)}$ for any $a \in K^\times$, which extends continuously to the completion $K_\mathfrak{p}$ of K (with respect to $v_\mathfrak{p}$).

For any embedding $\sigma : K \to \mathbb{C}$, denote by $|\cdot|_\sigma$ the absolute value on K such that $|a|_\sigma = |\sigma(a)|$, where $|\cdot|$ is the usual absolute value on $\mathbb{C}$.

3. By *arithmetic projective variety* we mean an integral projective $\mathcal{O}_K$-scheme which is flat over $\operatorname{Spec}\mathcal{O}_K$.

4. Let X be an arithmetic projective variety. A *Hermitian vector bundle* on X is a pair $\overline{E} = \big(E, (\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}\big)$, where E is a locally free $\mathcal{O}_X$-module of finite rank, and for any embedding $\sigma : K \to \mathbb{C}$, $\|\cdot\|_\sigma$ is a continuous Hermitian metric on $E_\sigma(\mathbb{C})$, invariant under the action of the complex conjugation. The *rank* of $\overline{E}$ is defined to be the rank of E, denoted by $\operatorname{rk}(E)$. A Hermitian vector bundle of rank 1 is called a *Hermitian line bundle*.

5. Let $\overline{E}$ be a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$. The *Arakelov degree* of $\overline{E}$ is defined as[3)]

$$\widehat{\deg}(\overline{E}) := \log \#\big(E/(\mathcal{O}_K s_1 + \cdots + \mathcal{O}_K s_r)\big) - \frac{1}{2}\sum_{\sigma:K\to\mathbb{C}} \log\det(\langle s_i, s_j\rangle_\sigma),$$

where $(s_1, \ldots, s_r) \in E^r$ forms a basis of E_K over K, $r = \operatorname{rk}(E)$.

6. Let $\overline{E}$ be a non-zero Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$. The *slope* of $\overline{E}$ is

$$\hat{\mu}(\overline{E}) := \frac{1}{[K:\mathbb{Q}]}\frac{\widehat{\deg}(\overline{E})}{\operatorname{rk} E}.$$

Denote by $\hat{\mu}_{\max}(\overline{E})$ the maximal value of slopes of all non-zero Hermitian subbundles (i.e., submodule of E equipped with the induced metrics) of $\overline{E}$ and by $\hat{\mu}_{\min}(\overline{E})$ the minimal value

3) By the product formula, this definition does not depend on the choice of $(s_1, \ldots, s_r)$.

of slopes of all non-zero Hermitian quotient bundles (i.e., projective quotient module of E equipped with the quotient metrics) of $\bar{E}$.

7. Let $\bar{E}$ and $\bar{F}$ be two non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$, and $\phi : E_K \to F_K$ be a non-zero homomorphism. The *height* of ϕ is defined as

$$h(\phi) := \frac{1}{[K:\mathbb{Q}]}\Bigl(\sum_{\mathfrak{p}} \log\|\phi\|_{\mathfrak{p}} + \sum_{\sigma:K\to\mathbb{C}} \log\|\phi\|_{\sigma}\Bigr),$$

where $\|\phi\|_{\mathfrak{p}}$ and $\|\phi\|_{\sigma}$ are respectively the operator norms of $\phi_{K_{\mathfrak{p}}} : E_{K_{\mathfrak{p}}} \to F_{K_{\mathfrak{p}}}$ and $\phi_{\sigma,\mathbb{C}} : E_{\sigma,\mathbb{C}} \to F_{\sigma,\mathbb{C}}$.

8. For any integer $n \geqq 1$ and any n-tuple $(\bar{E}_1, \ldots, \bar{E}_n)$ of non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$, denote by $\varrho(\bar{E}_1, \ldots, \bar{E}_n)$ the difference

$$\hat{\mu}_{\max}(\bar{E}_1 \otimes \cdots \otimes \bar{E}_n) - \sum_{i=1}^{n} \hat{\mu}_{\max}(\bar{E}_i).$$

For simplifying notation, we use the expression $\varrho^{(n)}(\bar{E})$ to denote $\varrho(\underbrace{\bar{E}, \ldots, \bar{E}}_{n \text{ copies}})$.

2. Slope inequalities

We firstly recall the basic ingredients of Bost's slope theory (for references, see [3], [5], [6], [17]), then discuss several slope (in)equalities and their arithmetic consequences. We begin with the following classical slope (in)equalities relating the source and the target of a homomorphism between Hermitian vector bundles (see [3], Appendix A).

Proposition 2.1. *Assume that $\bar{E}$ and $\bar{F}$ are two Hermitian vector bundles on* $\operatorname{Spec}\mathcal{O}_K$ *and $\phi : E_K \to F_K$ is a non-zero K-linear homomorphism.*

(i) *If ϕ is injective, then $\hat{\mu}_{\max}(\bar{E}) \leqq \hat{\mu}_{\max}(\bar{F}) + h(\phi)$.*

(ii) *If ϕ is surjective, then $\hat{\mu}_{\min}(\bar{E}) \leqq \hat{\mu}_{\min}(\bar{F}) + h(\phi)$.*

(iii) *If ϕ is an isomorphism, then $\hat{\mu}(\bar{E}) = \hat{\mu}(\bar{F}) + \dfrac{1}{\operatorname{rk}(E)} h(\Lambda^{\operatorname{rk}(E)}\phi)$.*

2.1. A slope equality. We give below a variant of the slope equality in Proposition 2.1 (iii) where the target Hermitian vector bundle can be written as a direct sum of Hermitian line bundles.

Proposition 2.2. *Let $\bar{E}$ be a Hermitian vector bundle of rank $r > 0$ on* $\operatorname{Spec}\mathcal{O}_K$ *and $(\bar{L}_i)_{i\in I}$ be a family of Hermitian line bundles on* $\operatorname{Spec}\mathcal{O}_K$. *If $\phi : E_K \to \bigoplus_{i\in I} L_{i,K}$ is an injective homomorphism, then there exists a subset I_0 of cardinal r of I such that the following equality* *holds*:

$$\hat{\mu}(\overline{E}) = \frac{1}{r}\left[\sum_{i\in I_0}\hat{\mu}(\overline{L}_i) + h\big(\Lambda^r(\mathrm{pr}_{I_0}\circ\phi)\big)\right], \tag{1}$$

where $\mathrm{pr}_{I_0} : \bigoplus_{i\in I} L_{i,K} \to \bigoplus_{i\in I_0} L_{i,K}$ *is the projection.*

Proof. Since ϕ is injective, there exists $I_0 \subset I$ of cardinal r such that $\mathrm{pr}_{I_0}\circ\phi$ is an isomorphism. Therefore, Proposition 2.1 (iii) implies

$$\hat{\mu}(\overline{E}) = \hat{\mu}\Big(\bigoplus_{i\in I_0}\overline{L}_i\Big) + \frac{1}{r}h\big(\Lambda^r(\mathrm{pr}_{I_0}\circ\phi)\big) = \frac{1}{r}\left[\sum_{i\in I_0}\hat{\mu}(\overline{L}_i) + h\big(\Lambda^r(\mathrm{pr}_{I_0}\circ\phi)\big)\right]. \qquad \square$$

2.2. Tensor product and image. Let $(\overline{E}_i)_{i=1}^n$ be a family of non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$. One always has (see [20], Corollary 2.5)

$$\hat{\mu}_{\max}(\overline{E}_1\otimes\cdots\otimes\overline{E}_n) \geqq \sum_{i=1}^n \hat{\mu}_{\max}(\overline{E}_i). \tag{2}$$

The inverse inequality is a conjecture of Bost [4], which is still an open problem.

Recall (see Notation 8) that if $(\overline{E}_i)_{i=1}^n$ is a family of non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$, then $\varrho(\overline{E}_1,\dots,\overline{E}_n)$ denotes the difference

$$\varrho(\overline{E}_1,\dots,\overline{E}_n) := \hat{\mu}_{\max}(\overline{E}_1\otimes\cdots\otimes\overline{E}_n) - \sum_{i=1}^n \hat{\mu}_{\max}(\overline{E}_i).$$

If all $\overline{E}_i$ are equal to the same Hermitian vector bundle $\overline{E}$, we write $\varrho^{(n)}(\overline{E})$ instead of $\varrho(\overline{E},\dots,\overline{E})$.

If $\overline{L}$ is a Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$, then for any non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec}\mathcal{O}_K$, one has

$$\hat{\mu}_{\max}(\overline{E}\otimes\overline{L}) = \hat{\mu}_{\max}(\overline{E}) + \hat{\mu}(\overline{L}) = \hat{\mu}_{\max}(\overline{E}) + \hat{\mu}_{\max}(\overline{L}).$$

Hence $\varrho(\overline{E},\overline{L}) = \varrho(\overline{L},\overline{E}) = 0$. More generally, for any family $(\overline{E}_i)_{i=1}^n$ of non-zero Hermitian vector bundles, one has $\varrho(\overline{E}_1,\overline{E}_2,\dots,\overline{E}_n) = 0$ if all Hermitian vector bundles except at most one among $\overline{E}_1,\dots,\overline{E}_n$ are direct sums of Hermitian line bundles.

Let $(\overline{E}_i)_{i=1}^n$ and $(\overline{F}_j)_{j=1}^m$ be two families of non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$. By (2), one has

$$\varrho(\overline{E}_1,\dots,\overline{E}_n,\overline{F}_1,\dots,\overline{F}_m) \geqq \varrho(\overline{E}_1,\dots,\overline{E}_n) + \varrho(\overline{F}_1,\dots,\overline{F}_m). \tag{3}$$

In particular, for any non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec}\mathcal{O}_K$, one has

$$\forall n,m\in\mathbb{N}^*, \quad \varrho^{(n+m)}(\overline{E}) \geqq \varrho^{(n)}(\overline{E}) + \varrho^{(m)}(\overline{E}). \tag{4}$$

Moreover, (2) implies that $\varrho^{(m)}(\bar{E}) \geqq 0$ for any $m \in \mathbb{N}^*$. Hence by (4), the sequence $\big(\varrho^{(n)}(\bar{E})\big)_{n \geqq 1}$ is increasing.

Remark 2.3. By the duality between maximal slope and minimal slope $\hat{\mu}_{\min}(\bar{E}) = -\hat{\mu}_{\max}(\bar{E}^\vee)$, one has

$$(5)\qquad \varrho(\bar{E}_1^\vee, \dots, \bar{E}_n^\vee) = \sum_{i=1}^n \hat{\mu}_{\min}(\bar{E}_i) - \hat{\mu}_{\min}(\bar{E}_1 \otimes \cdots \otimes \bar{E}_n)$$

Bost's conjecture can be reformulated as: $\varrho(\bar{E}_1, \dots, \bar{E}_n) \equiv 0$ *for any n-tuple* $(\bar{E}_i)_{i=1}^n$ *of non-zero Hermitian vector bundles over* $\operatorname{Spec} \mathcal{O}_K$. For estimations of ϱ, see [23], [8], [20]. By a result of Bost, one has[4)]

$$(6)\qquad \varrho(\bar{E}_1, \dots, \bar{E}_n) \leqq \frac{1}{2} \sum_{i=1}^n \log(\operatorname{rk} E_i).$$

Let $\bar{E}$ and $\bar{F}$ be two Hermitian vector bundles on $\operatorname{Spec} \mathcal{O}_K$, and $\bar{G}$ be a Hermitian vector subbundle of $\bar{F} \otimes \bar{E}$. We call *image* of $\bar{G}$ in $\bar{E}$ the smallest sub-$\mathcal{O}_K$-module H of E such that $F \otimes H$ contains G, equipped with the induced metrics. The minimal slope of $\overline{H}$ is estimated in Proposition 2.4 below, the proof of which uses the first part of Lemma A.1 in the Appendix.

Proposition 2.4. *With the above notation, one has*

$$(7)\qquad \hat{\mu}_{\min}(\overline{H}) \geqq \hat{\mu}_{\min}(\bar{G}) - \hat{\mu}_{\max}(\bar{F}) - \varrho(\bar{F}, \bar{G}^\vee) - \frac{1}{2} \log(\operatorname{rk} F).$$

Proof. Let ψ be the composed homomorphism $F^\vee \otimes G \to F^\vee \otimes F \otimes E \to E$, where the last arrow is induced by the trace homomorphism $F^\vee \otimes F \to \mathcal{O}_K$. The image of ψ identifies with H. By Lemma A.1 (i), the height of ψ equals $\frac{1}{2} \log(\operatorname{rk} F)$. Thus Proposition 2.1 (ii) implies

$$\begin{aligned} \hat{\mu}_{\min}(\overline{H}) &\geqq \hat{\mu}_{\min}(\bar{F}^\vee \otimes \bar{G}) - h(\psi) \\ &= \hat{\mu}_{\min}(\bar{G}) + \hat{\mu}_{\min}(\bar{F}^\vee) - \varrho(\bar{F}, \bar{G}^\vee) - \frac{1}{2} \log(\operatorname{rk} F). \qquad \square \end{aligned}$$

Remark 2.5. Assume in the above proposition that $\bar{F}$ can be written as a tensor product $\bar{F}_1 \otimes \cdots \otimes \bar{F}_n$. Then the same method gives the following variant of (7):

$$(8)\qquad \hat{\mu}_{\min}(\overline{H}) \geqq \hat{\mu}_{\min}(\bar{G}) - \sum_{i=1}^n \hat{\mu}_{\max}(\bar{F}_i) - \varrho(\bar{F}_1, \dots, \bar{F}_n, \bar{G}^\vee) - \frac{1}{2} \log(\operatorname{rk} F).$$

2.3. Applications. We give several applications of the slope (in)equalities established in previous subsections. More applications will be discussed in §3.

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4) Learned from a personal note of J.-B. Bost, also obtained by Y. André and É. Gaudron. The proof of a weaker version (with the coefficient 1 instead of $\frac{1}{2}$) can be found in [20].

In Arakelov theory, the slope inequalities are often applied on *evaluation maps* to obtain arithmetic results. See [6] for a survey of this method. Classically the evaluation map means the evaluation of some polynomials at one or several points of an affine space. The choice of the evaluation map is a crucial step in a typical proof of Diophantine approximation. Note that in Heath-Brown's determinant argument, there appears also this procedure. Evaluation maps in Arakelov geometry are quite similar to classical ones, but their construction is of geometrical nature. Let X be a projective variety over $\operatorname{Spec} K$ and L be an ample line bundle on X. Let Y be a closed subscheme of X and $i : Y \to X$ be the inclusion morphism. The evaluation map (of global sections of L) on Y is the K-linear mapping from $H^0(X, L)$ to $H^0(Y, i^*L)$ defined by restriction of sections.

To apply the slope method, we also need a metric structure. Let $n \geqq 1$ be an integer and $\overline{\mathcal{E}}$ be a Hermitian vector bundle of rank $n+1$ on $\operatorname{Spec} \mathcal{O}_K$. Denote by $\pi : \mathbb{P}(\mathcal{E}) \to \operatorname{Spec} \mathcal{O}_K$ the structural morphism. Let $\mathcal{L} := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the universal quotient of $\pi^*\mathcal{E}$. The Hermitian metrics on $\mathcal{E}$ induce by quotient a structure of Hermitian metrics (i.e. Fubini–Study metrics) on $\mathcal{L}$ which define a Hermitian line bundle $\overline{\mathcal{L}}$ on $\mathbb{P}(\mathcal{E})$. For any integer $D \geqq 1$, let $E_D = H^0\big(\mathbb{P}(\mathcal{E}), \mathcal{L}^{\otimes D}\big)$ and let $r(D)$ be its rank over $\mathcal{O}_K$, which is equal to $\binom{n+D}{D}$. For any $\sigma \in \Sigma_\infty$, denote by $\|\cdot\|_{\sigma,\sup}$ the norm on $E_{D,\sigma} := E_D \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$ such that

$$\forall s \in E_{D,\sigma}, \quad \|s\|_{\sigma,\sup} = \sup_{x \in \mathbb{P}(\mathcal{E}_K)_\sigma(\mathbb{C})} \|s(x)\|_\sigma.$$

Let $\|\cdot\|_{\sigma,J}$ be the Hermitian metric of John (cf. [34], see also [26], Definition-Theorem 2.4) associated to the norm $\|\cdot\|_{\sigma,\sup}$. Recall that, for any $s \in E_{D,\sigma}$, the following inequalities hold:

$$\|s\|_{\sigma,\sup} \leqq \|s\|_{\sigma,J} \leqq \sqrt{r(D)}\,\|s\|_{\sigma,\sup}, \tag{9}$$

where $r(D)$ is the rank of E_D. The $\mathcal{O}_K$-module E_D, equipped with the Hermitian metric $\|\cdot\|_{\sigma,J}$, forms a Hermitian vector bundle $\overline{E}_D$ on $\operatorname{Spec} \mathcal{O}_K$.

Remark 2.6. As an $\mathcal{O}_K$-module, E_D is isomorphic to $S^D\mathcal{E}$. Thus for any $\sigma \in \Sigma_\infty$, the Hermitian metric on $\mathcal{E}_{\sigma,\mathbb{C}}$ induces by symmetric power a Hermitian metric $\|\cdot\|_{\sigma,\mathrm{sym}}$ on $S^D\mathcal{E}_{\sigma,\mathbb{C}}$. Denote by $S^D\overline{\mathcal{E}}$ the corresponding Hermitian vector bundle on $\operatorname{Spec} \mathcal{O}_K$. Note that for any $\sigma \in \Sigma_\infty$, both metrics $\|\cdot\|_{\sigma,J}$ and $\|\cdot\|_{\sigma,\mathrm{sym}}$ are invariant under the action of the unitary group $U(\mathcal{E}_{\sigma,\mathbb{C}}, \|\cdot\|_\sigma)$, therefore they are proportional and the ratio is independent of σ (see [7], the proof of Lemma 4.3.6). We denote by $C_0(D)$ the constant such that, for any $0 \neq s \in E_{D,\sigma}$,

$$\log\|s\|_{\sigma,J} = \log\|s\|_{\sigma,\mathrm{sym}} + C_0(D). \tag{10}$$

One has

$$\hat{\mu}_{\min}(\overline{E}_D) = \hat{\mu}_{\min}(S^D\overline{\mathcal{E}}) - C_0(D).$$

Proposition 2.7. *With the above notation, the following inequalities hold*:

$$0 \leqq C_0(D) \leqq \log\sqrt{r(D)}, \quad \textit{where } r(D) = \operatorname{rk}(E_D). \tag{11}$$

Proof. Let s be a non-zero section in $H^0(\mathbb{P}(\mathcal{E}_{\sigma,\mathbb{C}}),\mathcal{L}_{\sigma,\mathbb{C}})$. By definition, one has $\|s\|_{\sigma,\sup} = \|s\|_\sigma = \|s\|_{\sigma,\mathrm{sym}}$. Hence

$$\|s^D\|_{\sigma,\sup} = \|s\|_{\sigma,\sup}^D = \|s\|_{\sigma,\mathrm{sym}}^D = \|s^D\|_{\sigma,\mathrm{sym}}.$$

As $C_0(D) = \log\|s^D\|_{\sigma,J} - \log\|s^D\|_{\sigma,\mathrm{sym}}$, by (9), we obtain (11). □

Remark 2.8. One has $r(D) = \binom{n+D}{D} \leqq (n+1)^D$, hence

$$C_0(D) \leqq \frac{D}{2}\log(n+1). \tag{12}$$

The following proposition gives an explicit lower bound of the minimal slope of $\bar{E}_D$.

Proposition 2.9. *For any integer $D \geqq 1$,*

$$\hat{\mu}_{\min}(\bar{E}_D) \geqq D\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \varrho^{(D)}(\bar{\mathcal{E}}^\vee) - C_0(D). \tag{13}$$

Proof. By definition, one has (see (5))

$$\hat{\mu}_{\min}(\bar{\mathcal{E}}^{\otimes D}) = D\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \varrho^{(D)}(\bar{\mathcal{E}}^\vee).$$

Moreover, $S^D\bar{\mathcal{E}}$ is a quotient of $\bar{\mathcal{E}}^{\otimes D}$, so $\hat{\mu}_{\min}(S^D\bar{\mathcal{E}}) \geqq \hat{\mu}_{\min}(\bar{\mathcal{E}}^{\otimes D})$. Hence we obtain

$$\hat{\mu}_{\min}(\bar{E}_D) = \hat{\mu}_{\min}(S^D\bar{\mathcal{E}}) - C_0(D) = D\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \varrho^{(D)}(\bar{\mathcal{E}}^\vee) - C_0(D). \tag{14}$$ □

Definition 2.10. If P is a rational point of $\mathbb{P}(\mathcal{E}_K)$, it extends in a unique way to a section $\mathcal{P}$ of π. The *height* of the point P with respect to $\bar{\mathcal{L}}$ is by definition the slope (see Notation 6) of the Hermitian line bundle $\mathcal{P}^*(\bar{\mathcal{L}})$ on $\operatorname{Spec}\mathcal{O}_K$, denoted by $h_{\bar{\mathcal{L}}}(P)$.

Proposition 2.11. *Let $D \geqq 1$ be an integer and $\bar{I}$ be a Hermitian vector subbundle of $\bar{E}_D$. Let $\mathcal{Y}$ be the subscheme of $\mathbb{P}(\mathcal{E})$ defined by annihilation of I. Suppose that P is a rational point of $\mathbb{P}(\mathcal{E}_K)$ which is not in $\mathcal{Y}(K)$. Denote by $\mathcal{P}$ the $\mathcal{O}_K$-point of $\mathbb{P}(\mathcal{E})$ extending P. For any finite place $\mathfrak{p}$ of K, let $\alpha_\mathfrak{p}$ be as follows*:

$$\alpha_\mathfrak{p} = \begin{cases} 1, & (\mathcal{P} \bmod \mathfrak{p}) \in \mathcal{Y}(\mathbb{F}_\mathfrak{p}), \\ 0, & \textit{else}. \end{cases}$$

Then, for any real number $N_0 > 0$, the following inequality holds:

$$\#\{\mathfrak{p} \mid \alpha_\mathfrak{p} = 1, N_\mathfrak{p} \geqq N_0\} \leqq \frac{Dh_{\bar{\mathcal{L}}}(P) - \hat{\mu}_{\min}(\bar{I})}{(\log N_0)/[K:\mathbb{Q}]}. \tag{15}$$

Proof. Let $\eta : I \to \mathcal{P}^*\mathcal{L}^{\otimes D}$ be the homomorphism induced by the evaluation map $E_D \to \mathcal{P}^*\mathcal{L}^{\otimes D}$. Since $P \notin \mathcal{Y}(K)$, the homomorphism η_K is surjective. By the slope inequality

(Proposition 2.1 (ii)), one has

$$Dh_{\bar{\mathcal{L}}}(P) \geqq \hat{\mu}_{\min}(\bar{I}) - h(\eta_K) \geqq \hat{\mu}_{\min}(\bar{I}) + \frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}} \alpha_{\mathfrak{p}} \log N_{\mathfrak{p}}.$$

Therefore the inequality (15) holds. □

Let X be an integral closed subscheme of $\mathbb{P}(\mathcal{E}_K)$ and $\mathscr{X}$ be its Zariski closure in $\mathbb{P}(\mathcal{E})$. Denote by

$$\eta_{X,D} : E_{D,K} = H^0\big(\mathbb{P}(\mathcal{E}_K), \mathcal{L}_K^{\otimes D}\big) \to H^0(X, \mathcal{L}|_X^{\otimes D})$$

the evaluation map on X and by F_D the saturation of $\mathrm{Im}(\eta_{X,D})$ in $H^0(\mathscr{X}, \mathcal{L}|_{\mathscr{X}}^{\otimes D})$. Namely F_D is the largest sub-$\mathcal{O}_K$-module of $H^0(\mathscr{X}, \mathcal{L}|_{\mathscr{X}}^{\otimes D})$ such that $F_{D,K} = \mathrm{Im}(\eta_{X,D})$. Note that, for sufficiently large D, the homomorphism $\eta_{X,D}$ is surjective, and therefore $F_D = H^0(\mathscr{X}, \mathcal{L}|_{\mathscr{X}}^{\otimes D})$.

The following result shows that the evaluation on a collection of rational points with small heights cannot be injective. Let $Z = (P_i)_{i \in I}$ be a collection of distinct rational points of X. The evaluation map

$$\eta_{Z,D} : H^0\big(\mathbb{P}(\mathcal{E}_K), \mathcal{L}_K^{\otimes D}\big) \to \bigoplus_{i \in I} P_i^* \mathcal{L}_K^{\otimes D}$$

factorizes through $\eta_{X,D}$. Denote by

$$\phi_{Z,D} : F_{D,K} \to \bigoplus_{i \in I} P_i^* \mathcal{L}_K^{\otimes D} \tag{16}$$

the homomorphism such that $\phi_{Z,D}\eta_{X,D} = \eta_{Z,D}$.

We equip F_D with quotient metrics (from that of $\bar{E}_D$) so that $\bar{F}_D$ becomes a Hermitian vector bundle on $\mathrm{Spec}\,\mathcal{O}_K$. Note that the quantity $\widehat{\deg}(\bar{F}_D)$ is the normalized version of the "*D-th height*" of X, defined and studied in [43], §2.2.

Proposition 2.12. *Assume that*

$$\sup_{i \in I} h_{\bar{\mathcal{L}}}(P_i) < \frac{\hat{\mu}_{\max}(\bar{F}_D)}{D} - \frac{1}{2D}\log r_1(D), \quad \textit{where } r_1(D) = \mathrm{rk}(F_D).$$

Then the homomorphism $\phi_{Z,D}$ cannot be injective.

Proof. Assume that $\phi_{Z,D}$ is injective. There exists a subset $I_0 \subset I$ of cardinal $r_1(D)$ such that $\mathrm{pr}_{I_0} \circ \phi_{Z,D}$ is injective. By Proposition 2.1 (i), we obtain that

$$\hat{\mu}_{\max}(\bar{F}_D) \leqq \max_{i \in I_0} Dh_{\bar{\mathcal{L}}}(P_i) + h(\mathrm{pr}_{I_0} \circ \phi_{Z,D}).$$

Note that

$$h(\mathrm{pr}_{I_0} \circ \phi_{Z,D}) \leqq \frac{1}{2}\log r_1(D).$$

This leads to a contradiction. □

3. The complexity of singular locus

In this section, we consider the following problem. Let K be a number field. Given a closed subvariety X of a projective space $\mathbb{P}_K^n$, we ask how to describe the complexity of the singular locus of X by the arithmetic invariants of X. When X is a hypersurface in $\mathbb{P}^n$ defined by a homogeneous polynomial $F(T_0, \dots, T_n)$ of degree d, the singular locus of X is determined by the equations

$$F = \frac{\partial}{\partial T_0} F = \cdots = \frac{\partial}{\partial T_n} F = 0.$$

Therefore the ideal of $\mathrm{Sing}(X)$ is generated by $n+2$ polynomials of height $\leqq dh(F)$. In the general case, the singular locus can be described by using the Jacobian criterion, provided a system of generators of the ideal of X.

Given a subvariety $X \subset \mathbb{P}^n$ of dimension d and of degree δ, a method to construct explicitly a system of polynomials defining X is to use the Chow form. The Chow form Φ_X of X is a multi-homogeneous polynomial of multi-degree $(\delta, \dots, \delta)$ on the multi-projective space $(\mathbb{P}^n)^{d+1}$. The general theory of Chow and van der Waerden [21] asserts that set-theoretically any subvariety of dimension d and of degree δ of $\mathbb{P}^n$ is uniquely determined by its Chow form.

Philippon [39] has defined the height of an arithmetic variety as that of its Chow form and applied his height theory on criteria of algebraic independence. The Philippon height can be compared to the Arakelov height [47], [41], [7]. As mentioned in the Introduction, one can construct explicitly a system of generators of a projective variety from its Chow form. This permits us in principle to understand the complexity of the singular locus of the projective variety by using the Jacobian criterion. However, the polynomials in the generating system obtained from the Chow form usually have degrees much higher than δ. For example, if the projective variety is a hypersurface in a projective space defined by a homogeneous equation F of degree δ, then the generating system obtained from the Chow form will be the linear space of equations of the form FG, where G runs over all homogeneous polynomials of degree δd. Therefore, if we try to estimate the complexity of the singular locus of the variety by using this linear system, supplementary errors will occur in the procedure of differential and also in that of taking the determinant.

In this article, we adopt the point of view of Cayley form. This approach is inspired by [27], [16]. The construction of Cayley form is quite similar to that of Chow form. The only difference is that, in the construction of Chow form, we use Stiefel coordinates; while in that of Cayley form, we use Plücker coordinates.

In the following, we recall the definition of Chow form and Cayley form, the computation of their heights, and the estimation of the complexity of the singular locus of a variety by using its Cayley form. In the rest of this section, let $n \in \mathbb{N} \setminus \{0\}$ and $\bar{\mathcal{E}}$ be a Hermitian vector bundle of rank $n+1$. Denote by $\bar{\mathcal{L}}$ the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ equipped with the Fubini–Study metrics. By a sub-variety of $\mathbb{P}(\mathcal{E}_K)$ we mean a closed integral subscheme of $\mathbb{P}(\mathcal{E}_K)$.

3.1. Chow form and Cayley form. Let W be the product $\mathbb{P}(\mathcal{E}_K) \times \mathbb{P}(\mathcal{E}_K^\vee)^{d+1}$ and Γ be the incidence subvariety of W which classifies all points $(\xi, u_0, \dots, u_d)$ such that

$\xi(u_0) = \cdots = \xi(u_d) = 0$. Denote by $p : W \to \mathbb{P}(\mathcal{E}_K)$ and $q : W \to \mathbb{P}(\mathcal{E}_K^\vee)^{d+1}$ the two projections.

Proposition 3.1. *Let $X \subset \mathbb{P}(\mathcal{E}_K)$ be a subvariety. Then the set-theoretical intersection $\Gamma \cap p^{-1}(X)$ is irreducible. Furthermore, if we consider $\Gamma \cap p^{-1}(X)$ as a reduced subvariety of W, then the scheme-theoretical image $q(\Gamma \cap p^{-1}(X))$ is a hypersurface of multi-degree $(\delta, \ldots, \delta)$.*

See [39] for an algebraic proof of this result in its generalized form, see [7], §4.3, for a geometric proof.

The hypersurface in Proposition 3.1 corresponds to a subspace of rank one of $S^\delta(\mathcal{E}_K^\vee)^{\otimes(d+1)}$ whose saturation (see Notation 1) in $S^\delta(\mathcal{E}^\vee)^{\otimes(d+1)}$ determines a Hermitian line subbundle $\bar{\Phi}_X$ of $S^\delta(\bar{\mathcal{E}}^\vee)^{\otimes(d+1)}$. The generic fibre $\Phi_{X,K}$ is called the *Chow form* of X.

Remark 3.2. The *Philippon height* of the arithmetic variety X is defined as

$$h_{\mathrm{Ph}}(X) := \frac{1}{[K:\mathbb{Q}]}\left(\sum_{\mathfrak{p}} \log\|\phi_X\|_{\mathfrak{p}} + \sum_{\sigma:K\to\mathbb{C}} \log M_\sigma(\phi_X)\right),$$

where ϕ_X is a non-zero element in Φ_X, and for any embedding σ, M_σ is the integral operator with respect to the Mahler measure associated to σ. By [7], Theorem 4.3.8 (generalizing some results in [47], [40]), the Philippon height of X is compared to the Arakelov height of X with respect to $\bar{\mathcal{L}}$. Recall that the (relative) Arakelov height is defined as

$$h_{\bar{\mathcal{L}}}(X) := \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\big(\hat{c}_1(\bar{\mathcal{L}})^{d+1}\cdot[\mathscr{X}]\big),$$

where $\mathscr{X}$ is the Zariski closure of X in $\mathbb{P}(\mathcal{E})$. One has

$$h_{\mathrm{Ph}}(X) = h_{\bar{\mathcal{L}}}(X) - \frac{1}{2}\delta(d+1)\mathcal{H}_n,$$

where $\mathcal{H}_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the n-th partial sum of the harmonic series.

Moreover, one has the relation (cf. [7], Proposition 4.3.5 and Theorem 4.3.8)

$$0 \leqq h_{\bar{\mathcal{L}}}(X) + \hat{\mu}(\bar{\Phi}_X) + \frac{1}{2}(d+1)\log\binom{n+\delta}{\delta} \leqq \frac{1}{2}\delta(d+1)\mathcal{H}_n. \tag{17}$$

Let $s^{(i)}$ $(0 \leqq i \leqq d)$ be variables taking values in the space of antisymmetric homomorphisms from $\mathcal{E}_K^\vee$ to $\mathcal{E}_K$, and ξ be a variable valued in $\mathcal{E}_K^\vee$. The mapping $(s^{(0)}, \ldots, s^{(d)}, \xi) \mapsto \Phi_{X,K}(s^{(0)}\xi, \ldots, s^{(d)}\xi)$ is a multihomogeneous polynomial of degree δ in each $s^{(i)}$ and of degree $(d+1)\delta$ in ξ. By specifying $s^{(i)}$, one obtains a linear system $J_{X,K}$ of homogeneous polynomials of degree $(d+1)\delta$ in $\xi \in \mathcal{E}_K^\vee$. The heights of these equations can be estimated by the height of X. The linear system $J_{X,K}$ defines a subscheme $\tilde{X}$ of $\mathbb{P}(\mathcal{E}_K)$

containing X. By [38], Lemma 11, it coincides with X on an open subscheme containing the regular subscheme X_{reg}. Furthermore, one has $\tilde{X}_{\mathrm{red}} = X$.

In the following, we introduce a variant of the Chow form, called the *Cayley form* of X. The advantage of the Cayley form is that we can construct from it a system of generators of X which are of degree δ.

Recall that in the construction of the Chow form, one has actually used the Stiefel coordinates of the Grassmannian. If we use Plücker coordinates instead, the same procedure leads to the so-called *Cayley form*. Let $\check{G} = \mathrm{Gr}(d+1, \mathcal{E}_K^\vee)$ be the Grassmannian which classifies all quotients of rank $d+1$ of $\mathcal{E}_K^\vee$ (or equivalently, all subspaces of rank $d+1$ of $\mathcal{E}_K$). Denote by Γ' the incidence subvariety of $\mathbb{P}(\mathcal{E}_K) \times \check{G}$ which classifies all points (ξ, U) such that $\xi(U) = 0$ (here we consider U as a subspace of $\mathcal{E}_K$). Let $p' : \mathbb{P}(\mathcal{E}_K) \times \check{G} \to \mathbb{P}(\mathcal{E}_K)$ and $q' : \mathbb{P}(\mathcal{E}_K) \times \check{G} \to \check{G}$ be the two projections.

Proposition 3.3 (see [27], §3.2.B). *Let $X \subset \mathbb{P}(\mathcal{E}_K)$ be a subvariety of dimension d and of degree δ. The set-theoretical intersection $\Gamma' \cap p'^{-1}(X)$ is irreducible. Furthermore, if we consider $\Gamma' \cap p'^{-1}(X)$ as a reduced subvariety of W', the scheme-theoretical image $q'\big(\Gamma' \cap p'^{-1}(X)\big)$ is a hypersurface of degree δ of $\check{G}$.*

Proof. The incidence variety Γ' is a fibration on $\mathbb{P}(\mathcal{E}_K)$ in Grassmannian varieties. Since X is irreducible, also is $\Gamma' \cap p'^{-1}(X) = p'|_{\Gamma'}^{-1}(X)$. Denote by $Y = \Gamma' \cap p'^{-1}(X)$, considered as a subvariety of Γ'. The projection q' being proper, the image $Z = q'(Y)$ is a closed integral subscheme of $\check{G}$. Let $\xi = \operatorname{Spec} K'$ be a geometric generic point of Z, which corresponds to a subspace V of rank $d+1$ of $\mathcal{E}_{K'}$. The fibre Y_ξ coincides with the subscheme of $X_{K'}$ defined by vanishing on V. Note that the dimension of $X_{K'}$ is d. So q' maps Y birationally to Z and hence $\dim Z = \dim Y = \dim \check{G} - 1$.

To calculate the degree of Z in $\check{G}$, we consider the following equality of cycle classes:

$$[Z] = (q'|_{\Gamma'})_*(p'|_{\Gamma'})^*[X] = \delta(q'|_{\Gamma'})_*(p'|_{\Gamma'})^*[U],$$

where U is the projective space associated to an arbitrary quotient space of rank $d+1$ of $\mathcal{E}_K$. Note that $(q'|_{\Gamma'})_*(p'|_{\Gamma'})^*[U]$ is just the first Schubert class in the Grassmannian $\check{G}$ (see [25], §14.7). Therefore the degree of Z is δ. $\square$

By Plücker's morphism $\check{G} \to \mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee)$, the coordinate algebra $B(\check{G}) = \bigoplus_{D \geqq 0} B_D(\check{G})$ of $\check{G}$ is a homogeneous quotient algebra of $\bigoplus_{D \geqq 0} S^D(\Lambda^{d+1}\mathcal{E}_K^\vee)$. To explain the role played by the Plücker coordinates, we consider the following construction. Denote by

$$\theta : \mathcal{E}_K^\vee \otimes (\Lambda^{d+1}\mathcal{E}_K) \to \Lambda^d \mathcal{E}_K$$

the subtraction homomorphism which sends $\xi \otimes (x_0 \wedge \cdots \wedge x_d)$ to

$$\sum_{i=0}^{d} (-1)^i \xi(x_i) x_0 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_d.$$

Let $\tilde{\Gamma}$ be the subvariety of $\mathbb{P}(\mathcal{E}_K) \times \mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee)$ which classifies the points (ξ, α) such that $\theta(\xi \otimes \alpha) = 0$. Let

$$\tilde{p} : \mathbb{P}(\mathcal{E}_K) \times \mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee) \to \mathbb{P}(\mathcal{E}_K) \quad \text{and} \quad \tilde{q} : \mathbb{P}(\mathcal{E}_K) \times \mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee) \to \mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee)$$

be the two projections.

Proposition 3.4. *Let $X \subset \mathbb{P}(\mathcal{E}_K)$ be a subvariety of dimension d and of degree δ. The set-theoretical intersection $\tilde{\Gamma} \cap \tilde{p}^{-1}(X)$ is irreducible. Moreover, if we consider $\tilde{\Gamma} \cap \tilde{p}^{-1}(X)$ as a reduced subvariety of $\mathbb{P}(\mathcal{E}_K) \times \mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee)$, then the scheme-theoretical image $\tilde{q}\big(\tilde{\Gamma} \cap \tilde{p}^{-1}(X)\big)$ is a hypersurface of degree δ of $\mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee)$.*

Denote by $\Psi_{X,K}$ the one-dimensional subspace of $S^\delta(\Lambda^{d+1}\mathcal{E}_K^\vee)$ which defines the hypersurface in Proposition 3.4. We call it the *Cayley form* of X. The saturation of $\Psi_{X,K}$ in $S^\delta(\Lambda^{d+1}\bar{\mathcal{E}})$, equipped with induced metrics, is called the *Cayley form of* X. Note that the incidence variety Γ' of $\mathbb{P}(\mathcal{E}_K) \times \check{G}$ is just the intersection of $\tilde{\Gamma}$ with $\mathbb{P}(\mathcal{E}_K) \times \check{G}$ (embedded in $\mathbb{P}(\mathcal{E}_K) \times \mathbb{P}(\Lambda^{d+1}\mathcal{E}_K^\vee)$ via the Plücker morphism).

The relationship between the Plücker and the Stiefel coordinates (see [27], p. 101, for details) leads to the following observation. Let ψ_X be a representing element of $\Psi_{X,K}$, considered as a homogeneous polynomial of degree δ on $\Lambda^{d+1}\mathcal{E}_K^\vee$. Then the multihomogeneous polynomial ϕ_X of multidegree $(\underbrace{\delta, \ldots, \delta}_{d+1 \text{ copies}})$ defined as

$$\phi_X(x_0, \ldots, x_d) := \psi_X(x_0 \wedge \cdots \wedge x_d)$$

spans the Chow form $\Phi_{X,K}$ of X.

Remark 3.5. Similarly to [7], Theorem 4.3.2, the following quantity can be compared to the Arakelov height of X with respect to $\bar{\mathcal{L}}$:

$$\tilde{h}_{\mathrm{Ph}}(X) := \frac{1}{[K:\mathbb{Q}]}\bigg(\sum_{\mathfrak{p}} \log\|\psi_X\|_{\mathfrak{p}} + \sum_{\sigma: K \to \mathbb{C}} \log M_\sigma(\psi_X)\bigg).$$

One has, by [7], Lemma 4.3.4,

$$\tilde{h}_{\mathrm{Ph}}(X) = h_{\bar{\mathcal{L}}}(X) - \frac{1}{2}\delta\mathcal{H}_N,$$

where

$$N = \mathrm{rk}(\Lambda^{d+1}\mathcal{E}_K) - 1 = \binom{n+1}{d+1} - 1 \quad \text{and} \quad \mathcal{H}_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}.$$

Moreover, by [7], Lemma 4.3.6, Remark 1.4.3 and Corollary 1.4.3, we obtain the following relation:

$$0 \leqq \tilde{h}_{\bar{\mathcal{L}}}(X) + \hat{\mu}(\overline{\Psi}_X) + \frac{1}{2}\log\binom{N+\delta}{\delta} \leqq \frac{1}{2}\delta\mathcal{H}_N.$$

By using the estimate

$$\log\binom{N+\delta}{\delta} \leqq \delta\log(N+1) = \delta\log\binom{n+1}{d+1}$$
$$\leqq \delta(d+1)\log(n+1) - \delta\log\big((d+1)!\big),$$

one obtains the relation

$$\hat{\mu}(\overline{\Psi}_X) \geqq -h_{\bar{\mathcal{L}}}(X) - \frac{\delta}{2}(d+1)\log(n+1) + \frac{\delta}{2}\log\big((d+1)!\big). \tag{18}$$

We construct a system of generators of X from $\Psi_{X,K}$. Choose a representative element ψ_X in $\Psi_{X,K}$ and consider it as a homogeneous polynomial of degree δ on $\Lambda^{d+1}\mathcal{E}_K^\vee$. Let $x, y_0, \dots, y_d$ be variables valued in $\mathcal{E}_K$ and let ξ be a variable valued in $\mathcal{E}_K^\vee$. For any $i = 0, 1, \dots, d$, let $z_i = \xi(x)y_i - \xi(y_i)x$. As

$$\begin{aligned}
&z_0 \wedge \cdots \wedge z_d \\
&\quad = \xi(x)^{d+1} y_0 \wedge \cdots \wedge y_d - \sum_{i=0}^{d} \xi(x)^d \xi(y_i)\, y_0 \wedge \cdots \wedge y_{i-1} \wedge x \wedge y_{i+1} \wedge \cdots \wedge y_d \\
&\quad = \xi(x)^d \Big(\xi(x)\, y_0 \wedge \cdots \wedge y_d - \sum_{i=0}^{d} \xi(y_i)\, y_0 \wedge \cdots \wedge y_{i-1} \wedge x \wedge y_{i+1} \wedge \cdots \wedge y_d \Big),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\psi_{X,K}(z_0 \wedge \cdots \wedge z_d) \\
&\quad = \xi(x)^{\delta d} \psi_{X,K}\Big(\xi(x)\, y_0 \wedge \cdots \wedge y_d - \sum_{i=0}^{d} \xi(y_i)\, y_0 \wedge \cdots \wedge y_{i-1} \wedge x \wedge y_{i+1} \wedge \cdots \wedge y_d \Big).
\end{aligned}$$

By specifying $x, y_0, \dots, y_d$ in

$$\psi_{X,K}\Big(\xi(x)\, y_0 \wedge \cdots \wedge y_d - \sum_{i=0}^{d} \xi(y_i)\, y_0 \wedge \cdots \wedge y_{i-1} \wedge x \wedge y_{i+1} \wedge \cdots \wedge y_d \Big),$$

we obtain a linear system $I_{X,K}$ of polynomials of degree δ on $\mathcal{E}_K^\vee$, which also defines the subscheme $\tilde{X}$ of $\mathbb{P}(\mathcal{E}_K)$. In fact, an antisymmetric homomorphism $\mathcal{E}_K^\vee \to \mathcal{E}_K$ acting on an element ξ in $\mathcal{E}_K^\vee$ can be written as a linear combination over K of elements of the form $\xi(x)y - \xi(y)x$, where x and y are elements in $\mathcal{E}_K$.

Let $\bar{I}_X$ be the saturated Hermitian vector subbundle of $S^\delta\bar{\mathcal{E}}$ whose generic fibre coincides with $I_{X,K}$. We are interested in estimating the complexity of $\bar{I}_X$, for which we need the following tensoriel construction of I_X.

Consider the $\mathcal{O}_K$-linear homomorphism $\mathcal{E} \otimes \mathcal{E}^{\otimes(d+1)} \otimes \mathcal{E}^\vee \to \Lambda^{d+1}\mathcal{E}$ sending

$$x \otimes y_0 \otimes \cdots \otimes y_d \otimes \xi$$

to

$$\xi(x)y_0 \wedge \cdots \wedge y_d - \sum_{i=0}^{d} \xi(y_i)y_0 \wedge \cdots \wedge y_{i-1} \wedge x \wedge y_{i+1} \wedge \cdots \wedge y_d,$$

which induces a homomorphism

$$\Gamma^\delta(\mathcal{E}) \otimes \Gamma^\delta(\mathcal{E})^{\otimes(d+1)} \otimes \Gamma^\delta(\mathcal{E}^\vee) \to \Gamma^\delta(\Lambda^{d+1}\mathcal{E}), \tag{19}$$

where for any projective $\mathcal{O}_K$-module of finite type F, $\Gamma^\delta(F)$ is the sub-$\mathcal{O}_K$-module of $F^{\otimes\delta}$ consisting of all elements which are invariant by the action of the symmetric group $\mathfrak{S}_d$. By the canonical isomorphism $\Gamma^\delta(F)^\vee \cong S^\delta(F^\vee)$, we obtain from (19) a homomorphism

$$S^\delta(\Lambda^{d+1}\mathcal{E}^\vee) \to S^\delta(\mathcal{E}^\vee) \otimes S^\delta(\mathcal{E}^\vee)^{\otimes(d+1)} \otimes S^\delta(\mathcal{E})$$

by duality. Denote by f_X the composed homomorphism

$$\Psi_X \to S^\delta(\Lambda^{d+1}\mathcal{E}^\vee) \to S^\delta(\mathcal{E}^\vee) \otimes S^\delta(\mathcal{E}^\vee)^{\otimes(d+1)} \otimes S^\delta(\mathcal{E}),$$

where Ψ_X is the submodule of $S^\delta(\Lambda^{d+1}\mathcal{E}^\vee)$ corresponding to the Cayley form. Then $\bar{I}_X$ is just the saturation of the image (see the paragraph below (6) for definition) of $\overline{f_X(\Psi_X)}$ (with induced metrics) in $S^\delta(\bar{\mathcal{E}})$.

Proposition 3.6. *With the above notation, the following inequality holds*:

$$\hat{\mu}_{\min}(\bar{I}_X) \geqq -h_{\bar{\mathcal{L}}}(X) - C_1, \tag{20}$$

where the constant $C_1 = C_1(\bar{\mathcal{E}}, d, \delta)$ *is defined as*

$$\begin{aligned} C_1 = {} & (d+2)\hat{\mu}_{\max}\big(S^\delta(\bar{\mathcal{E}}^\vee)\big) + \frac{1}{2}(d+2)\log\binom{n+\delta}{\delta} + \varrho^{(d+2)}\big(\Gamma^\delta(\bar{\mathcal{E}})\big) \\ & + \frac{\delta}{2}\log\big((d+2)(n-d)\big) + \frac{\delta}{2}(d+1)\log(n+1). \end{aligned} \tag{21}$$

Proof. By Proposition 2.1 and Lemma A.1 (ii), the slope of $\overline{f_X(\Psi_X)}$ is estimated as follows:

$$\hat{\mu}\big(\overline{f_X(\Psi_X)}\big) \geqq \hat{\mu}(\bar{\Psi}_X) - h(f_X) \geqq \hat{\mu}(\bar{\Psi}_X) - \frac{\delta}{2}\log\big((d+2)! \cdot (n-d)\big).$$

By (18), this implies

$$\hat{\mu}\big(\overline{f_X(\Psi_X)}\big) \geqq -h_{\bar{\mathcal{L}}}(X) - \frac{\delta}{2}\log\big((d+2)(n-d)\big) - \frac{\delta}{2}(d+1)\log(n+1). \tag{22}$$

By Proposition 2.4 (see also Remark 2.5), we obtain

$$\hat{\mu}_{\min}(\bar{I}_X) \geqq \hat{\mu}\big(\overline{f_X(\Psi_X)}\big) - (d+2)\hat{\mu}_{\max}\big(S^\delta(\bar{\mathcal{E}}^\vee)\big) - \varrho^{(d+2)}\big(\Gamma^\delta(\bar{\mathcal{E}})\big) - \frac{1}{2}(d+2)\log \mathrm{rk}(S^\delta\mathcal{E}). \tag{23}$$

Combining (22) with (23) (note that $\mathrm{rk}(S^\delta\mathcal{E}) = \binom{n+\delta}{\delta}$), one obtains (20). □

Remark 3.7. By [5] (see also [30], [26]), one obtains $\hat{\mu}_{\max}\big(S^\delta(\bar{\mathcal{E}}^\vee)\big) \ll_{\bar{\mathcal{E}}} \delta$. Furthermore, one has $\varrho^{(d+2)}\big(\Gamma^\delta(\bar{\mathcal{E}})\big) \ll \log \mathrm{rk}(S^\delta\mathcal{E}) \leqq \delta\log(n+1)$. Therefore, $C_1 \ll_{\bar{\mathcal{E}},d} \delta$.

3.2. Complexity of the singular locus. In the previous subsection, we have constructed explicitly a linear system $I_{X,K}$ which defines a subscheme $\tilde{X}$ of $\mathbb{P}(\mathcal{E}_K)$ containing X. Since $\tilde{X}_{\mathrm{red}} = X$ (see [38], Lemma 11), we obtain $\tilde{X}_{\mathrm{reg}} \subset X_{\mathrm{reg}}$, where $\tilde{X}_{\mathrm{reg}}$ and X_{reg} are respectively the open subschemes of all regular points of $\tilde{X}$ and of X. Moreover, since $\tilde{X}$ coincides with X on a dense open subset (loc. cit.), $\tilde{X}_{\mathrm{reg}}$ is a dense open subscheme of X. By using the Jacobian criterion, we shall construct from $I_{X,K}$ a linear system defining the singular locus $\tilde{X}_{\mathrm{sing}}$ of $\tilde{X}$, which contains X_{sing}, the singular locus of X. Before discussing the complexity of $\tilde{X}$, we treat a slightly general case where we consider a Hermitian subbundle of certain symmetric power of $\bar{\mathcal{E}}$ and estimate the complexity of linear systems constructed from minors of its Jacobian matrix.

For any integer $a \geqq 1$, denote by $D_a : S^a\mathcal{E} \to \mathcal{E} \otimes S^{a-1}\mathcal{E}$ the homomorphism of derivation which sends $x_1 \cdots x_a$ to $\sum_{i=1}^{a} x_i \otimes (x_1 \cdots x_{i-1}x_{i+1} \cdots x_a)$. Suppose that $\bar{I}$ is a Hermitian subbundle of $S^a\bar{\mathcal{E}}$ and that r is an integer such that $r \geqq 1$. We denote by $g_I^{(r)}$ the following composed homomorphism:

$$I^{\otimes r} \longrightarrow S^a(\mathcal{E})^{\otimes r} \xrightarrow{D_a^{\otimes r}} \mathcal{E}^{\otimes r} \otimes S^{a-1}(\mathcal{E})^{\otimes r} \longrightarrow \Lambda^r\mathcal{E} \otimes S^{(a-1)r}(\mathcal{E}),$$

where the last arrow is induced by canonical homomorphisms $\mathcal{E}^{\otimes r} \to \Lambda^r\mathcal{E}$ and $S^{a-1}(\mathcal{E})^{\otimes r} \to S^{(a-1)r}\mathcal{E}$. Let $\bar{F}_I^{(r)}$ be the image of $g_I^{(r)}$, equipped with induced metrics. Denote by $\bar{I}^{(r)}$ the image of $\bar{F}_I^{(r)}$ in $S^{(a-1)r}\bar{\mathcal{E}}$.

Theorem 3.8. *With the above notation, the following inequality holds*:

$$\hat{\mu}_{\min}(\bar{I}^{(r)}) \geqq r\hat{\mu}_{\min}(\bar{I}) - C_2, \tag{24}$$

where the constant $C_2 = C_2(\bar{\mathcal{E}}, r, a)$ *is defined as*

$$\hat{\mu}_{\max}(\Lambda^r\bar{\mathcal{E}}) + r\log \mathrm{rk}(S^a\mathcal{E}) + \log \mathrm{rk}(\Lambda^r\mathcal{E}) + \log\sqrt{r!} + r\log a. \tag{25}$$

Proof. By Lemma A.1 (iii) and (iv), the height of $g_I^{(r)}$ is bounded from above by $\log\sqrt{r!} + r\log a$. Therefore, Proposition 2.1 (ii) shows that

$$\begin{aligned}\hat{\mu}_{\min}(\bar{F}_I^{(r)}) &\geqq \hat{\mu}_{\min}(\bar{I}^{\otimes r}) - \log\sqrt{r!} - r\log a \\ &= r\hat{\mu}_{\min}(\bar{I}) - \varrho^{(r)}(\bar{I}^\vee) - \log\sqrt{r!} - r\log a,\end{aligned} \tag{26}$$

where $\bar{F}_I^{(r)}$ is the image of $g_I^{(r)}$, equipped with induced metrics. Note that (see (6))

$$\varrho^{(r)}(\bar{I}^\vee) \leqq \frac{r}{2}\log(\operatorname{rk} I) \leqq \frac{r}{2}\log(\operatorname{rk} S^a\mathcal{E}).$$

By Proposition 2.4, one has

$$\hat{\mu}_{\min}(\bar{I}^{(r)}) \geqq \hat{\mu}_{\min}(\bar{F}_I^{(r)}) - \hat{\mu}_{\max}(\Lambda^r\bar{\mathcal{E}}) - \varrho(\Lambda^r\bar{\mathcal{E}}^\vee, \bar{F}_I^{(r)}) - \frac{1}{2}\log(\operatorname{rk}\Lambda^r\mathcal{E}). \tag{27}$$

Therefore the required estimation follows from (26), (27) and the inequality

$$\varrho(\Lambda^r\bar{\mathcal{E}}^\vee, \bar{F}_I^{(r)}) \leqq \frac{1}{2}\log(\operatorname{rk}\Lambda^r\mathcal{E}) + \frac{1}{2}r\log(\operatorname{rk} S^a\mathcal{E}). \quad \square$$

Remark 3.9. (i) One has $C_2(\bar{\mathcal{E}}, r, a) \ll_{\bar{\mathcal{E}},r} a$.

(ii) When $\bar{\mathcal{E}}$ is a direct sum of Hermitian line bundles, the term $\varrho(\Lambda^r\bar{\mathcal{E}}^\vee, \bar{F}_I^{(r)})$ vanishes. Hence we can choose C_2 to be

$$\hat{\mu}_{\max}(\Lambda^r\bar{\mathcal{E}}) + \frac{r}{2}\log\operatorname{rk}(S^a\mathcal{E}) + \frac{1}{2}\log\operatorname{rk}(\Lambda^r\mathcal{E}) + \log\sqrt{r!} + r\log a.$$

(iii) If Bost's conjecture (see §2.2) is true, then we can choose

$$C_2 = \hat{\mu}_{\max}(\Lambda^r\bar{\mathcal{E}}) + \frac{1}{2}\log\operatorname{rk}(\Lambda^r\mathcal{E}) + \log\sqrt{r!} + r\log a.$$

We apply Theorem 3.8 on $I_X \subset S^\delta\mathcal{E}$ and on $r = n-d$. By using the estimate (18), we obtain the following result:

Theorem 3.10. *Let $X \subset \mathbb{P}(\mathcal{E}_K)$ be a subvariety of dimension d and of degree δ. Denote by $\mathscr{X}$ the Zariski closure of X in $\mathbb{P}(\mathcal{E})$. There exists a Hermitian vector subbundle $\overline{M}$ of $S^{(\delta-1)(n-d)}\bar{\mathcal{E}}$ satisfying*

$$\hat{\mu}_{\min}(\overline{M}) \geqq -(n-d)h_{\bar{\mathcal{L}}}(X) - C_3 \tag{28}$$

and such that the subscheme of $\mathbb{P}(\mathcal{E})$ defined by the vanishing of M contains the singular loci of fibres of $\mathscr{X}$ but not the generic point of $\mathscr{X}$, where the constant $C_3 = C_3(\bar{\mathcal{E}}, d, \delta)$ is defined as

$$(n-d)C_1(\bar{\mathcal{E}}, d, \delta) + C_2(\bar{\mathcal{E}}, n-d, \delta).$$

Moreover, one has $C_3(\bar{\mathcal{E}}, d, \delta) \ll_{\bar{\mathcal{E}},d} \delta$.

Proof. We take $\overline{M} = \bar{I}_X^{(n-d)}$, where $\bar{I}_X$ is defined in the paragraph below (19). Let $\tilde{X}$ be the subvariety of $\mathbb{P}(\mathcal{E}_K)$ defined by the vanishing of $I_{X,K}$ and $\tilde{\mathscr{X}}$ be its Zariski closure in $\mathbb{P}(\mathcal{E})$ (which is defined by vanishing of I_X since I_X is saturated). By the Jacobian criterion, the subscheme of $\mathbb{P}(\mathcal{E})$ defined by vanishing of M coincides with the locus of singular points of fibres of $\tilde{\mathscr{X}}$, which contains the locus of singular points of fibres of $\mathscr{X}$. The inequality (28) is a consequence of (24) and (20). The last assertion results from Remarks 3.7 and 3.9 (i). □

4. Estimations of Hilbert–Samuel functions

In this section, we discuss the estimations of the geometric and arithmetic Hilbert–Samuel functions. We fix in this section a Hermitian vector bundle $\bar{\mathcal{E}}$ of rank $n+1$ over $\operatorname{Spec}\mathcal{O}_K$ and a subvariety $X \subset \mathbb{P}(\mathcal{E}_K)$ which is of dimension $d \geqq 1$ and of degree δ. Denote by $\mathscr{X}$ the Zariski closure of X in $\mathbb{P}(\mathcal{E})$. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the universal line bundle. We equip it with the Fubini–Study metrics to obtain a Hermitian line bundle $\bar{\mathcal{L}}$ on $\mathbb{P}(\mathcal{E})$. For any integer $D \geqq 1$, let $E_D := H^0\big(\mathbb{P}(\mathcal{E}), \mathcal{L}^{\otimes D}\big)$ and $r(D)$ be its rank; let F_D be the saturation of the image of E_D in $H^0(\mathscr{X}, \mathcal{L}|_{\mathscr{X}}^{\otimes D})$ by the homomorphism of restriction of sections and let $r_1(D) := \operatorname{rk} F_D$.

4.1. Estimations of the geometric Hilbert–Samuel function. In this section, we recall several known results on explicit estimations of the geometric Hilbert–Samuel function. Let $X \subset \mathbb{P}(\mathcal{E}_K)$ be a closed subvariety of dimension d and of degree δ. We assume that $d < n$. The (geometric) Hilbert–Samuel function of X is by definition the function on $\mathbb{N}\backslash\{0\}$ which sends $D \in \mathbb{N}\backslash\{0\}$ to the rank of $H^0(X, \mathcal{L}|_X^{\otimes D})$. By the asymptotic Riemann–Roch Theorem, one has the following relation:

$$\operatorname{rk} H^0(X, \mathcal{L}|_X^{\otimes D}) = \frac{\delta}{d!} D^d + O(D^{d-1}).$$

However, here our concern is to obtain the upper and lower bounds of this quantity which hold for any D in $\mathbb{N}\backslash\{0\}$ except an explicit finite subset. In this direction there is a result of Kollár and Matsusaka [36] which asserts that when X is normal, one has

$$\left| \operatorname{rk} H^0(X, \mathcal{L}|_X^{\otimes D}) - \frac{\delta}{d!} D^d + \frac{(K_X \cdot \mathcal{L}|_X^{d-1})}{2(d-1)!} D^{d-1} \right| \leqq C \cdot D^{d-2},$$

where C is an explicitly computable constant depending only on δ and $(K_X \cdot \mathcal{L}|_X^{d-1})$, K_X being the dualizing line bundle of X. However, here we need the estimates independent of the dualizing sheaf (but asymptotically less precise than that of Kollár and Matsusaka). For the upper bound, we refer to the following result of Chardin [18] (see also [46], [1]).

Proposition 4.1. *For any integer $D \geqq 1$, one has*

$$\operatorname{rk} H^0(X, \mathcal{L}|_X^{\otimes D}) \leqq \delta \binom{D+d-1}{d} + \binom{D+d-1}{d-1}. \tag{29}$$

The proof relies on the generic hyperplane intersection of X and proceeds by induction on the dimension d. For details, see [1], §1.2.

Remark 4.2. Chardin has actually proved the following upper bound for the function $r_1(D)$:

$$(30) \qquad \forall D \geqq 1, \quad r_1(D) = \operatorname{rk} F_D \leqq \delta \binom{D+d}{d}.$$

As for the lower bound of $\operatorname{rk} H^0(X, \mathcal{L}|_X^{\otimes D})$, the following is an elementary result, which can be found in the book of Kollár [35].

Proposition 4.3. *For any integer $D \geqq \delta$, one has*

$$\operatorname{rk} H^0(X, \mathcal{L}|_X^{\otimes D}) \geqq \frac{\delta}{d!}(D+1-\delta)^d.$$

The proof consists of projecting generically X to a hypersurface of degree δ, which we refer to [35], p. 92.

In [46], Sombra has proved the following (optimal) lower bound for the function $r_1(D)$, which holds for all D.

Proposition 4.4. *For any integer $D \geqq 1$, one has*

$$(31) \qquad r_1(D) \geqq \binom{D+d+1}{d+1} - \binom{D-\delta+d+1}{d+1}.$$

4.2. Lower bound of the arithmetic Hilbert–Samuel function. In this subsection, we reformulate a result of David and Philippon [22] on an explicit lower bound of the arithmetic Hilbert–Samuel function in the framework of the slope method. Note that their argument relies on the higher Chow forms introduced by Philippon [39]. We begin with a reminder on it.

Let $m \geqq 1$ be an integer. Denote by W_m the product variety $\mathbb{P}(\mathcal{E}_K) \times \mathbb{P}\big(S^m(\mathcal{E}_K)^\vee\big)^{d+1}$. Let $\Gamma_m \subset W_m$ be the incidence subvariety classifying all points $(\alpha, u_0, \ldots, u_d)$ such that $\alpha^{\otimes m}(u_0) = \cdots = \alpha^{\otimes m}(u_d) = 0$, where we have considered a quotient of rank one of $S^m(\mathcal{E}_K)^\vee$ as a subspace of rank one of $S^m(\mathcal{E}_K)$. Denote by $p_m : W_m \to \mathbb{P}(\mathcal{E}_K)$ and $q_m : W_m \to \mathbb{P}\big(S^m(\mathcal{E}_K^\vee)\big)^{d+1}$ the two projections.

The following proposition asserts the existence of the higher Chow forms, which generalizes Proposition 3.1. This result has also been proved in [39]. See [7] for a geometric proof.

Proposition 4.5. *Let $X \subset \mathbb{P}(\mathcal{E}_K)$ be a subvariety. Then the set-theoretical intersection $\Gamma_m \cap p_m^{-1}(X)$ is irreducible. Furthermore, if we consider $\Gamma_m \cap p_m^{-1}(X)$ as a reduced subvariety of W, then the scheme-theoretical image $q_m\big(\Gamma_m \cap p_m^{-1}(X)\big)$ is a hypersurface of multidegree $(\delta m^d, \ldots, \delta m^d)$.*

Denote by $\overline{\Phi}_X^{[m]}$ the Hermitian line subbundle of $S^{\delta m^d}\big(S^m(\overline{\mathcal{E}})^\vee\big)^{\otimes(d+1)}$ corresponding to the hypersurface in Proposition 4.5. By definition, one has $\overline{\Phi}_X = \overline{\Phi}_X^{[1]}$.

Remark 4.6. By [7], Proposition 4.3.5 and Theorem 4.3.8, the following relation holds:

$$0 \leqq \hat{\mu}(\overline{\Phi}_X^{[m]}) + m^{d+1} h_{\bar{\mathcal{L}}}(X) + \frac{1}{2}(d+1)\log\binom{N_m + m^d\delta}{m^d\delta} \leqq \frac{1}{2} m^d \delta (d+1) \mathcal{H}_{N_m},$$

where

$$N_m = \mathrm{rk}\big(S^m(\mathcal{E}_K^\vee)\big) - 1 = \binom{n+m}{n} - 1 \quad \text{and} \quad \mathcal{H}_{N_m} = 1 + \frac{1}{2} + \cdots + \frac{1}{N_m}.$$

In particular, one has

$$\hat{\mu}(\overline{\Phi}_X^{[m]}) \leqq -m^{d+1} h_{\bar{\mathcal{L}}}(X) - \frac{1}{2}(d+1)\log\binom{N_m + m^d\delta}{m^d\delta} + \frac{1}{2} m^d \delta (d+1) \mathcal{H}_{N_m}.$$

By using the estimate

$$\log\binom{N_m + m^d\delta}{m^d\delta} \geqq m^d\delta \log(N_m + 1) - \log\big((m^d\delta)!\big),$$

we obtain the inequality

$$\hat{\mu}(\overline{\Phi}_X^{[m]}) \leqq -m^{d+1} h_{\bar{\mathcal{L}}}(X) + \frac{1}{2}(d+1) m^d \delta \log(m^d\delta). \tag{32}$$

In order to obtain an effective estimate of the arithmetic Hilbert–Samuel function, we need the following algebraic construction of $\Phi_X^{[m]}$ given by Philippon in [39].

Denote by A the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_K}(\mathcal{E})$. The algebra A identifies with $\bigoplus_{D \geqq 0} H^0\big(\mathbb{P}(\mathcal{E}), \mathcal{L}^{\otimes D}\big)$. For any integer $m \geqq 1$, define

$$\mathcal{O}_K^{[m]} := \mathrm{Sym}_{\mathcal{O}_K}\big(S^m(\mathcal{E})^{\vee \oplus (d+1)}\big), \tag{33}$$

$$A^{[m]} := \mathrm{Sym}_{\mathcal{O}_K}\big(\mathcal{E} \oplus S^m(\mathcal{E})^{\vee \oplus (d+1)}\big) \cong \mathrm{Sym}_{\mathcal{O}_K^{[m]}}(\mathcal{O}_K^{[m]} \otimes_{\mathcal{O}_K} \mathcal{E}). \tag{34}$$

As a symmetric algebra, the $\mathcal{O}_K$-algebra $\mathcal{O}_K^{[m]}$ is naturally graded. We equip $A^{[m]}$ with the grading which is induced from that of A, or equivalently the natural grading corresponding to the symmetric $\mathcal{O}_K^{[m]}$-algebra structure of $\mathrm{Sym}_{\mathcal{O}_K^{[m]}}(\mathcal{O}_K^{[m]} \otimes_{\mathcal{O}_K} \mathcal{E})$.

For $i \in \{0, 1, \ldots, d\}$, let tr_i be the image of the trace element of $S^m(\mathcal{E}) \otimes S^m(\mathcal{E})^\vee$ in $A^{[m]}$ via the $(i+1)$-th component of $S^m(\mathcal{E})^{\vee \oplus (d+1)}$. It is a homogeneous element of degree m in $A^{[m]}$. Recall that the trace element corresponds to $\mathrm{Id} : S^m(\mathcal{E}) \to S^m(\mathcal{E})$ through the natural isomorphism $S^m(\mathcal{E}) \otimes S^m(\mathcal{E})^\vee \cong \mathrm{Hom}_{\mathcal{O}_K}\big(S^m(\mathcal{E}), S^m(\mathcal{E})\big)$. Let $\mathfrak{I}$ be the kernel of the restriction homomorphism $A \to \bigoplus_{D \geqq 0} H^0(\mathscr{X}, \mathcal{L}^{\otimes D})$. It is a homogeneous ideal of A. Denote

$$\mathfrak{I}^{[m]} = A^{[m]}\mathfrak{I} + A^{[m]}\,\mathrm{tr}_0 + \cdots + A^{[m]}\,\mathrm{tr}_d. \tag{35}$$

It is a homogeneous ideal of $A^{[m]}$.

Proposition 4.7. (i) *The ideal*

$$\mathfrak{C}^{[m]}\mathfrak{I} := \bigcup_{D \geqq 0} (\mathfrak{I}^{[m]} :_{\mathcal{O}_K^{[m]}} A_D)$$

of $\mathcal{O}_K^{[m]}$ *is principal, and is generated by* $\Phi_X^{[m]}$.

(ii) *Assume that* $D \geqq (n-d)(\delta-1)+1$. *Then for any integer* $m \geqq 1$, *one has*

$$(\mathfrak{I}^{[m]} :_{\mathcal{O}_K^{[m]}} A_{D+m(d+1)-d}) \neq 0.$$

See [39], Proposition 1.5, for the proof of (i), and [22], Proposition 4.2, for that of (ii); see also [22], p. 528.

By using Proposition 4.7, we obtain the following lower bound of the arithmetic Hilbert–Samuel function, which reformulates [22], Proposition 4.10, in the language of the slope theory.

Theorem 4.8. *Let* $X \subset \mathbb{P}(\mathcal{E}_K)$ *be a closed subvariety of dimension* d *and of degree* δ, $\mathcal{X}$ *be the Zariski closure of* X *in* $\mathbb{P}(\mathcal{E})$. *For any integer* $D \geqq 1$ *let* $\bar{A}_D = S^D\bar{\mathcal{E}}$ *and let* $\mathfrak{I}_D$ *be the kernel of the restriction homomorphism* $A_D \to H^0(\mathcal{X}, \mathcal{L}^{\otimes D})$. *Then, for any integer* $D \geqq (n-d)(\delta-1)+1$, *one has*

$$\begin{aligned}(36)\qquad &\hat{\mu}(\overline{A_D/\mathfrak{I}_D})\\ &\geqq \frac{\big(D-(n-d)(\delta-1)-1\big)^{d+1}}{(d+1)^{d+1}r_1(D)}[h_{\bar{\mathcal{L}}}(X)-\delta(d+1)\hat{\mu}_{\min}(\bar{\mathcal{E}})]\\ &\quad -\frac{\delta}{2}\frac{\big(D-(n-d)(\delta-1)-1+d\big)^{d}}{(d+1)^{d-1}r_1(D)}\log\left(\left(\frac{D-(n-d)(\delta)-1+d}{d+1}\right)^{d}\delta\right)\\ &\quad -\frac{1}{2}D\log(n+1)-\frac{\varrho^{(Dr_1(D))}(\bar{\mathcal{E}}^\vee)}{r_1(D)},\end{aligned}$$

where $r_1(D) = \operatorname{rk} F_D = \operatorname{rk}(A_D/\mathfrak{I}_D)$.

Proof. Let $m \in \mathbb{N}\setminus\{0\}$ be a parameter which will be chosen in the end of the proof. Let $\mathcal{O}_K^{[m]}$, $A^{[m]}$ and $\mathfrak{I}^{[m]}$ be as in (33), (34) and (35) respectively. For any integer $D \geqq 1$, $A_D^{[m]}$ is a projective $\mathcal{O}_K^{[m]}$-module of rank $\binom{D+n}{n}$. The D-th homogeneous component $\mathfrak{I}_D^{[m]}$ of $\mathfrak{I}^{[m]}$ can be considered as a sub-$\mathcal{O}_K^{[m]}$-module of $A_D^{[m]}$. By definition, one has

$$\mathfrak{I}_D^{[m]} = \mathcal{O}_K^{[m]}\mathfrak{I}_D \oplus \mathcal{O}_K^{[m]}A_{D-m}\,\mathrm{tr}_0 \oplus \cdots \oplus \mathcal{O}_K^{[m]}A_{D-m}\,\mathrm{tr}_d.$$

Consider $M_D^{[m]} := \Lambda_{\mathcal{O}_K^{[m]}}^{\binom{D+n}{n}}\mathfrak{I}_D^{[m]}$. It is a sub-$\mathcal{O}_K^{[m]}$-module of

$$(37)\qquad \Lambda_{\mathcal{O}_K^{[m]}}^{\binom{D+n}{n}}A_D^{[m]} \cong \mathcal{O}_K^{[m]} \otimes_{\mathcal{O}_K} \det(A_D),$$

so it determines an ideal of $\mathcal{O}_K^{[m]}$ by twisting the module by $\det(A_D)^\vee$. Note that here $\det(A_D)$ is defined as $\Lambda_{\mathcal{O}_K}^{\mathrm{rk}(A_D)} A_D$. Let $I_D^{[m]} = \mathfrak{I}_D \oplus A_{D-m}\,\mathrm{tr}_0 \oplus \cdots \oplus A_{D-m}\,\mathrm{tr}_d$. One has $M_D^{[m]} = \mathcal{O}_K^{[m]} \otimes_{\mathcal{O}_K} \Lambda_{\mathcal{O}_K}^{\binom{D+n}{n}} I_D^{[m]}$. By Proposition 4.7 (ii), if

$$D \geqq (n-d)(\delta-1)+1+m(d+1)-d$$

(which we always assume in the rest of the proof), then $(\mathfrak{I}_D^{[m]} :_{\mathcal{O}_K^{[m]}} A_D) \neq 0$. Therefore $M_D^{[m]} \neq 0$, and hence the canonical image of

$$\det \mathfrak{I}_D \otimes \Lambda_{\mathcal{O}_K}^{r_1(D)}(A_{D-m}\,\mathrm{tr}_0 \oplus \cdots \oplus A_{D-m}\,\mathrm{tr}_d)$$

in $\Lambda_{\mathcal{O}_K}^{\mathrm{rk}\,A_D} I_D^{[m]}$ is non-zero. By definition of the Chow form, the canonical homomorphism

$$\det(A_D)^\vee \otimes \det(\mathfrak{I}_D) \otimes \Lambda_{\mathcal{O}_K}^{\mathrm{rk}(A_D/I_D)}(A_{D-m}\,\mathrm{tr}_0 \oplus \cdots \oplus A_{D-m}\,\mathrm{tr}_d) \to S^{r_1(D)}\big(S^m(\mathcal{E})^\vee\big)$$

factors through

$$\Phi_X^{[m]} \otimes \bigoplus_{i_0+\cdots+i_d=r_{D,m}} \bigotimes_{j=0}^{d} S^m(\mathcal{E})^{\vee\otimes i_j},$$

where $r_{D,m} = r_1(D) - \delta(d+1)m^d$. Hence the slope inequality implies

$$\begin{aligned}(38)\qquad -r_1(D)\cdot\hat{\mu}(\overline{A_D/\mathfrak{I}_D}) &+ \hat{\mu}_{\min}\big(\Lambda_{\mathcal{O}_K}^{r_1(D)}(\bar{A}_{D-m}^{\oplus(d+1)})\big)\\ &\leqq \hat{\mu}(\bar{\Phi}_X^{[m]}) + \max_{i_0+\cdots+i_d=r_{D,m}} \hat{\mu}_{\max}\left(\bigotimes_{j=0}^{d} S^m(\mathcal{E})^{\vee\otimes i_j}\right),\end{aligned}$$

where A_D is equipped with symmetric product metrics.

By Proposition 2.1 (ii) and Lemma A.1 (iv), one obtains

$$\begin{aligned}(39)\qquad \hat{\mu}_{\min}\big(\Lambda_{\mathcal{O}_K}^{r_1(D)}(\bar{A}_{D-m}^{\oplus(d+1)})\big) &\geqq \hat{\mu}_{\min}\big((\bar{A}_{D-m}^{\oplus(d+1)})^{\otimes r_1(D)}\big) - \frac{1}{2}r_1(D)\log r_1(D)\\ &= \hat{\mu}_{\min}(\bar{A}_{D-m}^{\otimes r_1(D)}) - \frac{1}{2}r_1(D)\log r_1(D).\end{aligned}$$

Note that $\bar{A}_{D-m}$ is a quotient of $\bar{\mathcal{E}}^{\otimes D-m}$. Hence

$$\begin{aligned}(40)\qquad \hat{\mu}_{\min}(\bar{A}_{D-m}^{\otimes r_1(D)}) &\geqq \hat{\mu}_{\min}(\bar{\mathcal{E}}^{\otimes(D-m)r_1(D)})\\ &\geqq r_1(D)(D-m)\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \varrho^{((D-m)r_1(D))}(\bar{\mathcal{E}}^\vee).\end{aligned}$$

Furthermore, if $i_0,\ldots,i_d$ are positive integers such that $i_0+\cdots+i_d = r_{D,m}$, then

$$\begin{aligned}(41)\qquad \hat{\mu}_{\max}\left(\bigotimes_{j=0}^{d} S^m(\bar{\mathcal{E}})^{\vee\otimes i_j}\right) &= -\hat{\mu}_{\min}\left(\bigotimes_{j=0}^{d} S^m(\bar{\mathcal{E}})^{\otimes i_j}\right)\\ &\leqq -\hat{\mu}_{\min}(\mathcal{E}^{\otimes m r_{D,m}}) \leqq -m r_{D,m}\hat{\mu}_{\min}(\bar{\mathcal{E}}) + \varrho^{(m r_{D,m})}(\bar{\mathcal{E}}^\vee),\end{aligned}$$

where we remind that $r_{D,m} := r_1(D) - \delta(d+1)m^d$.

Combining the inequalities (38)–(41), we obtain

$$\hat{\mu}(\overline{A_D/\mathfrak{I}_D}) \geqq -\frac{\hat{\mu}(\overline{\Phi}_X^{[m]})}{r_1(D)} - \frac{1}{2}\log\big(r_1(D)\big) + D\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \delta(d+1)\frac{m^{d+1}}{r_1(D)}\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \frac{\varrho^{(Dr_1(D))}(\bar{\mathcal{E}}^\vee)}{r_1(D)}, \tag{42}$$

where we have applied (4) and used the fact that $\big(\varrho^{(n)}(\bar{\mathcal{E}}^\vee)\big)_{n\geqq 1}$ is an increasing sequence. By (32) and the estimate $r_1(D) \leqq \mathrm{rk}(A_D) \leqq (n+1)^D$, we obtain

$$\hat{\mu}(\overline{A_D/\mathfrak{I}_D}) \geqq \frac{m^{d+1}}{r_1(D)}[h_{\bar{\mathcal{L}}}(X) - \delta(d+1)\hat{\mu}_{\min}(\bar{\mathcal{E}})] - \frac{1}{2}(d+1)\frac{m^d\delta\log(m^d\delta)}{r_1(D)} - \frac{1}{2}D\log(n+1) + D\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \frac{\varrho^{(Dr_1(D))}(\bar{\mathcal{E}}^\vee)}{r_1(D)}. \tag{43}$$

Since the inequality (43) holds for any m satisfying $D \geqq (n-d)(\delta-1)+1+m(d+1)-d$, and since the term $h_{\bar{\mathcal{L}}}(X) - \delta(d+1)\hat{\mu}_{\min}(\bar{\mathcal{E}})$ is non-negative (see [7], Proposition 3.2.4), by taking

$$m = \left\lfloor \frac{D-(n-d)(\delta-1)-1+d}{d+1} \right\rfloor \geqq \frac{D-(n-d)(\delta-1)-1}{d+1},$$

we obtain the theorem. □

Remark 4.9. As an $\mathcal{O}_K$-module, A_D is isomorphic to E_D. However, the symmetric product metrics on A_D differ from those of $\bar{E}_D$ (see §2.3, notably Remark 2.6). Thus, if we equip F_D with the quotient metric of those of $\bar{E}_D$, then for any integer $D \geqq (n-d)(\delta-1)+1$, one has

$$\hat{\mu}(\bar{F}_D) \geqq \frac{\big(D-(n-d)(\delta-1)-1\big)^{d+1}}{r_1(D)(d+1)^{d+1}}[h_{\bar{\mathcal{L}}}(X) - \delta(d+1)\hat{\mu}_{\min}(\bar{\mathcal{E}})] - \frac{\delta}{2}\frac{\big(D-(n-d)(\delta-1)-1+d\big)^d}{r_1(D)(d+1)^{d-1}}\log\left(\left(\frac{D-(n-d)(\delta)-1+d}{d+1}\right)^d\delta\right) - D\log(n+1) + D\hat{\mu}_{\min}(\bar{\mathcal{E}}) - \frac{\varrho^{(Dr_1(D))}(\bar{\mathcal{E}}^\vee)}{r_1(D)}, \tag{44}$$

where we have used the estimate $\mathrm{rk}(E_D) \leqq (n+1)^D$. Recall the following explicit estimates of the rank of F_D by functions on D, δ, and d (see Remark 4.2):

$$r_1(D) \leqq \delta\binom{D+d}{d} \leqq \frac{\delta}{d!}(D+d)^d.$$

Moreover, for $D \geqq \delta$, one has (by [46])

$$r_1(D) \geqq \binom{D+d+1}{d+1} - \binom{D-\delta+d+1}{d+1} = \sum_{j=1}^{\delta} \binom{D-\delta+d+j}{d} \geqq \frac{\delta(D-\delta+2)^d}{d!}.$$

Combining (44), we obtain that the following inequality holds for any integer D such that $D \geqq (n-d)(\delta-1)+1$:

$$\frac{1}{D}\hat{\mu}(\bar{F}_D) \geqq C_4(D,d,\delta)[h_{\mathrm{Ph}}(X) - \delta(d+1)\hat{\mu}_{\min}(\bar{\mathcal{E}})] - C_5(D,n,\bar{\mathcal{E}}), \tag{45}$$

where the constants C_4 and C_5 are defined as

$$C_4(D,d,\delta) = \frac{d!}{\delta(d+1)^{d+1}}\left(\frac{D-(n-d)(\delta-1)-1}{D+d}\right)^{d+1}, \tag{46}$$

$$\begin{aligned} C_5(D,n,\bar{\mathcal{E}}) = \log(n+1) - \hat{\mu}_{\min}(\bar{\mathcal{E}}) - \frac{\varrho^{(Dr_1(D))}(\bar{\mathcal{E}}^\vee)}{Dr_1(D)} \\ + \frac{d!}{2(d+1)^{d-1}}\left(\frac{D-(n-d)(\delta-1)-1+d}{D-\delta+2}\right)^d \\ \cdot D^{-1}\log\left(\left(\frac{D-(n-d)(\delta-1)-1+d}{d+1}\delta^d\right)\right). \end{aligned} \tag{47}$$

Note that one has

$$C_4(D,d,\delta) \geqq d!/\delta(2d+2)^{d+1}$$

and

$$C_5(D,n,\bar{\mathcal{E}}) \leqq \log(n+1) - \hat{\mu}_{\min}(\bar{\mathcal{E}}) - \frac{\varrho^{(Dr_1(D))}(\bar{\mathcal{E}}^\vee)}{Dr_1(D)} + 2^d$$

once $D \geqq 2(n-d)(\delta-1)+d+2$.

4.3. Upper bound of the arithmetic Hilbert–Samuel function. We show that a variant of Proposition 2.1 permits us to obtain an upper bound of the (normalized) arithmetic Hilbert–Samuel function $\hat{\mu}(\bar{F}_D)$. We actually find an explicit upper bound of $\hat{\mu}_{\max}(\bar{F}_D)$ which holds for any $D \geqq 1$. Let us begin by a reminder on the essential minimum.

Denote by $\bar{K}$ an algebraic closure of K. Let X be a subvariety of $\mathbb{P}(\mathcal{E}_K)$. The *essential minimum* of X (relatively to the Hermitian line bundle $\bar{\mathcal{L}}$) is by definition

$$\hat{\mu}_{\mathrm{ess}}(X) := \sup_{\substack{\emptyset \neq U \subset X \\ U \text{ open in } X}} \inf_{P \in U(\bar{K})} h_{\bar{\mathcal{L}}}(P).$$

By [48], Lemma 6.5, the essential minimum $\hat{\mu}_{\mathrm{ess}}(X)$ is finite, and one has the following estimate:

$$\hat{\mu}_{\mathrm{ess}}(X) \leqq \frac{h_{\bar{\mathcal{L}}}(X)}{\delta}, \tag{48}$$

where $h_{\bar{\mathcal{L}}}(X)$ is the Arakelov height of X with respect to $\bar{\mathcal{L}}$, and δ is the degree of X.

Theorem 4.10. *For any integer $D \geqq 1$, one has*

$$\hat{\mu}_{\max}(\bar{F}_D) \leqq D\hat{\mu}_{\mathrm{ess}}(X) + \frac{1}{2}\log r_1(D). \tag{49}$$

Proof. Let t be a real number such that $t > \hat{\mu}_{\mathrm{ess}}(X)$. Denote by $\mathcal{B}_t$ the class of algebraic points P of X such that $h_{\bar{\mathcal{L}}}(P) \leqq t$. Let φ_D be the evaluation map

$$F_{D,\bar{K}} \to \bigoplus_{P \in \mathcal{B}_t} P^* \mathcal{L}_{\bar{K}}^{\otimes D}.$$

By definition, the family $\mathcal{B}_t$ is Zariski dense in X, so φ_D is injective. By Proposition 2.12, one has

$$\hat{\mu}_{\max}(\bar{F}_D) \leqq Dt + \frac{1}{2}\log r_1(D).$$

Since $t > \hat{\mu}_{\mathrm{ess}}(X)$ is arbitrary, we obtain the assertion. □

Remark 4.11. The inequality (49), combined with the estimates (48) and the trivial estimate $r_1(D) \leqq (n+1)^D$, gives an explicit upper bound for $\hat{\mu}_{\max}(\bar{F}_D)$ in terms of the degree, the dimension and the Arakelov height of X:

$$\forall D \geqq 1, \quad \hat{\mu}_{\max}(\bar{F}_D) \leqq \frac{h_{\bar{\mathcal{L}}}(X)}{\delta} D + \frac{D}{2}\log(n+1). \tag{50}$$

A. Computation of norms of linear operators

In this appendix, we compute the operator norms of several operators acting on tensor powers of a Hermitian vector space. These computations have been useful in the application of the slope inequalities, notably in the estimation of the heights of K-linear homomorphisms.

Lemma A.1. *Let $m \in \mathbb{N}$ and V be a Hermitian space of dimension m.*

(i) *The canonical homomorphism $\alpha : V \otimes V^\vee \to \mathbb{C}$ has norm $\sqrt{m}$.*

(ii) *For any $d \in \{0, \ldots, m-1\}$, denote by $\beta_d : V \otimes V^{\otimes(d+1)} \otimes V^\vee \to \Lambda^{d+1} V$ the homomorphism which sends $x \otimes y_0 \otimes \cdots \otimes y_d \otimes \xi$ to*

$$\xi(x) y_0 \wedge \cdots \wedge y_d - \sum_{i=0}^{d} \xi(y_i) y_0 \wedge \cdots \wedge y_{i-1} \wedge x \wedge y_{i+1} \wedge \cdots \wedge y_d.$$

Then the norm of β_d is $\sqrt{(d+2)!(m-d-1)}$.

(iii) *For any integer a such that* $a \geqq 1$, *denote by* $D_a : S^a V \to V \otimes S^{a-1} V$ *the homomorphism which sends* $v_1 \cdots v_a$ *to*

$$\sum_{i=1}^{a} v_i \otimes (v_1 \cdots v_{i-1} v_{i+1} \cdots v_a).$$

Then the norm of D_a *is a.*

(iv) *Let r be an integer such that* $1 \leqq r \leqq m$. *Denote by* $\gamma_r : V^{\otimes r} \to \Lambda^r V$ *the canonical homomorphism. The norm of* γ_r *is* $\sqrt{r!}$.

Proof. Let $(e_i)_{i=1}^m$ be an orthonormal basis of V and let $(e_i^\vee)_{i=1}^m$ be its dual basis, which is an orthonormal basis of $V^\vee$.

(i) The homomorphism α sends

$$\sum_{1 \leqq i,j \leqq m} \lambda_{ij} e_i \otimes e_j^\vee \quad \text{to} \quad \sum_{i=1}^{m} \lambda_{ii}.$$

Hence $\|\alpha\| = \sqrt{m}$.

(ii) Note that β_d sends $\sum_{i,\underline{j},k} \lambda_{i,\underline{j},k} e_i \otimes e_{j_0} \otimes \cdots \otimes e_{j_d} \otimes e_k^\vee$ to

$$\sum_{i,\underline{j},k} \lambda_{i,\underline{j},k} \left(\delta_{ik} e_{j_0} \wedge \cdots \wedge e_{j_d} - \sum_{a=0}^{d} \delta_{j_a k} e_{j_0} \wedge \cdots \wedge e_{j_{a-1}} \wedge e_i \wedge e_{j_{a+1}} \wedge \cdots \wedge e_{j_d} \right), \tag{51}$$

where $\underline{j}$ stands for $(j_0, \ldots, j_d)$, and $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ else. Let $u_0, \ldots, u_d$ be integers such that $1 \leqq u_0 < \cdots < u_d \leqq m$ and $\underline{u} = (u_0, \ldots, u_d)$. The symmetric group $\mathfrak{S}_{d+1}$ acts on $\{1, \ldots, m\}^{d+1}$ by permuting the components. In other words, $\sigma \in \mathfrak{S}_{d+1}$ sends $(v_0, \ldots, v_d)$ to $(v_{\sigma(0)}, \ldots, v_{\sigma(d)})$. Denote by $\operatorname{sgn} : \mathfrak{S}_{d+1} \to \{\pm 1\}$ the sign function. If we write (51) as a linear combination in the basis $(e_{v_0} \wedge \cdots \wedge e_{v_d})_{1 \leqq v_0 < \cdots < v_d \leqq m}$, then the coefficient of $e_{u_0} \wedge \cdots \wedge e_{u_d}$ is

$$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}_{d+1}} \sum_{i=1}^{m} \operatorname{sgn}(\sigma) \lambda_{i,\sigma(\underline{u}),i} - \sum_{a=0}^{d} \sum_{\sigma \in \mathfrak{S}_{d+1}} \sum_{k=1}^{m} \operatorname{sgn}(\sigma) \lambda_{u_{\sigma(a)}, \sigma^{(a,k)}(\underline{u}), k} \\ &\quad = \sum_{\sigma \in \mathfrak{S}_{d+1}} \sum_{\substack{1 \leqq i \leqq m \\ i \neq u_{\sigma(0)}, \ldots, u_{\sigma(d)}}} \operatorname{sgn}(\sigma) \lambda_{i,\sigma(\underline{u}),i} - \sum_{a=0}^{d} \sum_{\sigma \in \mathfrak{S}_{d+1}} \sum_{\substack{1 \leqq k \leqq m \\ k \neq u_{\sigma(a)}}} \operatorname{sgn}(\sigma) \lambda_{u_{\sigma(a)}, \sigma^{(a,k)}(\underline{u}), k}, \end{aligned} \tag{52}$$

where $\sigma^{(a,k)}(\underline{u}) = (u_{\sigma(0)}, \ldots, u_{\sigma(a-1)}, k, u_{\sigma(a+1)}, \ldots, u_{(\sigma(d))})$. If a and b are two integers such that $0 \leqq a \neq b \leqq d$, and if $\sigma \in \mathfrak{S}_{d+1}$, we denote by $\sigma_{a,b}$ an element of $\mathfrak{S}_{d+1}$ such that $\sigma_{a,b}(c) = \sigma(c)$ for any $c \in \{0, \ldots, d\} \setminus \{a, b\}$ and that $\sigma_{a,b}(a) = \sigma(b)$, $\sigma_{a,b}(b) = \sigma(a)$. Note that, with this notation, the equality

$$\lambda_{u_{\sigma(a)}, \sigma^{(a,k)}(\underline{u}), k} = \lambda_{u_{\sigma_{a,b}(b)}, \sigma_{a,b}^{(b,k)}(\underline{u}), k}$$

holds provided that $k = u_{\sigma(b)}$. Moreover, one has $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma_{a,b})$. Therefore, the formula (52) may be simplified as

$$\sum_{\sigma\in\mathfrak{S}_{d+1}} \sum_{\substack{1\leqq i\leqq m\\ i\neq u_{\sigma(0)},\dots,u_{\sigma(d)}}} \operatorname{sgn}(\sigma)\lambda_{i,\sigma(\underline{u}),i} - \sum_{a=0}^{d}\sum_{\sigma\in\mathfrak{S}_{d+1}} \sum_{\substack{1\leqq k\leqq m\\ k\neq u_{\sigma(0)},\dots,u_{\sigma(d)}}} \operatorname{sgn}(\sigma)\lambda_{u_{\sigma(a)},\sigma^{(a,k)}(\underline{u}),k}.$$

Hence the norm of β_d is equal to $\sqrt{(d+2)!(m-d-1)}$.

(iii) For any $J = (J_l)_{l=1}^m \in \mathbb{N}^m$, let

$$|J| = J_1 + \cdots + J_m, \quad J! = J_1!\cdots J_m! \quad \text{and} \quad e^J = e_1^{J_1}\cdots e_m^{J_m} \in S^{|J|}H.$$

Then $(e^J)_{|J|=a}$ is an orthogonal base of S^aV. Note that the norm of e^J is $\sqrt{J!/a!}$. For any integer $l = 1,\dots,m$, let α_l be the element in $\mathbb{N}^m$ whose l-th coordinate is 1 and whose other coordinates are zero. If $x = \sum_{|J'|=a} \lambda_{J'}e^{J'}$ is an element of S^aV, the homomorphism D_a sends x to

$$\sum_{|J'|=a} \lambda_{J'} \sum_{l=1}^m J'_l e_l \otimes e^{J'-\alpha^{(l)}} = \sum_{|J|=a-1}\sum_{l=1}^m (J_l+1)\lambda_{J+\alpha^{(l)}} e_l \otimes e^J,$$

where we have used the convention $e^J = 0$ if $J \notin \mathbb{N}^m$. Therefore

$$\|D_a(x)\|^2 = \sum_{|J|=a-1}\sum_{l=1}^m (J_l+1)^2\lambda_{J+\alpha^{(l)}}^2 \frac{J!}{(a-1)!}$$

$$= \sum_{|J|=a-1}\sum_{l=1}^m (J_l+1)\lambda_{J+\alpha^{(l)}}^2 \frac{(J+\alpha^{(l)})!}{(a-1)!} = a\sum_{|J'|=a}\lambda_{J'}^2\frac{J'!}{a!}\sum_{l=1}^m J'_l = a^2\|x\|^2.$$

Hence the norm of D_a is a.

(iv) The homomorphism γ_r sends $\sum_{\underline{i}} \lambda_{\underline{i}} e_{i_1}\otimes\cdots\otimes e_{i_r}$ to $\sum_{\underline{i}} \lambda_{\underline{i}} e_{i_1}\wedge\cdots\wedge e_{i_r}$, where $\underline{i} = (i_1,\dots,i_r) \in \{1,\dots,m\}^r$. The symmetric group $\mathfrak{S}_r$ acts on $\{1,\dots,m\}^r$ such that $\sigma\in\mathfrak{S}_r$ sends $(i_1,\dots,i_r)$ to $(i_{\sigma(1)},\dots,i_{\sigma(r)})$. With this notation, $\sum_{\underline{i}} \lambda_{\underline{i}} e_{i_1}\wedge\cdots\wedge e_{i_r}$ is simplified as

$$\sum_{1\leqq i_1<\cdots<i_r\leqq m}\Big(\sum_{\sigma\in\mathfrak{S}_r}\operatorname{sgn}(\sigma)\lambda_{\sigma(\underline{i})}\Big) e_{i_1}\wedge\cdots\wedge e_{i_r}.$$

As $(e_{i_1}\wedge\cdots\wedge e_{i_r})_{1\leqq i_1<\cdots<i_r\leqq m}$ is an orthonormal basis of $\Lambda^r V$, we obtain that $\|\gamma_r\| = \sqrt{\#\mathfrak{S}_r} = \sqrt{r!}$. □

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