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MAXIMAL SLOPE OF TENSOR PRODUCT OF HERMITIAN VECTOR BUNDLES

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Abstract

We give an upper bound for the maximal slope of the tensor product of several non-zero Hermitian vector bundles on the spectrum of an algebraic integer ring. By Minkowski's First Theorem, we need to estimate the Arakelov degree of an arbitrary Hermitian line subbundle $\overline{M}$ of the tensor product. In the case where the generic fiber of M is semistable in the sense of geometric invariant theory, the estimation is established by constructing, through the classical invariant theory, a special polynomial which does not vanish on the generic fibre of M. Otherwise we use an explicit version of a result of Ramanan and Ramanathan to reduce the general case to the former one.

1. Introduction

It is well known that on a projective and smooth curve defined over a field of characteristic 0, the tensor product of two semistable vector bundles is still semistable. This result was first proved by Narasimhan and Seshadri [22] by using analytic method in the complex algebraic geometry framework. Then it has been reestablished by Ramanan and Ramanathan [23] in purely algebraic context, through the geometric invariant theory. Their method is based on a result of Kempf [20], which has also been independently obtained by Rousseau [25], generalizing the Hilbert–Mumford criterion [21] of semistability in the sense of geometric invariant theory. By reformulating the results of Kempf and Ramanan–Ramanathan, Totaro [27] (see also [10] for a review) has given a new proof of a conjecture due to Fontaine [13], which was first proved by Faltings [12], asserting that the tensor product of two semistable admissible filtered isocristals is still semistable.

Let us go back to the case of vector bundles. Consider a smooth projective curve C defined over a field k. For any non-zero vector bundle E on C, the *slope* of E is defined as the quotient of its degree by its rank and is denoted

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by $\mu(E)$. The *maximal slope* $\mu_{\max}(E)$ of E is the maximal value of slopes of all non-zero subbundles of E. By definition, $\mu(E) \le \mu_{\max}(E)$. We say that E is *semistable* if the equality $\mu(E) = \mu_{\max}(E)$ holds. If E and F are two non-zero vector bundles on C, then $\mu(E \otimes F) = \mu(E) + \mu(F)$. The result of Ramanan–Ramanathan [23] implies that, if k is of characteristic 0, then the equality holds for maximal slopes, i.e., $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$. When the characteristic of k is positive, this equality is not true in general (see [16] for a counter-example). Nevertheless, there always exists a constant a which only depends on C such that

$$\mu_{\max}(E) + \mu_{\max}(F) \le \mu_{\max}(E \otimes F) \le \mu_{\max}(E) + \mu_{\max}(F) + a. \tag{1}$$

Hermitian vector bundles play in Arakelov geometry the role of vector bundles in algebraic geometry. Let K be a number field and $\mathcal{O}_K$ be its integer ring. We denote by Σ_∞ the set of all embeddings of K into $\mathbb{C}$. A Hermitian vector bundle $\overline{E} = (E, h)$ on $\operatorname{Spec}\mathcal{O}_K$ is by definition a projective $\mathcal{O}_K$-module of finite type E together with a family of Hermitian metrics $h = (\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty}$, where for any $\sigma \in \Sigma_\infty$, $\|\cdot\|_\sigma$ is a Hermitian norm on $E \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$, subject to the condition that the data $(\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty}$ is invariant by the complex conjugation. That is, for any $e \in E$, $z \in \mathbb{C}$ and $\sigma \in \Sigma_\infty$, we have $\|e \otimes \overline{z}\|_{\overline{\sigma}} = \|e \otimes z\|_\sigma$.

The (normalized) *Arakelov degree* of a Hermitian vector bundle $\overline{E}$ of rank r on $\operatorname{Spec}\mathcal{O}_K$ is defined as

$$\widehat{\deg}_n \overline{E} = \frac{1}{[K:\mathbb{Q}]}\Big(\log \#(E/\mathcal{O}_K s_1 + \cdots + \mathcal{O}_K s_r) - \frac{1}{2}\sum_{\sigma\in\Sigma_\infty} \log\det(\langle s_i, s_j\rangle_\sigma)\Big),$$

where $(s_1, \cdots, s_r)$ is an arbitrary element in E^r which defines a basis of E_K over K. This definition does not depend on the choice of $(s_1, \cdots, s_r)$. The *slope* of a non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec}\mathcal{O}_K$ is defined as the quotient $\widehat{\mu}(\overline{E}) := \widehat{\deg}_n(\overline{E})/\operatorname{rk}(E)$ (for more details, see [3], [5], and [7]).

We say that a non-zero Hermitian vector bundle $\overline{E}$ is *Arakelov semistable* if the *maximal slope* $\widehat{\mu}_{\max}(\overline{E})$ of $\overline{E}$, defined as the maximal value of slopes of its non-zero Hermitian subbundles, equals its slope. If $\overline{E}$ is a non-zero Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$, Stuhler [26] and Grayson [19] have proved that there exists a unique Hermitian subbundle $\overline{E}_{\mathrm{des}}$ of $\overline{E}$ having $\widehat{\mu}_{\max}(\overline{E})$ as its slope and containing all Hermitian subbundle $\overline{F}$ of $\overline{E}$ such that $\widehat{\mu}(\overline{F}) = \widehat{\mu}_{\max}(\overline{E})$. Clearly $\overline{E}$ is Arakelov semistable if and only if $\overline{E} = \overline{E}_{\mathrm{des}}$. If it is **not** the case, then $\overline{E}_{\mathrm{des}}$ is said to be the Hermitian subbundle which *destabilizes* $\overline{E}$.

In a lecture at Oberwolfach, J.-B. Bost [4] has conjectured that the tensor product of two Arakelov semistable Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$ is Arakelov semistable. This conjecture is equivalent to the assertion that for any non-zero Hermitian vector bundles $\overline{E}$ and $\overline{F}$ on $\operatorname{Spec}\mathcal{O}_K$,

$$\widehat{\mu}_{\max}(\overline{E}\otimes\overline{F})=\widehat{\mu}_{\max}(\overline{E})+\widehat{\mu}_{\max}(\overline{F}).$$

We always have the inequality $\widehat{\mu}_{\max}(\overline{E}\otimes\overline{F})\geq\widehat{\mu}_{\max}(\overline{E})+\widehat{\mu}_{\max}(\overline{F})$. But the inverse inequality remains open. Several special cases of this conjecture have been proved. Some estimations of type (1) have been established with error terms depending on the ranks of the vector bundles and on the number field K. We resume some known results on this conjecture.

(1) By definition of maximal slope, if $\overline{E}$ is a non-zero Hermitian vector bundle and if $\overline{L}$ is a Hermitian line bundle, that is, a Hermitian vector bundle of rank one, then

$$\widehat{\mu}_{\max}(\overline{E}\otimes\overline{L})=\widehat{\mu}_{\max}(\overline{E})+\widehat{\deg}_n(\overline{L})=\widehat{\mu}_{\max}(\overline{E})+\widehat{\mu}_{\max}(\overline{L}).$$

The geometric counterpart of this equality is also true for the positive characteristic case.

(2) De Shalit and Parzanovski [11] have proved that, if $\overline{E}$ and $\overline{F}$ are two Arakelov semistable Hermitian vector bundles on $\operatorname{Spec}\mathbb{Z}$ such that $\operatorname{rk}E+\operatorname{rk}F\leq 5$, then $\overline{E}\otimes\overline{F}$ is Arakelov semistable.

(3) In [3] (see also [18]), using the comparison of a Hermitian vector bundle to a direct sum of Hermitian line bundles, Bost has proved that

$$\widehat{\mu}_{\max}(\overline{E}_1\otimes\cdots\otimes\overline{E}_n)\leq\sum_{i=1}^{n}\Big(\widehat{\mu}_{\max}(\overline{E}_i)+3\operatorname{rk}E_i\log(\operatorname{rk}E_i)\Big)$$

for any family of non-zero Hermitian vector bundles $(\overline{E}_i)_{i=1}^n$ on $\operatorname{Spec}\mathcal{O}_K$.

(4) Recently, Bost and Künnemann [6] have proved that, if K is a number field and if $\overline{E}$ and $\overline{F}$ are two non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$, then

$$\widehat{\mu}_{\max}(\overline{E}\otimes\overline{F})\leq\widehat{\mu}_{\max}(\overline{E})+\widehat{\mu}_{\max}(\overline{F})+\frac{1}{2}\big(\log\operatorname{rk}E+\log\operatorname{rk}F\big)+\frac{\log|\Delta_K|}{2[K:\mathbb{Q}]},$$

where Δ_K is the discriminant of K.

We state the main result of this article as follows:

Theorem 1.1. *Let K be a number field and $\mathcal{O}_K$ be its integer ring. If $(\overline{E}_i)_{i=1}^n$ is a family of non-zero Hermitian vector bundles on* $\operatorname{Spec}\mathcal{O}_K$*, then*

$$\widehat{\mu}_{\max}(\overline{E}_1\otimes\cdots\otimes\overline{E}_n)\leq\sum_{i=1}^{n}\Big(\widehat{\mu}_{\max}(\overline{E}_i)+\log(\operatorname{rk}E_i)\Big). \tag{2}$$

The idea goes back to an article of Bost [2] inspired by Bogomolov [24], Gieseker [17] and Cornalba–Harris [9]. In an article of Gasbarri [14] a similar idea also appears. By Minkowski's First Theorem, we reduce our problem to finding an upper bound for the Arakelov degree of an arbitrary Hermitian line subbundle $\overline{M}$ of $\overline{E}_1 \otimes \cdots \otimes \overline{E}_n$. In the case where M_K is semistable (in the sense of geometric invariant theory) for the action of $\mathbb{GL}(E_{1,K}) \times \cdots \times \mathbb{GL}(E_{n,K})$, the classical invariant theory gives invariant polynomials with coefficients in $\mathbb{Z}$ whose Archimedean norms are "small". The general case can be reduced to the former one using an explicit version of a result of Ramanan–Ramanathan [23].

The structure of the rest of this article is as follows. In the second section we fix the notation and present some preliminary results. In the third section we recall the first principal theorem in classical invariant theory and discuss some generalizations in the case of several vector spaces. We then establish in the fourth section an upper bound for the Arakelov degree of a Hermitian line subbundle under semistable hypothesis. The fifth section is contributed to some basic properties of filtrations in the category of vector spaces. Then in the sixth section, we state an explicit version of a result of Ramanan–Ramanathan in our context and, following the method of Totaro, give a proof of it. In the seventh section a criterion of Arakelov semistability is presented, which is an arithmetic analogue of a result of Bogomolov. In the eighth section, we explain how to use the results in previous sections to reduce the majoration of the Arakelov degree of an arbitrary Hermitian line subbundle to the case with semistability hypothesis, which has already been discussed in the fourth section. Finally, we give the proof of Theorem 1.1 in the ninth section.

2. Notation and preliminary results

Throughout this article, if K is a field and if V is a vector space of finite rank over K, we denote by $\mathbb{P}(V)$ the K-scheme which represents the functor

$$\begin{array}{ccc} \textbf{Schemes}/K & \longrightarrow & \textbf{Sets} \\ (p : S \to \operatorname{Spec} K) & \longmapsto & \left\{ \begin{array}{c} \text{locally free quotient} \\ \text{of rank 1 of } p^*V \end{array} \right\} \end{array} \tag{3}$$

In particular, $\mathbb{P}(V)(K)$ classifies all hyperplanes in V, or equivalently, all lines in $V^\vee$. We denote by $\mathcal{O}_V(1)$ the canonical line bundle on $\mathbb{P}(V)$. In other words, if $\pi : \mathbb{P}(V) \to \operatorname{Spec} K$ is the structural morphism, then $\mathcal{O}_V(1)$ is the quotient of π^*V defined as the universal object of the representable functor (3). For any integer $m \geq 1$, we use the expression $\mathcal{O}_V(m)$ to denote the line bundle $\mathcal{O}_V(1)^{\otimes m}$.

Let G be an algebraic group over $\operatorname{Spec} K$ and X be a projective variety over $\operatorname{Spec} K$. Suppose that G acts on X and that L is an ample G-linearized line bundle on X. We say that a rational point x of X is *semistable* for the action of G relatively to L if there exists an integer $D \geq 1$ and a section $s \in H^0(X, L^{\otimes D})$ invariant by the action of G such that x lies in the open subset of X defined by the non-vanishing of s. Clearly x is semistable for the action of G relatively to L if and only if it is semistable for the action of G relatively to any strictly positive tensor power of L.

In particular, if $G(K)$ acts linearly on a vector space V of finite rank over K, then the action of G on V induces naturally an action of G on $\mathbb{P}(V)$, and $\mathcal{O}_V(1)$ becomes a G-linearized line bundle. Let R be a vector subspace of rank 1 of $V^\vee$, which is viewed as a point in $\mathbb{P}(V)(K)$. Then R is semistable for the action of G relatively to $\mathcal{O}_V(1)$ if and only if there exists an integer $m \geq 1$ and a non-zero section $s \in H^0(\mathbb{P}(V), \mathcal{O}_V(m)) = S^m V$ which is invariant by the action of $G(K)$ such that the composed homomorphism $R^{\otimes m} \longrightarrow (S^m V)^\vee \xrightarrow{\ s\ } K$ is non-zero. Here we have identified $(S^m V)^\vee$ with the subspace of $V^{\vee \otimes m}$ of vectors which are invariant by the action of the symmetric group $\mathfrak{S}_m$.

We present some estimations for maximal slopes in the geometric case. Let k be an arbitrary field and C be a smooth projective curve of genus g defined over k. Let $b = \min\{\deg(L) \mid L \in \mathrm{Pic}(C),\ L \text{ is ample}\}$ and $a = b + g - 1$.

Lemma 2.1. *Let E be a non-zero vector bundle on C. If $H^0(C, E) = 0$, then $\mu_{\max}(E) \leq g - 1$.*

Proof. Since $H^0(C, E) = 0$, for any non-zero subbundle F of E, we also have $H^0(C, F) = 0$. Recall that the Riemann–Roch theorem asserts that

$$\mathrm{rk}_k\, H^0(C, F) - \mathrm{rk}_k\, H^1(C, F) = \deg(F) + \mathrm{rk}(F)(1 - g).$$

Therefore $\deg(F) + \mathrm{rk}(F)(1 - g) \leq 0$, which implies $\mu(F) \leq g - 1$. Since F is arbitrary, $\mu_{\max}(E) \leq g - 1$. □

Proposition 2.2. *For any non-zero vector bundles E and F on C, we have the inequality*

$$\mu_{\max}(E) + \mu_{\max}(F) \leq \mu_{\max}(E \otimes F) \leq \mu_{\max}(E) + \mu_{\max}(F) + a,$$

where $a = b + g - 1$ only depends on C.

Proof. (1) Let E_1 be a subbundle of E such that $\mu(E_1) = \mu_{\max}(E)$ and let F_1 be a subbundle of F such that $\mu(F_1) = \mu_{\max}(F)$. Since $E_1 \otimes F_1$ is a subbundle of $E \otimes F$, we obtain

$$\mu_{\max}(E) + \mu_{\max}(F) = \mu(E_1) + \mu(F_1) = \mu(E_1 \otimes F_1) \leq \mu_{\max}(E \otimes F),$$

which is the first inequality.

(2) We first prove that if E' and E'' are two non-zero vector bundles on C such that $\mu_{\max}(E') + \mu_{\max}(E'') < 0$, then $\mu_{\max}(E' \otimes E'') \leq g - 1$. In fact, if $\mu_{\max}(E' \otimes E'') > g - 1$, then by Lemma 2.1, $H^0(C, E' \otimes E'') \neq 0$. Therefore,

there exists a non-zero homomorphism φ from $E'^{\vee}$ to E''. Let G be the image of φ, which is non-zero since φ is non-zero. The vector bundle G is a subbundle of E'' and a quotient bundle of $E'^{\vee}$. Hence $G^{\vee}$ is a subbundle of $E'^{\vee\vee} \cong E'$. Therefore, we have $\mu(G) \leq \mu_{\max}(E'')$ and $\mu(G^{\vee}) = -\mu(G) \leq \mu_{\max}(E')$. By the taking the sum, we obtain $\mu_{\max}(E') + \mu_{\max}(E'') \geq 0$.

We now prove the second inequality in the proposition. By the definition of b, there exists a line bundle M such that $-b \leq \mu_{\max}(E)+\mu_{\max}(F)+\deg(M) = \mu_{\max}(E\otimes M)+\mu_{\max}(F) < 0$. Then, by combining the previously proved result, we obtain $\mu_{\max}(E \otimes M \otimes F) \leq g - 1$. Therefore,

$$\mu_{\max}(E \otimes F) \leq g - 1 - \deg(M) \leq \mu_{\max}(E) + \mu_{\max}(F) + g + b - 1.$$

□

We now recall some classical results in Arakelov theory, which will be useful afterwards. We begin by introducing the notation.

Let $\overline{E}$ be a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$. For any finite place $\mathfrak{p}$ of K, we denote by $K_{\mathfrak{p}}$ the completion of K with respect to $\mathfrak{p}$, equipped with the absolute value $|\cdot|_{\mathfrak{p}}$ which is normalized as $|\cdot|_{\mathfrak{p}} = \#(\mathcal{O}_K/\mathfrak{p})^{-v_{\mathfrak{p}}(\cdot)}$ with $v_{\mathfrak{p}}$ being the discrete valuation associated to $\mathfrak{p}$. The structure of the $\mathcal{O}_K$-module on E naturally induces a norm $\|\cdot\|_{\mathfrak{p}}$ on $E_{K_{\mathfrak{p}}} := E \otimes_K K_{\mathfrak{p}}$ such that $E_{K_{\mathfrak{p}}}$ becomes a Banach space over $K_{\mathfrak{p}}$.

If $\overline{L}$ is a Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$ and if s is an arbitrary non-zero element in L, then

$$\widehat{\deg}_n(\overline{L}) = \frac{1}{[K:\mathbb{Q}]}\Big(\log\#(L/\mathcal{O}_K s) - \sum_{\sigma:K\to\mathbb{C}} \log\|s\|_{\sigma}\Big),$$

which can also be written as

$$\widehat{\deg}_n(\overline{L}) = -\frac{1}{[K:\mathbb{Q}]}\Big(\sum_{\mathfrak{p}} \log\|s\|_{\mathfrak{p}} + \sum_{\sigma:K\to\mathbb{C}} \log\|s\|_{\sigma}\Big). \tag{4}$$

Note that (4) is analogous to the degree function of a line bundle on a smooth projective curve. Similarly to the geometric case, for any Hermitian vector bundle $\overline{E}$ of rank r on $\operatorname{Spec}\mathcal{O}_K$, we have

$$\widehat{\deg}_n(\overline{E}) = \widehat{\deg}_n(\Lambda^r\overline{E}) \tag{5}$$

where $\Lambda^r\overline{E}$ is the r^{th} exterior power of $\overline{E}$, that is, the *determinant* of $\overline{E}$, which is a Hermitian line bundle. Furthermore, if

$$0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$$

is a short exact sequence of Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$, the following equality holds:

$$\widehat{\deg}_n(\overline{E}) = \widehat{\deg}_n(\overline{E}') + \widehat{\deg}_n(\overline{E}''). \tag{6}$$

Lemma 2.3. *If $\overline{E}$ and $\overline{F}$ are respectively two Hermitian vector bundles of ranks r_1 and r_2 on $\operatorname{Spec}\mathcal{O}_K$, then*

$$\widehat{\deg}_n(\overline{E}\otimes\overline{F}) = \operatorname{rk}(E)\widehat{\deg}_n(\overline{F}) + \operatorname{rk}(F)\widehat{\deg}_n(\overline{E}). \tag{7}$$

Proof. The determinant Hermitian line bundle $\Lambda^{r_1+r_2}(\overline{E}\otimes\overline{F})$ is isomorphic to $(\Lambda^{r_1}\overline{E})^{\otimes r_2}\otimes(\Lambda^{r_2}\overline{F})^{\otimes r_1}$. Taking Arakelov degree and using (5) we obtain (7). □

We establish below the arithmetic analogue to the first inequality in Proposition 2.2.

Proposition 2.4. *Let $\overline{E}$ and $\overline{F}$ be two non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$. Then*

$$\widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F}) \le \widehat{\mu}_{\max}(\overline{E}\otimes\overline{F}).$$

Proof. Let $\overline{E}_1$ and $\overline{F}_1$ be, respectively, a non-zero Hermitian vector subbundle of $\overline{E}$ and $\overline{F}$. Then $\overline{E}_1\otimes\overline{F}_1$ is a Hermitian vector subbundle of $\overline{E}\otimes\overline{F}$. Therefore

$$\widehat{\mu}(\overline{E}_1) + \widehat{\mu}(\overline{F}_1) = \widehat{\mu}(\overline{E}_1\otimes\overline{F}_1) \le \widehat{\mu}_{\max}(\overline{E}\otimes\overline{F}),$$

where the equality results from (7). Since $\overline{E}_1$ and $\overline{F}_1$ are arbitrary, we obtain $\widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F}) \le \widehat{\mu}_{\max}(\overline{E}\otimes\overline{F})$. □

Corollary 2.5. *Let $(\overline{E}_i)_{1\le i\le n}$ be a finite family of non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$. Then the following equality holds:*

$$\widehat{\mu}_{\max}(\overline{E}_1) + \cdots + \widehat{\mu}_{\max}(\overline{E}_n) \le \widehat{\mu}_{\max}(\overline{E}_1\otimes\cdots\otimes\overline{E}_n). \tag{8}$$

Let $\overline{E}$ and $\overline{F}$ be two Hermitian vector bundles and $\varphi : E_K \to F_K$ be a non-zero K-linear homomorphism. For any finite place $\mathfrak{p}$ of K, we denote by $h_{\mathfrak{p}}(\varphi)$ the real number $\log\|\varphi_{\mathfrak{p}}\|$, where $\varphi_{\mathfrak{p}} : E_{K_{\mathfrak{p}}} \to F_{K_{\mathfrak{p}}}$ is induced from φ by scalar extension. Note that if φ is induced by an $\mathcal{O}_K$-homomorphism from E to F, then $h_{\mathfrak{p}}(\varphi)\le 0$ for any finite place $\mathfrak{p}$. Similarly, for any embedding $\sigma : K\to\mathbb{C}$, we define $h_\sigma(\varphi) = \log\|\varphi_\sigma\|$, where $\varphi_\sigma : E_{\sigma,\mathbb{C}} \to F_{\sigma,\mathbb{C}}$ is given by the scalar extension σ. Finally, we define the *height* of φ as

$$h(\varphi) = \frac{1}{[K:\mathbb{Q}]}\Big(\sum_{\mathfrak{p}} h_{\mathfrak{p}}(\varphi) + \sum_{\sigma:K\to\mathbb{C}} h_\sigma(\varphi)\Big).$$

Proposition 2.6 (see [3])**.** *Let $\overline{E}$ and $\overline{F}$ be two non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$ and $\varphi : E_K\to F_K$ be a K-linear homomorphism.*

(1) *If φ is injective, then*

$$\widehat{\mu}(\overline{E}) \le \widehat{\mu}_{\max}(\overline{F}) + h(\varphi). \tag{9}$$

(2) *If φ is non-zero, then*

$$\widehat{\mu}_{\min}(\overline{E}) \le \widehat{\mu}_{\max}(\overline{E}) + h(\varphi) \tag{10}$$

where $\widehat{\mu}_{\min}(\overline{E})$ is the minimal value of slopes of all non-zero Hermitian quotient bundles of $\overline{E}$.

For any non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec}\mathcal{O}_K$, let $\widehat{\operatorname{udeg}}_n(\overline{E})$ be the maximal degree of line subbundles of $\overline{E}$. We recall a result of Bost and Künnemann comparing the maximal degree and the maximal slope of $\overline{E}$, which is a variant of Minkowski's First Theorem.

Proposition 2.7 ([6, (3.27)]). *Let $\overline{E}$ be a non-zero Hermitian vector bundle on* $\operatorname{Spec}\mathcal{O}_K$. *Then*

$$\widehat{\operatorname{udeg}}_n(\overline{E}) \le \widehat{\mu}_{\max}(\overline{E}) \le \widehat{\operatorname{udeg}}_n(\overline{E}) + \frac{1}{2}\log(\operatorname{rk} E) + \frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}, \tag{11}$$

where Δ_K is the discriminant of K.

3. Reminder on invariant theory

In this section we recall some known results in classical invariant theory. We fix K to be a field of characteristic 0. If V is a vector space over K and if $u \in \mathbb{N}$, then the expression $V^{\otimes(-u)}$ denotes the space $V^{\vee\otimes u}$.

Let V be a finite dimensional non-zero vector space over K. For any $u \in \mathbb{N}$, we denote by $J_u : \operatorname{End}_K(V)^{\otimes u} \to \operatorname{End}_K(V^{\otimes u})$ the K-linear homomorphism (of vector spaces) which sends the tensor product $T_1 \otimes \cdots \otimes T_u$ of u elements in $\operatorname{End}_K(V)$ to their tensor product as an endomorphism of $V^{\otimes u}$. The mapping J_u is actually a homomorphism of K-algebras. Furthermore, as a homomorphism of vector spaces, J_u can be written as the composition of the following natural isomorphisms:

$$\operatorname{End}_K(V)^{\otimes u} \longrightarrow (V^\vee \otimes V)^{\otimes u} \longrightarrow (V^{\otimes u})^\vee \otimes V^{\otimes u} \longrightarrow \operatorname{End}_K(V^{\otimes u}),$$

so it is itself an isomorphism. Moreover, there is an action of the symmetric group $\mathfrak{S}_u$ on $V^{\otimes u}$ by permuting the factors. This representation of $\mathfrak{S}_u$ defines a homomorphism from the group algebra $K[\mathfrak{S}_u]$ to $\operatorname{End}_K(V^{\otimes u})$. The elements of $\mathfrak{S}_u$ act by conjugation on $\operatorname{End}_K(V^{\otimes u})$. If we identify $\operatorname{End}_K(V^{\otimes u})$ with $\operatorname{End}_K(V)^{\otimes u}$ by the isomorphism J_u, then the corresponding $\mathfrak{S}_u$-action is just the permutation of factors in the tensor product. Finally, the group $\operatorname{GL}_K(V)$ acts diagonally on $V^{\otimes u}$.

When $u = 0$, J_0 reduces to the identical homomorphism $\operatorname{Id} : K \to K$, and $\mathfrak{S}_0$ reduces to the group of one element. The "diagonal" action of $\operatorname{GL}_K(V)$ on $V^{\otimes 0} \cong K$ is trivial.

We recall below the "first principal theorem" of classical invariant theory (cf. [28, Chapter III]; see also [1, Appendix 1] for a proof).

Theorem 3.1. *Let V be a finite dimensional non-zero vector space over K. Let $u \in \mathbb{N}$ and $v \in \mathbb{Z}$. If T is a non-zero element in $V^{\vee \otimes u} \otimes V^{\otimes v}$, which is invariant by the action of $\mathrm{GL}_K(V)$, then $u = v$, and T is a linear combination of permutations in $\mathfrak{S}_u$ acting on V (here we identify $V^{\vee \otimes u} \otimes V^{\otimes u}$ with $\mathrm{End}_K(V^{\otimes u})$).*

We now present a generalization of Theorem 3.1 to the case of several linear spaces. In the rest of this section, we fix a family $(V_i)_{1 \leq i \leq n}$ of finite dimensional non-zero vector spaces over K. For any $i \in \{1, \cdots, n\}$, let r_i be the rank of V_i over K. For any mapping $\alpha : \{1, \cdots, n\} \to \mathbb{Z}$, we shall use the notation

$$V^\alpha := V_1^{\otimes \alpha(1)} \otimes \cdots \otimes V_n^{\otimes \alpha(n)} \tag{12}$$

to simplify the writing. Denote by G the algebraic group $\mathbb{GL}_K(V_1) \times_K \cdots \times_K \mathbb{GL}_K(V_n)$. Then $G(K)$ is the group $\mathrm{GL}_K(V_1) \times \cdots \times \mathrm{GL}_K(V_n)$. For any mapping $\alpha : \{1, \cdots, n\} \to \mathbb{N}$ with natural integer values, we denote by $\mathfrak{S}_\alpha$ the product $\mathfrak{S}_{\alpha(1)} \times \cdots \times \mathfrak{S}_{\alpha(n)}$ of symmetric groups. We have a natural isomorphism of K-algebras from $\mathrm{End}_K(V^\alpha)$ to $\mathrm{End}_K(V_1)^{\otimes \alpha(1)} \otimes_K \cdots \otimes_K \mathrm{End}_K(V_n)^{\otimes \alpha(n)}$. The group $G(K)$ acts naturally on V^α and the group $\mathfrak{S}_\alpha$ acts on V^α by permutating tensor factors. By induction on n, Theorem 3.1 implies the following corollary:

Corollary 3.2. *With the notation above, if $\alpha : \{1, \cdots, n\} \to \mathbb{N}$ and $\beta : \{1, \cdots, n\} \to \mathbb{Z}$ are two mappings and if T is a non-zero element in $(V^\alpha)^\vee \otimes V^\beta$ which is invariant by the action of $G(K)$, then $\alpha = \beta$, and T is a linear combination of elements in $\mathfrak{S}_\alpha$ acting on V^α.*

Let $\mathcal{A}$ be a finite non-empty family of mappings from $\{1, \cdots, n\}$ to $\mathbb{N}$ and $(b_i)_{1 \leq i \leq n}$ be a family of integers such that r_i divides b_i for any i. We denote by W the vector space $\bigoplus_{\alpha \in \mathcal{A}} V^\alpha$. Note that the group $G(K)$ acts naturally on W. Let L be the $G(K)$-module $(\det V_1)^{\otimes b_1/r_1} \otimes \cdots \otimes (\det V_n)^{\otimes b_n/r_n}$. For any integer $D \geq 1$ and any element $\underline{\alpha} = (\alpha_j)_{1 \leq j \leq D} \in \mathcal{A}^D$, let

$$\mathrm{pr}_{\underline{\alpha}} : W^{\otimes D} \longrightarrow V^{\alpha_1} \otimes \cdots \otimes V^{\alpha_D}$$

be the canonical projection. Finally, let $\pi : \mathbb{P}(W^\vee) \to \operatorname{Spec} K$ be the canonical morphism.

Theorem 3.3. *With the notation above, if m is a strictly positive integer and if R is a vector subspace of rank 1 of W (considered as a rational point of $\mathbb{P}(W^\vee)$) which is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m) \otimes \pi^* L$, then there exists an integer $D \geq 1$ and a family $\underline{\alpha} = (\alpha_j)_{1 \leq j \leq mD}$ of elements in $\mathcal{A}$ such that, by noting $A = \alpha_1 + \cdots + \alpha_{mD}$, we have $A(i) = Db_i$ and hence $b_i \geq 0$ for any i.*

Furthermore, there exists an element $\sigma \in \mathfrak{S}_A$ *such that the composition of homomorphisms*

$$\begin{array}{ccccc}
R^{\otimes mD} \otimes L^{\vee \otimes D} & \longrightarrow & W^{\otimes mD} \otimes L^{\vee\otimes D} & \xrightarrow{\mathrm{pr}_{\underline{\alpha}} \otimes \mathrm{Id}} & V^A \otimes L^{\vee\otimes D} \\
 & & & & \Big\downarrow {\scriptstyle \sigma\otimes\mathrm{Id}} \\
 & & & & V^A \otimes L^{\vee\otimes D} \\
 & & & & \Big\downarrow {\scriptstyle \det_{V_1}^{\otimes Db_1/r_1} \otimes\cdots\otimes \det_{V_n}^{\otimes Db_n/r_n} \otimes \mathrm{Id}} \\
 & & & & L^{\otimes D} \otimes L^{\vee\otimes D} \cong K
\end{array}$$

does not vanish, where the first arrow is induced by the canonical inclusion of $R^{\otimes mD}$ *in* $W^{\otimes mD}$.

Proof. Since R is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m)\otimes\pi^*L$, there exists an integer $D \geq 1$ and an element $s \in S^{mD}(W^\vee)\otimes L^{\otimes D}$ which is invariant by the action of $G(K)$ such that the composition of homomorphisms

$$R^{\otimes mD} \otimes L^{\vee\otimes D} \longrightarrow S^{mD}(W^\vee)^\vee \otimes L^{\vee\otimes D} \xrightarrow{\ s\ } K$$

does not vanish, the first arrow being the canonical inclusion.

As K is of characteristic 0, $S^{mD}(W^\vee)$ is a direct factor as a $\mathrm{GL}(W)$-module of $W^{\vee\otimes mD}$. Hence $S^{mD}(W^\vee)\otimes L^{\otimes D}$ is a direct factor as a $G(K)$-module of $W^{\vee\otimes mD}\otimes L^{\otimes D}$. So we can choose $s' \in W^{\vee\otimes mD}\otimes L^{\otimes D}$ invariant by the action of $G(K)$ such that the class of s' in $S^{mD}(W^\vee)\otimes L^{\otimes D}$ coincides with s. There then exists $\underline{\alpha} = (\alpha_j)_{1\leq j\leq mD} \in \mathcal{A}^D$ such that the composition

$$R^{\otimes mD} \otimes L^{\vee\otimes D} \longrightarrow W^{\otimes mD} \otimes L^{\vee\otimes D} \xrightarrow{\mathrm{pr}_{\underline{\alpha}}\otimes\mathrm{Id}} V^A \otimes L^{\vee\otimes D} \xrightarrow{\ s'_{\underline{\alpha}}\ } K$$

is non-zero, where $A = \alpha_1 + \cdots + \alpha_{mD}$ and $s'_{\underline{\alpha}}$ is the component of index $\underline{\alpha}$ of s'. Let $B : \{1,\cdots,n\} \to \mathbb{Z}$ be the mapping which sends i to Db_i. Note that for any i, $\Lambda^{r_i}V_i = \det V_i$ is naturally a direct factor of $V_i^{\otimes r_i}$. We can therefore choose a preimage $s''_{\underline{\alpha}}$ of $s'_{\underline{\alpha}}$ in $(V^A)^\vee \otimes V^B$ which is invariant by $G(K)$. By Corollary 3.2, $A = B$ and $s''_{\underline{\alpha}}$ is a linear combination of permutations acting on V. Therefore the theorem is proved. □

4. Degree of a semistable line subbundle

Let K be a number field and $\mathcal{O}_K$ be its integer ring. Consider a family $(\overline{E}_i)_{1\leq i\leq n}$ of non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$. Let $\mathcal{A}$ be a non-empty and finite family of non-identically zero mappings from $\{1,\cdots,n\}$

to $\mathbb{N}$. We define a new Hermitian vector bundle over $\operatorname{Spec}\mathcal{O}_K$ as follows:

$$\overline{E} := \bigoplus_{\alpha\in\mathcal{A}} \overline{E}_1^{\otimes\alpha(1)} \otimes\cdots\otimes \overline{E}_n^{\otimes\alpha(n)}.$$

In this section, we shall use the ideas in [2] to obtain an upper bound for the Arakelov degree of a Hermitian line subbundle $\overline{M}$ of $\overline{E}$ under the hypothesis of semistability (in the sense of geometric invariant theory) for M_K. This upper bound is crucial because, as we shall see later, the general case can be reduced to this special one through an argument of Ramanan and Ramanathan [23].

For any $i\in\{1,\cdots,n\}$, let r_i be the rank of E_i and V_i be the vector space $E_{i,K}$. Let $W=E_K$ and $\pi:\mathbb{P}(W^\vee)\to\operatorname{Spec}K$ be the canonical morphism. By definition $W=\bigoplus_{\alpha\in\mathcal{A}}V^\alpha$, where V^α is defined in (12). We denote by G the algebraic group $\mathbb{GL}_K(V_1)\times\cdots\times\mathbb{GL}_K(V_n)$ which acts naturally on $\mathbb{P}(W^\vee)$. Let $(b_i)_{1\le i\le n}$ be a family of strictly positive integers such that r_i divides b_i. Finally, let

$$\overline{L} = (\Lambda^{r_1}\overline{E}_1)^{\otimes b_1/r_1}\otimes\cdots\otimes(\Lambda^{r_n}\overline{E}_n)^{\otimes b_n/r_n}.$$

Lemma 4.1. *Let H be a Hermitian space of dimension $d>0$. Then the norm of the homomorphism* $\det : H^{\otimes d}\to\Lambda^d H$ *equals* $\sqrt{d!}$.

Proof. Let $(e_i)_{1\le i\le d}$ be an orthonormal basis of H and $(e_i^\vee)_{1\le i\le d}$ be its dual basis in $H^\vee$. If we identify $\Lambda^d H$ with $\mathbb{C}$ via the basis $e_1\wedge\cdots\wedge e_d$, then the homomorphism det, viewed as an element in $H^{\vee\otimes d}$, can be written as

$$\sum_{\sigma\in\mathfrak{S}_d}\operatorname{sign}(\sigma)e^\vee_{\sigma(1)}\otimes\cdots\otimes e^\vee_{\sigma(d)},$$

which is the sum of $d!$ orthogonal vectors of norm 1 in $H^{\vee\otimes d}$. So its norm is $\sqrt{d!}$. □

Theorem 4.2. *With the notation above, if $m\ge 1$ is an integer and if $\overline{M}$ is a Hermitian line subbundle of $\overline{E}$ such that M_K is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m)\otimes\pi^*L_K$, then*

$$\widehat{\deg}(\overline{M})\le\frac{1}{m}\widehat{\deg}(\overline{L})+\frac{1}{2m}\sum_{i=1}^n b_i\log(\operatorname{rk}E_i)=\sum_{i=1}^n\frac{b_i}{m}\Big(\widehat{\mu}(\overline{E}_i)+\frac12\log(\operatorname{rk}E_i)\Big).$$

Proof. By Theorem 3.3, we get, by combining the slope inequality (9) and Lemma 4.1,

$$\begin{aligned}mD\widehat{\deg}(\overline{M})-D\widehat{\deg}(\overline{L}) &= mD\widehat{\deg}(\overline{M})-\sum_{i=1}^n Db_i\widehat{\mu}(\overline{E}_i)\\ \le\sum_{i=1}^n\frac{A(i)\log(r_i!)}{2r_i} &= \sum_{i=1}^n\frac{Db_i\log(r_i!)}{2r_i}\le\frac12 D\sum_{i=1}^n b_i\log r_i,\end{aligned}$$

where we have used the evident estimation $r! \leq r^r$ to obtain the last inequality. Finally, we divide the inequality by mD and obtain

$$\widehat{\deg}(\overline{M}) \leq \frac{1}{m}\widehat{\deg}(\overline{L}) + \frac{1}{2m}\sum_{i=1}^{n} b_i \log r_i = \sum_{i=1}^{n}\frac{b_i}{m}\Big(\widehat{\mu}(\overline{E}_i) + \frac{\log r_i}{2}\Big).$$

□

Let m be a strictly positive integer which is divisible by all r_i. We apply Theorem 4.2 to the special case where $\mathcal{A}$ contains a single map $\alpha \equiv 1$ and where $b_i = m$ for any i. Then we get the following upper bound:

Corollary 4.3. *If $\overline{M}$ is a Hermitian line subbundle of $\overline{E}_1 \otimes \cdots \otimes \overline{E}_n$ such that M_K is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m) \otimes \pi^* L_K$, then we have*

$$\widehat{\deg}(\overline{M}) \leq \sum_{i=1}^{n}\Big(\widehat{\mu}(\overline{E}_i) + \frac{1}{2}\log(\operatorname{rk} E_i)\Big). \tag{13}$$

5. Filtrations of vector spaces

In this section, we introduce some basic notation and results on $\mathbb{R}$-filtrations of vector spaces, which we shall use in the sequel. We fix a field K.

5.1. Definition of filtrations. Let V be a non-zero vector space of finite rank r over K. We call $\mathbb{R}$-*filtration* of V any family $\mathcal{F} = (\mathcal{F}_\lambda V)_{\lambda \in \mathbb{R}}$ of subspaces of V such that

(1) $\mathcal{F}_\lambda V \supset \mathcal{F}_{\lambda'} V$ for all $\lambda \leq \lambda'$,
(2) $\mathcal{F}_\lambda V = 0$ for λ sufficiently positive,
(3) $\mathcal{F}_\lambda V = V$ for λ sufficiently negative, and
(4) the function $x \mapsto \operatorname{rk}_K(\mathcal{F}_x V)$ on $\mathbb{R}$ is left continuous.

A filtration $\mathcal{F}$ of V is equivalent to the data of a flag

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_d = 0 \tag{14}$$

of V together with a strictly increasing sequence of real numbers $(\lambda_i)_{0 \leq i < d}$. In fact, we have the relation $\mathcal{F}_\lambda V = \bigcup_{\lambda_i \geq \lambda} V_i$. We define the *expectation* of $\mathcal{F}$ to be

$$\mathbb{E}[\mathcal{F}] := \sum_{i=0}^{d-1} \frac{\operatorname{rk}_K(V_i/V_{i+1})}{\operatorname{rk}_K V}\lambda_i. \tag{15}$$

Furthermore, we define a function $\lambda_{\mathcal{F}} : V \to \mathbb{R} \cup \{+\infty\}$ such that

$$\lambda_{\mathcal{F}}(x) = \sup\{a \in \mathbb{R} \mid x \in \mathcal{F}_a V\}. \tag{16}$$

The function $\lambda_{\mathcal{F}}$ takes values in $\{\lambda_0, \cdots, \lambda_{d-1}\} \cup \{+\infty\}$ and is finite on $V \setminus \{0\}$.

5.2. Spaces of filtrations. Let Z be a subset of $\mathbb{R}$. We say that $\mathcal{F}$ is *supported by* Z if $\{\lambda_i \mid 0 \le i < d\} \subset Z$. We say that a basis $\mathbf{e}$ of V is *compatible* with $\mathcal{F}$ if it is compatible with the flag (14). That is, $\#(V_i \cap \mathbf{e}) = \mathrm{rk}(V_i)$.

We denote by $\mathbf{Fil}_V$ the set of all filtrations of V. For any non-empty subset Z of $\mathbb{R}$, denote by $\mathbf{Fil}_V^Z$ the set of all filtrations of V supported by Z. Finally, for any basis $\mathbf{e}$, we use the expression $\mathbf{Fil}_{\mathbf{e}}$ to denote the set of all filtrations of V with which $\mathbf{e}$ is compatible, and we denote by $\mathbf{Fil}_{\mathbf{e}}^Z$ the subset of $\mathbf{Fil}_{\mathbf{e}}$ of filtrations supported by Z.

Proposition 5.1. *Let $\mathbf{e} = (e_1, \cdots, e_r)$ be a basis of V and Z be a non-empty subset of $\mathbb{R}$. The mapping $\Phi_{\mathbf{e}} : \mathbf{Fil}_{\mathbf{e}}^Z \to Z^r$ defined by*

$$\Phi_{\mathbf{e}}(\mathcal{F}) = (\lambda_{\mathcal{F}}(e_1), \cdots, \lambda_{\mathcal{F}}(e_r)) \tag{17}$$

is a bijection.

By Proposition 5.1, if F is a subfield of $\mathbb{R}$, then the set $\mathbf{Fil}_{\mathbf{e}}^F$ can be viewed as a vector space of rank r over F via the bijection $\Phi_{\mathbf{e}}$.

Proposition 5.2. *Let v be a non-zero vector in V, F be a subfield of $\mathbb{R}$ and $\mathbf{e}$ be a basis of V. Then the function $\mathcal{F} \mapsto \lambda_{\mathcal{F}}(v)$ from $\mathbf{Fil}_{\mathbf{e}}^F$ to $\mathbb{R}$ can be written as the minimal value of a finite number of F-linear forms.*

Proof. Let $v = \sum_{i=1}^r a_i e_i$ be the decomposition of v in the basis $\mathbf{e}$, then for any filtration $\mathcal{F} \in \mathbf{Fil}_{\mathbf{e}}^F$, we have

$$\lambda_{\mathcal{F}}(v) = \min_{\substack{1 \le i \le n \\ a_i \ne 0}} \lambda_{\mathcal{F}}(e_i).$$

□

5.3. Construction of filtrations. For any real number $\varepsilon > 0$, we define the *dilation* of $\mathcal{F}$ by ε as the filtration

$$\psi_\varepsilon \mathcal{F} := (\mathcal{F}_{\varepsilon\lambda} V)_{\lambda \in \mathbb{R}} \tag{18}$$

of V. We have

$$\mathbb{E}[\psi_\varepsilon \mathcal{F}] = \varepsilon \mathbb{E}[\mathcal{F}] \qquad \text{and} \qquad \lambda_{\psi_\varepsilon \mathcal{F}} = \varepsilon \lambda_{\mathcal{F}}. \tag{19}$$

Let $(V^{(i)})_{1 \le i \le n}$ be a family of non-zero vector spaces of finite rank over K and $V = \bigoplus_{i=1}^n V^{(i)}$ be their direct sum. For each $i \in \{1, \cdots, n\}$, let $\mathcal{F}^{(i)}$ be a filtration of $V^{(i)}$. We construct a filtration $\mathcal{F}$ of V such that

$$\mathcal{F}_\lambda V = \bigoplus_{i=1}^n \mathcal{F}_\lambda^{(i)} V^{(i)}.$$

The filtration $\mathcal{F}$ is called the *direct sum* of $\mathcal{F}^{(i)}$ and is denoted by $\mathcal{F}^{(1)} \oplus \cdots \oplus \mathcal{F}^{(n)}$. If $\mathbf{e}^{(i)}$ is a basis of $V^{(i)}$ which is compatible with $\mathcal{F}^{(i)}$, then the disjoint union $\mathbf{e}^{(1)} \amalg \cdots \amalg \mathbf{e}^{(n)}$, which is a basis of $V^{(1)} \oplus \cdots \oplus V^{(n)}$, is compatible with

$\mathcal{F}^{(1)} \oplus \cdots \oplus \mathcal{F}^{(n)}$. Similarly, if $W = \bigotimes_{i=1}^n V^{(i)}$ is the tensor product of $V^{(i)}$, we construct a filtration $\mathcal{G}$ of W such that

$$\mathcal{G}_\lambda W = \sum_{\lambda_1+\cdots+\lambda_n \geq \lambda} \bigotimes_{i=1}^n \mathcal{F}^{(i)}_{\lambda_i} V^{(i)},$$

called the *tensor product* of $\mathcal{F}^{(i)}$ and denoted by $\mathcal{F}^{(1)} \otimes \cdots \otimes \mathcal{F}^{(n)}$. If $\mathbf{e}^{(i)}$ is a basis of $V^{(i)}$ which is compatible with the filtration $\mathcal{F}^{(i)}$, then the basis

$$\mathbf{e}^{(1)} \otimes \cdots \otimes \mathbf{e}^{(n)} := \{e_1 \otimes \cdots \otimes e_n \,|\, \forall 1 \leq i \leq n,\ e_i \in \mathbf{e}^{(i)}\}$$

of $V^{(1)} \otimes \cdots \otimes V^{(n)}$ is compatible with $\mathcal{F}^{(1)} \otimes \cdots \otimes \mathcal{F}^{(n)}$. Finally, for any $\varepsilon > 0$,

$$\psi_\varepsilon(\mathcal{F}^{(1)} \otimes \cdots \otimes \mathcal{F}^{(n)}) = \psi_\varepsilon \mathcal{F}^{(1)} \otimes \cdots \otimes \psi_\varepsilon \mathcal{F}^{(n)}. \tag{20}$$

5.4. Scalar product on the space of filtrations. Let V be a non-zero vector space of finite rank r over K. If $\mathcal{F}$ is a filtration of V corresponding to the flag (14) and the increasing sequence $(\lambda_i)_{0 \leq i < d}$, we define a real number

$$\|F\| = \Big(\frac{1}{r}\sum_{i=0}^{d-1} \mathrm{rk}(V_i/V_{i+1})\lambda_i^2\Big)^{\frac{1}{2}},$$

called the *norm* of the filtration $\mathcal{F}$. If $\mathbf{e} = (e_1, \cdots, e_r)$ is a basis of V which is compatible with $\mathcal{F}$, then the equality $\|\mathcal{F}\|^2 = \dfrac{1}{r}\displaystyle\sum_{i=1}^r \lambda_{\mathcal{F}}(e_i)^2$ holds. Note that $\|\mathcal{F}\| = 0$ if and only if $\mathcal{F}$ is supported by $\{0\}$. In this case, we say that the filtration $\mathcal{F}$ is *trivial.*

If $\mathcal{F}$ and $\mathcal{G}$ are two filtrations of V, then by Bruhat's decomposition, there always exists a basis $\mathbf{e}$ of V which is compatible simultaneously with $\mathcal{F}$ and $\mathcal{G}$. We define the *scalar product* of $\mathcal{F}$ and $\mathcal{G}$ as

$$\langle \mathcal{F}, \mathcal{G} \rangle := \frac{1}{r}\sum_{i=1}^r \lambda_{\mathcal{F}}(e_i)\lambda_{\mathcal{G}}(e_i). \tag{21}$$

This definition does not depend on the choice of $\mathbf{e}$. Furthermore, we have $\langle \mathcal{F}, \mathcal{F} \rangle = \|\mathcal{F}\|^2$.

Proposition 5.3. *Let* $\mathbf{e}$ *be a basis of* V*. Then the function*

$$(x, y) \mapsto r\langle \Phi_{\mathbf{e}}^{-1}(x), \Phi_{\mathbf{e}}^{-1}(y)\rangle$$

on $\mathbb{R}^r \times \mathbb{R}^r$ *coincides with the usual Euclidean product on* $\mathbb{R}^r$*, where* $\Phi_{\mathbf{e}} : \mathbf{Fil}_{\mathbf{e}} \to \mathbb{R}^r$ *is the bijection defined in* (17)*.*

5.5. Construction of filtration from subquotients. Let V be a non-zero vector space of finite rank over K and let $\mathcal{F}$ be a filtration of V corresponding to the flag $V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_d = 0$ together with the sequence $(\lambda_j)_{0 \le j < d}$. For any $j \in \{0, \cdots, d-1\}$, we pick a basis $\mathbf{e}^j$ of the subquotient V_j/V_{j+1}. After choosing a preimage of $\mathbf{e}^j$ in V_j and taking the disjoint union of the preimages, we get a basis $\mathbf{e} = (e_1, \cdots, e_r)$ of V which is compatible with the filtration $\mathcal{F}$. The basis $\mathbf{e}$ defines a natural isomorphism Ψ from V to $\bigoplus_{j=0}^{d-1}(V_j/V_{j+1})$ which sends e_i to its class in $V_{\tau(i)}/V_{\tau(i)+1}$, where $\tau(i) = \max\{j \,|\, e_i \in V_j\}$.

Let $\mathcal{G}^j$ be a filtration of V_j/V_{j+1} with which $\mathbf{e}^j$ is compatible. We construct a filtration $\mathcal{G}$ on V which is the direct sum via Ψ of $(\mathcal{G}^j)_{0 \le j \le d-1}$. Note that the basis $\mathbf{e}$ is compatible with the new filtration $\mathcal{G}$. If e_i is an element in $\mathbf{e}$, then $\lambda_{\mathcal{G}}(e_i) = \lambda_{\mathcal{G}^{\tau(i)}}(\Psi(e_i))$. Therefore we have

$$\begin{aligned}\mathbb{E}[\mathcal{G}] &= \frac{1}{r}\sum_{j=0}^{d-1}\mathbb{E}[\mathcal{G}^j]\mathrm{rk}_K(V_j/V_{j+1}),\\ \langle \mathcal{F}, \mathcal{G}\rangle &= \frac{1}{r}\sum_{j=0}^{d-1}\lambda_j\mathbb{E}[\mathcal{G}^j]\,\mathrm{rk}_K(V_j/V_{j+1}).\end{aligned} \tag{22}$$

6. More facts in geometric invariant theory

In this section, we shall establish the explicit version of a result of Ramanan and Ramanathan [23] (Proposition 1.12), for our particular purpose, along the path indicated by Totaro [27] in his proof of Fontaine's conjecture.

Let K be a perfect field. If G is a reductive group over $\operatorname{Spec} K$, we call the *one-parameter subgroup* of G any morphism of K-group schemes from $\mathbb{G}_{\mathrm{m},K}$ to G. Let X be a K-scheme on which G acts. If x is a rational point of X and if h is a one-parameter subgroup of G, then we get a K-morphism from $\mathbb{G}_{\mathrm{m},K}$ to X given by the composition

$$\mathbb{G}_{\mathrm{m},K} \xrightarrow{\ h\ } G \xrightarrow{\ \sim\ } G \times_K \operatorname{Spec} K \xrightarrow{\mathrm{Id}\times x} G \times_K X \xrightarrow{\ \sigma\ } X\ ,$$

where σ is the action of the group. If in addition X is proper over $\operatorname{Spec} K$, this morphism extends in the unique way to a K-morphism $f_{h,x}$ from $\mathbb{A}^1_K$ to X. We denote by 0 the unique element in $\mathbb{A}^1(K) \setminus \mathbb{G}_{\mathrm{m}}(K)$. The morphism $f_{h,x}$ sends the point 0 to a rational point of X which is invariant by the action of $\mathbb{G}_{\mathrm{m},K}$. If L is a G-linearized line bundle on X, then the action of $\mathbb{G}_{\mathrm{m},K}$ on $L|_{f_{h,x}(0)}$ defines a character of $\mathbb{G}_{\mathrm{m},K}$ of the form

$$t \mapsto t^{\mu(x,h,L)},\ \text{where } \mu(x,h,L) \in \mathbb{Z}.$$

Furthermore, if we denote by $\mathrm{Pic}^G(X)$ the group of isomorphism classes of all G-linearized line bundles, then $\mu(x, h, \cdot)$ is a homomorphism of groups from $\mathrm{Pic}^G(X)$ to $\mathbb{Z}$.

Remark 6.1. In [21], the authors have defined the μ-invariant with a minus sign.

We now recall a well-known result which gives a semistability criterion for rational points in a projective variety equipped with an action of a reductive group.

Theorem 6.1 (Hilbert–Mumford–Kempf–Rousseau)**.** *Let G be a reductive group which acts on a projective variety X over* $\operatorname{Spec} K$*, L be an ample G-linearized line bundle on X and $x \in X(K)$ be a rational point. The point x is semistable for the action of G relatively to L if and only if $\mu(x, h, L) \geq 0$ for any one-parameter subgroup h of G.*

This theorem was originally proved by Mumford (see [21]) for the case where K is algebraically closed. Then it has been independently proved in all generality by Kempf [20] and Rousseau [25], where Kempf's approach has been revisited by Ramanan and Ramanathan [23] to prove that the tensor product of two semistable vector bundles on a smooth curve (over a perfect field) is also semistable. The idea of Kempf is to choose a special one-parameter subgroup h_0 of G destabilizing x, which minimizes a certain function. The uniqueness of his construction allows us to descend to a smaller field. Later, Totaro [27] introduced a new approach to Kempf's construction and thus found an elegant proof of Fontaine's conjecture.

In the rest of this section, we recall Totaro's approach of the Hilbert–Mumford criterion in our setting. We begin by explicitly calculating the number $\mu(x, h, L)$ using filtrations introduced in the previous section.

Let V be a vector space of finite rank over K and $\rho : G \to \mathbb{GL}(V)$ be a representation of G on V. If $h : \mathbb{G}_{\mathrm{m},K} \to G$ is a one-parameter subgroup, then the multiplicative group $\mathbb{G}_{\mathrm{m},K}$ acts on V via h and ρ. Hence we can decompose V into the direct sum of eigenspaces. More precisely, we have the decomposition $V = \bigoplus_{i \in \mathbb{Z}} V(i)$, where the action of $\mathbb{G}_{\mathrm{m},K}$ on $V(i)$ is given by the composition

$$\mathbb{G}_{\mathrm{m},K} \times_K V(i) \xrightarrow{(t \mapsto t^i) \times \mathrm{Id}} \mathbb{G}_{\mathrm{m},K} \times_K V(i) \longrightarrow V(i) \ ,$$

the second arrow being the scalar multiplication structure on $V(i)$. We then define a filtration $\mathcal{F}^{\rho,h}$ (supported by $\mathbb{Z}$) of V such that

$$\mathcal{F}^{\rho,h}_{\lambda} V = \sum_{i \geq \lambda} V(i) \qquad \text{where } \lambda \in \mathbb{R},$$

called the *filtration associated to* h relatively to the representation ρ. If there is no ambiguity on the representation, we also write $\mathcal{F}^h$ instead of $\mathcal{F}^{\rho,h}$ to simplify the notation. If $G = \mathbb{GL}(V)$ and if ρ is the canonical representation, then for any filtration $\mathcal{F}$ of V supported by $\mathbb{Z}$, there exists a one-parameter subgroup h of G such that the filtration associated to h equals $\mathcal{F}$.

From the scheme-theoretical point of view, the algebraic group G acts via the representation ρ on the projective space $\mathbb{P}(V^\vee)$.

The following result is in [21, Proposition 2.3]. Here we work on the dual space $V^\vee$.

Proposition 6.2. *Let x be a rational point of $\mathbb{P}(V^\vee)$, viewed as a one-dimensional subspace of V and let v_x be an arbitrary non-zero vector in x. Then*

$$\mu(x, h, \mathcal{O}_{V^\vee}(1)) = -\lambda_{\mathcal{F}^{\rho,h}}(v_x),$$

where the function $\lambda_{\mathcal{F}^{\rho,h}}$ is defined in (16).

Proof. Let $v_x = \sum_{i\in\mathbb{Z}} v_x(i)$ be the canonical decomposition of v_x. Let $i_0 = \lambda_{\mathcal{F}^{\rho,h}}(v_x)$. By definition, it is the maximal index i such that $v_x(i)$ is non-zero. Furthermore, $f_{h,x}(0)$ is just the rational point x_0 which corresponds to the subspace of V generated by $v_x(i_0)$. The restriction of $\mathcal{O}_{V^\vee}(1)$ on x_0 identifies with the quotient $(Kv_x(i_0))^\vee$ of $V^\vee$. Since the action of $\mathbb{G}_{\mathrm{m},K}$ on $v_x(i_0)$ via h is the multiplication by t^{i_0}, its action on $(Kv_x(i_0))^\vee$ is then the multiplication by t^{-i_0}. Therefore, $\mu(x, h, \mathcal{O}_{V^\vee}(1)) = -i_0 = -\lambda_{\mathcal{F}^{\rho,h}}(v_x)$. $\square$

Let $(V_i)_{1\le i\le n}$ be a finite family of non-zero vector spaces of finite rank over K. For any $i \in \{1, \cdots, n\}$, let r_i be the rank of V_i. Let G be the algebraic group $\mathbb{GL}(V_1) \times \cdots \times \mathbb{GL}(V_n)$. We suppose that the algebraic group G acts on a vector space V. Let $\pi : \mathbb{P}(V^\vee) \to \operatorname{Spec} K$ be the canonical morphism. For each i, we choose an integer m_i which is divisible by r_i. Let M be the G-linearized line bundle on $\mathbb{P}(V^\vee)$ defined as

$$M := \bigotimes_{i=1}^{n} \pi^*(\Lambda^{r_i} V_i)^{\otimes m_i/r_i}.$$

It is a trivial line bundle on $\mathbb{P}(V^\vee)$ with possibly non-trivial G-action. Note that any one-parameter subgroup of G is of the form $h = (h_1, \cdots, h_n)$, where h_i is a one-parameter subgroup of $\mathbb{GL}(V_i)$. Let $\mathcal{F}^{h_i}$ be the filtration of V_i associated to h_i relatively to the canonical representation of $\mathbb{GL}(V_i)$ on V_i. The action of $\mathbb{G}_{\mathrm{m},K}$ via h_i on $\Lambda^{r_i} V_i$ is nothing but the multiplication by $t^{r_i \mathbb{E}[\mathcal{F}^{h_i}]}$. Thus we obtain the following result.

Proposition 6.3. *With the notation above, for any rational point x of $\mathbb{P}(V^\vee)$, we have*

$$\mu(x, h, M) = \sum_{i=1}^{n} m_i \mathbb{E}[\mathcal{F}^{h_i}].$$

We now introduce Kempf's destabilizing flag for the action of a finite product of general linear groups. Consider a family $(V^{(i)})_{1\le i\le n}$ of finite dimensional non-zero vector space over K. Let W be the tensor product $V^{(1)}\otimes_K\cdots\otimes_K V^{(n)}$ and G be the algebraic group $\mathbb{GL}(V^{(1)})\times\cdots\times\mathbb{GL}(V^{(n)})$. For any $i\in\{1,\cdots,n\}$, let $r^{(i)}$ be the rank of $V^{(i)}$. The group G acts naturally on W and hence on $\mathbb{P}(W^\vee)$. We denote by $\pi:\mathbb{P}(W^\vee)\to\operatorname{Spec}K$ the canonical morphism. Let m be a strictly positive integer which is divisible by all $r^{(i)}$ and let L be a G-linearized line bundle on $\mathbb{P}(W^\vee)$ as follows:

$$L:=\mathcal{O}_{W^\vee}(m)\otimes\bigotimes_{i=1}^n\pi^*(\det V^{(i)})^{\otimes(m/r^{(i)})}. \tag{23}$$

For any rational point x of $\mathbb{P}(W^\vee)$, we define a function $\Lambda_x:\mathbf{Fil}^{\mathbb{Q}}_{V^{(1)}}\times\cdots\times\mathbf{Fil}^{\mathbb{Q}}_{V^{(n)}}\to\mathbb{R}$ such that

$$\Lambda_x(\mathcal{G}^{(1)},\cdots,\mathcal{G}^{(n)})=\frac{\mathbb{E}[\mathcal{G}^{(1)}]+\cdots+\mathbb{E}[\mathcal{G}^{(n)}]-\lambda_{\mathcal{G}^{(1)}\otimes\cdots\otimes\mathcal{G}^{(n)}}(v_x)}{(\|\mathcal{G}^{(1)}\|^2+\cdots+\|\mathcal{G}^{(n)}\|^2)^{\frac12}} \tag{24}$$

if at least one filtration among the $\mathcal{G}^{(i)}$'s is non-trivial, and $\Lambda_x(\mathcal{G}^{(1)},\cdots,\mathcal{G}^{(n)})=0$ otherwise. We recall that in (24), v_x is an arbitrary non-zero element in x. Note that the function Λ_x is invariant by dilation. In other words, for any positive number $\varepsilon>0$,

$$\Lambda_x(\psi_\varepsilon\mathcal{G}^{(1)},\cdots,\psi_\varepsilon\mathcal{G}^{(n)})=\Lambda_x(\mathcal{G}^{(1)},\cdots,\mathcal{G}^{(n)}),$$

where the dilation ψ_ε is defined in (18).

Proposition 6.4. *Let x be a rational point of $\mathbb{P}(W^\vee)$. Then the point x is* **not** *semistable for the action of G relatively to L if and only if the function Λ_x defined above takes at least one strictly negative value.*

Proof. By Propositions 6.2 and 6.3, for any rational point x of $\mathbb{P}(W^\vee)$,

$$\mu(x,h,L)=m\Big(\sum_{i=1}^n\mathbb{E}[\mathcal{F}^{h_i}]-\lambda_{\mathcal{F}^h}(v_x)\Big). \tag{25}$$

"$\Longrightarrow$": By the Hilbert–Mumford criterion (Theorem 6.1), there exists a one-parameter subgroup $h=(h_1,\cdots,h_n)$ of G such that $\mu(x,h,L)<0$. The filtration $\mathcal{F}^h$ of W associated to h coincides with the tensor product filtration $\mathcal{F}^{h_1}\otimes\cdots\otimes\mathcal{F}^{h_n}$, where $\mathcal{F}^{h_i}$ is the filtration of $V^{(i)}$ associated to h_i. Therefore,

$$\Lambda_x(\mathcal{F}^{h_1},\cdots,\mathcal{F}^{h_n})=\frac{\mu(x,h,L)}{m(\|\mathcal{F}^{h_1}\|^2+\cdots+\|\mathcal{F}^{h_n}\|^2)^{\frac12}}<0.$$

"$\Longleftarrow$": Suppose that $(\mathcal{G}^{(1)},\cdots,\mathcal{G}^{(n)})$ is an element in $\mathbf{Fil}^{\mathbb{Q}}_{V^{(1)}}\times\cdots\times\mathbf{Fil}^{\mathbb{Q}}_{V^{(n)}}$ such that $\Lambda_x(\mathcal{G}^{(1)},\cdots,\mathcal{G}^{(n)})<0$. By equalities (19), (20) and the invariance of Λ_x by dilation, we can assume that $\mathcal{G}^{(1)},\cdots,\mathcal{G}^{(n)}$ are all supported by $\mathbb{Z}$. In this case, there exists, for each $i\in\{1,\cdots,n\}$, a one-parameter subgroup

h_i of $\mathbb{GL}(V^{(i)})$ such that $\mathcal{F}^{h_i} = \mathcal{G}^{(i)}$. Let $h = (h_1, \cdots, h_n)$. By combining the negativity of $\Lambda_x(\mathcal{F}^{h_1}, \cdots, \mathcal{F}^{h_n})$ with (25), we obtain $\mu(x, h, L) < 0$, so x is not semistable. □

Proposition 6.6 below generalizes Proposition 2 of [27]. The proof uses Lemma 6.5, which is equivalent to Lemma 3 of [27], or Lemma 1.1 of [23]; see [23] for the proof of the lemma.

Lemma 6.5. *Let $n \geq 1$ be an integer and let $\mathscr{T}$ be a finite non-empty family of linear forms on $\mathbb{R}^n$. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ such that $\Lambda(y) = \|y\|^{-1} \max_{l \in \mathscr{T}} l(y)$ for $y \neq 0$, and that $\Lambda(0) = 0$. Suppose that the function Λ takes at least one strictly negative value. Then*

1. *the function Λ attains its minimal value, furthermore, all points in $\mathbb{R}^n$ minimizing Λ are proportional;*
2. *if c is the minimal value of Λ and if $y_0 \in \mathbb{R}^n$ is a minimizing point of Λ, then for any $y \in \mathbb{R}^n$,*

$$\Lambda(y) \geq c \frac{\langle y_0, y \rangle}{\|y_0\| \cdot \|y\|}; \tag{26}$$

3. *if, in addition, all linear forms in $\mathscr{T}$ are of rational coefficients, then there exists a point in $\mathbb{Q}^n$ which minimizes Λ.*

Proposition 6.6. *With the notation of Proposition 6.4, if x is not semistable for the action of G relatively to L, then the function Λ_x attains its minimal value. Furthermore, the element in $\mathbf{Fil}^{\mathbb{Q}}_{V^{(1)}} \times \cdots \times \mathbf{Fil}^{\mathbb{Q}}_{V^{(n)}}$ minimizing Λ_x is unique up to dilatation. Finally, if $(\mathcal{F}^{(1)}, \cdots, \mathcal{F}^{(n)})$ is an element in $\mathbf{Fil}^{\mathbb{Q}}_{V^{(1)}} \times \cdots \times \mathbf{Fil}^{\mathbb{Q}}_{V^{(n)}}$ minimizing Λ_x and if c is the minimal value of Λ_x, then for any element $(\mathcal{G}^{(1)}, \cdots, \mathcal{G}^{(n)})$ in $\mathbf{Fil}^{\mathbb{Q}}_{V^{(1)}} \times \cdots \times \mathbf{Fil}^{\mathbb{Q}}_{V^{(n)}}$, the following inequality holds:*

$$\sum_{i=1}^{n} \mathbb{E}[\mathcal{G}^{(i)}] - \lambda_{\mathcal{G}^{(1)} \otimes \cdots \otimes \mathcal{G}^{(n)}}(v_x) \geq c \frac{\langle \mathcal{F}^{(1)}, \mathcal{G}^{(1)} \rangle + \cdots + \langle \mathcal{F}^{(n)}, \mathcal{G}^{(n)} \rangle}{(\|\mathcal{F}^{(1)}\|^2 + \cdots + \|\mathcal{F}^{(n)}\|^2)^{\frac{1}{2}}}. \tag{27}$$

Proof. For each $i \in \{1, \cdots, n\}$, let $\mathbf{e}^{(i)} = (e_j^{(i)})_{1 \leq j \leq r^{(i)}}$ be a basis of $V^{(i)}$. Let $\mathbf{e} = (\mathbf{e}^{(i)})_{1 \leq i \leq n}$. Denote by $\Lambda_x^{\mathbf{e}}$ the restriction of Λ_x on $\mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(1)}} \times \cdots \times \mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(n)}}$. The space $\mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(1)}} \times \cdots \times \mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(n)}}$ is canonically embedded in $\mathbf{Fil}_{\mathbf{e}^{(1)}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}$, which can be identified as an Euclidian space with $\mathbb{R}^{r^{(1)}} \times \cdots \times \mathbb{R}^{r^{(n)}}$ through $\Phi_{\mathbf{e}^{(1)}} \times \cdots \times \Phi_{\mathbf{e}^{(n)}}$ (see Proposition 5.3). We extend naturally $\Lambda_x^{\mathbf{e}}$ to a function $\Lambda_x^{\mathbf{e},\dagger}$ on $\mathbf{Fil}_{\mathbf{e}^{(1)}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}$, whose numerator part is the maximal value of a finite number of linear forms with rational coefficients (see Proposition 5.2) and whose denominator part is just the norm of vector in the Euclidean space. Then by Lemma 6.5, the function $\Lambda_x^{\mathbf{e},\dagger}$ attains its minimal value, and there exists an element in $\mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(1)}} \times \cdots \times \mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(n)}}$ which minimizes

$\Lambda_x^{\mathbf{e},\dagger}$. By definition the same element also minimizes $\Lambda_x^{\mathbf{e}}$. Since the function $\Lambda_x^{\mathbf{e}}$, viewed as a function on $\mathbb{R}^{r^{(1)}+\cdots+r^{(n)}}$, only depends on the set

$$\Big\{S\subset\prod_{i=1}^n\{1,\cdots,r^{(i)}\}\;\Big|\;v_x\in\sum_{(j_1,\cdots,j_n)\in S}Ke_{j_1}^{(1)}\otimes\cdots\otimes e_{j_n}^{(n)}\Big\},$$

there are only a finite number of functions on Euclidian space of dimension $r^{(1)}+\cdots+r^{(n)}$ of the form $\Lambda_x^{\mathbf{e}}$. Thus we deduce that the function Λ_x attains globally its minimal value, and the minimizing element of Λ_x could be chosen in $\mathbf{Fil}^{\mathbb{Q}}_{V^{(1)}}\times\cdots\times\mathbf{Fil}^{\mathbb{Q}}_{V^{(n)}}$.

Suppose that there are two elements in $\mathbf{Fil}^{\mathbb{Q}}_{V^{(1)}}\times\cdots\times\mathbf{Fil}^{\mathbb{Q}}_{V^{(n)}}$ which minimizes Λ_x. By Bruhat's decomposition, we can choose $\mathbf{e}$ as above such that both elements lie in $\mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(1)}}\times\cdots\times\mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(n)}}$. Therefore, by Lemma 6.5 they differ only by a dilation. Finally to prove inequality (27), it suffices to choose $\mathbf{e}$ such that $(\mathcal{F}^{(1)},\cdots,\mathcal{F}^{(n)})$ and $(\mathcal{G}^{(1)},\cdots,\mathcal{G}^{(n)})$ are both in $\mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(1)}}\times\cdots\times\mathbf{Fil}^{\mathbb{Q}}_{\mathbf{e}^{(n)}}$, and then apply Lemma 6.5(2). □

Although the minimizing filtrations $(\mathcal{F}^{(1)},\cdots,\mathcal{F}^{(n)})$ in Proposition 6.6 are *a priori* supported by $\mathbb{Q}$, it is always possible to choose them to be supported by $\mathbb{Z}$ after a dilation.

In the rest of the section, let x be a rational point of $\mathbb{P}(W^\vee)$ which is **not** semistable for the action of G relatively to L. We fix an element $(\mathcal{F}^{(1)},\cdots,\mathcal{F}^{(n)})$ in $\mathbf{Fil}^{\mathbb{Z}}_{V^{(1)}}\times\cdots\times\mathbf{Fil}^{\mathbb{Z}}_{V^{(n)}}$ minimizing Λ_x. Let $c=\Lambda_x(\mathcal{F}^{(1)},\cdots,\mathcal{F}^{(n)})$ and define

$$\widetilde{c}:=\frac{c}{(\|\mathcal{F}^{(1)}\|^2+\cdots+\|\mathcal{F}^{(n)}\|^2)^{\frac12}}. \tag{28}$$

Note that $\widetilde{c}<0$. Moreover, it is a rational number since the following equality holds:

$$\widetilde{c}=\frac{\mathbb{E}[\mathcal{F}^{(1)}]+\cdots+\mathbb{E}[\mathcal{F}^{(n)}]-\lambda_{\mathcal{F}^{(1)}\otimes\cdots\otimes\mathcal{F}^{(n)}}(v_x)}{\|\mathcal{F}^{(1)}\|^2+\cdots+\|\mathcal{F}^{(n)}\|^2}.$$

We suppose that $\mathcal{F}^{(i)}$ corresponds to the flag $\mathscr{D}^{(i)}:V^{(i)}=V_0^{(i)}\supsetneq V_1^{(i)}\supsetneq\cdots\supsetneq V^{(i)}_{d^{(i)}}=0$ and the strictly increasing sequence of integers $\lambda^{(i)}=(\lambda_j^{(i)})_{0\le j<d^{(i)}}$. Let $\widetilde{G}$ be the algebraic group

$$\widetilde{G}:=\prod_{i=1}^n\prod_{j=0}^{d^{(i)}-1}\mathbb{GL}(V_j^{(i)}/V_{j+1}^{(i)}).$$

Let $\mathcal{F}=\mathcal{F}^{(1)}\otimes\cdots\otimes\mathcal{F}^{(n)}$ and $\beta=\lambda_{\mathcal{F}}(v_x)$, which is the largest integer i such that $v_x\in\mathcal{F}_iW$. Let $\widetilde{W}:=\mathcal{F}_iW/\mathcal{F}_{i+1}W$ and let $\widetilde{v}_x$ be the canonical image of

v_x in $\widetilde{W}$. Notice that

$$\widetilde{W} = \sum_{\lambda_{j_1}^{(1)}+\cdots+\lambda_{j_n}^{(n)}\geq\beta} \bigotimes_{i=1}^{n} V_{j_i}^{(i)} \Big/ \sum_{\lambda_{j_1}^{(1)}+\cdots+\lambda_{j_n}^{(n)}>\beta} \bigotimes_{i=1}^{n} V_{j_i}^{(i)}$$
$$\cong \bigoplus_{\lambda_{j_1}^{(1)}+\cdots+\lambda_{j_n}^{(n)}=\beta} \bigotimes_{i=1}^{n} \Big(V_{j_i}^{(i)}/V_{j_i+1}^{(i)}\Big).$$

So the algebraic group $\widetilde{G}$ acts naturally on $\widetilde{W}$. Let $\widetilde{x}$ be the rational point of $\mathbb{P}(\widetilde{W}^\vee)$ corresponding to the subspace of $\widetilde{W}$ generated by $\widetilde{v}_x$.

For all integers i,j such that $1 \leq i \leq n$ and $0 \leq j < d^{(i)}$, let $r_j^{(i)}$ be the rank of $V_j^{(i)}/V_{j+1}^{(i)}$ over K. We choose a strictly positive integer N divisible by all $r^{(i)} = \operatorname{rk}_K V^{(i)}$ and such that the number

$$a_j^{(i)} := -\frac{N\widetilde{c}\lambda_j^{(i)}}{r^{(i)}}$$

is an integer. This is always possible since $\widetilde{c} \in \mathbb{Q}$. The sequence $(\lambda_j^{(i)})_{0\leq j<d^{(i)}}$ is strictly increasing, so is $\mathbf{a}^{(i)} := (a_j^{(i)})_{0\leq j<d^{(i)}}$. Finally we define $b_j^{(i)} := \dfrac{N}{r^{(i)}} + a_j^{(i)}$.

We are now able to establish an explicit version of Proposition 1.12 in [23] for the product of general linear groups.

Proposition 6.7. *Let* $\widetilde{\pi} : \mathbb{P}(\widetilde{W}^\vee) \to \operatorname{Spec} K$ *be the canonical morphism and let*

$$\widetilde{L} := \mathcal{O}_{\widetilde{W}^\vee}(N) \otimes \Big(\bigotimes_{i=1}^{n}\bigotimes_{j=0}^{d^{(i)}-1} \widetilde{\pi}^*\big(\Lambda^{r_j^{(i)}}(V_j^{(i)}/V_{j+1}^{(i)})\big)^{\otimes b_j^{(i)}}\Big).$$

Then the rational point $\widetilde{x}$ *of* $\mathbb{P}(\widetilde{W}^\vee)$ *is semistable for the action of* $\widetilde{G}$ *relatively to the* G*-linearized line bundle* $\widetilde{L}$*.*

Proof. We choose an arbitrary filtration $\mathcal{G}^{(i),j}$ of $V_j^{(i)}/V_{j+1}^{(i)}$ supported by $\mathbb{Z}$. We have explained in Subsection 5.5 how to construct a new filtration $\mathcal{G}^{(i)}$ of $V^{(i)}$ from $\mathcal{G}^{(i),j}$. Let

$$\mathcal{G} = \bigotimes_{i=1}^{n} \mathcal{G}^{(i)}, \qquad \widetilde{\mathcal{G}} = \bigoplus_{\lambda_{j_1}^{(1)}+\cdots+\lambda_{j_n}^{(n)}=\beta} \bigotimes_{i=1}^{n} \mathcal{G}^{(i),j_i}.$$

From the construction we know that $\lambda_{\mathcal{G}}(v_x) = \lambda_{\widetilde{\mathcal{G}}}(\widetilde{v}_x)$. Using (22), the inequality (27) implies

$$\sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1} \frac{r_j^{(i)}}{r^{(i)}}\mathbb{E}[\mathcal{G}^{(i),j}] - \sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1} \frac{\widetilde{c}\lambda_j^{(i)} r_j^{(i)}}{r^{(i)}}\mathbb{E}[\mathcal{G}^{(i),j}] - \lambda_{\widetilde{\mathcal{G}}}(\widetilde{v}_x) \geq 0, \tag{29}$$

where the constant $\widetilde{c}$ is defined in (28). Hence

$$\sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1} b_j^{(i)} r_j^{(i)} \mathbb{E}[\mathcal{G}^{(i),j}] - N\lambda_{\widetilde{\mathcal{G}}}(\widetilde{v}_x) \geq 0. \tag{30}$$

Let h be an arbitrary one-parameter subgroup of $\widetilde{G}$ corresponding to filtrations $\mathcal{G}^{(i),j}$. By Propositions 6.2 and 6.3, together with the fact that $\mu(\widetilde{x}, h, \cdot)$ is a homomorphism of groups, we obtain

$$\begin{aligned}\mu(\widetilde{x}, h, \widetilde{L}) &= \mu(\widetilde{x}, h, \mathcal{O}_{\widetilde{W}^\vee}(N)) + \sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1} b_j^{(i)} r_j^{(i)} \mathbb{E}[\mathcal{G}^{(i),j}] \\ &= -N\lambda_{\widetilde{\mathcal{G}}}(\widetilde{v}_x) + \sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1} b_j^{(i)} r_j^{(i)} \mathbb{E}[\mathcal{G}^{(i),j}] \geq 0.\end{aligned}$$

By Hilbert-Mumford criterion, the point $\widetilde{x}$ is semistable for the action of $\widetilde{G}$ relatively to $\widetilde{L}$. □

Finally we point out the following consequence of the inequality (30).

Proposition 6.8. *The minimizing filtrations* $(\mathcal{F}^{(1)}, \cdots, \mathcal{F}^{(n)})$ *satisfy*

$$\mathbb{E}[\mathcal{F}^{(1)}] = \cdots = \mathbb{E}[\mathcal{F}^{(n)}] = 0.$$

In other words, the equality $\sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} = 0$ *holds, or equivalently,*

$$\sum_{j=0}^{d^{(i)}-1} \lambda_j^{(i)} r_j^{(i)} = 0 \textit{ for any } i \in \{1, \cdots, n\}.$$

Proof. Let $(u_i)_{1\leq i\leq n}$ be an arbitrary sequence of integers. Let $\mathcal{G}^{(i),j}$ be the filtration of $V_j^{(i)}/V_j^{(i+1)}$ which is supported by $\{u_i\}$. Note that in this case $\widetilde{\mathcal{G}}$ is supported by $\{u_1 + \cdots + u_n\}$. The inequality (30) gives

$$\sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1} b_j^{(i)} r_j^{(i)} u_i - N\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i \sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} \geq 0.$$

Since $(u_i)_{1\leq i\leq n}$ is arbitrary, $\sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} = 0$ and $\sum_{j=0}^{d^{(i)}-1} \lambda_j^{(i)} r_j^{(i)} = 0$. □

7. A criterion of Arakelov semistability

We shall give a semistability criterion for Hermitian vector bundles, which is the arithmetic analogue of a result due to Bogomolov in geometric framework (see [24]).

Let $\overline{E}$ be a non-zero Hermitian vector bundle over $\operatorname{Spec}\mathcal{O}_K$ and let $V = E_K$. We denote by r its rank. If $\mathscr{D} : V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_d = 0$ is a flag of V, it induces a strictly decreasing sequence of saturated sub-$\mathcal{O}_K$-modules $E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d = 0$ of E. For any $j \in \{0, \cdots, d-1\}$, let r_j be the rank of E_j/E_{j+1}. If $\mathbf{a} = (a_j)_{0\le j<d}$ is an element in $r\mathbb{Z}^d$, we denote by $\overline{\mathscr{L}}_{\mathscr{D}}^{\mathbf{a}}$ the Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$ as follows

$$\overline{\mathscr{L}}_{\mathscr{D}}^{\mathbf{a}} := \bigotimes_{j=0}^{d-1}\Big((\Lambda^{r_j}(\overline{E}_j/\overline{E}_{j+1}))^{\otimes a_j} \otimes (\Lambda^r\overline{E})^{\vee\otimes\frac{r_j a_j}{r}}\Big). \tag{31}$$

If $\mathbf{a} = (a_j)_{0\le j<d} \in \mathbb{Z}^d$ satisfies $\sum_{j=0}^{d-1} r_j a_j = 0$, we define

$$\overline{\mathscr{L}}_{\mathscr{D}}^{\mathbf{a}} := \bigotimes_{j=0}^{d-1}(\Lambda^{r_j}(\overline{E}_j/\overline{E}_{j+1}))^{\otimes a_j}.$$

Proposition 7.1. *If the Hermitian vector bundle $\overline{E}$ is Arakelov semistable, then for any integer $d \ge 1$, any flag $\mathscr{D}$ of length d of V, and any strictly increasing sequence $\mathbf{a} = (a_j)_{0\le j<d}$ of integers either in $r\mathbb{Z}^d$, or such that $\sum_{j=0}^{d-1} r_j a_j = 0$, we have $\widehat{\deg}(\overline{\mathscr{L}}_{\mathscr{D}}^{\mathbf{a}}) \le 0$.*

Proof. By definition,

$$\begin{aligned}\widehat{\deg}(\overline{\mathscr{L}}_{\mathscr{D}}^{\mathbf{a}}) &= \sum_{j=0}^{d-1} a_j\left[-\frac{\operatorname{rk}(E_j)-\operatorname{rk}(E_{j+1})}{r}\widehat{\deg}(\overline{E}) + \widehat{\deg}(\overline{E}_j) - \widehat{\deg}(\overline{E}_{j+1})\right]\\ &= \sum_{j=0}^{d-1} a_j\left[\operatorname{rk}(E_j)\Big(\widehat{\mu}(\overline{E}_j)-\widehat{\mu}(\overline{E})\Big) - \operatorname{rk}(E_{j+1})\Big(\widehat{\mu}(\overline{E}_{j+1})-\widehat{\mu}(\overline{E})\Big)\right]\\ &= \sum_{j=1}^{d-1}(a_j - a_{j-1})\operatorname{rk}(E_j)\big(\widehat{\mu}(\overline{E}_j)-\widehat{\mu}(\overline{E})\big).\end{aligned}$$

If $\overline{E}$ is Arakelov semistable, then $\widehat{\mu}(\overline{E}_j) \le \widehat{\mu}(\overline{E})$ for any j. Hence $\widehat{\deg}(\overline{\mathscr{L}}_{\mathscr{D}}^{\mathbf{a}}))$ ≤ 0. □

Remark 7.1. The converse of Proposition 7.1 is also true. Let E_1 be a saturated sub-$\mathcal{O}_K$-module of E. Consider the flag $\mathscr{D} : V \supsetneq E_{1,K} \supsetneq 0$ and the integer sequence $\mathbf{a} = (0, r)$. Then $\widehat{\deg}(\overline{\mathscr{L}}_{\mathscr{D}}^{\mathbf{a}}) = r\operatorname{rk}(E_1)\big(\widehat{\mu}(\overline{E}) - \widehat{\mu}(\overline{E}_1)\big)$. Therefore $\widehat{\mu}(\overline{E}_1) \le \widehat{\mu}(\overline{E})$. Since E_1 is arbitrary, the Hermitian vector bundle $\overline{E}$ is Arakelov semistable.

8. Degree of a general line subbundle

In this section, we shall give an upper bound for the Arakelov degree of a Hermitian line subbundle of a finite tensor product of Hermitian vector bundles. As explained in Section 1, we shall use the results established in Section 6 to reduce our problem to the case with semistability condition (in geometric invariant theory sense), which has already been discussed in Section 4. We point out that, in order to obtain the same estimation as (13) in full generality, we should assume that all Hermitian vector bundles $\overline{E}_i$ are Arakelov semistable, as a price paid for removing the semistability condition for M_K.

We denote by K a number field and by $\mathcal{O}_K$ its integer ring. Let $(\overline{E}^{(i)})_{1\le i\le n}$ be a family of **Arakelov semistable** Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$. For any $i\in\{1,\cdots,n\}$, let $r^{(i)}$ be the rank of $E^{(i)}$ and $V^{(i)}=E^{(i)}_K$. Let $\overline{E}=\overline{E}^{(1)}\otimes\cdots\otimes\overline{E}^{(n)}$ and $W=E_K$. We denote by $\pi:\mathbb{P}(W^\vee)\rightarrow\operatorname{Spec}K$ the natural morphism. The algebraic group $G:=\mathbb{GL}_K(V^{(1)})\times_K\cdots\times_K\mathbb{GL}_K(V^{(n)})$ acts naturally on $\mathbb{P}(W^\vee)$. Let $\overline{M}$ be a Hermitian line subbundle of $\overline{E}$ and m be a strictly positive integer which is divisible by all $r^{(i)}$'s.

Proposition 8.1. *For any Hermitian line subbundle $\overline{M}$ of $\overline{E}^{(1)}\otimes\cdots\otimes\overline{E}^{(n)}$, we have*

$$\widehat{\deg}(\overline{M})\le\sum_{i=1}^n\Big(\widehat{\mu}(\overline{E}^{(i)})+\frac{1}{2}\log(\operatorname{rk}E^{(i)})\Big).$$

Proof. We have proved in Corollary 4.3 that if M_K is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m)\otimes\pi^*\Big(\bigotimes_{i=1}^n(\Lambda^{r^{(i)}}V^{(i)})^{\otimes m/r^{(i)}}\Big)$, where m is a strictly positive integer which is divisible by all $r^{(i)}$, then the following inequality holds:

$$\widehat{\deg}(\overline{M})\le\sum_{i=1}^n\Big(\widehat{\mu}(\overline{E}_i)+\frac{1}{2}\log r^{(i)}\Big).$$

If this hypothesis of semistability is not fulfilled, by Proposition 6.7, there exist two strictly positive integers N and β, and for any $i\in\{1,\cdots,n\}$,

(1) a flag $\mathscr{D}^{(i)}:V^{(i)}=V^{(i)}_0\supsetneq V^{(i)}_1\supsetneq\cdots\supsetneq V^{(i)}_{d^{(i)}}=0$ of $V^{(i)}$ corresponding to the sequence $E^{(i)}=E^{(i)}_0\supsetneq E^{(i)}_1\supsetneq\cdots\supsetneq E^{(i)}_{d^{(i)}}=0$ of saturated sub-$\mathcal{O}_K$-modules of E,

(2) two strictly increasing sequences $\lambda^{(i)}=(\lambda^{(i)}_j)_{0\le j<d^{(i)}}$ and $\mathbf{a}^{(i)}=(a^{(i)}_j)_{0\le j<d^{(i)}}$ of integers,

such that

(i) N is divisible by all $r^{(i)}$'s,

(ii) $\sum_{j=0}^{d^{(i)}-1}a^{(i)}_jr^{(i)}_j=0$, where $r^{(i)}_j=\operatorname{rk}(V^{(i)}_j/V^{(i)}_{j+1})$,

(iii) the inclusion of M in E factorizes through

$$\sum_{\lambda_{i_1}^{(1)}+\cdots\lambda_{i_n}^{(n)}\geq\beta} E_{i_1}^{(1)}\otimes\cdots\otimes E_{i_n}^{(n)},$$

(iv) the canonical image of M_K in

$$\begin{aligned}\widetilde{W} &:= \sum_{\lambda_{j_1}^{(1)}+\cdots+\lambda_{j_n}^{(n)}\geq\beta}\bigotimes_{i=1}^{n} V_{j_i}^{(i)}\Bigg/\sum_{\lambda_{j_1}^{(1)}+\cdots+\lambda_{j_n}^{(n)}>\beta}\bigotimes_{i=1}^{n} V_{j_i}^{(i)}\\ &\cong \bigoplus_{\lambda_{j_1}^{(1)}+\cdots+\lambda_{j_n}^{(n)}=\beta}\bigotimes_{i=1}^{n}\Big(V_{j_i}^{(i)}/V_{j_i+1}^{(i)}\Big)\end{aligned}$$

is non-zero, and is semistable for the action of the group

$$\widetilde{G} := \prod_{i=1}^{n}\prod_{j=0}^{d^{(i)}-1}\mathbb{GL}(V_j^{(i)}/V_{j+1}^{(i)})$$

relatively to

$$\mathcal{O}_{\widetilde{W}^\vee}(N)\otimes\Big(\bigotimes_{i=1}^{n}\bigotimes_{j=0}^{d^{(i)}-1}\widetilde{\pi}^*\big(\Lambda^{r_j^{(i)}}(V_j^{(i)}/V_{j+1}^{(i)})\big)^{\otimes b_j^{(i)}}\Big),$$

where $\widetilde{\pi} : \mathbb{P}(\widetilde{W}^\vee)\to\operatorname{Spec}K$ is the canonical morphism, and $b_j^{(i)} = N/r^{(i)} + a_j^{(i)}$.

Note that $\bigotimes_{j=0}^{d^{(i)}-1}\big(\Lambda^{r_j^{(i)}}(\overline{E}_j^{(i)}/\overline{E}_{j+1}^{(i)})\big)^{\otimes a_j^{(i)}}$ is nothing other than $\overline{\mathscr{L}}_{\mathscr{D}^{(i)}}^{\mathbf{a}^{(i)}}$ defined in Section 7.

Applying Theorem 4.2, we get

$$\begin{aligned}\widehat{\deg}(\overline{M}) &\leq \frac{1}{N}\sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1}\frac{N}{r^{(i)}}\big(\widehat{\deg}(\overline{E}_j^{(i)})-\widehat{\deg}(\overline{E}_{j+1}^{(i)})\big)\\ &\qquad+\frac{1}{N}\sum_{i=1}^{n}\widehat{\deg}\overline{\mathscr{L}}_{\mathscr{D}^{(i)}}^{\mathbf{a}^{(i)}}+\sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1}\frac{r_j^{(i)}b_j^{(i)}}{2N}\log r_j^{(i)}\\ &=\sum_{i=1}^{n}\widehat{\mu}(\overline{E}^{(i)})+\frac{1}{N}\sum_{i=1}^{n}\widehat{\deg}\overline{\mathscr{L}}_{\mathscr{D}^{(i)}}^{\mathbf{a}^{(i)}}+\sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1}\frac{r_j^{(i)}b_j^{(i)}}{2N}\log r_j^{(i)}\\ &\leq\sum_{i=1}^{n}\widehat{\mu}(\overline{E}^{(i)})+\sum_{i=1}^{n}\sum_{j=0}^{d^{(i)}-1}\frac{r_j^{(i)}b_j^{(i)}}{2N}\log r_j^{(i)},\end{aligned}$$

where the last inequality is because $\overline{E}^{(i)}$'s are Arakelov semistable (see Proposition 7.1). By Theorem 3.3, the semistability of the canonical image of M_K

implies that $b_j^{(i)} \geq 0$. Therefore

$$\widehat{\deg}(\overline{M}) \leq \sum_{i=1}^{n} \widehat{\mu}(\overline{E}^{(i)}) + \sum_{i=1}^{n} \sum_{j=0}^{d^{(i)}-1} \frac{r_j^{(i)} b_j^{(i)}}{2N} \log r^{(i)}.$$

Since $\sum_{j=0}^{d^{(i)}-1} r_j^{(i)} a_j^{(i)} = 0$ (see Proposition 6.8), we have proved the proposition. □

Corollary 8.2. *The following inequality is verified:*

$$\widehat{\mu}_{\max}(\overline{E}^{(1)} \otimes \cdots \otimes \overline{E}^{(n)}) \leq \sum_{i=1}^{n} \Big(\widehat{\mu}(\overline{E}^{(i)}) + \log(\operatorname{rk} E^{(i)}) \Big) + \frac{\log |\Delta_K|}{2[K:\mathbb{Q}]}. \tag{32}$$

Proof. Since the Hermitian line bundle $\overline{M}$ in Proposition 8.1 is arbitrary, we obtain

$$\widehat{\operatorname{udeg}}_n(\overline{E}^{(1)} \otimes \cdots \otimes \overline{E}^{(n)}) \leq \sum_{i=1}^{n} \Big(\widehat{\mu}(\overline{E}^{(i)}) + \frac{1}{2} \log(\operatorname{rk} E^{(i)}) \Big).$$

Combining with (11) we obtain (32). □

9. Proof of Theorem 1.1

We finally give the proof of Theorem 1.1.

Lemma 9.1. *Let K be a number field and $\mathcal{O}_K$ be its integer ring. Let $(\overline{E}_i)_{1 \leq i \leq n}$ be a finite family of non-zero Hermitian vector bundles (non-necessarily Arakelov semistable) and $\overline{E} = \overline{E}_1 \otimes \cdots \otimes \overline{E}_n$. Then*

$$\widehat{\mu}_{\max}(\overline{E}) \leq \sum_{i=1}^{n} \Big(\widehat{\mu}_{\max}(E_i) + \log(\operatorname{rk} E_i) \Big) + \frac{\log |\Delta_K|}{2[K:\mathbb{Q}]}.$$

Proof. Let F be a sub-$\mathcal{O}_K$-module of E. By taking Harder–Narasimhan flags of E_i's (cf. [3]), there exists, for any $i \in \{1, \cdots, n\}$, an Arakelov semistable subquotient $\overline{F}_i/\overline{G}_i$ of E_i such that

(1) $\widehat{\mu}(\overline{F}_i/\overline{G}_i) \leq \widehat{\mu}_{\max}(\overline{E}_i)$,
(2) the inclusion homomorphism from F to E factorizes through $F_1 \otimes \cdots \otimes F_n$,
(3) the canonical image of F in $(F_1/G_1) \otimes \cdots \otimes (F_n/G_n)$ does not vanish.

Combining with the slope inequality (10), Corollary 8.2 implies that

$$\begin{aligned}\widehat{\mu}_{\min}(\overline{F}) &\leq \sum_{i=1}^{n}\Big(\widehat{\mu}(\overline{F}_i/\overline{G}_i)+\log(\mathrm{rk}(F_i/G_i))\Big)+\frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}\\ &\leq \sum_{i=1}^{n}\Big(\widehat{\mu}_{\max}(\overline{E}_i)+\log(\mathrm{rk}\,E_i)\Big)+\frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}.\end{aligned}$$

Since F is arbitrary, the lemma is proved. □

Proof of Theorem 1.1. Let $N \geq 1$ be an arbitrary integer. On one hand, by Lemma 9.1 we have, by considering $\overline{E}^{\otimes N}$ as $\underbrace{\overline{E}_1\otimes\cdots\otimes\overline{E}_1}_{N \text{ copies}}\otimes\cdots\otimes\underbrace{\overline{E}_n\otimes\cdots\otimes\overline{E}_n}_{N \text{ copies}}$, that

$$\widehat{\mu}_{\max}(\overline{E}^{\otimes N}) \leq \sum_{i=1}^{n} N\Big(\widehat{\mu}_{\max}(\overline{E}_i)+\log(\mathrm{rk}\,E_i)\Big)+\frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}.$$

On the other hand, by Corollary 2.5, $\widehat{\mu}_{\max}(\overline{E}^{\otimes N}) \geq N\widehat{\mu}_{\max}(\overline{E})$. Hence

$$\widehat{\mu}_{\max}(\overline{E}) \leq \sum_{i=1}^{n}\Big(\widehat{\mu}_{\max}(\overline{E}_i)+\log(\mathrm{rk}\,E_i)\Big)+\frac{\log|\Delta_K|}{2N[K:\mathbb{Q}]}.$$

Since N is arbitrary, we obtain by taking $N \to +\infty$,

$$\widehat{\mu}_{\max}(\overline{E}) \leq \sum_{i=1}^{n}\Big(\widehat{\mu}_{\max}(\overline{E}_i)+\log(\mathrm{rk}\,E_i)\Big),$$

which completes the proof. □

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