THE APPLICATIONS OF OKOUNKOV BODIES TO ARITHMETIC PROBLEMS

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Since the seminal works of Okounkov [10], Kaveh and Khovanskii [6], Lazarsfeld and Muastaţă [7], the theory of Okounkov bodies has been shown to be an efficient tool to describe geometric invariants in birational algebraic geometry. The typical example is the volume function on the group of Cartier divisors on an integral projective variety. Recently, the arithmetic analogue of Okounkov bodies has been discovered in the framework of Arakelov geometry, and has led to interesting applications. In some situations, the application of Okounkov bodies in the arithmetic problem is crucial because the arithmetic analogue of classical methods is still missing or is very sophisticated.

From the point of view of Arakelov geometry, the arithmetic varieties should be considered as the analogue of algebraic varieties fibered over a smooth projective curve (the function field setting). In the number theory case, it is $\operatorname{Spec}\mathbb{Z}$ that plays the role of the base curve. By definition, an arithmetic projective variety refers to a projective and flat morphism $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ from an integral scheme $\mathscr{X}$ to $\operatorname{Spec}\mathbb{Z}$. A major obstruction to study such objects is that the base scheme is not "compact". For example, the principal divisor on $\operatorname{Spec}\mathbb{Z}$ need not have degree zero. It is a natural idea to compactify $\operatorname{Spec}\mathbb{Z}$ by the usual absolute value of $\mathbb{Q}$ (called the *infinite place*). Then the situation becomes analogous to the function field case since the closed points of a regular projective curve correspond to the valuations of the function field of the curve whose restriction on the base field is trivial. The compactness of the augmented object is justified by the following product formula

$$\forall\, a \in \mathbb{Q} \setminus \{0\}, \quad |a| \cdot \prod_p |a|_p = 1,$$

where p runs over the set of all prime numbers, and $|.|_p$ is the p-adic absolute value on $\mathbb{Q}$. However, it turns out that no scheme structure can be defined for this augmented object and one cannot find the direct analogue of projective varieties in the arithmetic setting.

The genuine idea of Arakelov is to introduce analytic object to "compactify" an arithmetic variety. Let $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic projective variety. One can imagine that we attach to the arithmetic projective variety the complex analytic space associated to the $\mathbb{C}$-scheme $\mathscr{X}_{\mathbb{C}}$ as the "fiber

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over the infinite place". The algebraic objects in the algebraic geometry setting correspond to the similar algebraic objects on the arithmetic variety $\mathscr{X}$ equipped with additional structures (often metrics) on the induced object on the analytic space. For example, the notion of line bundles in the geometric setting corresponds to the notion of hermitian line bundles in the arithmetic framework as follows. Let $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic projective variety, a *hermitian line bundle* is defined as any couple $\overline{\mathscr{L}} = (\mathscr{L}, \|.\|)$, where $\mathscr{L}$ is a line bundle on the scheme $\mathscr{X}$, and $\|.\|$ is a continuous metric on the pull-back of $\mathscr{L}$ on the analytic space associated to $\mathscr{L}_{\mathbb{C}}$, invariant under the action of the complex conjugation.

Given a hermitian line bundle $\overline{\mathscr{L}}$ on an arithmetic projective variety $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$, one can construct a lattice in a normed vector space as follows. We denote by $\pi_*(\mathscr{L})$ the $\mathbb{Z}$-module $H^0(\mathscr{X}, \mathscr{L})$, whose rank identifies with the dimension of the vector space $H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}})$ over $\mathbb{Q}$. The vector space $\pi_*(\mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}$, which can be considered as a vector subspace of $H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}})$, is naturally equipped with sup norm

$$\forall\, s \in \pi_*(\mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \|s\|_{\sup} := \sup_{x \in X(\mathbb{C})} \|s\|(x).$$

We shall use the expression $\pi_*(\overline{\mathscr{L}})$ to denote the lattice $(\pi_*(\mathscr{L}), \|.\|_{\sup})$. We say that a section $s \in \pi_*(\mathscr{L})$ is *small* if $\|s\|_{\sup} \leqslant 1$. We denote by $\widehat{H}^0(\overline{\mathscr{L}})$ the set of all small sections of $\overline{\mathscr{L}}$. The set $\widehat{H}^0(\overline{\mathscr{L}})$ is necessarily finite. This notion is analogous to the space of global sections of a line bundle in the algebraic geometry setting. Motivated by this observation, Moriwaki [8, 9] has introduced the notion of arithmetic volume function for hermitian line bundles (or more generally, for arithmetic $\mathbb{R}$-Cartier divisors) as follows

$$\widehat{\operatorname{vol}}(\overline{\mathscr{L}}) := \limsup_{n \to +\infty} \frac{\ln \#\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n})}{n^{\dim(\mathscr{X})} / \dim(\mathscr{X})!}.$$

This function has soon been proved to be quite useful in the arithmetic geometry. Moreover, it shares many good properties as its avatar in algebraic geometry, as shown by the works of Moriwaki mentioned above, and also by the work [11] of Yuan.

Despite the similitude of definitions, the study of the arithmetic volume function is by no means identical to that of the classical volume function and often much more difficult. In fact the small section set $\widehat{H}^0(\overline{\mathscr{L}})$ is not stable by the addition in general. The classical method in the study of graded linear series do not work in the arithmetic setting. Although tools from the complex analytic geometry can be used to remedy the defeat due to the lack of the algebraic structure, the implementation of these tools is often very sophisticated and demand extra hypotheses (smoothness, positivity, etc.) on the metric of the hermitian line bundle.

Under this circumstance, the theory of Okounkov bodies has been applied to the study of the arithmetic volume function and has let to interesting results such as the arithmetic version of Fujita's approximation theorem.

There are essentially two approaches on the arithmetic analogue of Okounknov bodies in the literature : the one developed in [12] constructs a convex body attached to the sets $(\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}))_{n\in\mathbb{N}}$ in a way similar to the classical approach of Okounkov bodies and requires a fine study on this family of sets; the one developed in [2, 3, 1] uses $\mathbb{R}$-filtrations to interpret the arithmetic volume function as the integral of certain level function on the geometric Okounkov body of the generic fiber L and relies on the theory of Okounkov bodies of graded linear series.

In the following, we will give a brief introduction to the $\mathbb{R}$-filtration approach mentioned above. Consider a lattice $\overline{E}=(E,\|.\|)$ in a normed finite dimensional real vector space. Here E denotes a free $\mathbb{Z}$-module of finite rank and $\|.\|$ is a norm on the real vector space $E_{\mathbb{R}}=E\otimes\mathbb{R}$. We can then introduce a decreasing $\mathbb{R}$-filtration $\mathcal{F}$ on $E_{\mathbb{Q}}$ as follows :

$$\forall\, t\in\mathbb{R},\quad \mathcal{F}^t(E_{\mathbb{Q}})=\mathrm{Vect}_{\mathbb{Q}}(\{s\in E\,:\,\|s\|\leqslant e^{-t}\}).$$

The jump points of the filtration are nothing but the logarithmic version of the successive minima of the lattice. The Minkowski's second theorem leads to the following estimation

$$\ln\#\widehat{H}^0(\overline{E})=\int_0^{+\infty}\mathrm{rk}(\mathcal{F}^t(E_{\mathbb{Q}}))\,\mathrm{d}t+O(r\ln(r)),$$

where $\widehat{H}^0(\overline{E})=\{s\in E\,:\,\|s\|\leqslant 1\}$, $r=\mathrm{rk}_{\mathbb{Z}}(E)$, and the implicit constant is absolute.

We now consider an arithmetic projective variety $\pi:\mathscr{X}\to\operatorname{Spec}\mathbb{Z}$ and a hermitian line bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$. We denote by $X=\mathscr{X}_{\mathbb{Q}}$ the generic fiber of π and by L the restriction of $\mathscr{L}$ on X. We assume that L is big. It turns out that the lattice structure of $\pi_*(\overline{\mathscr{L}}^{\otimes n})$ induces as above an $\mathbb{R}$-filtration $\mathcal{F}$ on the vector space $V_n=H^0(X,L^{\otimes n})$. The fundamental idea of the $\mathbb{R}$-filtration approach is that the direct sum

$$V_\bullet^t=\bigoplus_{n\geqslant 0}\mathcal{F}^{nt}(V_n)$$

is actually a graded linear series of L. The theory of Okounkov bodies then allows to attach to this graded linear series a convex body $\Delta(V_\bullet^t)$ in $\mathbb{R}^d$ (with $d=\dim(X)$) such that

$$\mathrm{vol}(\Delta(V_\bullet^t))=\lim_{n\to+\infty}\frac{\mathrm{rk}(V_n)}{n^d}.$$

A direct computation shows that

$$\int_0^{+\infty}\mathrm{rk}(\mathcal{F}^t(V_n))\,\mathrm{d}t=n\int_0^{+\infty}\mathrm{rk}(V_n^t)=\bigg(\int_0^{+\infty}\mathrm{vol}(\Delta(V_\bullet^t))\,\mathrm{d}t\bigg)n^{d+1}+o(n^{d+1}).$$

Therefore Minkowski's second theorem stated as above leads to

$$\lim_{n\to+\infty}\frac{\ln\#\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n})}{n^{d+1}}=\int_0^{+\infty}\mathrm{vol}(\Delta(V_\bullet^t))\,\mathrm{d}t.$$

In particular, if one denotes by $\widehat{\Delta}(\overline{\mathscr{L}})$ the convex body

$$\{(x,t)\,:\,t \geqslant 0,\, x \in \Delta(V_\bullet^t)\} \subset \mathbb{R}^{d+1},$$

then one can interpret the arithmetic volume $\widehat{\mathrm{vol}}(\overline{\mathscr{L}})$ as $(d+1)!\mathrm{vol}(\widehat{\Delta}(\overline{\mathscr{L}}))$. One can also introduce a level function $\varphi_{\overline{\mathscr{L}}}$ on the Okounkov body $\Delta(L)$ of the total graded linear series of L with

$$\varphi_{\overline{\mathscr{L}}}(x) = \sup\{t \in \mathbb{R}\,:\, x \in \Delta(V_\bullet^t)\}.$$

Then $\mathrm{vol}(\widehat{\Delta}(\overline{\mathscr{L}}))$ identifies with the the integral of the function $\max(\varphi_{\overline{\mathscr{L}}}, 0)$ on the Okounkov body $\Delta(L)$ with respect to the Lebesgue measure.

The $\mathbb{R}$-filtration approach is very flexible. It allows to separate difficulties arising from different structure of the problems and reduce the problems of divers natures to the study of graded linear series in the classical algebraic geometry setting. We refer the readers to [4, 5] for further applications of this approach in different settings.

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