OKOUNKOV BODIES: AN APPROACH OF FUNCTION FIELD ARITHMETIC

Huayi Chen

Abstract. — By using the method of function field arithmetic, we associate, to each graded linear series which is birational and of sub-finite type, a convex body whose Lebesgue measure identifies with the volume of the graded linear series. This approach allows to remove the hypothesis on the existence of a regular rational point, which appears for example in the construction of Lazarsfeld and Mustaţă. Moreover, it requires less non-intrinsic parameters of the projective variety.

(MSC2010 classification: 14G40, 11G30)

Contents

1. Introduction

The theory of Okounkov bodies, initialized by Okounkov [**16, 17**] and then developed by Lazarsfeld and Mustaţă [**15**], and Kaveh and Khovanskii [**11, 10**] respectively, is an efficient tool to study the asymptotic behaviour of graded linear series on projective varieties. Let X be an integral projective scheme of Krull dimension d over a field k and L be an invertible $\mathcal{O}_X$-module. This theory associates to each graded linear series of L a convex body (called Okounkov body) of $\mathbb{R}^d$ whose Lebesgue measure is equal to the volume of the graded linear series divided by $d!$ (under mild conditions on the graded linear series). The construction relies on the choice of a valuation of the rational function field $k(X)$ valued in the group $\mathbb{Z}^d$ equipped with a monomial order, such that the residue field of the valuation coincides with k. This valuation can be constructed through the choice of a flag of smooth subschemes of X containing a regular rational point (see [**15**]). From the point of view of birational geometry, the dependence of the Okounkov body on the choice of the valuation is a subtle problem. We refer the readers to [**13**] for a detailed discussion on the convex bodies appearing as Okounkov bodies of divisors, see also [**15**, §5]. Moreover, the existence of a regular rational point actually implies that the k-scheme is geometrically irreducible. Although interesting consequences of the Okounkov convex body approach, such as Fujita approximation and the volume as a limite, can still be obtained by using fine technics of local multiplicities (see [**9**]), the construction of covex bodies becomes more complicated and relies on even more non-intrinsic choices.

The purpose of this article is to introduce a new approach of relating graded linear series to convex bodies in Euclidean spaces, which is based on function field arithmetics. The construction only depends on the choice of a flag of successive field extensions

$$k = K_0 \subset K_1 \subset \ldots \subset K_d = k(X),$$

such that each extension K_i/K_{i-1} is of transcendental degree 1 (the extensions K_i/K_{i-1} are necessarily finitely generated, see [**4**] Chapter V, §14, no.7 Corollary 3). Thus we can consider K_i as the field of rational functions of a regular projective curve C_i over $\mathrm{Spec}(K_{i-1})$. Given a graded linear series $V_\bullet$ of a Cartier divisor on X, we let $V_{\bullet,K_i}$ be the graded sub-K_i-algebra of $\bigoplus_{n\in\mathbb{N}} k(X)$ generated by $V_\bullet$. It generates a graded $\mathcal{O}_{C_i}$-algebra of vector bundles $E^i_\bullet$ on the curve C_i. In the case where the graded linear series $V_\bullet$ is birational (see Definition 3.5), we show that the volume of $V_{\bullet,K_i}$ is equal to the arithmetic volume of the graded-$\mathcal{O}_{C_i}$-algebra $E^i_\bullet$ and we construct by induction a family of convex bodies $(\Delta^i)_{i=0}^d$, where Δ^i is a convex body in $\mathbb{R}^i$, such that Δ^i is delimited by the graph of a non-negative concave function on Δ^{i-1} which comes from the function field arithmetic of $E^i_\bullet$ by using the method of Fujita approximation for arithmetic graded linear series (see [**6, 2**]).

From the point of view of birational geometry, the above procedure holds for the general setting of graded linear series of a finitely generated extension K of the base field k and it is not necessary to fix a projective model of K. For any $d \in \mathbb{N}$, $d \geqslant 1$, we denote by $\mathcal{C}_d$ the set of d-uplets $(\Delta^i)_{i=0}^d$ where each Δ^i is a convex body in $\mathbb{R}^d$ and such that the projection of Δ^i on $\mathbb{R}^{i-1}$ by the $i-1$ first coordinates is contained in Δ^{i-1}.

Theorem 1.1. — *Let k be a field and K/k be a finitely generated extension of transcendental degree $d \geqslant 1$. We fix a flag*

$$k = K_0 \subset K_1 \subset \ldots \subset K_d = K \tag{1}$$

of field extensions such that K_i/K_{i-1} is of transcendental degree 1. Let $\mathcal{A}(K/k)$ be the set of all birational graded linear series of sub-finite type of K/k (see Definition 3.5). There exists a map $(\Delta^i)_{i=0}^d$ from $\mathcal{A}(K/k)$ to $\mathcal{C}_d$, only depending on the flag (1), which verifies the following conditions:

(a) For all graded linear series $V_\bullet$ and $V'_\bullet$ in $\mathcal{A}(K/k)$ such that $V_n \subset V'_n$ for sufficiently positif n, one has $\Delta^i(V_\bullet) \subset \Delta^i(V'_\bullet)$ for any $i \in \{0, \ldots, d\}$.

(b) For any graded linear series $V_\bullet$ in $\mathcal{A}(K/k)$, any integer $m \geqslant 1$ and any $i \in \{0, \ldots, d\}$, one has $\Delta^i(V_\bullet^{(m)}) = m\Delta^i(V_\bullet)$, where $V_\bullet^{(m)} := \bigoplus_{n \in \mathbb{N}} V_{mn}$.

(c) For all graded linear series $V_\bullet$ and $W_\bullet$ in $\mathcal{A}(K/k)$, one has

$$\Delta^i(V_\bullet) + \Delta^i(W_\bullet) \subset \Delta^i(V_\bullet \cdot W_\bullet),$$

for any $i \in \{0, \ldots, d\}$, where "+" denotes the Minkowski sum.

(d) For any graded linear series $V_\bullet$ in $\mathcal{A}(K/k)$ and any $i \in \{0, \ldots, d\}$, the mass of $\Delta^i(V_\bullet)$ with respect to the Lebesgue measure is equal to the volume of the graded linear series $V_{\bullet, K_{d-i}}$ of K/K_{d-i} divided by $i!$.

Moreover, the graded linear series $V_\bullet$ satisfies the Fujita approximation property (see Definition 3.7).

As explained above, for any $V_\bullet \in \mathcal{A}(K/k)$, the family of convex bodies $\Delta^i(V_\bullet)$ are constructed in a recursive way. More precisely, it can be shown that the graded linear series $V_{\bullet,K_1}$ belongs to $\mathcal{A}(K/K_1)$ (see Remark 3.6). We let $(\Delta^0(V_\bullet), \ldots, \Delta^{d-1}(V_\bullet))$ be the family $(\Delta^0(V_{\bullet,K_1}), \ldots, \Delta^{d-1}(V_{\bullet,K_1}))$, where by abuse of notation we use the expression $(\Delta^i)_{i=0}^{d-1}$ to denote the map $\mathcal{A}(K/K_1) \to \mathcal{C}_{d-1}$ predicted by the theorem as an induction hypothesis. Finally, the extension K_1/k corresponds to a regular projective curve C over $\operatorname{Spec} k$ whose rational function field is K_1. The convex body $\Delta^d(V_\bullet)$ is then constructed as the arithmetic Okonkov body of the graded $\mathcal{O}_C$-algebra on vector bundles generated by $V_\bullet$.

Compared to the classic approach of Okounkov bodies, the above construction relies on fewer parameters, which are closely related to the birational geometry of the extension K/k. Moreover, the convex bodies $\Delta^i(V_\bullet)$, $i \in \{0, \ldots, d\}$ are related by linear projections, which reflects interesting geometric information about the graded linear series $V_{\bullet,K_i}$.

The rest of the article is organised as follows. In the second section, we recall some classic results on function field arithmetic of vector bundles. In the third section, we discuss several properties of graded linear series. Finally, in the fourth section, we explain the construction of convex bodies associated to graded linear series and prove the main theorem.

2. Function field arithmetic of vector bundles

In this section, we let k be a field and C be a regular projective curve over $\operatorname{Spec} k$ (namely an integral regular projective scheme of dimension 1 over $\operatorname{Spec} k$). By vector bundle over C, we mean a locally free $\mathcal{O}_C$-module E of finite rank.

2.1. Degree function. — Recall that Riemann-Roch formula shows that, for any vector bundle E on C, one has

$$h^0(E) - h^1(E) = \deg(E) + \operatorname{rk}(E)(1 - g(C/k)), \tag{2}$$

where $h^0(E)$ and $h^1(E)$ are respectively the ranks of $H^0(C, E)$ and $H^1(C, E)$, $\deg(E) = \deg(c_1(E) \cap [C])$ is the degree of E, and $g(C/k)$ is the genus of the curve C relatively to k, which is equal to the dimension of $H^0(C, \omega_{C/k})$ over k, $\omega_{C/k}$ being the relative dualizing sheaf. Note that by Serre duality one has $h^1(E) = h^0(E^\vee \otimes \omega_{C/k})$.

Let $K = k(C)$ be the field of rational functions on C. Recall that any closed point x of C determines a discrete valuation $\operatorname{ord}_x(\cdot)$ on the field K. We denote by $|\cdot|_x$ the absolute value on K defined as $|\cdot|_x := \mathrm{e}^{-\operatorname{ord}_x(\cdot)}$. We denote by K_x the completion of K with respect to $|\cdot|_x$, on which the absolute value $|\cdot|_x$ extends by continuity. If E is a vector bundle on C, the $\mathcal{O}_C$-module structure on E determines, for each closed point $x \in C$, a norm $\|\cdot\|_x$ on E_{K_x} defined as

$$\|s\|_x := \inf\{|a| \,:\, a \in K_x^\times,\ a^{-1}s \in E \otimes_{\mathcal{O}_C} \mathfrak{o}_x\},$$

where $\mathfrak{o}_x$ is the valuation ring of K_x. Recall that the degree of E can also be computed as

$$\deg(E) = -\sum_{\text{closed point } x \in C} [k(x) : k] \log \|s_1 \wedge \cdots \wedge s_r\|_{x,\det}, \tag{3}$$

where $(s_1, \ldots, s_r)$ is an arbitrary basis of E_K over K, $k(x)$ is the residue field of the closed point x, and $\|\cdot\|_{x,\det}$ is the determinant norm on $\det(E_{K_x})$ induced by $\|\cdot\|_x$. Similarly, for any non-zero vector s in E_K, we define

$$\deg(s) = -\sum_{\text{closed point } x \in C} [k(x) : k] \log \|s\|_x. \tag{4}$$

Note that, if E is an invertible $\mathcal{O}_C$-module, then for any non-zero vector $s \in E_K$, one has $\deg(s) = \deg(E)$.

Proposition 2.1. — *Let C be a regular projective curve over a field k and E be a vector bundle on C. For any non-zero section $s \in H^0(C, E)$ viewed as a vector in E_K, one has* $\deg(s) \geqslant 0$.

Proof. — Since s is a global section of E, for any closed point $x \in C$, one has

$$-\log\|s\|_x = \mathrm{ord}_x(s) \geqslant 0.$$

Hence we obtain from (4) the relation $\deg(s) \geqslant 0$. □

Proposition 2.2. — *Let C be a regular projective curve over a field k and L be an invertible $\mathcal{O}_C$-module. If $\deg(L) \geqslant 0$, then $h^0(L) \leqslant \deg(L) + 1$.*

Proof. — The inequality is trivial if $h^0(L) = 0$. In the following, we assume that L admits a non-zero global section s, which defines an injective homomorphism from $L^\vee$ to $\mathcal{O}_C$. Therefore one has

$$h^0(L^\vee \otimes \omega_{C/k}) \leqslant h^0(\omega_{C/k}) = g(C/k),$$

where $g(C/k)$ is the genus of C. By Riemann-Roch formula (2) we obtain

$$h^0(L) = h^0(L^\vee \otimes \omega_{C/k}) + \deg(L) + 1 - g(C/k) \leqslant \deg(L) + 1.$$

□

2.2. Successive minima. — Let k be a field, C be a regular projective curve over $\operatorname{Spec} k$ and E be a vector bundle on C. For any $i \in \{1, \ldots, \mathrm{rk}(E)\}$, let $\lambda_i(E)$ be the supremum of $\lambda \in \mathbb{R}$ such that the K-vector subspace of E_K generated by the vectors $s \in E_K \setminus \{0\}$ verifying $\deg(s) \geqslant \lambda$ has rank $\geqslant i$. By definition one has

$$\lambda_1(E) \geqslant \ldots \geqslant \lambda_r(E), \quad \text{with } r = \mathrm{rk}(E).$$

These invariants are similar to (the minus logarithmic version of) successive minima in geometry of numbers.

Proposition 2.3. — *Let C be a regular projective curve over a field k and E be a vector bundle on C. If E is generated by global sections, then for any $i \in \{1, \ldots, \mathrm{rk}(E)\}$ one has $\lambda_i(E) \geqslant 0$.*

Proof. — Let $K = k(C)$ be the field of rational functions on C. Since E is generated by global sections, there exist non-zero global sections $s_1, \ldots, s_r$ of E which form a basis of E_K over K. By Proposition 2.1, one has $\deg(s_i) \geqslant 0$ for any $i \in \{1, \ldots, n\}$, which implies the assertion of the proposition. □

By Hadamard's inequality, if $(s_1, \ldots, s_r)$ is a basis of E_K over K, then for any closed point $x \in C$ one has

$$\log\|s_1 \wedge \cdots \wedge s_r\|_{x,\det} \leqslant \sum_{i=1}^r \log\|s_i\|_x.$$

Thus we obtain

$$\lambda_1(E) + \cdots + \lambda_r(E) \leqslant \deg(E). \tag{5}$$

Roy and Thunder have proved the following converse inequality (see [**18**, Theorem 2.1]):

$$\lambda_1(E)+\cdots+\lambda_r(E)\geqslant \deg(E)-\mathrm{rk}(E)\ell_C(g(C/k)), \tag{6}$$

where ℓ_C is a non-negative function on $\mathbb{R}_+$ depending only on the curve C.

The successive minima are related to the $\mathbb{R}$-filtration by minima. Let E be a vector bundle on C. For any $t\in\mathbb{R}$, let $\mathcal{F}^t(E_K)$ be the K-vector subspace of E_K generated by non-zero vectors $s\in E_K$ such that $\deg(s)\geqslant t$. Then $(\mathcal{F}^t(E_K))_{t\in\mathbb{R}}$ is a decreasing $\mathbb{R}$-filtration of E_K. Note that the function $(t\in\mathbb{R})\mapsto \dim_K(\mathcal{F}^t(E_K))$ is left continuous, and one has

$$\sum_{i=1}^{\mathrm{rk}(E)}\delta_{\lambda_i(E)}=-\frac{\mathrm{d}}{\mathrm{d}t}\dim_K(\mathcal{F}^t(E_K)),$$

where $\delta_{\lambda_i(E)}$ denotes the Dirac measure on $\lambda_i(E)$.

2.3. Positive degree. — Let k be a field, C be a regular projective curve and E be a vector bundle on C. We denote by $\deg_+(E)$ the suprmum of the degrees of all vector subbundles of E, called the positive degree of E. Recall that one has (see [**8**, Theorem 2.4])

$$\left|h^0(E)-\deg_+(E)\right|\leqslant \mathrm{rk}(E)\max(g(C/k)-1,1), \tag{7}$$

where $h^0(E)$ is the rank of $H^0(C,E)$ over k. Moreover, by Hadamard's inequality, one has

$$\sum_{i=1}^{\mathrm{rk}(E)}\max(\lambda_i(E),0)\leqslant \deg_+(E), \tag{8}$$

which is similar to (5). Also we can deduce from (6) the following inequality (we refer the readers to [**8**, Proposition 8.1] for a proof)

$$\deg_+(E)\leqslant\sum_{i=1}^{\mathrm{rk}(E)}\max(\lambda_i(E),0)+\mathrm{rk}(E)\ell_C(g(C/k)). \tag{9}$$

Combining the inequalities (7)–(9), we obtain the following result.

Proposition 2.4. — *Let C be a regular projective curve over a field k and E be a vector bundle on C. One has*

$$\left|h^0(E)-\sum_{i=1}^{\mathrm{rk}(E)}\max(\lambda_i(E),0)\right|\leqslant \mathrm{rk}(E)\widetilde{\ell}_C(g(C/k)), \tag{10}$$

where $\widetilde{\ell}_C(x)=\ell_C(x)+\max(x-1,1)$ for any $x\in\mathbb{R}_+$.

3. Graded linear series

3.1. Graded linear series of a divisor. — Let k be a field and $\pi : X \to \operatorname{Spec} k$ be an integral projective k-scheme. Let K be the field of rational functions on X. For any Cartier divisor D on X, we denote by $H^0(D)$ the k-vector space

$$H^0(D) = \{f \in K \,:\, D + \operatorname{div}(f) \geqslant 0\} \cup \{0\}.$$

Denote by $\mathcal{O}_X(D)$ the sub-$\mathcal{O}_X$-module of the constant sheaf $\pi^*(K)$ generated by $-D$. It is an invertible $\mathcal{O}_X$-module. For any finite dimensional k-vector subspace V of K which is contained in $H^0(D)$, the canonical homomorphism $\pi^*(V) \to \pi^*(K)$ factors through $\mathcal{O}_X(D)$. The locus where the homomorphism $\pi^*(V) \to \mathcal{O}_X(D)$ is not surjective is called the *base locus* of V with respect to the Cartier divisor D, denoted by $\mathrm{B}_D(V)$. It is a Zariski closed subset of X. If V is non-zero, then one has $\mathrm{B}_D(V) \subsetneq X$. Moreover, the homomorphism $\pi^*(V) \to \mathcal{O}_X(D)$ induces a k-morphism $j_{V,D} : X \setminus \mathrm{B}_D(V) \to \mathbb{P}(V)$ such that $j_{V,D}^*(\mathcal{O}_V(1)) \cong \mathcal{O}_X(D)|_{X \setminus \mathrm{B}_D(V)}$, where $\mathcal{O}_V(1)$ denotes the universal invertible sheaf of $\mathbb{P}(V)$. Note that the rational morphism $X \dashrightarrow \mathbb{P}(V)$ determined by $j_{V,D}$ does not depend on the choice of the Cartier divisor D (such that $H^0(D) \supset V$). We denote by j_V this rational morphism.

Definition 3.1. — Let V be a finite dimensional k-vector subspace of K. We say that V is *birational* if the rational morphism $j_V : X \dashrightarrow \mathbb{P}(V)$ maps X birationally to its image. Note that this condition is equivalent to the condition $K = k(V)$, where $k(V)$ denotes the sub-extension of K/k generated by elements of the form a/b in K, a and b being elements in V, $b \neq 0$. By definition, if V is birational and if f is a non-zero element of K, then $fV := \{fg \,|\, g \in V\}$ is also birational.

Remark 3.2. — Let V be a finite dimensional k-vector subspace of K. If there exists a very ample Cartier divisor A on X such that $H^0(A) \subset V$, then the rational morphisme $j_V : X \dashrightarrow \mathbb{P}(V)$ maps X birationally to its image, namely V is birational. More generally, if V is a finite dimensional k-vector subspace of K which is birational and if W is another finite dimensional k-vector subspace of K containing V, then W is also birational.

Let D be a Cartier divisor on X. We denote by $V_\bullet(D) := \bigoplus_{n \in \mathbb{N}} H^0(nD)$. This is a graded k-algebra. We call *graded linear series* of D any graded sub-k-algebra of $V_\bullet$. If $V_\bullet$ is a graded linear series of D, its *volume* is defined as

$$\operatorname{vol}(V_\bullet) := \limsup_{n \to \infty} \frac{\dim_k(V_n)}{n^d/d!},$$

where d is the Krull dimension of X. In particular, if $V_\bullet$ is the total graded linear series $V_\bullet(D)$, its volume is also called the *volume of* D, denoted by vol(D). The divisor D is said to be *big* if vol(D) > 0. Note that if D is an ample divisor, then its volume is positive, and can be written in terms of the self-intersection number (D^d) (see [**14**, §2.2.C] for more details).

Definition 3.3. — Let D be a Cartier divisor on X and $V_\bullet$ be a graded linear series of D. Following [**15**, Definition 2.5], we say that the graded linear series $V_\bullet$ is *birational* if for sufficiently positive integer n, the rational map $j_{V_n} : X \dashrightarrow \mathbb{P}(V_n)$ maps X birationally to its image.

Remark 3.4. —
(i) Let D be a Cartier divisor on X and $V_\bullet$ be a birational graded linear series of D. If $\nu : X' \to X$ is a birational projective k-morphism from an integral projective k-scheme X' to X, then $V_\bullet$ is also a birational graded linear series of $\nu^*(D)$.
(ii) Let D be a Cartier divisor on X and $V_\bullet$ be a graded linear series. We assume that $V_\bullet$ *contains an ample divisor*, namely $V_n \neq \{0\}$ for sufficiently positive integer n, and there exist an ample Cartier divisor A on X and an integer $p \geqslant 1$ such that $V_n(A) \subset V_{np}$ for $n \in \mathbb{N}_{\geqslant 1}$. Then $V_\bullet$ is a birational graded linear series of D.

3.2. Graded linear series of a finitely generated extension. — Let k be a field and K be a field extension of k which is finitely generated over k.

Definition 3.5. — By *linear series* of K/k we mean any finite dimensional k-vector subspace V of K. We say that a linear series V of K/k is *birational* if $K = k(V)$.

We call *graded linear series* of K/k any graded sub-k-algebra $V_\bullet$ of $\bigoplus_{n\in\mathbb{N}} K$ (equipped with the polynomial graded ring structure) such that each homogeneous component V_n is a linear series of K/k for any $n \in \mathbb{N}$. If $V_\bullet$ is a graded linear series of K, its *volume* is defined as

$$\mathrm{vol}(V_\bullet) := \limsup_{n\to\infty} \frac{\dim_k(V_n)}{n^d/d!} \in [0, +\infty],$$

where d is the transcendental degree of K over k.

Let $V_\bullet$ be a graded linear series of K/k. We say that $V_\bullet$ is *of finite type* if it is finitely generated as a k-algebra. We say that $V_\bullet$ is of *sub-finite type* if it is contained in a graded linear series of finite type. We say that $V_\bullet$ *birational* if $k(V_n) = K$ for sufficiently positive n.

We denote by $\mathcal{A}(K/k)$ the set of all birational graded linear series of sub-finite type of K/k.

Remark 3.6. —
(i) Let $V_\bullet$ and $V'_\bullet$ be two graded linear series of K/k. Denote by $V_\bullet \cdot V'_\bullet$ the graded linear series $\bigoplus_{n\in\mathbb{N}}(V_n \cdot V'_n)$ of K/k, where $V_n \cdot V'_n$ is the k-vector space generated by $\{ff' \mid f \in V_n,\ f' \in V'_n\}$. If both

graded linear series $V_\bullet$ and $V'_\bullet$ are of finite type (resp. of sub-finite type, birational), then also is $V_\bullet \cdot V'_\bullet$.

(ii) Let $V_\bullet$ be a graded linear series of K/k. If k'/k is a field extension such that $k' \subset K$, we denote by $V_{\bullet,k'}$ the graded linear series $\bigoplus_{n\in\mathbb{N}} V_{n,k'}$ of K/k', where $V_{n,k'}$ is the k'-vector subspace of K generated by V_n. If $V_\bullet$ is of finite type (resp. of sub-finite type, birational), then also is $V_{\bullet,k'}$.

(iii) Let $V_\bullet$ be a graded linear series of K/k which is of sub-finite type. There then exists a birational graded linear series of finite type $W_\bullet$ of K/k such that $V_n \subset W_n$ for any $n \in \mathbb{N}$. Without loss of generality we may assume that $1 \in W_1$ and that $W_\bullet$ is generated as W_0-algebra by W_1. Then the scheme $X = \mathrm{Proj}(W_\bullet)$ is a projective model of the field K over k (namely an integral projective k-scheme such that $k(X) = K$). Moreover, the element $1 \in W_1$ defines an ample Cartier divisor A on X such that $H^0(nA) = W_n$ for sufficiently positive integer n. Therefore, in the asymptotic study of the behaviour of V_n when $n \to \infty$, we may assume without loss of generality that $V_\bullet$ is a graded linear series of a Cartier divisor on an integral projective scheme over k which is a projective model of K over k.

Definition 3.7. — Let $V_\bullet$ be a graded linear series of K/k. We say that $V_\bullet$ satisfies the *Fujita approximation property* if the relation

$$\sup_{W_\bullet \subset V_\bullet} \mathrm{vol}(W_\bullet) = V_\bullet$$

holds, where $W_\bullet$ runs over the set of graded sub-k-algebra of finite type of $V_\bullet$.

Remark 3.8. — If a graded linear series $V_\bullet$ verifies the Fujita approximation property, then the "limsup" in the definition of its volume is actually a limit, provided that $V_n \neq \{0\}$ for sufficiently positive n. Let $W_\bullet$ be a graded sub-k-algebra of finite type of $V_\bullet$. For sufficiently divisible integer $m \geqslant 1$, the graded k-algebra

$$W_\bullet^{(m)} := k \oplus \bigoplus_{n\in\mathbb{N},\, n\geqslant 1} W_{nm}$$

is generated by W_m (see [**3**, III.§1, no.3, Lemma 2]) and, by the classic theory of Hilbert-Samuel functions (see [**5**, VIII.§4]), the sequence

$$(d!\,\dim_k(W_{nm})/(nm)^d)_{n\geqslant 1}$$

converges to the volume of $W_\bullet$ (even though d may differ from the Krull dimension of the algebra $W_\bullet$, the sequence still converges in $[0, +\infty]$). By the assumption that $V_n \neq \{0\}$ for sufficiently positive n, we obtain that

$$\liminf_{n\to\infty} \frac{\dim_k(V_n)}{n^d/d!} \geqslant \mathrm{vol}(W_\bullet),$$

which implies the convergence of the sequence $(d!\dim_k(V_n)/n^d)_{n\geqslant 1}$ if $V_\bullet$ satisfies the Fujita approximation property.

3.3. Construction of vector bundles from linear series. — Let C be a regular projective curve over a field k and $k(C)$ be the field of rational functions on C. Denote by η the generic point of C.

Definition 3.9. — Let M be a vector space over $k(C)$ and V be a finite dimensional k-vector subspace of M. For any affine open subset U of C, we let $E(U)$ be the sub-$\mathcal{O}_C(U)$-module of M generated by V. These modules defines a torsion-free coherent sheaf E on C which is a vector bundle since C is a regular curve. We say that E is the vector bundle on C generated by the couple (M,V).

Remark 3.10. — By definition any element $s\in V$ defines a global section of E over C, which is non-zero when $s\neq 0$. Hence we can consider V as a k-vector subspace of $H^0(C,E)$. Moreover, the vector bundle E is generated by global sections, and hence $\lambda_i(E)\geqslant 0$ for any $i\in\{1,\ldots,\mathrm{rk}(E)\}$ (see Proposition 2.3).

We now consider the particular case where V is a linear series. Let K/k be a finitely generated extension of fields such that K contains $k(C)$. Suppose given a projective model X of the field K (namely X is an integral projective k-scheme such that $k(X)\cong K$) equipped with a projective surjective k-morphism $\pi: X\rightarrow C$ and a Cartier divisor D on X. Let V be a finite dimensional k-vector subspace of $H^0(D)$. Note that the vector bundle E generated by (K,V) identifies with the vector subbundle of $\pi_*(\mathcal{O}_X(D))$ generated by V. Moreover, the generic fibre E_η of E is the $k(C)$-vector subspace of K generated by V (with the notation of Remark 3.6 (ii), $E_\eta=V_{k(C)}$). In particular, if V is a birational linear series of K/k, then E_η is a birational linear series of $K/k(C)$.

4. Construction of convex bodies

In this section, we prove the main theorem (Theorem 1.1) of the article. Throughout the section, we fix a field k and a finitely generated extension K/k and we let d be the transcendental degree of K over k. In the case where $d=0$, we assume that $K=k$, and the Lebesgue measure on $\mathbb{R}^0=\{0\}$ is assumed to be the Dirac measure on 0 by convention. We fix also a flag

$$k=K_0\subsetneq K_1\subsetneq\ldots\subsetneq K_d=K \tag{11}$$

of sub-extensions of K/k such that each extension K_i/K_{i-1} is of transcendental degree 1. We denote by $\mathcal{A}(K/k)$ the set of graded linear series $V_\bullet$ of K/k which are of sub-finite type and birational (see Definition 3.5). In the following, we will construct by induction a map Δ^d from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$ which only depends on the choice of the flag (11), which satisfies the following properties:

(a) if $V_\bullet$ and $V'_\bullet$ are two elements of $\mathcal{A}(K/k)$ such that $V_n \subset V'_n$ for sufficiently positive n, then one has $\Delta^d(V_\bullet) \subset \Delta^d(V'_\bullet)$;

(b) for any $V_\bullet \in \mathcal{A}(K/k)$ and any $m \in \mathbb{N}$, $m \geqslant 1$, one has
$$\Delta^d(V_\bullet^{(m)}) = m\Delta^d(V_\bullet);$$

(c) if $V_\bullet$ and $V'_\bullet$ are elements in $\mathcal{A}(K/k)$, one has
$$\Delta^d(V_\bullet \cdot V'_\bullet) \supset \Delta^d(V_\bullet) + \Delta^d(V'_\bullet).$$

(d) for any graded linear series $V_\bullet \in \mathcal{A}(K/k)$, the Lebesgue measure of $\Delta^d(V_\bullet)$ is equal to $\mathrm{vol}(V_\bullet)/d!$.

We also prove the following assertion by induction on d.

Proposition 4.1. — *Any graded linear series $V_\bullet \in \mathcal{A}(K/k)$ satisfies the Fujita approximation property (see Definition 3.7). In particular, the sequence*
$$\frac{\dim_k(V_n)}{n^d/d!}, \quad n \in \mathbb{N},\ n \geqslant 1$$
converges to $\mathrm{vol}(V_\bullet)$ *when n tends to the infinity.*

4.1. Case of $d = 0$. — In the case where $d = 0$, one has $V_n = k$ for sufficiently positive n since K is assumed to be k. Moreover, the graded linear series $V_\bullet$ is of finite type and one has $\mathrm{vol}(V_\bullet) = 1$. Clearly $V_\bullet$ satisfies the Fujita approximation property. We let $\Delta^0(V_\bullet)$ be $\{0\} = \mathbb{R}^0$. Its mass with respect to the Lebesgue measure (which is the Dirac measure on 0 by convention) is 1, which is equal to $\mathrm{vol}(V_\bullet)/0!$.

4.2. Induction hypothesis. — We assume that $d \geqslant 1$ and that the construction of convex bodies associated to birational graded linear series of finite type has been defined and that Proposition 4.1 has been established for field extensions of transcendental degree $d-1$. In particular, we assume that a map Δ^{d-1} from $\mathcal{A}(K/K_1)$ to the set of convex bodies in $\mathbb{R}^{d-1}$ has been constructed, which satisfies the conditions (a)–(d) above.

4.3. Construction of convex bodies. — Since the extension K_1/k (which is of finite type) is of transcendental degree 1, there exists a regular projective curve C over $\operatorname{Spec} k$ such that $k(C) = K_1$. We denote by η the generic point of the curve C. In the following, we consider a graded linear series $V_\bullet$ in $\mathcal{A}(K/k)$. Note that $V_{\bullet,K_1}$ is a graded linear series in $\mathcal{A}(K/K_1)$ (see Remark 3.6 (ii)). For any $n \in \mathbb{N}$, we let E_n be the vector bundle on C generated by (K, V_n) (see Definition 3.9). Note that one has $E_{n,\eta} = V_{n,K_1}$. Moreover, the direct sum $E_\bullet = \bigoplus_{n\in\mathbb{N}} E_n$ forms a graded $\mathcal{O}_C$-algebra, and one has $V_{\bullet,K_1} \cong E_\bullet \otimes_{\mathcal{O}_C} K_1$

as graded-K_1-algebras. We call $E_\bullet$ the *graded system of vector bundles on C generated by $V_\bullet$*.

By Remark 3.6 (iii), there exists a birational projective model X of the field K over k and and ample Cartier divisor D on X such that $V_\bullet$ identifies with a graded linear series of D. The inclusion $K_1 \to K$ defines a rational k-morphism from X to C. By replacing X by a birational modification of X, we may assume that the rational k-morphism $X \dashrightarrow C$ extends to a flat projective k-morphism $\pi : X \to C$. In particular, we can consider $E_\bullet$ as a graded $\mathcal{O}_C$-sub-algebra of $\bigoplus_{n\in\mathbb{N}} \pi_*(\mathcal{O}(nD))$, see §3.3 for more details.

For each integer $n \in \mathbb{N}$, the vector space V_{n,K_1} is equipped with the $\mathbb{R}$-filtration by minima as follows

$$\forall\, t \in \mathbb{R}, \quad \mathcal{F}^t(V_{n,K_1}) = \mathrm{Vect}_{K_1}(\{s \in V_{n,K_1} \mid s \neq 0,\ \deg(s) \geqslant t\}),$$

where in the computation of degree, we consider the vector bundle structure of E_n. In particular, the filtration is multiplicative. In other words, for all $(n, m) \in \mathbb{N}^2$ and $(t_1, t_2) \in \mathbb{R}^2$, one has

$$\mathcal{F}^{t_1}(V_{n,K_1})\mathcal{F}^{t_2}(V_{m,K_1}) \subset \mathcal{F}^{t_1+t_2}(V_{n+m,K_1}). \tag{12}$$

For any $t \in \mathbb{R}$, let $V^t_{\bullet,K_1}$ be the graded sub-K_1-algebra of $V_{\bullet,K_1}$ defined as

$$\bigoplus_{n\in\mathbb{N}} \mathcal{F}^{nt}(V_{n,K_1}).$$

Clearly, the graded linear series $V^t_{\bullet,K_1}$ is of sub-finite type. Let

$$\lambda^{\mathrm{asy}}_{\max}(E_\bullet) := \sup_{n\in\mathbb{N},\, n\geqslant 1} \frac{\lambda_1(E_n)}{n} = \lim_{n\to+\infty} \frac{\lambda_1(E_n)}{n},$$

where the second equality comes from the fact that the sequence $(\lambda_1(E_n))_{n\geqslant 1}$ is super-additive. Since E_n is a vector subbundle of $\pi_*(\mathcal{O}(D))$, we obtain that

$$\lambda_1(E_n) \leqslant \mu_{\max}(\pi_*(\mathcal{O}(nD))),$$

where for any vector bundle F on C, $\mu_{\max}(F)$ denote the maximal slope of F, defined as

$$\sup_{F'\subset F} \deg(F')/\operatorname{rk}(F'),$$

with F' running over the set of all non-zero vector subbundles of F. By [**7**, Theorem 4.3.6], we obtain that $\lambda^{\mathrm{asy}}_{\max}(E_\bullet)$ is finite. The following lemma is similar to [**1**, Lemma 1.6].

Lemma 4.2. — *For any real number $t < \lambda^{\mathrm{asy}}_{\max}(E_\bullet)$, the graded linear series $V^t_{\bullet,K_1}$ belongs to $\mathcal{A}(K/K_1)$.*

Proof. — Since $V^t_{\bullet,K_1}$ is a graded sub-K_1-algebra of $V_{\bullet,K_1}$, it is of sub-finite type (see Remark 3.6 (ii) for the fact that $V_{\bullet,K_1}$ is of sub-finite type). It remains to verify that $V^t_{\bullet,K_1}$ is birational. By Remark 3.6 (ii), the graded linear series $V_{\bullet,K_1}$ is birational. Let $N_0 \in \mathbb{N}$ such that the linear series V_{N_0,K_1} is birational. Let $\varepsilon > 0$ such that $t + \varepsilon < \lambda^{\mathrm{asy}}_{\max}(E_\bullet)$. for sufficiently positive integer n, there exists an element $s_n \in V_{n,K_1} \setminus \{0\}$ such that $\deg(s_n) > (t+\varepsilon)n$. For any section $s \in V_{N_0,K_1}$ such that $\deg(s) \geqslant 0$, one has

$$\deg(ss_n) \geqslant \deg(s) + \deg(s_n) > (t + \varepsilon)n \tag{13}$$

which is bounded from below by $t(n + N_0)$ when $n \geqslant tN_0\varepsilon^{-1}$, where the first inequality (13) comes from the fact that (since $E_\bullet$ is a graded $\mathcal{O}_C$-algebra)

$$\forall \text{ closed point } x \in C, \quad \|ss_n\|_x \leqslant \|s\|_x \cdot \|s_n\|_x.$$

We then deduce that the linear series V^t_{n,K_1} is birational for sufficiently positive n, namely the graded linear series $V^t_\bullet$ is birational. □

By the induction hypothesis, for any real number $t < \lambda^{\mathrm{asy}}_{\max}(E_\bullet)$ the convex body $\Delta^{d-1}(V^t_{\bullet,K_1})$ is well defined. Since E_n is generated by global sections, one has $V^t_{\bullet,K_1} = V_{\bullet,K_1}$ for $t \leqslant 0$. Moreover, for any couple (t_1, t_2) of real numbers such that $t_1 \leqslant t_2 \leqslant \lambda^{\mathrm{asy}}_{\max}(E_\bullet)$ one has $\Delta^{d-1}(V^{t_1}_{\bullet,K_1}) \supset \Delta^{d-1}(V^{t_2}_{\bullet,K_1})$. We define a function $G_{E_\bullet} : \Delta^{d-1}(V_{\bullet,K_1}) \to [0, \lambda^{\mathrm{asy}}_{\max}(E_\bullet)]$ such that

$$G_{E_\bullet}(x) = \sup\{t \,|\, x \in \Delta^{d-1}(V^t_{\bullet,K_1})\}.$$

By convention, if $\{t \,|\, x \in \Delta^{d-1}(V^t_{\bullet,K_1})\}$ is empty, then $G_{E_\bullet}(x)$ is defined as $\lambda^{\mathrm{asy}}_{\max}(E_\bullet)$. By definition, for any $t \in [0, \lambda^{\mathrm{asy}}_{\max}(E_\bullet)]$, one has

$$\{x \in \Delta^{d-1}(V_{\bullet,K_1}) \,|\, G_{E_\bullet}(x) \geqslant t\} = \bigcap_{\varepsilon>0} \Delta^{d-1}(V^t_{\bullet,K_1}).$$

Therefore, the function $G_{E_\bullet}$ is upper semicontinuous. Moreover, by the condition (12) and the induction hypothesis (notably the conditions (b) and (c)), we obtain that, for any rational number $\lambda \in [0, 1]$ and all real numbers t_1 and t_2 bounded from above by $\lambda^{\mathrm{asy}}_{\max}(E_\bullet)$, one has

$$\Delta(V^{\lambda t_1+(1-\lambda)t_2}_{\bullet,K_1}) \supset \lambda\Delta(V^{t_1}_{\bullet,K_1}) + (1 - \lambda)\Delta(V^{t_2}_{\bullet,K_1}).$$

By Sierpiński's theorem (see for example [**12**, Theorem 9.4.2]), we obtain that the function $G_{E_\bullet}$ is concave and is continuous on the interior of $\Delta(V_{\bullet,K_1})$.

Definition 4.3. — We let $\Delta^d(V_\bullet)$ be the convex body in $\mathbb{R}^d$ delimited by the concave function $G_{E_\bullet}$, namely $\Delta^d(V_\bullet)$ is by definition the closure of the set

$$\{(x, t) \,|\, x \in \Delta^{d-1}(V_{\bullet,K_1}),\ 0 \leqslant t \leqslant G_{E_\bullet}(x)\}.$$

By the same method of [**2**, Theorem 1.11 and Corollary 1.13], we deduce from the induction hypothesis (notably the limit property predicted in Proposition

4.1, which replaces the condition of containing an ample series in *loc. cit.*) on $V^t_{\bullet,K_1}$ ($t < \lambda^{\mathrm{asy}}_{\max}(E_\bullet)$) that

$$
\begin{aligned}
\mathrm{vol}(\Delta^d(V_\bullet)) &= \int_{t\in[0,\lambda^{\mathrm{asy}}_{\max}(E_\bullet)[} \mathrm{vol}(\Delta^{d-1}(V^t_{\bullet,K_1}))\,\mathrm{d}t \\
&= \lim_{n\to+\infty} \frac{1}{n^d} \sum_{\substack{1\leqslant i\leqslant \dim_{K_1}(V_{n,K_1}) \\ \lambda_i(E_n)\geqslant 0}} \lambda_i(E_n) = \mathrm{vol}(E_\bullet),
\end{aligned} \tag{14}
$$

where by definition

$$\mathrm{vol}(E_\bullet) := \lim_{n\to+\infty} \frac{\dim_k H^0(C, E_n)}{n^d},$$

and the last inequality of (14) comes from (10) and the fact that (which is also a consequence of the induction hypothesis) and the sub-finiteness condition

$$\dim_{K_1}(V_{n,K_1}) = O(n^{d-1}), \quad n\to+\infty.$$

Remark 4.4. — Similarly to [**2**, Theorem 1.14], the graded system $E_\bullet$ of vector bundles on $\mathcal{C}$ satisfies the "arithmetic Fujita approximation property", namely

$$\sup_{p\geqslant 1} \frac{\mathrm{vol}(E^{[p]}_\bullet)}{p^d} = \mathrm{vol}(E_\bullet), \tag{15}$$

where $E^{[p]}_\bullet$ is the graded sub-$\mathcal{O}_C$-algebra of $E^{(p)}_\bullet$ generated by E_p.

4.4. Verification of the properties. — In the previous subsection, we have constructed a map Δ^d from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$. It remains to verify that the map Δ^d satisfies the properties (a)–(d) described in the beginning of the section and that any graded linear series in $\mathcal{A}(K/k)$ satisfies the Fujita approximation property.

Let $V_\bullet$ and $V'_\bullet$ be two graded linear series in $\mathcal{A}(K/k)$. Let $E_\bullet$ and $E'_\bullet$ be graded systems of vector bundles on C generated by $V_\bullet$ and $V'_\bullet$ respectively. Assume that, for sufficiently positive integer n, one has $V_n \subset V'_n$, which implies that E_n is a vector subbundle of E'_n. In particular, for any $t\in\mathbb{R}$, one has $V^t_n \subset V'^t_n$ for sufficiently positive t. Thus by the induction hypothesis one has $\Delta^{d-1}(V^t_\bullet) \subset \Delta^{d-1}(V'^t_\bullet)$ for any $t < \lambda^{\mathrm{asy}}_{\max}(E_\bullet)$, which implies that $\Delta^d(V_\bullet) \subset \Delta^d(V'_\bullet)$. Therefore, the map Δ^d satisfies the condition (a).

Let $V_\bullet$ be a graded linear series in $\mathcal{A}(K/k)$ and $E_\bullet$ be the graded system of vector bundles on C generated by $V_\bullet$. For any $m\in\mathbb{N}$, $m\geqslant 1$, let $E^{(m)}_\bullet := \bigoplus_{n\in\mathbb{N}} E_{mn}$, which is the graded system of vector bundles on C generated by $V^{(m)}_\bullet$. By definition one has $\lambda^{\mathrm{asy}}_{\max}(E^{(m)}_\bullet) = m\lambda^{\mathrm{asy}}_{\max}(E_\bullet)$. Moreover, for any $t <$

$\lambda_{\max}^{\mathrm{asy}}(E_\bullet)$, one has $V_{\bullet,K_1}^{(m),mt} = V_{\bullet,K_1}^{t,(m)}$. Therefore one has $\Delta^d(V_\bullet^{(m)}) = m\Delta^d(V_\bullet)$. In other words, the map Δ^d satisfies the condition (b).

Let $V_\bullet$ and $V'_\bullet$ be two graded linear series in $\mathcal{A}(K/k)$, and $E_\bullet$ and $E'_\bullet$ be graded system of vector bundles on C generated by $V_\bullet$ and $V'_\bullet$, respectively. Let $F_\bullet$ be the graded system of vector bundles on C generated by $V_\bullet \cdot V'_\bullet$. For any $n \in \mathbb{N}$, one has a homomorphisme of $\mathcal{O}_C$-modules $E_n \otimes_{\mathcal{O}_C} E'_n \to F_n$ induced by the canonical k-linear map $V_n \otimes_k V'_n \to V_n \cdot V'_n$. In particular, for any $(t,t') \in \mathbb{R}^2$ one has

$$\mathcal{F}^t(V_{n,K_1}) \cdot \mathcal{F}^{t'}(V'_{n,K_1}) \subset \mathcal{F}^{t+t'}(V_{n,K_1} \cdot V_{n,K_1}).$$

This implies that, for any $(x,y) \in \Delta^{d-1}(V_\bullet) \times \Delta^{d-1}(V'_\bullet)$, one has

$$G_{F_\bullet}(x+y) \geqslant G_{E_\bullet}(x) + G_{E'_\bullet}(y).$$

Hence $\Delta^d(V_\bullet \cdot V'_\bullet) \supset \Delta^d(V_\bullet) + \Delta^d(V'_\bullet)$. Namely the map Δ^d satisfies the condition (c).

The condition (d) and the Fujita approximation property of $V_\bullet$ follow from (14) and the following lemma, which concludes the proof of Theorem 1.1.

Lemma 4.5. — *Let $V_\bullet$ be a graded linear series in $\mathcal{A}(K/k)$ and $E_\bullet$ be the graded system of vector bundles on C generated by $V_\bullet$. One has*

$$\mathrm{vol}(V_\bullet) = \mathrm{vol}(E_\bullet) := \lim_{n\to+\infty} \frac{\dim_k H^0(C,E_n)}{n^d/d!}. \tag{16}$$

Moreover, the graded linear series $V_\bullet$ satisfies the Fujita approximation property.

Proof. — For any $n \in \mathbb{N}$, one has (see Remark 3.10)

$$\dim_k(V_n) \leqslant \dim_k H^0(C,E_n).$$

Therefore $\mathrm{vol}(V_\bullet) \leqslant \mathrm{vol}(E_\bullet)$.

In the following, we prove the converse inequality. By Remark 3.6 (iii), we may assume without loss of generality that K is the rational function field of a normal projective k-scheme X and that $V_\bullet$ is a birational graded linear series of a very ample Cartier divisor D on X. Moreover, the inclusion $k(C) \subset X$ defines a rational k-morphism from X to C. By replacing X by its blowing-up along the locus where the rational k-morphism is not defined, we may suppose that the rational morphism $X \dashrightarrow C$ extends to a flat projective k-morphism $\pi : X \to C$. We can thus identify each vector bundle E_n with the vector subbundle of $\pi_*(\mathcal{O}_X(D))$ generated by V_n.

We denote by $\varphi : C \to \operatorname{Spec} k$ the structural morphism. Let $p \geqslant 1$ be an integer such that the rational morphism $j_p : X \dashrightarrow \mathbb{P}(V_p)$ defined by the linear series V_p maps X birationally to its image. Let $u_p : X_p \to X$ be the blowing-up of X along the base locus of j_p, namely

$$X_p := \operatorname{Proj}\Big(\bigoplus_{m\geqslant 0}(\varphi\pi)^*(\operatorname{Sym}^m(V_p)) \longrightarrow \mathcal{O}_X(pmD)\Big).$$

Then the rational morphism $j_p : X \dashrightarrow \mathbb{P}(V_p)$ gives rise to a projective k-morphism $f_p : X_p \to \mathbb{P}(V_p)$ which maps X_p birationally to its image. Let L_p be the pull-back of the tautological invertible sheaf $\mathcal{O}_{V_p}(1)$ by j_p. Hence one has

$$\operatorname{vol}(L_p) = \operatorname{vol}(\mathcal{O}_{V_p}(1)|_{f_p(X_p)})$$

since f_p maps birationally X_p to its image. Therefore the volume of L_p is equal to that of the graded linear series

$$V_\bullet^{[p]} := \bigoplus_{n\geqslant 0}\operatorname{Im}(\operatorname{Sym}^n_k(V_p) \longrightarrow V_{np}).$$

Let

$$E_\bullet^{[p]} = \bigoplus_{n\geqslant 0}\operatorname{Im}(\operatorname{Sym}^n_{\mathcal{O}_C}(E_p) \longrightarrow E_{np}).$$

For sufficiently positive integer n, one has $E_n^{[p]} \subset (\pi u_p)_*(L_p^{\otimes n})$. Hence

$$\operatorname{vol}(V_\bullet) \geqslant \frac{\operatorname{vol}(V_\bullet^{[p]})}{p^d} = \frac{\operatorname{vol}(L_p)}{p^d} \geqslant \frac{\operatorname{vol}(E_\bullet^{[p]})}{p^d}. \tag{17}$$

By (15) we obtain $\operatorname{vol}(V_\bullet) \geqslant \operatorname{vol}(E_\bullet)$. The equality (16) is thus proved. Moreover, the inequality (17) and the equality (16) also imply that

$$\sup_{p\geqslant 1}\frac{\operatorname{vol}(V_\bullet^{[p]})}{p^d} = \operatorname{vol}(V_\bullet).$$

Hence the graded linear series $V_\bullet$ satisfies the Fujita approximation property. □

5. General graded linear series of sub-finite type

In this section, we consider graded linear series of sub-finite type, which are not necessarily birational. We still fix a field extension K/k which is finitely generated and we let d be the transcendental degree of the extension.

Definition 5.1. — Let $V_\bullet$ be a graded linear series of K/k. We call *Iitaka dimension* of $V_\bullet$ the transcendental degree of $k(V_\bullet)$ over k, where $k(V_\bullet)$ denotes the smallest sub-extension of K/k containing all $k(V_n)$, ($n \in \mathbb{N}$, $n \geqslant 1$). We denote by $\kappa(V_\bullet)$ the Iitaka dimension of $V_\bullet$ over k.

Proposition 5.2. — *Let $V_\bullet$ be a graded linear series of K/k. We assume that V_n does not reduce to $\{0\}$ for sufficiently positive integer n. Then for sufficiently positive integer n, one has $k(V_n) = k(V_\bullet)$.*

Proof. — Since $k(V_\bullet)/k$ is a sub-extension of a finitely generated extension, it is also finitely generated (see [**4**] Chapter V, §14, no.7 Corollary 3). In particular, there exists $m \in \mathbb{N}$, $m \geqslant 1$, such that $k(V_\bullet)$ is generated by $k(V_1), \ldots, k(V_m)$. Now let N_0 be an integer, $N_0 \geqslant 1$ such that $V_n \neq \{0\}$ for $n \geqslant N_0$. Let n be an integer such that $n \geqslant N_0 + m$. For any $i \in \{1, \ldots, m\}$, we pick an element $f_i \in V_{n-i}$. Then V_n contains $\bigcup_{i=1}^m f_i V_i$. Therefore $k(V_n)$ contains $k(f_i V_i) = k(V_i)$ for any $i \in \{1, \ldots, n\}$, which implies that $k(V_n) = k(V_\bullet)$. □

Remark 5.3. — Let $V_\bullet$ be a graded linear series of K/k such that $V_n \neq \{0\}$ for sufficiently positive n. We assume that $1 \in V_1$ (and hence $1 \in V_n$ for any $n \geqslant 1$). Then we can consider $V_\bullet$ as a graded linear series of $k(V_\bullet)/k$. As a graded linear series of $k(V_\bullet)/k$, $V_\bullet$ is birational by definition. Therefore, if it is of sub-finite type (as a graded linear series of $k(V_\bullet)/k$), then the construction in the previous section allows to define a convex body $\Delta(V_\bullet)$ in $\mathbb{R}^\kappa$, where κ is the Iitaka dimension of $V_\bullet$, such that

$$\lim_{n \to +\infty} \frac{\mathrm{rk}_k(V_n)}{n^\kappa/\kappa!}.$$

However, suppose that $V_\bullet$ is of sub-finite type as graded linear series of K/k, I do not know if it is of sub-finite type as graded linear series of $k(V_\bullet)/k$.

Acknowledgement: I would like to thank Hideaki Ikoma for his valuable comments.

References

[1] S. Boucksom & H. Chen – "Okounkov bodies of filtered linear series", à paraître dans *Compositio Mathematica*, 2009.

[2] S. Boucksom & H. Chen – "Okounkov bodies of filtered linear series", *Compositio Mathematica* **147** (2011), no. 4, p. 1205–1229.

[3] N. Bourbaki – *Éléments de mathématique*, Actualités Scientifiques et Industrielles, No. 1293, Hermann, Paris, 1961, Algèbre commutative. Chapitre 3: Graduations, filtra- tions et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire.

[4] N. Bourbaki – *Éléments de mathématique*, vol. 864, Masson, Paris, 1981, Algèbre. Chapitres 4 à 7.

[5] ______, *Éléments de mathématique*, Masson, Paris, 1983, Algèbre commutative. Chapitre 8. Dimension. Chapitre 9. Anneaux locaux noethériens complets.

[6] H. Chen – “Arithmetic Fujita approximation”, *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série* **43** (2010), no. 4, p. 555–578.

[7] ———, “Convergence des polygones de Harder-Narasimhan”, *Mémoires de la Société Mathématique de France* **120** (2010), p. 1–120.

[8] ———, “Majorations explicites des fonctions de Hilbert-Samuel géométrique et arithmétique”, *Mathematische Zeitschrift* **279** (2015), no. 1-2, p. 99–137.

[9] S. D. Cutkosky – “Asymptotic multiplicities of graded families of ideals and linear series”, *Advances in Mathematics* **264** (2014), p. 55–113.

[10] K. Kaveh & A. G. Khovanskii – “Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory”, *Ann. of Math. (2)* **176** (2012), no. 2, p. 925–978.

[11] K. Kaveh & A. Khovanskii – “Algebraic equations and convex bodies”, in *Perspectives in analysis, geometry, and topology*, Progr. Math., vol. 296, Birkhäuser/Springer, New York, 2012, p. 263–282.

[12] M. Kuczma – *An introduction to the theory of functional equations and inequalities*, second ed., Birkhäuser Verlag, Basel, 2009, Cauchy’s equation and Jensen’s inequality, Edited and with a preface by Attila Gilányi.

[13] A. Küronya, V. Lozovanu & C. Maclean – “Convex bodies appearing as Okounkov bodies of divisors”, *Advances in Mathematics* **229** (2012), no. 5, p. 2622–2639.

[14] R. Lazarsfeld – *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[15] R. Lazarsfeld & M. Mustaţă – “Convex bodies associated to linear series”, *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série* **42** (2009), no. 5, p. 783–835.

[16] A. Okounkov – “Brunn-Minkowski inequality for multiplicities”, *Inventiones Mathematicae* **125** (1996), no. 3, p. 405–411.

[17] ———, “Why would multiplicities be log-concave?”, in *The orbit method in geometry and physics (Marseille, 2000)*, Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, p. 329–347.

[18] D. Roy & J. L. Thunder – “An absolute Siegel’s lemma”, *Journal für die Reine und Angewandte Mathematik* **476** (1996), p. 1–26.

March 2, 2017

Huayi Chen, Université Paris Diderot, Institut de Mathématiques de Jussieu - Paris Rive Gauche, Bâtiment Sophie Germain, Boîte Courrier 7012, 75205 Paris Cedex 13
E-mail : `huayi.chen@imj-prg.fr` • *Url :* `webusers.imj-prg.fr/~huayi.chen`