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Arakelov Theory on Arithmetic Surfaces Over a Trivially Valued Field

Huayi Chen[1] and Atsushi Moriwaki[2,*]

[1]Université de Paris, Sorbonne Université, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75013 Paris, France and [2]Department of Mathematics, Faculty of Science, Kyoto University, Kyoto, 606-8502, Japan

**Correspondence to be sent to: e-mail: moriwaki@math.kyoto-u.ac.jp*

In this article, we consider an analogue of Arakelov theory of arithmetic surfaces over a trivially valued field. In particular, we establish an arithmetic Hilbert–Samuel theorem and study the effectivity up to $\mathbb{R}$-linear equivalence of pseudoeffective metrised $\mathbb{R}$-divisors.

1 Introduction

In Arakelov geometry, one considers an algebraic variety over the spectrum of a number field and studies various constructions and invariants on the variety such as metrised line bundles, intersection product, height functions, etc. Although these notions have some similarities to those in classic algebraic geometry, their construction is often more sophisticated and needs analytic tools.

Recently, an approach of $\mathbb{R}$-filtration has been proposed to study several invariants in Arakelov geometry, which allows one to get around analytic techniques in the study of some arithmetic invariants, see for example [6, 12, 13]. Let us recall briefly this approach in the setting of Euclidean lattices for simplicity. Let $\overline{E} = (E, \|\cdot\|)$ be a Euclidean lattice, namely a free $\mathbb{Z}$-module of finite type E equipped with a Euclidean norm $\|\cdot\|$ on $E_{\mathbb{R}} = E \otimes_{\mathbb{Z}} \mathbb{R}$. We construct a family of vector subspaces of $E_{\mathbb{Q}} = E \otimes_{\mathbb{Z}} \mathbb{Q}$ as

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follows. For any $t \in \mathbb{R}$, let $\mathcal{F}^t(\overline{E})$ be the $\mathbb{Q}$-vector subspace of $E_{\mathbb{Q}}$ generated by the lattice vectors s such that $\|s\| \leqslant \mathrm{e}^{-t}$. This construction is closely related to the successive minima of Minkowski. In fact, the i-th minimum of the lattice $\overline{E}$ is equal to

$$\exp\Big(-\sup\{t \in \mathbb{R} \mid \mathrm{rk}_{\mathbb{Q}}(\mathcal{F}^t(\overline{E})) \geqslant i\}\Big).$$

The family $(\mathcal{F}^t(\overline{E}))_{t\in\mathbb{R}}$ is therefore called the $\mathbb{R}$-*filtration by minima* of the Euclidean lattice $\overline{E}$.

Classically in Diophantine geometry, one focuses on the lattice points of small length, which are analogous to global sections of a vector bundle over a smooth projective curve. However, such points are in general not stable by addition. This phenomenon brings difficulties to the study of Arakelov geometry over a number field and prevents the direct transplantation of algebraic geometry methods in the arithmetic setting. In the $\mathbb{R}$-filtration approach, the arithmetic invariants are encoded in a family of vector spaces, which allows to apply directly algebraic geometry methods to study some problems in Arakelov geometry.

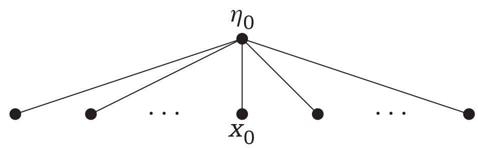
If we equipped $\mathbb{Q}$ with the trivial absolute value $|\cdot|_0$ such that $|a|_0 = 1$ if a belongs to $\mathbb{Q}\setminus\{0\}$ and $|0|_0 = 0$, then the above $\mathbb{R}$-filtration by minima can be considered as an ultrametric norm $\|\cdot\|_0$ on the $\mathbb{Q}$-vector space $E_{\mathbb{Q}}$ such that

$$\|s\|_0 = \exp(-\sup\{t \in \mathbb{R} \mid s \in \mathcal{F}^t(\overline{E})\}).$$

Interestingly, finite-dimensional ultrametrically normed vector spaces over a trivially valued field are also similar to vector bundles over a smooth projective curve. This method is especially successful in the study of the arithmetic volume function. Moreover, $\mathbb{R}$-filtrations, or equivalently ultrametric norms with respect to the trivial absolute value, are also closely related to the geometric invariant theory of the special linear group, as shown in [11, §6].

All these works suggest that there would be an Arakelov theory over a trivially valued field. From the philosophical point of view, the $\mathbb{R}$-filtration approach should be considered as a correspondence from the arithmetic geometry over a number field to that over a trivially valued field, which preserves some interesting arithmetic invariants. The purpose of this article is to build up such a theory for curves over a trivially valued field (which are actually analogous to arithmetic surfaces). Considering the simplicity of the trivial absolute value, one might expect such a theory to be simple. On the contrary, the arithmetic intersection theory for adelic divisors in this setting is already highly non-trivial, which has interesting interactions with the convex analysis on infinite trees.

Let k be a field equipped with the trivial absolute value and X be a regular irreducible projective curve over $\operatorname{Spec} k$. We denote by X^{an} the Berkovich analytic space associated with X, which identifies with a tree of length 1 whose leaves correspond to closed points of X (see [2, §3.5]).



In fact, the elements of X^{an} are of the form $\xi = (j(\xi), |\cdot|_\xi)$, where $j(\xi)$ is a scheme point of X and $|\cdot|_\xi$ is an absolute value on the residue field $\kappa(\xi)$ of the scheme point $j(\xi)$, which extends the absolute value $|\cdot|$ on k. Any regular function f over a Zariski open subset U of X determines a function $|f| : j^{-1}(U) \to \mathbb{R}$, which sends $\xi \in j^{-1}(U)$ to $|f(j(\xi))|_\xi$. The set X^{an} is then equipped with the most coarse topology, which makes continuous the map $j : X^{\mathrm{an}} \to X$ and function $|f|$, where f runs over the set of all regular functions on Zariski open subsets of X. We denote by η_0 the generic point of X together with the trivial absolute value on the function field of X. For each closed point x of X, we denote by x_0 the scheme point x together with the trivial absolute value on the residue field of x. Note that the closed point x also determines a discrete valuation $\operatorname{ord}_x(\cdot)$ on the function field of X. Thus, each positive real number t corresponds to an element of X^{an}, which consists of the generic point of X and the absolute value $\exp(-t \operatorname{ord}_x(\cdot))$. Such elements form an edge of the above infinite tree linking the points η_0 (which is the root) and x_0 (which is a leaf). We denote by $[\eta_0, x_0]$ the corresponding closed edge, which is parametrised by the interval $[0, +\infty]$ and we denote by $t(\cdot) : [\eta_0, x_0] \to [0, +\infty]$ the parametrisation map. Recall that an $\mathbb{R}$-divisor D on X can be viewed as an element in the free real vector space over the set $X^{(1)}$ of all closed points of X. We denote by $\operatorname{ord}_x(D)$ the coefficient of $x \in X^{(1)}$ in the writing of D into a linear combination of elements of $X^{(1)}$. We call Green function of D any continuous map $g : X^{\mathrm{an}} \to [-\infty, +\infty]$ such that there exists a continuous function $\varphi_g : X^{\mathrm{an}} \to \mathbb{R}$, which satisfies the following condition:

$$\forall\, x \in X^{(1)},\ \forall\, \xi \in [\eta_0, x_0[, \quad \varphi_g(\xi) = g(\xi) - \operatorname{ord}_x(D) t(\xi).$$

The couple $\overline{D} = (D, g)$ is called a metrised $\mathbb{R}$-divisor on X. Note that the set $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ of metrised $\mathbb{R}$-divisors on X actually forms a vector space over $\mathbb{R}$.

Let D be an $\mathbb{R}$-divisor on X. We denote by $H^0(D)$ the subset of the field $\operatorname{Rat}(X)$ of rational functions on X consisting of the zero rational function and all rational functions s such that $D + (s)$ is effective as an $\mathbb{R}$-divisor, where (s) denotes the principal divisor associated with s, whose coefficient of x is the order of s at x. The set $H^0(D)$ is actually a k-vector subspace of $\mathrm{Rat}(X)$. Moreover, the Green function g determines an ultrametric norm $\|\cdot\|_g$ on the vector space $H^0(D)$ such that

$$\|s\|_g = \exp\Big(-\inf_{\xi\in X^{\mathrm{an}}}(g+g_{(s)})(\xi)\Big),$$

where $g_{(s)}$ is the canonical Green function associated with the divisor (s) (see Definition 3.5 and Remark 5.2, (2)).

Let $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ be metrised $\mathbb{R}$-divisors on X such that φ_{g_1} and φ_{g_2} are absolutely continuous with square integrable densities, we define an intersection pairing of $\overline{D}_1$ and $\overline{D}_2$ as (see Section 3.3 for details)

$$\begin{aligned}(\overline{D}_1\cdot\overline{D}_2) := {} & g_1(\eta_0)\deg(D_1) + g_2(\eta_0)\deg(D_1)\\ & - \sum_{x\in X^{(1)}}[\kappa(x):k]\int_{\eta_0}^{x_0} g_1'(\xi)g_2'(\xi)\,\mathrm{d}t(\xi).\end{aligned} \tag{1.1}$$

One major contribution of the article is to describe the asymptotic behaviour of the system of ultrametrically normed vector spaces $(H^0(nD), \|\cdot\|_{ng})$ in terms of the intersection pairing, under the condition that the Green function g is plurisubharmonic (see Definition 6.14). More precisely, we obtain an analogue of the arithmetic Hilbert–Samuel theorem as follows (see Section 7 *infra*).

Theorem 1.1. Let $\overline{D} = (D, g)$ be a metrised $\mathbb{R}$-divisor on X. We assume that $\deg(D) > 0$ and g is plurisubharmonic. Then one has

$$\lim_{n\to+\infty}\frac{-\ln\|s_1\wedge\cdots\wedge s_{r_n}\|_{ng,\det}}{n^2/2} = (\overline{D}\cdot\overline{D}),$$

where $(s_i)_{i=1}^{r_n}$ is a basis of $H^0(nD)$ over k (with r_n being the dimension of the k-vector space $H^0(nD)$), $\|\cdot\|_{ng,\det}$ denotes the determinant norm associated with $\|\cdot\|_{ng}$, and $(\overline{D}\cdot\overline{D})$ is the self-intersection number of $\overline{D}$.

Note that the pairing (1.1) is similar to the local admissible pairing introduced in [31, §2] or, more closely, similar to the Arakelov intersection theory on arithmetic surfaces with L_1^2-Green functions (see [4, §5]). This construction is also naturally related to harmonic analysis on metrised graphes introduced in [1] (see also [17] for the capacity pairing in this setting), although the point η_0 is linked to infinitely many vertices. An alternative way to construct the intersection pairing (under diverse extra conditions on Green functions) is to introduce a base change to a field extension k' of k, which is equipped with a non-trivial absolute value extending the trivial absolute value on k. It is then possible to define a Monge–Ampère measure on $X_{k'}^{\mathrm{an}}$ for the pull-back of g_1, either by the theory of δ-forms [19, 20], or by the non-Archimedean Bedford–Taylor theory developed in [9], or more directly, by the method of Chambert–Loir measure [8, 10]. We also refer to the thesis of Thuillier [28] for a detailed construction in the case of curves. It turns out that the push-forward of this measure on X^{an} does not depend on the choice of valued extension k'/k (see [5, Lemma 7.2]). With this point of view, the intersection pairing could be considered as the height of D_2 with respect to (D_1, g_1) plus the integral of g_2 with respect to the push-forward of this Monge–Ampère measure (see Remark 7.2). We refer the readers to [15, §3.9] for more details. However, although it is expected that the two constructions lead to the same intersection number, only several particular cases are known and it remains an intriguing problem to explicitly compare the two approaches and to reprove certain results of the current article by using the base change to a non-trivially valued field. Our construction (1.1) has, however, several advantages. Despite of its flexibility on the choice of Green functions, the formula (1.1) is completely explicit. Moreover, compare to the classic formulation of Monge–Ampère measure, the L_1^2 inner product, combined with Legendre-type transform (see Proposition 2.6), reveals the deep connection between convex analysis and the geometry of graded linear series, which is a key argument to prove Theorem 1.1.

Diverse notions of positivity, such as bigness and pseudo-effectivity, are discussed in the article. We also study the effectivity up to $\mathbb{R}$-linear equivalence of pseudo-effective metrised $\mathbb{R}$-divisors. The analogue of this problem in algebraic geometry is very deep. It is the core of the non-vanishing conjecture, which has applications in the existence of log minimal models [3]. It is also related to Keel's conjecture (see [22, Question 0.9] and [25, Question 0.3]) for the ampleness of divisors on a projective surface over a finite field. In the setting of an arithmetic curve associated with a number field, this problem can actually be interpreted as Dirichlet's unit theorem in algebraic number theory. In the setting of higher dimensional arithmetic varieties, the above effectivity problem is very subtle. Both examples and obstructions were studied in the literature, see for example [16, 24] for more details.

In this article, we establish the following result.

Theorem 1.2. Let (D, g) be a metrised $\mathbb{R}$-divisor on X. For any $x \in X^{(1)}$, we let

$$\mu_{\inf,x}(g) := \inf_{\xi \in]\eta_0, x_0[} \frac{g(\xi)}{t(\xi)}.$$

Let

$$\mu_{\inf}(g) := \sum_{x \in X^{(1)}} \mu_{\inf,x}(g)[\kappa(x) : k].$$

Then the following assertions hold:

(1) (D, g) is pseudo-effective if and only if $\mu_{\inf}(g) \geqslant 0$.

(2) (D, g) is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor if and only if $\mu_{\inf,x}(g) \geqslant 0$ for all but finitely many $x \in X^{(1)}$ and if one of the following conditions holds:

  (a) $\mu_{\inf}(g) > 0$, and

  (b) $\sum_{x \in X^{(1)}} \mu_{\inf,x}(g)x$ is a principal $\mathbb{R}$-divisor.

The article is organised as follows. In the 2nd section, we discuss several properties of convex functions on a half line. In the 3rd section, we study Green functions on an infinite tree. The 4th section is devoted to a presentation of graded linear series on a regular projective curve. These sections prepare various tools to develop in the 5th section an Arakelov theory of metrised $\mathbb{R}$-divisors on a regular projective curve over a trivially valued field. In the 6th section, we discuss diverse notions of global and local positivity of metrised $\mathbb{R}$-divisors. Finally, in the 7th section, we prove the Hilbert–Samuel theorem for arithmetic surfaces in the setting of Arakelov geometry over a trivially valued field.

2 Asymptotically Linear Functions

2.1 Asymptotic linear functions on $\mathbb{R}_{>0}$

We say that a continuous function $g : \mathbb{R}_{>0} \to \mathbb{R}$ is *asymptotically linear* (at the infinity) if there exists a real number $\mu(g)$ such that the function

$$\varphi_g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}, \quad \varphi_g(t) := g(t) - \mu(g)t$$

extends to a continuous function on $[0, +\infty]$. The real number $\mu(g)$ satisfying this condition is unique. We call it the *asymptotic slope of g*. The set of asymptotically linear continuous functions forms a real vector space with respect to the addition and the multiplication by a scalar. The map $\mu(\cdot)$ is a linear form on this vector space.

We denote by $L^2_1(\mathbb{R}_{>0})$ the vector space of continuous functions φ on $\mathbb{R}_{>0}$ such that the derivative (in the sense of distribution) φ' is represented by a square-integrable function on $\mathbb{R}_{>0}$. We say that an asymptotically linear continuous function g on $\mathbb{R}_{>0}$ is *pairable* if the function φ_g belongs to $L^2_1(\mathbb{R}_{>0})$.

Remark 2.1. The functional space L_1^2 is a natural object of the potential theory on Riemann surfaces. In the classic setting of Arakelov geometry, it has been used in the intersection theory on arithmetic surfaces. We refer to [4, §3] for more details.

2.2 Convex function on $[0, +\infty]$

Let φ be a convex function on $\mathbb{R}_{>0}$. Then φ is continuous on $\mathbb{R}_{>0}$. Moreover, for any $t \in \mathbb{R}_{>0}$, the right derivative of φ at t, given by

$$\lim_{\varepsilon \downarrow 0} \frac{\varphi(t+\varepsilon) - \varphi(t)}{\varepsilon},$$

exists in $\mathbb{R}$. By abuse of notation, we denote by φ' the right derivative function of φ on $\mathbb{R}_{>0}$. It is a right continuous increasing function. We refer to [26, Theorem 1.26] for more details. Moreover, for any $(a, b) \in \mathbb{R}_{>0}^2$, one has

$$\varphi(b) - \varphi(a) = \int_{]a,b[} \varphi'(t)\,\mathrm{d}t. \tag{2.1}$$

See [26, Theorem 1.28] for a proof. In particular, the function φ' represents the derivative of φ in the sense of distribution.

Proposition 2.2. Let φ be a convex function on $\mathbb{R}_{>0}$, which is bounded.

(1) One has $\varphi' \leqslant 0$ on $\mathbb{R}_{>0}$ and $\lim_{t \to +\infty} \varphi'(t) = 0$. In particular, the function φ is decreasing and extends to a continuous function on $[0, +\infty]$.

(2) We extend φ continuously on $[0, +\infty]$. The function

$$(t \in \mathbb{R}_{>0}) \longmapsto \frac{\varphi(t) - \varphi(0)}{t}$$

is increasing. Moreover, one has

$$\lim_{t \downarrow 0} \frac{\varphi(t) - \varphi(0)}{t} = \lim_{t \downarrow 0} \varphi'(t) \in [-\infty, 0],$$

which is denoted by $\varphi'(0)$.

In addition, we have the following two propositions.

Proposition 2.3. Let φ and ψ be continuous functions on $[0, +\infty]$, which are convex on $\mathbb{R}_{>0}$. One has

$$\int_{]0,+\infty]} \varphi\,\mathrm{d}\psi' = -\int_{\mathbb{R}_{>0}} \psi'(t)\varphi'(t)\,\mathrm{d}t - \varphi(0)\psi'(0) \in [-\infty, +\infty[. \tag{2.2}$$

In particular, if $\varphi(0) = \psi(0) = 0$, then one has

$$\int_{]0,+\infty]} \varphi \,\mathrm{d}\psi' = \int_{]0,+\infty]} \psi \,\mathrm{d}\varphi'. \tag{2.3}$$

Proof. By (2.1), one has

$$\int_{]0,+\infty]} \varphi \,\mathrm{d}\psi' = \int_{]0,+\infty]} \int_{]0,x[} \varphi'(t)\,\mathrm{d}t\,\mathrm{d}\psi'(x) + \varphi(0) \int_{]0,+\infty]} \mathrm{d}\psi'.$$

By Fubini's theorem (by Proposition 2.2, one has $\varphi' \leqslant 0$ and hence Fubini's theorem applies, see for example [27, Theorem 8.8]), the double integral is equal to

$$\int_{\mathbb{R}_{>0}} \varphi'(t) \int_{]t,+\infty]} \mathrm{d}\psi'\,\mathrm{d}t = -\int_{\mathbb{R}_{>0}} \varphi'(t)\psi'(t)\,\mathrm{d}t.$$

Therefore, the equality (2.2) holds. In the case where $\varphi(0) = \psi(0) = 0$, one has

$$\int_{]0,+\infty]} \varphi\,\mathrm{d}\psi' = -\int_{\mathbb{R}_{>0}} \psi'(t)\varphi'(t)\,\mathrm{d}t = \int_{]0,+\infty]} \psi\,\mathrm{d}\varphi'.$$

■

Proposition 2.4. Let φ be a continuous function on $[0, +\infty]$, which is convex on $\mathbb{R}_{>0}$. One has

$$\int_{\mathbb{R}_{>0}} x\,\mathrm{d}\varphi'(x) = \varphi(0) - \varphi(+\infty). \tag{2.4}$$

Proof. This is a consequence of Fubini's theorem together with Proposition 2.2. ■

2.3 Transform of Legendre type

Definition 2.5. Let φ be a continuous function on $[0, +\infty]$, which is convex on $\mathbb{R}_{>0}$. We denote by φ^* the function on $[0, +\infty]$ defined as

$$\forall\,\lambda \in [0, +\infty], \quad \varphi^*(\lambda) := \inf_{x\in[0,+\infty]} (x\lambda + \varphi(x) - \varphi(0)).$$

Clearly, the function φ^* is increasing and non-positive. Moreover, one has

$$\varphi^*(0) = \inf_{x\in[0,+\infty]} \varphi(x) - \varphi(0) = \varphi(+\infty) - \varphi(0).$$

Therefore, for any $\lambda \in [0, +\infty]$, one has

$$\varphi(+\infty) - \varphi(0) \leqslant \varphi^*(\lambda) \leqslant 0.$$

The following proposition is important for the Hilbert–Samuel formula.

Proposition 2.6. Let φ be a continuous function on $[0,+\infty]$, which is convex on $\mathbb{R}_{>0}$. For $p \in \mathbb{R}_{>1}$, one has

$$\int_0^{+\infty} (-\varphi'(x))^p \mathrm{d}x = -(p-1)p \int_0^{+\infty} \lambda^{p-2}\varphi^*(\lambda)\mathrm{d}\lambda.$$

In particular,

$$\int_0^{+\infty} \varphi'(x)^2 \mathrm{d}x = -2 \int_0^{+\infty} \varphi^*(\lambda)\,\mathrm{d}\lambda. \tag{2.5}$$

Proof. Since φ' is increasing one has

$$\varphi^*(\lambda) = \inf_{x\in[0,+\infty[} \int_0^x (\lambda + \varphi'(t))\,\mathrm{d}t = \int_0^{+\infty} \min\{\lambda + \varphi'(t), 0\}\,\mathrm{d}t.$$

Therefore, by Fubini's theorem,

$$\begin{aligned}
\int_0^{+\infty} \lambda^{p-2}\varphi^*(\lambda)\,\mathrm{d}\lambda &= \int_0^{+\infty} \left(\int_0^{+\infty} \lambda^{p-2} \min\{\lambda + \varphi'(t), 0\}\,\mathrm{d}\lambda\right) \mathrm{d}t \\
&= \int_0^{+\infty} \left(\int_0^{-\varphi'(t)} \lambda^{p-2}(\lambda + \varphi'(t))\,\mathrm{d}\lambda\right) \mathrm{d}t \\
&= \int_0^{+\infty} \left[\frac{\lambda^p}{p} + \frac{\varphi'(t)\lambda^{p-1}}{p-1}\right]_0^{-\varphi'(t)} \mathrm{d}t \\
&= \frac{-1}{(p-1)p} \int_0^{+\infty} (-\varphi'(t))^p\,\mathrm{d}t,
\end{aligned}$$

as required. ∎

2.4 Convex envelope of asymptotically linear functions

Let $g : \mathbb{R}_{>0} \to \mathbb{R}$ be an asymptotically linear continuous function (see Section 2.1). We define the *convex envelope* of g as the largest convex function $\breve{g}$ on $\mathbb{R}_{>0}$, which is bounded from above by g. Note that $\breve{g}$ identifies with the supremum of all affine functions bounded from above by g.

Proposition 2.7. Let $g : \mathbb{R}_{>0} \to \mathbb{R}$ be an asymptotically linear continuous function. Then $\breve{g}$ is also an asymptotically linear continuous function. Moreover, one has $\mu(g) = \mu(\breve{g})$ and $g(0) = \breve{g}(0)$.

Proof. Let $\varphi_g : [0,+\infty] \to \mathbb{R}$ be the continuous function such that $\varphi_g(t) = g(t) - \mu(g)t$ on $\mathbb{R}_{>0}$. Let M be a real number such that $|\varphi_g(t)| \leqslant M$ for any $t \in [0,+\infty]$. One has

$$\mu(g)t - M \leqslant g(t) \leqslant \mu(g)t + M.$$

Therefore,

$$\mu(g)t - M \leqslant \breve{g}(t) \leqslant \mu(g)t + M.$$

By Proposition 2.2, the function

$$\varphi_{\breve{g}} : \mathbb{R}_{>0} \to \mathbb{R}, \quad \varphi_{\breve{g}}(t) := \breve{g}(t) - \mu(g)t$$

extends continuously on $[0,+\infty]$. It remains to show that $g(0) = \breve{g}(0)$. Let $\varepsilon > 0$. The function $t \mapsto (g(t) - g(0) + \varepsilon)/t$ is continuous on $]0,+\infty]$ and one has

$$\lim_{t \downarrow 0} \frac{g(t) - g(0) + \varepsilon}{t} = +\infty.$$

Therefore, this function is bounded from below by a real number α. Hence, the function g is bounded from below on $\mathbb{R}_{>0}$ by the affine function

$$t \longmapsto \alpha t + g(0) - \varepsilon,$$

which implies that $\breve{g}(0) \geqslant g(0) - \varepsilon$. Since $g \geqslant \breve{g}$ and since ε is arbitrary, we obtain $\breve{g}(0) = g(0)$. ■

3 Green Functions on a Tree of Length 1

The purpose of this section is to establish a framework of Green functions on a tree of length 1, which serves as a fundament of the arithmetic intersection theory of metrised $\mathbb{R}$-divisors on an arithmetic surface over a trivially valued field.

3.1 Tree of length 1 associated with a set

Let S be a non-empty set. We denote by $\mathcal{T}(S)$ the quotient set of the disjoint union $\coprod_{x \in S} [0,+\infty]$ obtained by gluing the points 0 in the copies of $[0,+\infty]$. The quotient map from $\coprod_{x \in S} [0,+\infty]$ to $\mathcal{T}(S)$ is denoted by π. For each $x \in S$, we denote by $\xi_x : [0,+\infty] \to \mathcal{T}(S)$ the restriction of the quotient map π to the copy of $[0,+\infty]$ indexed by x. The set $\mathcal{T}(S)$ is the union of $\xi_x([0,+\infty])$, $x \in S$.

Notation 3.1. Note that the images of 0 in $\mathcal{T}(S)$ by all maps ξ_x are the same, which we denote by η_0. The image of $+\infty$ by the map ξ_x is denoted by x_0. If a and b are elements of $[0,+\infty]$ such that $a < b$, the images of the intervals $[a,b]$, $[a,b[$, $]a,b]$, $]a,b[$ by ξ_x are denoted by $[\xi_x(a), \xi_x(b)]$, $[\xi_x(a), \xi_x(b)[$, $]\xi_x(a), \xi_x(b)]$, $]\xi_x(a), \xi_x(b)[$ respectively.

Definition 3.2. We denote by $t : \mathcal{T}(S) \to [0, +\infty]$ the map, which sends an element $\xi \in \xi_x([0, +\infty])$ to the unique number $a \in [0, +\infty]$ such that $\xi_x(a) = \xi$. In other words, for any $x \in S$, the restriction of $t(\cdot)$ to $[\eta_0, x_0]$ is the inverse of the injective map ξ_x. We call $t(\cdot)$ the *parametrisation map* of $\mathcal{T}(S)$.

Definition 3.3. We equip $\mathcal{T}(S)$ with the following topology. A subset U of $\mathcal{T}(S)$ is open if and only if the conditions below are simultaneously satisfied:

(1) for any $x \in S$, $\xi_x^{-1}(U)$ is an open subset of $[0, +\infty]$, and

(2) if $\eta_0 \in U$, then U contains $[\eta_0, x_0]$ for all but finitely many $x \in S$.

By definition, all maps $\xi_x : [0, +\infty] \to \mathcal{T}(S)$ are continuous. However, if S is an infinite set, then the parametrisation map $t(\cdot)$ is *not* continuous.

Note that the topological space $\mathcal{T}(S)$ is compact. We can visualise it as an infinite tree of length 1 whose root is η_0 and whose leaves are x_0 with $x \in S$.

3.2 Green functions

Let S be a non-empty set and $w : S \to \mathbb{R}_{>0}$ be a map. We call *Green function* on $\mathcal{T}(S)$ any continuous map g from $\mathcal{T}(S)$ to $[-\infty, +\infty]$ such that, for any $x \in S$, the composition of g with $\xi_x|_{\mathbb{R}_{>0}}$ defines an asymptotically linear function on $\mathbb{R}_{>0}$. For any $x \in S$, we denote by $\mu_x(g)$ the unique real number such that the function

$$(u \in \mathbb{R}_{>0}) \longmapsto g(\xi_x(u)) - \mu_x(g)u$$

extends to a continuous function on $[0, +\infty]$. We denote by $\varphi_g : \mathcal{T}(S) \to \mathbb{R}$ the continuous function on $\mathcal{T}(S)$ such that

$$\varphi_g(\xi) = g(\xi) - \mu_x(g)t(\xi) \text{ for any } \xi \in [\eta_0, x_0],\ x \in S.$$

Remark 3.4. Let g be a Green function on $\mathcal{T}(S)$. It takes finite values on $\mathcal{T}(S) \setminus \{x_0 : x \in S\}$. Moreover, for any $x \in S$, the value of g at x_0 is finite if and only if $\mu_x(g) = 0$. As g is a continuous map, $g^{-1}(\mathbb{R})$ contains all but finitely many x_0 with $x \in S$. In other words, for all but finitely many $x \in S$, one has $\mu_x(g) = 0$. Note that the Green function g is bounded if and only if $\mu_x(g) = 0$ for any $x \in S$.

Definition 3.5. Let g be a Green function on $\mathcal{T}(S)$. We denote by g_{can} the map from $\mathcal{T}(S)$ to $[-\infty, +\infty]$, which sends $\xi \in [\eta_0, x_0]$ to $\mu_x(g)t(\xi)$. Note that the composition of g_{can} with $\xi_x|_{\mathbb{R}_{>0}}$ is a linear function on $\mathbb{R}_{>0}$. We call it the *canonical Green function* associated with g. Note that there is a unique bounded Green function φ_g on $\mathcal{T}(S)$ such that $g = g_{\mathrm{can}} + \varphi_g$. We call it the *bounded Green function* associated with g. The formula $g = g_{\mathrm{can}} + \varphi_g$ is called the *canonical decomposition* of the Green function g. If $g = g_{\mathrm{can}}$, we say that the Green function g is *canonical*.

Proposition 3.6. Let g be a Green function on $\mathcal{T}(S)$. For all but countably many $x \in S$, the restriction of g on $[\eta_0, x_0]$ is a constant function.

Proof. For any $n \in \mathbb{N}$ such that $n \geqslant 1$, let U_n be set of $\xi \in \mathcal{T}(S)$ such that

$$|g(\xi) - g(\eta_0)| < n^{-1}.$$

This is an open subset of $\mathcal{T}(S)$, which contains η_0. Hence, there is a finite subset S_n of S such that $[\eta_0, x_0] \subset U_n$ for any $x \in S \setminus S_n$. Let $S' = \bigcup_{n \in \mathbb{N}, n \geqslant 1} S_n$. This is a countable subset of S. For any $x \in S \setminus S'$ and any $\xi \in [\eta_0, x_0]$, one has $g(\xi) = g(\eta_0)$ ■

Remark 3.7. It is clear that, if g is a Green function on $\mathcal{T}(S)$, for any $a \in \mathbb{R}$, the function $ag : \mathcal{T}(S) \to [-\infty, +\infty]$ is a Green function on $\mathcal{T}(S)$. Moreover, the canonical decomposition of Green functions allows to define the sum of two Green functions. Let g_1 and g_2 be two Green functions on $\mathcal{T}(S)$. We define $g_1 + g_2$ as $(g_{1,\mathrm{can}} + g_{2,\mathrm{can}}) + (\varphi_{g_1} + \varphi_{g_2})$.

Note that the set of all Green functions, equipped with the addition and the multiplication by a scalar, forms a vector space over $\mathbb{R}$.

3.3 Pairing of Green functions

Let S be a non-empty set and $w : S \to \mathbb{R}_{>0}$ be a map, called a *weight function*. We say that a Green function g on $\mathcal{T}(S)$ is *pairable with respect to* w if the following conditions are satisfied:

(1) for any $x \in S$, the function $\varphi_g \circ \xi_x|_{\mathbb{R}_{>0}}$ belongs to $L^2_1(\mathbb{R}_{>0})$ (see Section 2.1), and
(2) one has

$$\sum_{x \in S} w(x) \int_{\mathbb{R}_{>0}} (\varphi_g \circ \xi_x|_{\mathbb{R}_{>0}})'(u)^2 \, \mathrm{d}u < +\infty.$$

For each $x \in S$ we fix a representative of the function $(\varphi_g \circ \xi_x|_{\mathbb{R}_{>0}})'$ and we denote by

$$\varphi_g' : \bigcup_{x \in S}]\eta_0, x_0[\longrightarrow \mathbb{R}$$

the function, which sends $\xi \in]\eta_0, x_0[$ to $(\varphi_g \circ \xi_x|_{\mathbb{R}_{>0}})'(t(\xi))$.

We equip $\coprod_{x\in S}[0,+\infty]$ with the disjoint union of the weighted Lebesgue measure $w(x)\,\mathrm{d}u$, where $\mathrm{d}u$ denotes the Lebesgue measure on $[0,+\infty]$. We denote by $\nu_{S,w}$ the push-forward of this measure by the projection map

$$\coprod_{x\in S}[0,+\infty] \longrightarrow \mathcal{T}(S).$$

Then the function φ_g' is square-integrable with respect to the measure $\nu_{S,w}$.

Definition 3.8. Note that pairable Green functions form a vector subspace of the vector space of Green functions. Let g_1 and g_2 be pairable Green functions on $\mathcal{T}(S)$. We define the *pairing* of g_1 and g_2 as

$$\sum_{x\in S} w(x)\Big(\mu_x(g_1)g_2(\eta_0)+\mu_x(g_2)g_1(\eta_0)\Big)-\int_{\mathcal{T}(S)}\varphi_{g_1}'(\xi)\varphi_{g_2}'(\xi)\,\nu_{S,w}(\mathrm{d}\xi),$$

denoted by $\langle g_1,g_2\rangle_w$, called the *pairing* of Green functions g_1 and g_2. Note that $\langle\rangle_w$ is a symmetric bilinear form on the vector space of pairable Green functions.

3.4 Convex Green functions

Let S be a non-empty set. We say that a Green function g on $\mathcal{T}(S)$ is *convex* if, for any element x of S, the function $g\circ\xi_x$ on $\mathbb{R}_{>0}$ is convex.

Convention 3.9. If g is a convex Green function on $\mathcal{T}(S)$, by convention we choose, for each $x\in S$, the right derivative of $\varphi_g\circ\xi_x|_{\mathbb{R}_{>0}}$ to represent the derivative of $\varphi_g\circ\xi_x|_{\mathbb{R}_{>0}}$ in the sense of distribution. In other words, $\varphi_g'\circ\xi_x|_{\mathbb{R}_{>0}}$ is given by the right derivative of the function $\varphi_g\circ\xi_x|_{\mathbb{R}_{>0}}$. Moreover, for any $x\in S$, we denote by $\varphi_g'(\eta_0;x)$ the element $\varphi_{g\circ\xi_x}'(0)\in[-\infty,0]$ (see Proposition 2.2 (2)). We emphasise that $\varphi_{g\circ\xi_x}'(0)$ could differ when x varies.

Definition 3.10. Let g be a Green function on $\mathcal{T}(S)$. We call *convex envelope* of g and we denote by $\breve{g}$ the continuous map from $\mathcal{T}(S)$ to $[-\infty,+\infty]$ such that, for any $x\in S$, $\breve{g}\circ\xi_x|_{\mathbb{R}_{>0}}$ is the convex envelope of $g\circ\xi_x|_{\mathbb{R}_{>0}}$ (see Section 2.4). By Proposition 2.7, the function $\breve{g}$ is well defined and defines a convex Green function on $\mathcal{T}(S)$. Moreover, it is the largest convex Green function on $\mathcal{T}(S)$, which is bounded from above by g.

Proposition 3.11. Let g be a Green function on $\mathcal{T}(S)$. The following equalities hold:

$$g_{\mathrm{can}}=\breve{g}_{\mathrm{can}},\quad g(\eta_0)=\breve{g}(\eta_0),\quad \breve{\varphi}_g=\varphi_{\breve{g}}.$$

Proof. The 1st two equalities follow from Proposition 2.7. The 3rd equality comes from the 1st one and the fact that $\breve{g} = g_{\mathrm{can}} + \breve{\varphi}_g$. ■

3.5 Infimum slopes

Let S be a non-empty set and g be a Green function on $\mathcal{T}(S)$. For any $x \in S$, we denote by $\mu_{\inf,x}(g)$ the element

$$\inf_{\xi \in]\eta_0, x_0[} \frac{g(\xi)}{t(\xi)} \in [-\infty, \infty[.$$

Clearly, one has $\mu_{\inf,x}(g) \leqslant \mu_x(g)$. Therefore, by Remark 3.4 we obtain that $\mu_{\inf,x}(g) \leqslant 0$ for all but finitely many $x \in S$. If $w : S \to \mathbb{R}_{\geqslant 0}$ is a weight function, we define the *global infimum slope* of g with respect to w as

$$\sum_{x \in X^{(1)}} \mu_{\inf,x}(g) w(x) \in [-\infty, \infty[.$$

This element is well defined because $\mu_{\inf,x}(g) \leqslant 0$ for all but finitely many $x \in S$. If there is no ambiguity about the weight function (notably when S is the set of closed points of a regular projective curve, cf. Definition 6.6), the global infimum slope of g is also denoted by $\mu_{\inf}(g)$.

Proposition 3.12. Let g be a convex Green function on $\mathcal{T}(S)$. For any $x \in S$ one has

$$\mu_{\inf,x}(g - g(\eta_0)) = \mu_x(g) + \varphi_g'(\eta_0; x).$$

Proof. This is a direct consequence of Proposition 2.2 (2). ■

4 Graded Linear Series

Let k be a field and X be a regular projective curve over Spec$k$. We denote by $X^{(1)}$ the set of closed points of X.

Definition 4.1. By $\mathbb{R}$-*divisor* on X, we mean an element in the free $\mathbb{R}$-vector space generated by $X^{(1)}$. We denote by $\mathrm{Div}_{\mathbb{R}}(X)$ the $\mathbb{R}$-vector space of $\mathbb{R}$-divisors on X. If D is an element of $\mathrm{Div}_{\mathbb{R}}(X)$, the coefficient of x in the expression of D into a linear combination of elements of $X^{(1)}$ is denoted by $\mathrm{ord}_x(D)$. If $\mathrm{ord}_x(D)$ belongs to $\mathbb{Q}$ for any $x \in X^{(1)}$, we say that D is a $\mathbb{Q}$-*divisor*; if $\mathrm{ord}_x(D) \in \mathbb{Z}$ for any $x \in X^{(1)}$, we say that D is a *divisor* on X. The subsets of $\mathrm{Div}_{\mathbb{R}}(X)$ consisting of $\mathbb{Q}$-divisors and divisors are denoted by $\mathrm{Div}_{\mathbb{Q}}(X)$ and $\mathrm{Div}(X)$, respectively.

Definition 4.2. Let D be an $\mathbb{R}$-divisor on X. We define the *degree* of D to be

$$\deg(D) := \sum_{x \in X^{(1)}} [\kappa(x) : k] \operatorname{ord}_x(D), \tag{4.1}$$

where for $x \in X$, $\kappa(x)$ denotes the residue field of x. Denote by $\mathrm{Supp}(D)$ the set of all $x \in X^{(1)}$ such that $\operatorname{ord}_x(D) \neq 0$, called the *support* of the $\mathbb{R}$-divisor D. This is a finite subset of $X^{(1)}$.

Remark 4.3. Although $X^{(1)}$ is an infinite set, (4.1) is actually a finite sum: one has

$$\deg(D) = \sum_{x \in \mathrm{Supp}(D)} [\kappa(x) : k] \operatorname{ord}_x(D).$$

Definition 4.4. Denote by $\mathrm{Rat}(X)$ the field of rational functions on X. If f is a non-zero element of $\mathrm{Rat}(X)$, we denote by (f) the *principal divisor* associated with f, namely the divisor on X given by

$$\sum_{x \in X^{(1)}} \operatorname{ord}_x(f) x,$$

where $\operatorname{ord}_x(f) \in \mathbb{Z}$ denotes the valuation of f with respect to the discrete valuation ring $\mathcal{O}_{X,x}$. The map $\mathrm{Rat}(X)^\times \to \mathrm{Div}(X)$ is additive and hence induces an $\mathbb{R}$-linear map

$$\mathrm{Rat}(X)^\times_{\mathbb{R}} := \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathrm{Div}_{\mathbb{R}}(X),$$

which we still denote by $f \mapsto (f)$.

Definition 4.5. We say that an $\mathbb{R}$-divisor D is *effective* if $\operatorname{ord}_x(D) \geqslant 0$ for any $x \in X^{(1)}$. We denote by $D \geqslant 0$ the condition "*D is effective*". For any $\mathbb{R}$-divisor D on X, we denote by $H^0(D)$ the set

$$\{f \in \mathrm{Rat}(X)^\times \;:\; (f) + D \geqslant 0\} \cup \{0\}.$$

It is a finite-dimensional k-vector subspace of $\mathrm{Rat}(X)$. We denote by $\mathrm{genus}(X)$ the genus of the curve X relatively to k.

Remark 4.6. The theorem of Riemann–Roch implies that if D is a divisor such that $\deg(D) > 2\,\mathrm{genus}(X) - 2$, then one has

$$\dim_k(H^0(D)) = \deg(D) + 1 - \mathrm{genus}(X). \tag{4.2}$$

We refer the readers to [14, Lemma 2.2] for a proof.

Definition 4.7. Let D be an $\mathbb{R}$-divisor on X. We denote by $\Gamma(D)^{\times}_{\mathbb{R}}$ the set

$$\{f \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}} \,:\, (f) + D \geqslant 0\}.$$

This is an $\mathbb{R}$-vector subspace of $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$. Similarly, we denote by $\Gamma(D)^{\times}_{\mathbb{Q}}$ the $\mathbb{Q}$-vector subspace

$$\{f \in \mathrm{Rat}(X)^{\times}_{\mathbb{Q}} \,:\, (f) + D \geqslant 0\}$$

of $\mathrm{Rat}(X)^{\times}_{\mathbb{Q}}$. Note that one has

$$\Gamma(D)^{\times}_{\mathbb{Q}} = \bigcup_{n \in \mathbb{N},\, n \geqslant 1} \{f^{\frac{1}{n}} \,:\, f \in H^0(nD) \setminus \{0\}\}. \tag{4.3}$$

Definition 4.8. Let D be an $\mathbb{R}$-divisor on X. We denote by $\lfloor D \rfloor$ and $\lceil D \rceil$ the divisors on C such that

$$\mathrm{ord}_x(\lfloor D \rfloor) = \lfloor \mathrm{ord}_x(D) \rfloor, \quad \mathrm{ord}_x(\lceil D \rceil) = \lceil \mathrm{ord}_x(D) \rceil.$$

Clearly, one has $\deg(\lfloor D \rfloor) \leqslant \deg(D) \leqslant \deg(\lceil D \rceil)$. Moreover,

$$\deg(\lfloor D \rfloor) > \deg(D) - \sum_{x \in \mathrm{Supp}(D)} [\kappa(x) : k], \tag{4.4}$$

$$\deg(\lceil D \rceil) < \deg(D) + \sum_{x \in \mathrm{Supp}(D)} [\kappa(x) : k]. \tag{4.5}$$

Let $(D_i)_{i \in I}$ be a family of $\mathbb{R}$-divisors on X such that

$$\sup_{i \in I} \mathrm{ord}_x(D_i) = 0$$

for all but finitely many $x \in X^{(1)}$. We denote by $\sup_{i \in I} D_i$ the $\mathbb{R}$-divisor such that

$$\forall\, x \in X^{(1)}, \quad \mathrm{ord}_x\left(\sup_{i \in I} D_i\right) = \sup_{i \in I} \mathrm{ord}_x(D_i).$$

It is easy to see the following proposition (left as an exercise to the reader).

Proposition 4.9. Let D be an $\mathbb{R}$-divisor on X such that $\deg(D) \geqslant 0$. One has

$$\lim_{n \to +\infty} \frac{\dim_k(H^0(nD))}{n} = \deg(D). \tag{4.6}$$

Next we consider the following:

Proposition 4.10. Let D be an $\mathbb{R}$-divisor on X such that $\deg(D) > 0$. Then one has

$$\sup_{s\in\Gamma(D)_{\mathbb{Q}}^{\times}} (s^{-1}) = D. \tag{4.7}$$

Proof. For any $s \in \Gamma(D)_{\mathbb{Q}}^{\times}$ one has

$$\forall\, x \in X^{(1)}, \quad \mathrm{ord}_x(s) + \mathrm{ord}_x(D) \geqslant 0$$

and hence $\mathrm{ord}_x(s^{-1}) \leqslant \mathrm{ord}_x(D)$.

For any $x \in X^{(1)}$ and any $\varepsilon > 0$, we pick an $\mathbb{R}$-divisor $D_{x,\varepsilon}$ on X such that $D - D_{x,\varepsilon}$ is effective, $\mathrm{ord}_x(D_{x,\varepsilon}) = \mathrm{ord}_x(D)$ and $0 < \deg(D_{x,\varepsilon}) < \varepsilon$. Since $\deg(D_{x,\varepsilon}) > 0$, the set $\Gamma(D_{x,\varepsilon})_{\mathbb{Q}}^{\times}$ is not empty (see (4.3) and Proposition 4.9). This set is also contained in $\Gamma(D)_{\mathbb{Q}}^{\times}$ since $D_{x,\varepsilon} \leqslant D$. Let f be an element of $\Gamma(D_{x,\varepsilon})_{\mathbb{Q}}^{\times}$. One has

$$D_{x,\varepsilon} + (f) \geqslant 0 \quad \text{and} \quad \deg(D_{x,\varepsilon} + (f)) = \deg(D_{x,\varepsilon}) < \varepsilon.$$

Therefore,

$$\mathrm{ord}_x(D + (f)) = \mathrm{ord}_x(D_{x,\varepsilon} + (f)) \leqslant \frac{\varepsilon}{[\kappa(x) : k]},$$

which leads to

$$\mathrm{ord}_x(f^{-1}) \geqslant \mathrm{ord}_x(D) - \frac{\varepsilon}{[\kappa(x) : k]}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\sup_{s\in\Gamma(D)_{\mathbb{Q}}^{\times}} \mathrm{ord}_x(s^{-1}) = \mathrm{ord}_x(D).$$

■

Remark 4.11. Let D be an $\mathbb{R}$-divisor on X. Note that one has

$$\sup_{s\in\Gamma(D)_{\mathbb{R}}^{\times}} (s^{-1}) \leqslant D.$$

Therefore, the above proposition implies that, if $\deg(D) > 0$, then

$$\sup_{s\in\Gamma(D)_{\mathbb{R}}^{\times}} (s^{-1}) = D.$$

This equality also holds when $\deg(D) = 0$ and $\Gamma(D)^{\times}_{\mathbb{R}} \neq \emptyset$. In fact, if s is an element of $\Gamma(D)^{\times}_{\mathbb{R}}$, then one has $D + (s) \geqslant 0$. Moreover, since $\deg(D) = 0$, one has $\deg(D + (s)) = \deg(D) + \deg((s)) = 0$ and hence $D + (s) = 0$. Similarly, if D is an $\mathbb{R}$-divisor on X such that $\Gamma(D)^{\times}_{\mathbb{Q}} \neq \emptyset$, then the equality

$$\sup_{s \in \Gamma(D)^{\times}_{\mathbb{Q}}} (s^{-1}) = D$$

always holds.

Definition 4.12. Let $\mathrm{Rat}(X)$ be the field of rational functions on X. By *graded linear series* on X, we refer to a graded sub-k-algebra $V_{\bullet} = \bigoplus_{n \in \mathbb{N}} V_n T^n$ of $\mathrm{Rat}(X)[T] = \bigoplus_{n \in \mathbb{N}} \mathrm{Rat}(X) T^n$, which satisfies the following conditions:

(1) $V_0 = k$,
(2) there exists $n \in \mathbb{N}_{\geqslant 1}$ such that $V_n \neq \{0\}$, and
(3) there exists an $\mathbb{R}$-divisor D on X such that $V_n \subseteq H^0(nD)$ for any $n \in \mathbb{N}$.

If W is a k-vector subspace of $\mathrm{Rat}(X)$, we denote by $k(W)$ the extension

$$k(\{f/g \,:\, (f,g) \in (W \setminus \{0\})^2\})$$

of k. If $V_{\bullet}$ is a graded linear series on V, we set

$$k(V_{\bullet}) := k\Big(\bigcup_{n \in \mathbb{N}_{\geqslant 1}} \{f/g \,:\, (f,g) \in (V_n \setminus \{0\})^2\} \Big).$$

If $k(V_{\bullet}) = \mathrm{Rat}(X)$, we say that the graded linear series $V_{\bullet}$ is *birational*.

Example 4.13. Let D be an $\mathbb{R}$-divisor on X such that $\deg(D) > 0$. Then the total graded linear series $\bigoplus_{n \in \mathbb{N}} H^0(nD)$ is birational.

Proposition 4.14. Let $V_{\bullet}$ be a graded linear series on X. The set

$$\mathbb{N}(V_{\bullet}) := \{n \in \mathbb{N}_{\geqslant 1} \,:\, V_n \neq \{0\}\}$$

equipped with the additive law forms a sub-semigroup of $\mathbb{N}_{\geqslant 1}$. Moreover, for any $n \in \mathbb{N}(V_{\bullet})$, which is sufficiently positive, one has $k(V_{\bullet}) = k(V_n)$.

Proof. Let n and m be elements of $\mathbb{N}(V_{\bullet})$. If f and g are respectively non-zero elements of V_n and V_m, then fg is a non-zero element of V_{n+m}. Hence, $n + m$ belongs to $\mathbb{N}(V_{\bullet})$.

Therefore, $\mathbb{N}(V_\bullet)$ is a sub-semigroup of $\mathbb{N}_{\geqslant 1}$. In particular, if $d \geqslant 1$ is a generator of the subgroup of $\mathbb{Z}$ generated by $\mathbb{N}(V_\bullet)$, then there exists $N_0 \in \mathbb{N}_{\geqslant 1}$ such that $dn \in \mathbb{N}(V_\bullet)$ for any $n \in \mathbb{N}$, $n \geqslant N_0$.

Since $k \subseteq k(V_\bullet) \subseteq \mathrm{Rat}(X)$ and $\mathrm{Rat}(X)$ is finitely generated over k, the extension $k(V_\bullet)/k$ is finitely generated (see [7, Chapter V, §14, n°7, Corollary 3]). Therefore, there exist a finite family $\{n_1, \ldots, n_r\}$ of elements in $\mathbb{N}_{\geqslant 1}$, together with pairs $(f_i, g_i) \in (V_{dn_i} \setminus \{0\})^2$ such that $k(V_\bullet) = k(f_1/g_1, \ldots, f_r/g_r)$. Let $N \in \mathbb{N}$ such that

$$N - \max\{n_1, \ldots, n_r\} \geqslant N_0.$$

For any $i \in \{1, \ldots, r\}$ and $M \in \mathbb{N}_{\geqslant N}$, let $h_{M,i} \in V_{d(M-n_i)} \setminus \{0\}$. Then

$$(h_{M,i} f_i, h_{M,i} g_i) \in (V_{dM} \setminus \{0\})^2,$$

which shows that $k(V_\bullet) = k(V_{dM})$. ■

Definition 4.15. If $V_\bullet$ is a graded linear series, we define $\Gamma(V_\bullet)^\times_{\mathbb{Q}}$ as

$$\bigcup_{n \in \mathbb{N}_{\geqslant 1}} \{f^{\frac{1}{n}} \mid f \in V_n \setminus \{0\}\},$$

and let $D(V_\bullet)$ be the following $\mathbb{R}$-divisor

$$\sup_{s \in \Gamma(V_\bullet)^\times_{\mathbb{Q}}} (s^{-1}),$$

called the $\mathbb{R}$-*divisor generated by* $V_\bullet$. The conditions (2) and (3) in Definition 4.12 show that the $\mathbb{R}$-divisor $D(V_\bullet)$ is well defined and has non-negative degree.

Proposition 4.16. Let $V_\bullet$ be a birational graded linear series on X. One has

$$\lim_{n \in \mathbb{N},\, V_n \neq \{0\} n \to +\infty} \frac{\dim_k(V_n)}{n} = \deg(D(V_\bullet)) > 0. \tag{4.8}$$

Proof. By definition, for any $n \in \mathbb{N}$ one has $V_n \subseteq H^0(nD(V_\bullet))$. Therefore, Proposition 4.9 leads to

$$\limsup_{n \to +\infty} \frac{\dim_k(V_n)}{n} \leqslant \deg(D(V_\bullet)).$$

Let $p \in \mathbb{N}(V_\bullet)$ be a sufficiently positive integer (so that $\mathrm{Rat}(X) = k(V_p)$). Let

$$V_\bullet^{[p]} := \bigoplus_{n\in\mathbb{N}} \mathrm{Im}(S^n V_p \longrightarrow V_{np})T^n.$$

Clearly, one has $D(V_\bullet^{[p]}) \leqslant pD(V_\bullet)$. Conversely, one has

$$D(V_\bullet) = \sup_{\substack{p\in\mathbb{N}(V_\bullet)\\ f\in V_p\setminus\{0\}}} \frac{1}{p}(f^{-1}) \leqslant \sup_{p\in\mathbb{N}(V_\bullet)} \frac{1}{p}D(V_\bullet^{[p]}).$$

Hence, the following equality holds:

$$D(V_\bullet) = \sup_{p\in\mathbb{N}(V_\bullet)} \frac{1}{p}D(V_\bullet^{[p]}).$$

Moreover, since $\mathrm{Rat}(X) = k(V_p)$, X identifies with the normalisation of $\mathrm{Proj}(V_\bullet^{[p]})$ and the pull-back on X of the tautological line bundle on $\mathrm{Proj}(V_\bullet^{[p]})$ identifies with $\mathcal{O}(D(V_\bullet^{[p]}))$. This leads to

$$\frac{1}{p}\deg(D(V_\bullet^{[p]})) = \lim_{n\to+\infty} \frac{\dim_k(V_n^{[p]})}{pn} \leqslant \liminf_{\substack{n\in\mathbb{N},\, V_n\neq\{0\}\\ n\to+\infty}} \frac{\dim_k(V_n)}{n},$$

where in the inequality we have used the fact that $\dim_k(V_{np}) \leqslant \dim_k(V_{np+r})$ once $V_r \neq \{0\}$. As the map $p \mapsto \frac{1}{p}D(V_\bullet^{[p]})$ preserves the order if we consider the relation of divisibility on p, by the relation $D(V_\bullet) = \sup_p \frac{1}{p}D(V_\bullet^{[p]})$ we obtain that

$$\deg(D(V_\bullet)) = \sup_p \frac{1}{p}\deg(D(V_\bullet^{[p]})) \leqslant \liminf_{\substack{n\in\mathbb{N},\, V_n\neq\{0\}\\ n\to+\infty}} \frac{\dim_k(V_n)}{n}.$$

Therefore, the equality in (4.8) holds.

If p is a positive integer such that $\mathrm{Rat}(X) = k(V_p)$, then V_p admits an element s, which is transcendental over k. In particular, the graded linear series $V_\bullet^{[p]}$ contains a polynomial ring of one variable, which shows that

$$\liminf_{n\to+\infty} \frac{\dim_k(V_n)}{n} > 0.$$

■

5 Arithmetic Surface Over a Trivially Valued Field

In this section, we fix a commutative field k and we denote by $|\cdot|$ the trivial absolute value on k. Let X be a regular projective curve over Spec$k$. We denote by X^{an} the *Berkovich topological space* associated with X. Recall that, as a set X^{an} consists of couples of the form $\xi = (x, |\cdot|_\xi)$, where x is a scheme point of X and $|\cdot|_\xi$ is an absolute value on the residue field $\kappa(x)$ of x, which extends the trivial absolute value on k. We denote by $j : X^{\mathrm{an}} \to X$ the map sending any pair in X^{an} to its 1st coordinate. For any $\xi \in X^{\mathrm{an}}$, we denote by $\widehat{\kappa}(\xi)$ the completion of $\kappa(j(\xi))$ with respect to the absolute value $|\cdot|_\xi$, on which $|\cdot|_\xi$ extends in a unique way. For any regular function f on a Zariski open subset U of X, we let $|f|$ be the function on $j^{-1}(U)$ sending any ξ to the absolute value of $f(j(\xi)) \in \kappa(j(\xi))$ with respect to $|\cdot|_\xi$. The *Berkovich topology* on X^{an} is defined as the most coarse topology making the map j and all functions of the form $|f|$ continuous, where f is a regular function on a Zariski open subset of X.j

Remark 5.1. Let $X^{(1)}$ be the set of all closed points of X. The Berkovich topological space X^{an} identifies with the tree $\mathcal{T}(X^{(1)})$, where

(a) the root point η_0 of the tree $\mathcal{T}(X^{(1)})$ corresponds to the pair consisting of the generic point η of X and the trivial absolute value on the field of rational functions on X;

(b) for any $x \in X^{(1)}$, the leaf point $x_0 \in \mathcal{T}(X^{(1)})$ corresponds to the closed point x of X together with the trivial absolute value on the residue field $\kappa(x)$;

(c) for any $x \in X^{(1)}$, any $\xi \in \,]\eta_0, x_0[$ corresponds to the pair consisting of the generic point η of X and the absolute value $\mathrm{e}^{-t(\xi)\,\mathrm{ord}_x(\cdot)}$, where $\mathrm{ord}_x(\cdot)$ is the discrete valuation on the field of rational functions $\mathrm{Rat}(X)$ corresponding to x and $t(\cdot)$ is the parametrization map on $\mathcal{T}(X^{(1)})$ (see Definition 3.2).

5.1 Metrised divisors

We call *metrised* $\mathbb{R}$*-divisor* on X any pair (D, g), where D is an $\mathbb{R}$*-divisor* on X and g is a Green function on $\mathcal{T}(X^{(1)})$ such that $\mu_x(g) = \mathrm{ord}_x(D)$ for any $x \in X^{(1)}$ (see Section 3.2). If in addition D is a $\mathbb{Q}$*-divisor* (resp. divisor), we say that D is a *metrised* $\mathbb{Q}$*-divisor* (resp. *metrised divisor*).

If (D, g) is a metrised $\mathbb{R}$-divisor on X and a is a real number, then (aD, ag) is also a metrised $\mathbb{R}$-divisor, denoted by $a(D, g)$. Moreover, if (D_1, g_1) and (D_2, g_2) are two metrised $\mathbb{R}$-divisors on X, then $(D_1 + D_2, g_1 + g_2)$ is also a metrised $\mathbb{R}$-divisor, denoted by $(D_1, g_1) + (D_2, g_2)$. The set $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ of all metrised $\mathbb{R}$-divisors on X then forms a vector space over $\mathbb{R}$.

If (D,g) is a metrised $\mathbb{R}$-divisor on X, we say that g is a *Green function of the $\mathbb{R}$-divisor D*.

Remark 5.2.

(1) Let (D,g) be a metrised $\mathbb{R}$-divisor on X. Note that the $\mathbb{R}$-divisor part D is uniquely determined by the Green function g. Therefore, the study of metrised $\mathbb{R}$-divisors on X is equivalent to that of Green functions on the infinite tree $\mathcal{T}(X^{(1)})$. The notation of pair (D,g) facilitates however the presentation on the study of metrised linear series of (D,g).

(2) Let D be an $\mathbb{R}$-divisor on X, there is a unique canonical Green function on $\mathcal{T}(X^{(1)})$ (see Definition 3.5), denoted by g_D, such that (D,g_D) is an metrised $\mathbb{R}$-divisor. On each edge $[\eta_0,x_0]$ of the infinite tree X^{an}, this Green function is equal to $\mathrm{ord}_x(D)t(\cdot)$. For a general Green function g of D, the canonical Green function associated with g (see Definition 3.5) is equal to g_D. In particular, if (D,g) is a metrised $\mathbb{R}$-divisor such that D is the zero $\mathbb{R}$-divisor, then the Green function g is bounded.
Canonical Green functions are similar to Green functions arising from integral models in the classic setting of Arakelov geometry over number fields. Here the valuation ring of $(k,|\cdot|)$ identifies with k and hence the only "integral model" of (X,D) is (X,D) itself.

Definition 5.3. Let $\mathrm{Rat}(X)$ be the field of rational functions on X and $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ be the $\mathbb{R}$-vector space $\mathrm{Rat}(X)^{\times}\otimes_{\mathbb{Z}}\mathbb{R}$. For any ϕ in $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$, the couple $((\phi),g_{(\phi)})$ is called the *principal metrised $\mathbb{R}$-divisor* associated with ϕ and is denoted by $\widehat{(\phi)}$.

Definition 5.4. If (D,g) is a metrised $\mathbb{R}$-divisor, for any $\phi\in\Gamma(D)^{\times}_{\mathbb{R}}$, we define

$$\|\phi\|_g := \exp\Big(-\inf_{\xi\in\mathcal{T}(X^{(1)})}(g_{(\phi)}+g)(\xi)\Big). \tag{5.1}$$

By convention, $\|0\|_g$ is defined to be zero.

5.2 Ultrametrically normed vector spaces

Let E be a finite-dimensional vector space over k (equipped with the trivial absolute value). By *ultrametric norm* on E, we mean a map $\|\cdot\|:E\to\mathbb{R}_{\geqslant 0}$ such that

(1) for any $x\in E$, $\|x\|=0$ if and only if $x=0$,

(2) $\|ax\| = \|x\|$ for any $x \in E$ and $a \in k \setminus \{0\}$, and
(3) for any $(x, y) \in E \times E$, $\|x + y\| \leqslant \max\{\|x\|, \|y\|\}$.

Let r be the rank of E over k. We define the *determinant norm associated with* $\|\cdot\|$ *the norm* $\|\cdot\|_{\det}$ on $\det(E) = \Lambda^r(E)$ *such that*

$$\forall\, \eta \in \det(E), \quad \|\eta\| = \inf_{\substack{(s_1,\ldots,s_r)\in E^r\\ s_1\wedge\ldots\wedge s_r=\eta}} \|s_1\| \cdots \|s_r\|.$$

We define the *Arakelov degree* of $(E, \|\cdot\|)$ as

$$\widehat{\deg}(E, \|\cdot\|) = -\ln\|\eta\|_{\det}, \tag{5.2}$$

where η is a non-zero element of $\det(E)$. We then define the *positive Arakelov degree* as

$$\widehat{\deg}_+(E, \|\cdot\|) := \sup_{F\subset E} \widehat{\deg}(F, \|\cdot\|_F),$$

where F runs over the set of all vector subspaces of E, and $\|\cdot\|_F$ denotes the restriction of $\|\cdot\|$ to F.

Example 5.5. Let (D, g) be a metrised $\mathbb{R}$-divisor on X. Note that the restriction of $\|\cdot\|_g$ to $H^0(D)$ defines an ultrametric norm on the k-vector space $H^0(D)$.

Assume that $(E, \|\cdot\|)$ is a non-zero finite-dimensional ultrametrically normed vector space over k. We introduce a Borel probability measure $\mathbb{P}_{(E,\|\cdot\|)}$ on $\mathbb{R}$ such that, for any $t \in \mathbb{R}$,

$$\mathbb{P}_{(E,\|\cdot\|)}(]t, +\infty[) = \frac{\dim_k(\{s \in E \;:\; \|s\| < \mathrm{e}^{-t}\})}{\dim_k(E)}.$$

Then, for any random variable Z that follows $\mathbb{P}_{(E,\|\cdot\|)}$ as its probability law, one has

$$\frac{\widehat{\deg}(E, \|\cdot\|)}{\dim_k(E)} = \mathbb{E}[Z] = \int_{\mathbb{R}} t\,\mathbb{P}_{(E,\|\cdot\|)}(\mathrm{d}t) \tag{5.3}$$

and

$$\frac{\widehat{\deg}_+(E, \|\cdot\|)}{\dim_k(E)} = \mathbb{E}[\max(Z, 0)] = \int_0^{+\infty} t\,\mathbb{P}_{(E,\|\cdot\|)}(\mathrm{d}t). \tag{5.4}$$

5.3 Essential infima

Let (D, g) be a metrised $\mathbb{R}$-divisor on X such that $\Gamma(D)^\times_{\mathbb{R}}$ is not empty. We define

$$\lambda_{\mathrm{ess}}(D, g) := \sup_{\phi\in\Gamma(D)^\times_{\mathbb{R}}} \inf_{\xi\in X^{\mathrm{an}}} (g_{(\phi)} + g)(\xi),$$

called the *essential infimum* of the metrised $\mathbb{R}$-divisor (D,g). By (5.1), we can also write $\lambda_{\mathrm{ess}}(D,g)$ as

$$\sup_{\phi\in\Gamma(D)^{\times}_{\mathbb{R}}}\Big(-\ln\|\phi\|_g\Big).$$

The superadditivity of λ_{ess} is an easy consequence of its definition (left as an exercise to the reader).

Proposition 5.6. Let (D_1,g_1) and (D_2,g_2) be metrised $\mathbb{R}$-divisors such that $\Gamma(D_1)^{\times}_{\mathbb{R}}$ and $\Gamma(D_2)^{\times}_{\mathbb{R}}$ are non-empty. Then one has

$$\lambda_{\mathrm{ess}}(D_1+D_2,g_1+g_2)\geqslant\lambda_{\mathrm{ess}}(D_1,g_1)+\lambda_{\mathrm{ess}}(D_2,g_2). \tag{5.5}$$

Remark 5.7. In the literature, the essential infimum of height function is studied in the number field setting. We can consider its analogue in the setting of Arakelov geometry over a trivially valued field. For any closed point x of X, we define the height of x with respect to (D,g) as

$$h_{(D,g)}(x):=\varphi_g(x_0),$$

where $\varphi_g=g-g_{\mathrm{can}}$ is the bounded Green function associated with g (see Definition 3.5), and x_0 denotes the point of X^{an} corresponding to the closed point x equipped with the trivial absolute value on its residue field. In particular, for any element $x\in X^{(1)}$ outside of the support of D, one has

$$h_{(D,g)}(x)=g(x_0).$$

Then the *essential infimum* of the height function $h_{(D,g)}$ is defined as

$$\mu_{\mathrm{ess}}(D,g):=\sup_{Z\subsetneq X}\inf_{x\in X^{(1)}\setminus Z}h_{(D,g)}(x),$$

where Z runs over the set of closed subschemes of X, which are different from X (namely a finite subset of $X^{(1)}$). If $\Gamma(D)^{\times}_{\mathbb{R}}$ is not empty, one has

$$\lambda_{\mathrm{ess}}(D,g)\leqslant\sup_{\phi\in\Gamma(D)^{\times}_{\mathbb{R}}}\inf_{x\in X^{(1)}}(g_{(\phi)}+g)(x_0).$$

For each $\phi\in\Gamma(D)^{\times}_{\mathbb{R}}$, one has

$$\inf_{x\in X^{(1)}}(g_{(\phi)}+g(x_0))\leqslant\inf_{x\in X^{(1)}\setminus(\mathrm{Supp}(D)\cup\mathrm{Supp}((\phi)))}g(x_0),$$

which is clearly bounded from above by $\mu_{\text{ess}}(D,g)$. Therefore, one has

$$\lambda_{\text{ess}}(D,g) \leqslant \mu_{\text{ess}}(D,g). \tag{5.6}$$

The following proposition implies that $\lambda_{\text{ess}}(D,g)$ is actually finite.

Proposition 5.8. Let (D,g) be a metrised $\mathbb{R}$-divisor on X. One has $\mu_{\text{ess}}(D,g) = g(\eta_0)$, where η_0 denotes the point of X^{an} corresponding to the generic point of X equipped with the trivial absolute value on its residue field.

Proof. Let α be a real number that is $> g(\eta_0)$. The set

$$\{\xi \in X^{\text{an}} \,:\, g(\xi) < \alpha\}$$

is an open subset of X^{an} containing η_0 and hence there exists a finite subset S of $X^{(1)}$ such that $g(x_0) < \alpha$ for any $x \in X^{(1)} \setminus S$. Therefore, we obtain $\mu_{\text{ess}}(D,g) \leqslant \alpha$. Since $\alpha > g(\eta_0)$ is arbitrary, we get $\mu_{\text{ess}}(D,g) \leqslant g(\eta_0)$.

Conversely, if β is a real number such that $\beta < g(\eta_0)$, then

$$\{\xi \in X^{\text{an}} \,:\, g(\xi) > \beta\}$$

is an open subset of X^{an} containing η_0. Hence, there exists a finite subset S' of $X^{(1)}$ such that $g(x_0) > \beta$ for any $x \in X^{(1)} \setminus S'$. Hence, $\mu_{\text{ess}}(D,g) \geqslant \beta$. Since $\beta < g(\eta_0)$ is arbitrary, we obtain $\mu_{\text{ess}}(D,g) \geqslant g(\eta_0)$. ■

Lemma 5.9. Let $r \in \mathbb{N}_{\geqslant 1}$ and $s_1,\ldots,s_r$ be elements of $\text{Rat}(X)^{\times}_{\mathbb{Q}}$ and $a_1,\ldots,a_r$ be real numbers, which are linearly independent over $\mathbb{Q}$. Let $s := s_1^{a_1}\cdots s_r^{a_r} \in \text{Rat}(X)^{\times}_{\mathbb{R}}$. Then for any $i \in \{1,\ldots,r\}$ one has $\text{Supp}((s_i)) \subset \text{Supp}((s))$.

Proof. Let x be a closed point of X, which does not lie in the support of (s). One has

$$\sum_{i=1}^{r} \text{ord}_x(s_i)a_i = 0$$

and hence $\text{ord}_x(s_1) = \ldots = \text{ord}_x(s_r) = 0$ since $a_1,\ldots,a_r$ are linearly independent over $\mathbb{Q}$. ■

Lemma 5.10. Let n and r be two positive integers, $\ell_1,\ldots,\ell_n$ be linear forms on $\mathbb{R}^r$ of the form

$$\ell_j(\boldsymbol{y}) = b_{j,1}y_1 + \cdots + b_{j,r}y_r, \text{ where } (b_{j,1},\ldots,b_{j,r}) \in \mathbb{Q}^r$$

and $q_1, \ldots, q_n$ be non-negative real numbers. Let $\boldsymbol{a} = (a_1, \ldots, a_r)$ be an element of $\mathbb{R}^r_{>0}$, which forms a linearly independent family over $\mathbb{Q}$, and such that $\ell_j(\boldsymbol{a}) + q_j \geqslant 0$ for any $j \in \{1, \ldots, n\}$. Then, for any $\varepsilon > 0$, there exists a sequence

$$\boldsymbol{\delta}^{(m)} = (\delta_1^{(m)}, \ldots, \delta_r^{(m)}), \quad m \in \mathbb{N}$$

in $\mathbb{R}^r_{>0}$, which converges to $(0, \ldots, 0)$ and verifies the following conditions:

(1) for any $j \in \{1, \ldots, n\}$, one has $\ell_j(\boldsymbol{\delta}^{(m)}) + \varepsilon q_j \geqslant 0$, and
(2) for any $m \in \mathbb{N}$ and any $i \in \{1, \ldots, r\}$, one has $\delta_i^{(m)} + a_i \in \mathbb{Q}$.

Proof. Without loss of generality, we may assume that $q_1 = \cdots = q_d = 0$ and $\min\{q_{d+1}, \ldots, q_n\} > 0$. Since $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$, for $j \in \{1, \ldots, d\}$, one has $\ell_j(\boldsymbol{a}) > 0$. Hence, there exists an open convex cone C in $\mathbb{R}^r_{>0}$ that contains $\boldsymbol{a}$, such that $\ell_j(\boldsymbol{y}) > 0$ for any $\boldsymbol{y} \in C$ and $j \in \{1, \ldots, d\}$. Moreover, if we denote by $\|\cdot\|_{\sup}$ the norm on $\mathbb{R}^r$ (where $\mathbb{R}$ is equipped with its usual absolute value) defined as

$$\|(y_1, \ldots, y_r)\|_{\sup} := \max\{|y_1|, \ldots, |y_r|\},$$

then there exists $\lambda > 0$ such that, for any $\boldsymbol{z} \in C$ such that $\|z\|_{\sup} < \lambda$ and any $j \in \{d+1, \ldots, n\}$, one has $\ell_j(\boldsymbol{z}) + \varepsilon q_j \geqslant 0$. Let

$$C_\lambda = \{\boldsymbol{y} \in C \,:\, \|\boldsymbol{y}\|_{\sup} < \lambda\}.$$

It is a convex open subset of $\mathbb{R}^r$. For any $\boldsymbol{y} \in C_\lambda$ and any $j \in \{1, \ldots, n\}$, one has

$$\ell_j(\boldsymbol{y}) + \varepsilon q_j \geqslant 0.$$

Since C_λ is open and convex, so is its translation by $-\boldsymbol{a}$. Note that the set of rational points in a convex open subset of $\mathbb{R}^r$ is dense in the convex open subset. Therefore, the set of all points $\boldsymbol{\delta} \in C_\lambda$ such that $\boldsymbol{\delta} + \boldsymbol{a} \in \mathbb{Q}^r$ is dense in C_λ. Since $(0, \ldots, 0)$ lies on the boundary of C_λ, it can be approximated by a sequence $(\boldsymbol{\delta}^{(m)})_{m \in \mathbb{N}}$ of elements in C_λ such that $\boldsymbol{\delta}^{(m)} + \boldsymbol{a} \in \mathbb{Q}^r$ for any $m \in \mathbb{N}$. ■

Remark 5.11. We keep the notation and hypotheses of Lemma 5.10. For any $m \in \mathbb{N}$, and $j \in \{1, \ldots, n\}$ one has

$$\ell_j(\boldsymbol{a} + \boldsymbol{\delta}^{(m)}) + (1 + \varepsilon) q_j \geqslant 0,$$

or equivalently,

$$\ell_j\Big(\frac{1}{1+\varepsilon}(\boldsymbol{a}+\delta^{(m)})\Big)+q_j\geqslant 0.$$

Therefore, one can find a sequence $(\boldsymbol{a}^{(p)})_{p\in\mathbb{N}}$ of elements in $\mathbb{Q}^r$ that converges to $\boldsymbol{a}$ and such that

$$\ell_j(\boldsymbol{a}^{(p)})+q_j\geqslant 0$$

hods for any $j\in\{1,\dots,n\}$ and any $p\in\mathbb{N}$.

The following proposition says that, in order to define the essential infimum, we can replace $\mathbb{R}$ with $\mathbb{Q}$ under the assumption $\Gamma(D)^{\times}_{\mathbb{Q}}\neq\emptyset$.

Proposition 5.12. Let (D,g) be a metrised $\mathbb{R}$-divisor on X such that $\Gamma(D)^{\times}_{\mathbb{Q}}\neq\emptyset$. One has

$$\begin{aligned}\lambda_{\mathrm{ess}}(D,g)&=\sup_{\phi\in\Gamma(D)^{\times}_{\mathbb{Q}}}\inf_{\xi\in X^{\mathrm{an}}}(g_{(\phi)}+g)(\xi)=\sup_{\phi\in\Gamma(D)^{\times}_{\mathbb{Q}}}\Big(-\ln\|\phi\|_g\Big)\\&=\sup_{n\in\mathbb{N},\,n\geqslant 1}\frac{1}{n}\sup_{s\in H^0(nD)\setminus\{0\}}\Big(-\ln\|s\|_{ng}\Big).\end{aligned}\tag{5.7}$$

Proof. By definition one has

$$\Gamma(D)^{\times}_{\mathbb{Q}}=\bigcup_{n\in\mathbb{N},\,n\geqslant 1}\{s^{\frac{1}{n}}\;:\;s\in H^0(nD)\setminus\{0\}\}.$$

Moreover, for $\phi\in\Gamma(D)^{\times}_{\mathbb{Q}}$, one has

$$\inf_{\xi\in X^{\mathrm{an}}}(g_{(\phi)}+g)(\xi)=-\ln\|\phi\|_g.$$

Therefore, the 2nd and 3rd equalities of (5.7) hold. To show the 1st equality, we denote temporarily by $\lambda_{\mathbb{Q},\mathrm{ess}}(D,g)$ the 2nd term of (5.7).

Let a be an arbitrary positive rational number. The correspondence $\Gamma(D)^{\times}_{\mathbb{Q}}\to\Gamma(aD)^{\times}_{\mathbb{Q}}$ given by $\phi\mapsto\phi^a$ is a bijection. Moreover, for $\phi\in\Gamma(D)^{\times}_{\mathbb{Q}}$ one has $\|\phi^a\|_{ag}=\|\phi\|_g^a$. Hence, the equality

$$\lambda_{\mathbb{Q},\mathrm{ess}}(aD,ag)=a\lambda_{\mathbb{Q},\mathrm{ess}}(D,g)\tag{5.8}$$

holds.

By our assumption, we can choose $\phi \in \Gamma(D)^{\times}_{\mathbb{Q}}$. For $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$, the map

$$\alpha_{\psi} : \Gamma(D)^{\times}_{\mathbb{K}} \longrightarrow \Gamma(D + (\psi))^{\times}_{\mathbb{K}}, \quad \phi \longmapsto \phi\psi^{-1}$$

is a bijection. Moreover, for any $\phi \in \Gamma(D)^{\times}_{\mathbb{K}}$,

$$\|\phi\|_g = \|\alpha_{\psi}(\phi)\|_{g+g_{(\psi)}}.$$

Hence, one has

$$\lambda_{\mathbb{Q},\mathrm{ess}}(D, g) = \lambda_{\mathbb{Q},\mathrm{ess}}(D + (\psi), g + g_{(\psi)}), \tag{5.9}$$

$$\lambda_{\mathrm{ess}}(D, g) = \lambda_{\mathrm{ess}}(D + (\psi), g + g_{(\psi)}). \tag{5.10}$$

Furthermore, for any $c \in \mathbb{R}$, one has

$$\lambda_{\mathbb{Q},\mathrm{ess}}(D, g + c) = \lambda_{\mathbb{Q},\mathrm{ess}}(D, g) + c, \tag{5.11}$$

$$\lambda_{\mathrm{ess}}(D, g + c) = \lambda_{\mathrm{ess}}(D, g) + c. \tag{5.12}$$

Therefore, to prove the proposition, we may assume without loss of generality that D is effective and $\varphi_g \geqslant 0$.

By definition one has $\lambda_{\mathbb{Q},\mathrm{ess}}(D, g) \leqslant \lambda_{\mathrm{ess}}(D, g)$. To show the converse inequality, it suffices to prove that, for any $s \in \Gamma(D)^{\times}_{\mathbb{R}}$, one has

$$-\ln\|s\|_g \leqslant \lambda_{\mathbb{Q},\mathrm{ess}}(D, g).$$

We choose $s_1, \ldots, s_r$ in $\mathrm{Rat}(X)^{\times}_{\mathbb{Q}}$ and $a_1, \ldots, a_r$ in $\mathbb{R}_{>0}$ such that $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$ and that $s = s_1^{a_1} \cdots s_r^{a_r}$. By Lemma 5.9, for any $i \in \{1, \ldots, r\}$, the support of (s_i) is contained in that of (s). Assume that $\mathrm{Supp}((s)) = \{x_1, \ldots, x_n\}$. Since $s \in \Gamma(D)^{\times}_{\mathbb{R}}$, for $j \in \{1, \ldots, n\}$, one has

$$a_1 \operatorname{ord}_{x_j}(s_1) + \cdots + a_r \operatorname{ord}_{x_j}(s_r) + \operatorname{ord}_{x_j}(D) \geqslant 0. \tag{5.13}$$

By Lemma 5.10, for any rational number $\varepsilon > 0$, there exists a sequence

$$(\delta_1^{(m)}, \ldots, \delta_r^{(m)}), \quad m \in \mathbb{N}$$

in $\mathbb{R}^r$, which converges to $(0, \ldots, 0)$, and such that:

(1) for any $j \in \{1, \dots, n\}$ and any $m \in \mathbb{N}$, one has

$$\delta_1^{(m)} \operatorname{ord}_{x_j}(s_1) + \cdots + \delta_r^{(m)} \operatorname{ord}_{x_j}(s_r) + \varepsilon \operatorname{ord}_{x_j}(D) \geqslant 0;$$

(2) for any $i \in \{1, \dots, r\}$ and any $m \in \mathbb{N}$, $\delta_i^{(m)} + a_i \in \mathbb{Q}$.

For any $m \in \mathbb{N}$, let

$$s^{(m)} = s_1^{\delta_1^{(m)}} \cdots s_r^{\delta_r^{(m)}} \in \Gamma(\varepsilon D)_{\mathbb{R}}^{\times}.$$

The conditions (1) and (2) above imply that $s \cdot s^{(m)} \in \Gamma((1+\varepsilon)D)_{\mathbb{Q}}^{\times}$. Hence, one has

$$\inf_{\xi \in X^{\mathrm{an}}} \big((1+\varepsilon)g + g_{(s \cdot s^{(m)})}\big)(\xi) \leqslant \lambda_{\mathbb{Q},\mathrm{ess}}((1+\varepsilon)D, (1+\varepsilon)g).$$

Since D is effective and $\varphi_g \geqslant 0$ by $s^{(m)} \in \Gamma(\varepsilon D)_{\mathbb{R}}^{\times}$, one has

$$\varepsilon g + g_{(s^{(m)})} \geqslant \varepsilon \varphi_g \geqslant 0.$$

Therefore, we obtain

$$-\ln\|s\|_g = \inf_{\xi \in X^{\mathrm{an}}} (g + g_{(s)})(\xi) \leqslant \lambda_{\mathbb{Q},\mathrm{ess}}((1+\varepsilon)D, (1+\varepsilon)g) = (1+\varepsilon)\lambda_{\mathbb{Q},\mathrm{ess}}(D, g),$$

where the last equality comes from (5.8). Taking the limit when $\varepsilon \in \mathbb{Q}_{>0}$ tends to 0, we obtain the desired inequality. ■

5.4 χ-volume

Let (D, g) be a metrised $\mathbb{R}$-divisor on X. We define the χ-*volume* of (D, g) as

$$\widehat{\mathrm{vol}}_{\chi}(D, g) := \limsup_{n \to +\infty} \frac{\widehat{\deg}(H^0(nD), \|\cdot\|_{ng})}{n^2/2}.$$

This invariant is similar to the χ-volume function in the number field setting introduced in [29]. Note that, if $\deg(D) < 0$, then $H^0(D) = \{0\}$, so that $H^0(nD) = \{0\}$ for all $n \in \mathbb{Z}_{>0}$. Indeed, if $f \in H^0(D) \setminus \{0\}$, then $0 \leqslant \deg(D + (f)) = \deg(D) < 0$, which is a contradiction. Hence, $\widehat{\mathrm{vol}}_{\chi}(D, g) = 0$.

The following two propositions are easily proved by using the definition of $\widehat{\mathrm{vol}}_{\chi}$ (left as an exercise to the reader).

Proposition 5.13. Let D be an $\mathbb{R}$-divisor on X, and g and g' be Green functions of D. If $g \leqslant g'$, then $\widehat{\mathrm{vol}}_\chi(D,g) \leqslant \widehat{\mathrm{vol}}_\chi(D,g')$.

Proposition 5.14. Let (D,g) be a metrised $\mathbb{R}$-divisor such that $\deg(D) \geqslant 0$. For any $c \in \mathbb{R}$, one has

$$\widehat{\mathrm{vol}}_\chi(D,g+c) = 2c\deg(D) + \widehat{\mathrm{vol}}_\chi(D,g). \tag{5.14}$$

Definition 5.15. Let (D,g) be a metrised $\mathbb{R}$-divisor such that $\deg(D) > 0$. We denote by $\Gamma(D,g)^\times_{\mathbb{R}}$ the set of $s \in \Gamma(D)^\times_{\mathbb{R}}$ such that $\|s\|_g < 1$. Similarly, we denote by $\Gamma(D,g)^\times_{\mathbb{Q}}$ the set of $s \in \Gamma(D)^\times_{\mathbb{Q}}$ such that $\|s\|_g < 1$. For any $t \in \mathbb{R}$ such that $t < \lambda_{\mathrm{ess}}(D,g)$, we let $D_{g,t}$ be the $\mathbb{R}$-divisor

$$\sup_{s\in\Gamma(D,g-t)^\times_{\mathbb{Q}}} (s^{-1}).$$

For sufficiently negative number t such that $\|s\|_g < \mathrm{e}^{-t}$ for any $s \in \Gamma(D)^\times_{\mathbb{R}}$, one has

$$\Gamma(D,g-t)^\times_{\mathbb{Q}} = \Gamma(D)^\times_{\mathbb{Q}}$$

and hence, by Proposition 4.10, $D_{g,t} = D$. If $t \geqslant \lambda_{\mathrm{ess}}(D,g)$, by convention we let $D_{g,t}$ be the zero $\mathbb{R}$-divisor.

Proposition 5.16. Let (D,g) be a metrised $\mathbb{R}$-divisor such that $\deg(D) > 0$, and $t \in \mathbb{R}$ such that $t < \lambda_{\mathrm{ess}}(D,g)$. Let

$$V^t_\bullet(D,g) := \bigoplus_{n\in\mathbb{N}} \{s \in H^0(nD) \,:\, \|s\|_{ng} < \mathrm{e}^{-tn}\}T^n \subseteq K[T].$$

Then one has

$$\lim_{n\to+\infty} \frac{\dim_k(V^t_n(D,g))}{n} = \deg(D_{g,t}) > 0. \tag{5.15}$$

Proof. By Proposition 4.16, it suffices to show that the graded linear series $V^t_\bullet(D,g)$ is birational (see Definition 4.12). As $\deg(D) > 0$, there exists $m \in \mathbb{N}_{\geqslant 1}$ such that $k(H^0(mD)) = \mathrm{Rat}(X)$ (see Example 4.13 and Proposition 4.14). Note that the norm $\|\cdot\|_{mg}$ is a bounded function on $H^0(mD)$. In fact, if $(s_i)_{i=1}^{r_m}$ is a basis of $H^0(kmD)$, as the norm $\|\cdot\|_{mg}$ is ultrametric, for any $(\lambda_i)_{i=1}^{r_m} \in k^{r_m}$, one has

$$\|\lambda_1 s_1 + \cdots + \lambda_{r_m} s_{r_m}\|_{mg} \leqslant \max_{i\in\{1,\ldots,r_m\}} \|s_i\|_{mg}.$$

We choose $\varepsilon > 0$ such that $t + \varepsilon < \lambda_{\mathrm{ess}}(D, g)$. By (5.7) we obtain that there exist $n \in \mathbb{N}_{\geqslant 1}$ and $s \in H^0(nD)$ such that $\|s\|_{ng} \leqslant \mathrm{e}^{-n(t+\varepsilon)}$. Let d be a positive integer such that

$$d > \frac{1}{n\varepsilon}\Big(tm + \max_{i \in \{1,\ldots,r_m\}} \ln\|s_i\|_{mg}\Big).$$

Then, for any $s' \in H^0(mD)$, one has

$$\|s^d s'\|_{(dn+m)g} < \mathrm{e}^{-(dn+m)t},$$

which means that $s^d s' \in V^t_{dn+m}(D, g)$. Therefore, we obtain $k(V^t_{dn+m}(D, g)) = \mathrm{Rat}(X)$ since it contains $k(H^0(mD))$. The graded linear series $V^t_\bullet(D, g)$ is thus birational and (5.15) is proved. ∎

Theorem 5.17. Let (D, g) be a metrised $\mathbb{R}$-divisor such that $\deg(D) > 0$. Let $\mathbb{P}_{(D,g)}$ be the Borel probability measure on $\mathbb{R}$ such that

$$\mathbb{P}_{(D,g)}(]t, +\infty[) = \deg(D_{g,t}) \tag{5.16}$$

for $t < \lambda_{\mathrm{ess}}(D, g)$ and $\mathbb{P}_{(D,g)}(]t, +\infty[) = 0$ for $t \geqslant \lambda_{\mathrm{ess}}(D, g)$. Then one has

$$\frac{\widehat{\mathrm{vol}}_\chi(D, g)}{2\deg(D)} = \int_{\mathbb{R}} t\, \mathbb{P}_{(D,g)}(\mathrm{d}t). \tag{5.17}$$

Proof. For any $n \in \mathbb{N}$, let $\mathbb{P}_n$ be the Borel probability measure on $\mathbb{R}$ such that

$$\mathbb{P}_n(]t, +\infty[) = \frac{\dim_k(V^t_n(D, g))}{\dim_k(H^0(nD))}$$

for $t < \lambda_{\mathrm{ess}}(D, g)$ and $\mathbb{P}_n(]t, +\infty[) = 0$ for $t \geqslant \lambda_{\mathrm{ess}}(D, g)$. By Propositions 5.16 and 4.9, one has

$$\forall\, t \in \mathbb{R}, \quad \lim_{n \to +\infty} P_n(]t, +\infty[) = \mathbb{P}_{(D,g)}(]t, +\infty[).$$

Therefore, the sequence of probability measures $(\mathbb{P}_n)_{n \in \mathbb{N}}$ converges weakly to $\mathbb{P}$. Moreover, if we write g as $g_D + \varphi_g$, where φ_g is a continuous function on X^{an}, then the supports of the probability measures P_n are contained in $[\inf \varphi_g, g(\eta_0)]$. Therefore, one has

$$\lim_{n\to+\infty}\int_{\mathbb{R}} t\,\mathbb{P}_n(\mathrm{d}t)=\int_{\mathbb{R}} t\,\mathbb{P}_{(D,g)}(\mathrm{d}t).$$

By (5.3), for any $n\in\mathbb{N}_{\geqslant 1}$ such that $H^0(nD)\neq\{0\}$, one has

$$\int_{\mathbb{R}} t\,\mathbb{P}_n(\mathrm{d}t)=\frac{\widehat{\deg}(H^0(nD),\|\cdot\|_{ng})}{\dim_k(H^0(nD))}.$$

Therefore, we obtain (5.17). ■

Remark 5.18. Theorem 5.17 and Proposition 4.9 show that the sequence defining the χ-volume function has a limit. More precisely, if (D,g) is a metrised $\mathbb{R}$-divisor such that $\deg(D)>0$, then one has

$$\widehat{\mathrm{vol}}_\chi(D,g)=\lim_{n\to+\infty}\frac{\widehat{\deg}(H^0(nD),\|\cdot\|_{ng})}{n^2/2}.$$

Definition 5.19. Let (D,g) be a metrised $\mathbb{R}$-Cartier divisor on X such that $\deg(D)>0$. We denote by $G_{(D,g)}:[0,\deg(D)]\to\mathbb{R}$ the function sending $u\in[0,\deg(D)]$ to

$$\sup\{t\in\mathbb{R}_{<g(\eta)}\;:\;\deg(D_{g,t})>u\}.$$

For any $t<g(\eta_0)$ one has

$$\mathbb{P}_{(D,g)}(]G_{(D,g)}(\lambda),+\infty[)=\frac{\deg(D_{g,G_{(D,g)}(\lambda)})}{\deg(D)},$$

namely, the probability measure $\mathbb{P}_{(D,g)}$ coincides with the direct image of the uniform distribution on $[0,\deg(D)]$ by the map $G_{(D,g)}$.

Proposition 5.20. Let (D,g) be a metrised $\mathbb{R}$-divisor such that $\deg(D)>0$. For any $t\in\mathbb{R}$ such that $t<\lambda_{\mathrm{ess}}(D,g)$, one has

$$D_{g,t}=\sup_{s\in\Gamma(D,g-t)^\times_{\mathbb{R}}}(s^{-1}). \tag{5.18}$$

Proof. Since $\deg(D)>0$, the set $\Gamma(D)^\times_{\mathbb{Q}}$ is not empty. Let $\phi\in\Gamma(D)^\times_{\mathbb{Q}}$ and $(D',g')=(D,g)+\widehat{(\phi)}$. By (5.9), one has $\lambda_{\mathrm{ess}}(D,g)=\lambda_{\mathrm{ess}}(D',g')$. Moreover, the correspondence

$s \mapsto s \cdot \phi^{-1}$ defines a bijection from $\Gamma(D, g-t)_{\mathbb{K}}^{\times}$ to $\Gamma(D', g'-t)_{\mathbb{K}}^{\times}$ for $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Therefore, without loss of generality, we may assume that D is effective. Moreover, by replacing g by $g-t$ and t by 0 we may assume that $\lambda_{\mathrm{ess}}(D,g) > 0$ and $t=0$.

It suffices to check that $D_{g,0} \geqslant (s^{-1})$ for any $s \in \Gamma(D,g)_{\mathbb{R}}^{\times}$. We write s as $s_1^{a_1} \cdots s_r^{a_r}$, where $s_1, \ldots, s_r$ are elements of $\mathrm{Rat}(X)_{\mathbb{Q}}^{\times}$, and $a_1, \ldots, a_r$ are positive real numbers which are linearly independent over $\mathbb{Q}$. Assume that $\mathrm{Supp}((s)) = \{x_1, \ldots, x_n\}$. By Lemma 5.9, for any $i \in \{1, \ldots, r\}$, the support of (s_i) is contained in $\{x_1, \ldots, x_n\}$. For any $j \in \{1, \ldots, n\}$, one has

$$\mathrm{ord}_{x_j}(D) + \sum_{i=1}^{r} \mathrm{ord}_{x_j}(s_i) a_i \geqslant 0.$$

By Lemma 5.10 and Remark 5.11, there exists a sequence of vectors

$$\boldsymbol{a}^{(m)} = (a_1^{(m)}, \ldots, a_r^{(m)}), \quad m \in \mathbb{N}$$

in $\mathbb{Q}^r$ such that

$$\mathrm{ord}_{x_j}(D) + \sum_{i=1}^{r} \mathrm{ord}_{x_j}(s_i) a_i^{(m)} \geqslant 0 \tag{5.19}$$

and

$$\lim_{m \to +\infty} \boldsymbol{a}^{(m)} = (a_1, \ldots, a_r). \tag{5.20}$$

For any $m \in \mathbb{N}$, let

$$s^{(m)} = s_1^{a_1^{(m)}} \cdots s_r^{a_r^{(m)}}.$$

By (5.19) one has $s^{(m)} \in \Gamma(D)_{\mathbb{Q}}^{\times}$. Moreover, by (5.20) and the fact that $\|s\|_g < 1$, for sufficiently positive m, one has $\|s^{(m)}\|_g < 1$ and hence $D_{g,0} \geqslant ((s^{(m)})^{-1})$. By taking the limit when $m \to +\infty$, we obtain $D_{g,0} \geqslant (s^{-1})$. ■

Corollary 5.21. Let (D,g) be a metrised $\mathbb{R}$-Cartier divisor such that $\deg(D) > 0$. For any $a > 0$ one has

$$\widehat{\mathrm{vol}}_{\chi}(aD, ag) = a^2 \, \widehat{\mathrm{vol}}_{\chi}(D,g).$$

Proof. Let M be a sufficiently positive real number such that $D_{g,-M} = D$. By Proposition 5.20 one has

$$(aD)_{ag,at} = aD_{g,t}.$$

By (5.17) and integration by parts one has

$$\begin{aligned}\widehat{\mathrm{vol}}_\chi(aD,ag) &= -2\deg(aD)\int_{-M}^{\lambda_{\mathrm{ess}}(D,g)} at\,\mathrm{d}\deg((aD)_{ag,at})\\ &= 2\int_{-M}^{\lambda_{\mathrm{ess}}(D,g)}\deg((aD)_{ag,at})\,\mathrm{d}at + 2aM\deg(aD)\\ &= 2a^2\int_{-M}^{\lambda_{\mathrm{ess}}(D,g)}\deg(D_{g,t})\,\mathrm{d}t + 2a^2M\deg(D) = a^2\,\widehat{\mathrm{vol}}_\chi(D,g).\end{aligned}$$

■

Theorem 5.22. Let (D_1, g_1) and (D_2, g_2) be metrised $\mathbb{R}$-Cartier divisors such that $\deg(D_1) > 0$ and $\deg(D_2) > 0$. One has

$$\frac{\widehat{\mathrm{vol}}_\chi(D_1+D_2, g_1+g_2)}{\deg(D_1)+\deg(D_2)} \geqslant \frac{\widehat{\mathrm{vol}}_\chi(D_1,g_1)}{\deg(D_1)} + \frac{\widehat{\mathrm{vol}}_\chi(D_2,g_2)}{\deg(D_2)}$$

Proof. Let t_1 and t_2 be real numbers such that $t_1 < \lambda_{\mathrm{ess}}(D_1, g_2)$ and $t_2 < \lambda_{\mathrm{ess}}(D_2, g_2)$. For all $s_1 \in \Gamma(D_1, g_1 - t_1)^\times_{\mathbb{R}}$ and $s_2 \in \Gamma(D_2, g_2 - t_2)^\times_{\mathbb{R}}$ one has

$$s_1 s_2 \in \Gamma(D_1 + D_2, g_1 + g_2 - t_1 - t_2)^\times_{\mathbb{R}}.$$

Therefore, by Proposition 5.20 one has

$$(D_1 + D_2)_{g_1+g_2,t_1+t_2} \geqslant (D_1)_{g_1,t_1} + (D_2)_{g_2,t_2}. \tag{5.21}$$

As a consequence, for any $(\lambda_1, \lambda_2) \in [0, \deg(D_1)] \times [0, \deg(D_2)]$, one has

$$G_{(D_1+D_2,g_1+g_2)}(\lambda_1 + \lambda_2) \geqslant G_{(D_1,g_1)}(\lambda_1) + G_{(D_2,g_2)}(\lambda_2). \tag{5.22}$$

Let U be a random variable that follows the uniform distribution on $[0, \deg(D_1)]$. Let $f : [0, \deg(D_1)] \to [0, \deg(D_2)]$ be the linear map sending u to $u\deg(D_2)/\deg(D_1)$. By Theorem 5.17 one has

$$\frac{\widehat{\mathrm{vol}}_\chi(D_1+D_2, g_1+g_2)}{2(\deg(D_1)+\deg(D_2))} = \mathbb{E}[G_{(D_1+D_2,g_1+g_2)}(U + f(U))]$$

since $U + f(U)$ follows the uniform distribution on $[0, \deg(D_1) + \deg(D_2)]$. By (5.22) we obtain

$$\begin{aligned}\frac{\widehat{\mathrm{vol}}_\chi(D_1 + D_2, g_1 + g_2)}{2(\deg(D_1) + \deg(D_2))} &\geqslant \mathbb{E}[G_{(D_1,g_1)}(U)] + \mathbb{E}[G_{(D_2,g_2)}(f(U))] \\ &\geqslant \frac{\widehat{\mathrm{vol}}_\chi(D_1, g_1)}{2\deg(D_1)} + \frac{\widehat{\mathrm{vol}}_\chi(D_2, g_2)}{2\deg(D_2)}.\end{aligned}$$

The theorem is thus proved. ■

Finally, let us consider other properties of $\widehat{\mathrm{vol}}_\chi(\cdot)$. First of all, it is easy to see the following (left as an exercise to the reader).

Proposition 5.23. Let D be an $\mathbb{R}$-divisor on X such that $\deg(D) \geqslant 0$, and g and g' be Green functions of D. Then one has the following:

(1) $2\deg(D) \min_{\xi \in X^{\mathrm{an}}} \{\varphi_g(\xi)\} \leqslant \widehat{\mathrm{vol}}_\chi(D, g) \leqslant 2\deg(D) \max_{\xi \in X^{\mathrm{an}}} \{\varphi_g(\xi)\}$.

(2) $|\widehat{\mathrm{vol}}_\chi(D, g) - \widehat{\mathrm{vol}}_\chi(D, g')| \leqslant 2\|\varphi_g - \varphi_{g'}\|_{\sup} \deg(D)$.

(3) If $\deg(D) = 0$, then $\widehat{\mathrm{vol}}_\chi(D, g) = 0$.

Next let us consider the continuity of $\widehat{\mathrm{vol}}_\chi$.

Proposition 5.24. Let V be a finite-dimensional vector subspace of $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$. Then $\widehat{\mathrm{vol}}_\chi(\cdot)$ is continuous on V.

Proof. We denote by V_+ the subset of those (D, g) such that $\deg(D) > 0$. The function $V_+ \to \mathbb{R}$ given by $(D, g) \mapsto \widehat{\mathrm{vol}}_\chi(D, g)/\deg(D)$ is concave by Corollary 5.21 and Theorem 5.22, and hence it is continuous on V_+.

We fix $(D, g) \in V$. If $\deg(D) < 0$, then there exists a neighbourhood U of (D, g) in V such that $\deg(D') < 0$ for any $(D', g') \in U$. Hence, $\widehat{\mathrm{vol}}_\chi(\cdot)$ vanishes on U. If $\deg(D) > 0$, then the above observation shows the continuity at (D, g), so that we may assume that $\deg(D) = 0$. Then, by (3) of Proposition 5.23, $\widehat{\mathrm{vol}}_\chi(D, g) = 0$. Therefore, it is sufficient to show that

$$\lim_{(\varepsilon_{1,n}, \ldots, \varepsilon_{r,n}) \to (0, \ldots, 0)} \widehat{\mathrm{vol}}_\chi(\varepsilon_{1,n}(D_1, g_1) + \cdots + \varepsilon_{r,n}(D_r, g_r) + (D, g)) = 0,$$

where $(D_1,g_1),\ldots,(D_r,g_r)\in V$. By using (1) of Proposition 5.23,

$$
\begin{aligned}
&|\widehat{\mathrm{vol}}_\chi(\varepsilon_{1,n}(D_1,g_1)+\cdots+\varepsilon_{r,n}(D_r,g_r)+(D,g))|\\
&\qquad\leqslant 2\|\varepsilon_{1,n}\varphi_{g_1}+\cdots+\varepsilon_{r,n}\varphi_{g_r}+\varphi_g\|_{\sup}\deg(\varepsilon_{1,n}D_1+\cdots+\varepsilon_{r,n}D_r+D)
\end{aligned}
$$

On the other hand, note that

$$
\begin{cases}
\displaystyle\lim_{(\varepsilon_{1,n},\ldots,\varepsilon_{r,n})\to(0,\ldots,0)}\|\varepsilon_{1,n}\varphi_{g_1}+\cdots+\varepsilon_{r,n}\varphi_{g_r}+\varphi_g\|_{\sup}=\|\varphi_g\|_{\sup},\\
\displaystyle\lim_{(\varepsilon_{1,n},\ldots,\varepsilon_{r,n})\to(0,\ldots,0)}\deg(\varepsilon_{1,n}D_1+\cdots+\varepsilon_{r,n}D_r+D)=\deg(D)=0.
\end{cases}
$$

Thus, the assertion follows. ■

5.5 Volume function

Let (D,g) be a metrised $\mathbb{R}$-divisor on X. We define the *volume* of (D,g) as

$$
\widehat{\mathrm{vol}}(D,g):=\limsup_{n\to+\infty}\frac{\widehat{\deg}_+(nD,ng)}{n^2/2}.
$$

Note that this function is analogous to the arithmetic volume function introduced in [23].

Proposition 5.25. Let (D,g) be a metrised $\mathbb{R}$-divisor such that $\deg(D)>0$. Let $\mathbb{P}_{(D,g)}$ be the Borel probability measure on $\mathbb{R}$ defined in Theorem 5.17. Then one has

$$
\frac{\widehat{\mathrm{vol}}(D,g)}{2\deg(D)}=\int_{\mathbb{R}}\max\{t,0\}\,\mathbb{P}_{(D,g)}(\mathrm{d}t), \tag{5.23}
$$

$$
\widehat{\mathrm{vol}}(D,g)=\int_0^{+\infty}\deg(D_{g,t})\,\mathrm{d}t. \tag{5.24}
$$

Proof. We keep the notation introduced in the proof of Theorem 5.17. By (5.4), for any $n\in\mathbb{N}_{\geqslant 1}$ one has

$$
\frac{\widehat{\deg}_+(H^0(nD),\|\cdot\|_{ng})}{\dim_k(H^0(nD))}=\int_{\mathbb{R}}\max\{t,0\}\,\mathbb{P}_n(\mathrm{d}t).
$$

By passing to limit when $n\to+\infty$, we obtain the 1st equality. The 2nd equality comes from the 1st one and (5.16) by integration by part. ■

6 Positivity

The purpose of this section is to discuss several positivity conditions of metrised $\mathbb{R}$-divisors. We fix in this section a field k equipped with the trivial absolute value $|\cdot|$ and a regular integral projective curve X sur $\operatorname{Spec} k$.

6.1 Bigness and pseudo-effectivity

Let (D, g) be a metrised $\mathbb{R}$-divisor on X. If $\widehat{\operatorname{vol}}(D, g) > 0$, we say that (D, g) is *big*; if for any big metrised $\mathbb{R}$-divisor (D_0, g_0) on X, the metrised $\mathbb{R}$-divisor $(D + D_0, g + g_0)$ is big, we say that (D, g) is *pseudo-effective*.

Remark 6.1. Let (D, g) be a metrised $\mathbb{R}$-divisor. Let $n \in \mathbb{N}$, $n \geqslant 1$. If $H^0(nD) \neq \{0\}$, then $\Gamma(D)^{\times}_{\mathbb{Q}}$ is not empty. Moreover, for any non-zero element $s \in H^0(nD)$, one has

$$-\ln\|s\|_g \leqslant n\lambda_{\mathrm{ess}}(D, g)$$

by (5.7) and (5.6) and Proposition 5.8. In particular, one has

$$\widehat{\deg}_+(H^0(nD), \|\cdot\|_{ng}) \leqslant n \max\{\lambda_{\mathrm{ess}}(D, g), 0\} \dim_k(H^0(nD)).$$

Therefore, if $\widehat{\operatorname{vol}}(D, g) > 0$, then one has $\deg(D) > 0$ and $\lambda_{\mathrm{ess}}(D, g) > 0$. Moreover, in the case where (D, g) is big, one has

$$\frac{\widehat{\operatorname{vol}}(D, g)}{2\deg(D)} \leqslant \lambda_{\mathrm{ess}}(D, g). \tag{6.1}$$

Proposition 6.2. Let (D, g) be a metrised divisor on X. The following assertions are equivalent:

(1) (D, g) is big.
(2) $\deg(D) > 0$ and $\lambda_{\mathrm{ess}}(D, g) > 0$
(3) $\deg(D) > 0$ and there exists $s \in \Gamma(D)^{\times}_{\mathbb{R}}$ such that $\|s\|_g < 1$.
(4) $\deg(D) > 0$ and there exists $s \in \Gamma(D)^{\times}_{\mathbb{Q}}$ such that $\|s\|_g < 1$.

Proof. "(1) ⇔ (2)" We have seen in the above Remark that, if (D, g) is big, then $\deg(D) > 0$ and $\lambda_{\mathrm{ess}}(D, g) > 0$. The converse comes from the equality

$$\widehat{\operatorname{vol}}(D, g) = \int_0^{+\infty} \deg(D_{g,t})\,\mathrm{d}t$$

proved in Proposition 5.25. Note that the function $t \mapsto \deg(D_{g,t})$ is decreasing. Moreover, by Proposition 5.16, one has $\deg(D_{g,t}) > 0$ once $t < \lambda_{\mathrm{ess}}(D,g)$. Therefore, if $\lambda_{\mathrm{ess}}(D,g) > 0$, then $\widehat{\mathrm{vol}}(D,g) > 0$.

"(2)⇔(3)" comes from the definition of $\lambda_{\mathrm{ess}}(D,g)$.

"(2)⇔(4)" comes from Proposition 5.12. ■

Corollary 6.3.

(1) If (D,g) is a big metrised $\mathbb{R}$-divisor on X, then, for any positive real number ε, the metrised $\mathbb{R}$-divisor $\varepsilon(D,g) = (\varepsilon D, \varepsilon g)$ is big.

(2) If (D_1,g_1) and (D_2,g_2) are two metrised $\mathbb{R}$-divisor on X, which are big, then (D_1+D_2, g_1+g_2) is also big.

Proof. The 1st assertion follows from Proposition 6.2 and the equalities $\deg(\varepsilon D) = \varepsilon \deg(D)$ and $\lambda_{\mathrm{ess}}(\varepsilon(D,g)) = \varepsilon\lambda_{\mathrm{ess}}(D,g)$.

We then prove the 2nd assertion. Since (D_1,g_1) and (D_2,g_2) are big, one has $\deg(D_1) > 0$, $\deg(D_2) > 0$, $\lambda_{\mathrm{ess}}(D_1,g_1) > 0$, $\lambda_{\mathrm{ess}}(D_2,g_2) > 0$. Therefore, $\deg(D_1+D_2) = \deg(D_1) + \deg(D_2) > 0$. Moreover, by (5.5) one has

$$\lambda_{\mathrm{ess}}(D_1+D_2, g_1+g_2) \geqslant \lambda_{\mathrm{ess}}(D_1,g_1) + \lambda_{\mathrm{ess}}(D_2,g_2) > 0.$$

Therefore, (D_1+D_2, g_1+g_2) is big. ■

Corollary 6.4. Let (D,g) be a metrised $\mathbb{R}$-divisor on X such that $\deg(D) > 0$. Then (D,g) is pseudo-effective if and only if $\lambda_{\mathrm{ess}}(D,g) \geqslant 0$.

Proof. Suppose that (D,g) is pseudo-effective. Since $\deg(D) > 0$, by (5.12) there exists $c > 0$ such that $\lambda_{\mathrm{ess}}(D,g+c) > 0$ (and thus $(D,g+c)$ is big by Proposition 6.2). Hence, for any $\varepsilon \in \,]0,1[$,

$$(1-\varepsilon)(D,g) + \varepsilon(D,g+c) = (1-\varepsilon)\Big((D,g) + \frac{\varepsilon}{1-\varepsilon}(D,g+c)\Big)$$

is big. Therefore,

$$\lambda_{\mathrm{ess}}\big((1-\varepsilon)(D,g) + \varepsilon(D,g+c)\big) = \lambda_{\mathrm{ess}}(D,g+\varepsilon c) = \lambda_{\mathrm{ess}}(D,g) + \varepsilon c > 0.$$

Since $\varepsilon \in \,]0,1[$ is arbitrary, we obtain $\lambda_{\mathrm{ess}}(D,g) \geqslant 0$.

In the following, we assume that $\lambda_{\mathrm{ess}}(D,g) \geqslant 0$ and we prove that (D,g) is pseudo-effective. For any big metrised $\mathbb{R}$-divisor (D_1,g_1) one has

$$\deg(D+D_1) = \deg(D) + \deg(D_1) > 0$$

and, by (5.5),

$$\lambda_{\mathrm{ess}}(D+D_1, g+g_1) \geqslant \lambda_{\mathrm{ess}}(D,g) + \lambda_{\mathrm{ess}}(D_1,g_1) > 0.$$

Therefore, $(D+D_1, g+g_1)$ is big. ■

Proposition 6.5. Let (D,g) be a metrised $\mathbb{R}$-divisor on X, which is pseudo-effective. Then one has $\deg(D) \geqslant 0$ and $g(\eta_0) \geqslant 0$.

Proof. Let (D_1,g_1) be a big metrised $\mathbb{R}$-divisor. For any $\varepsilon > 0$, the metrised $\mathbb{R}$-divisor $(D+\varepsilon D_1, g+\varepsilon g_1)$ is big. Therefore, by Proposition 6.2, one has

$$\deg(D+\varepsilon D_1) = \deg(D) + \varepsilon \deg(D_1) > 0.$$

Moreover, by Proposition 6.2, the inequality (5.6), and Proposition 5.8, one has

$$g(\eta_0) + \varepsilon g_1(\eta_0) \geqslant \lambda_{\mathrm{ess}}(D+\varepsilon D_1, g+\varepsilon g_1) > 0.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\deg(D) \geqslant 0$ and $g(\eta_0) \geqslant 0$. ■

6.2 Criteria of effectivity up to $\mathbb{R}$-linear equivalence

Let (D,g) be a metrised $\mathbb{R}$-divisor on X. We say that (D,g) is *effective* if D is effective and g is a non-negative function. We say that two metrised $\mathbb{R}$-divisor are $\mathbb{R}$-*linearly equivalent* if there exists an element $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that

$$(D_2,g_2) = (D_1,g_1) + \widehat{(\varphi)}.$$

By Proposition 6.2, if (D,g) is big, then it is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor.

Definition 6.6. Let (D, g) be a metrised $\mathbb{R}$-divisor on X. We denote by $\mu_{\inf}(g)$ the value

$$\sum_{x\in X^{(1)}} \mu_{\inf,x}(g)[\kappa(x):k] \in [-\infty, \infty[,$$

where by definition (see Section 3.5)

$$\mu_{\inf,x}(g) = \inf_{\xi\in]\eta_0,x_0[} \frac{g(\xi)}{t(\xi)}.$$

Note that

$$\mu_{\inf,x}(g) \leqslant \lim_{\xi\to x_0} \frac{g(\xi)}{t(\xi)} = \mathrm{ord}_x(D).$$

Therefore,

$$\mu_{\inf}(g) \leqslant \sum_{x\in X^{(1)}} \mathrm{ord}_x(D)[\kappa(x):k] = \deg(D). \tag{6.2}$$

Moreover, if D_1 is an $\mathbb{R}$-divisor and g_{D_1} is the canonical Green function associated with D_1, then one has

$$\forall\, x \in X^{(1)}, \quad \mu_{\inf,x}(g + g_{D_1}) = \mu_{\inf,x}(g) + \mathrm{ord}_x(D_1) \tag{6.3}$$

and hence

$$\mu_{\inf}(g + g_{D_1}) = \mu_{\inf}(g) + \deg(D_1). \tag{6.4}$$

The invariant $\mu_{\inf}(\cdot)$ is closely related to the effectivity of a metrised $\mathbb{R}$-divisor.

Proposition 6.7. Let (D, g) be a metrised $\mathbb{R}$-divisor. Assume that there exists an element $\phi \in \Gamma(D)^{\times}_{\mathbb{R}}$ such that $g + g_{(\phi)} \geqslant 0$. Then for all but a finite number of $x \in X^{(1)}$ one has $\mu_{\inf,x}(g) = 0$. Moreover, $\mu_{\inf}(g) \geqslant 0$.

Proof. By (6.3) , for any $x \in X^{(1)}$ one has

$$\mu_{\inf,x}(g + g_{(\phi)}) = \mu_{\inf,x}(g) + \mathrm{ord}_x(\phi).$$

Therefore, for all but a finite number of $x \in X^{(1)}$, one has

$$\mu_{\inf,x}(g) = \mu_{\inf,x}(g + g_{(\phi)}) \geqslant 0.$$

Note that $\mu_{\inf,x}(g) \leqslant \operatorname{ord}_x(D)$ for any $x \in X^{(1)}$, and hence $\mu_{\inf,x}(g) \leqslant 0$ for $x \in X^{(1)} \setminus \operatorname{Supp}(D)$. We then deduce that $\mu_{\inf,x}(g)$ vanishes for all but finitely many $x \in X^{(1)}$. Moreover, by (6.4) one has

$$\mu_{\inf}(g) = \mu_{\inf}(g + g_{(\phi)}) \geqslant 0.$$

■

Proposition 6.8. Let (D, g) be a metrised $\mathbb{R}$-divisor on X.

(1) (D, g) is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor if and only if there exists $s \in \Gamma(D)^{\times}_{\mathbb{R}}$ with $\|s\|_g \leqslant 1$.
(2) If (D, g) is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor, then (D, g) is pseudo-effective.
(3) Assume that $\mu_{\inf,x}(g) \geqslant 0$ for all but finitely many $x \in X^{(1)}$ and $\mu_{\inf}(g) > 0$, then (D, g) is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor.
(4) Assume that $\mu_{\inf,x}(g) \geqslant 0$ for all but finitely many $x \in X^{(1)}$, and $\mu_{\inf}(g) = 0$, then (D, g) is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor if and only if the $\mathbb{R}$-divisor $\sum_{x \in X^{(1)}} \mu_{\inf,x}(g) x$ is principal.

Proof. (1) Let s be an element of $\Gamma(D)^{\times}_{\mathbb{R}}$, one has

$$(D, g) + \widehat{(s)} = (D + (s), g_{(s)} + g).$$

By definition, $D + (s)$ is effective. Moreover,

$$-\ln\|s\|_g = \inf(g_{(s)} + g).$$

Therefore, $\|s\|_g \leqslant 1$ if and only if $g_{(s)} + g \geqslant 0$.

(2) Since there exists $s \in \Gamma(D)^{\times}_{\mathbb{R}}$ such that $\|s\|_g \leqslant 1$, one has $\lambda_{\mathrm{ess}}(D, g) \geqslant 0$ and $\deg(D) \geqslant 0$. Let (D_1, g_1) be a big metrised $\mathbb{R}$-divisor. By Proposition 6.2, one has $\deg(D) > 0$ and $\lambda_{\mathrm{ess}}(D, g) > 0$. Therefore,

$$\deg(D + D_1) = \deg(D) + \deg(D_1) > 0,$$

and, by Proposition 5.6,

$$\lambda_{\mathrm{ess}}(D+D_1,g+g_1)\geqslant\lambda_{\mathrm{ess}}(D,g)+\lambda_{\mathrm{ess}}(D_1,g_1)>0.$$

Still by Proposition 6.2, we obtain that $(D+D_1,g+g_1)$ is big.

(3) Let S be a finite subset of $X^{(1)}$, which contains $\mathrm{Supp}(D)$ and all $x\in X^{(1)}$ such that $\mu_{\mathrm{inf},x}(g)<0$ and satisfies the inequality

$$\sum_{x\in S}\mu_{\mathrm{inf},x}(g)[\kappa(x):k]>0.$$

Since the $\mathbb{R}$-divisor $\sum_{x\in S}\mu_{\mathrm{inf},x}(g)x$ has a positive degree, there exists an element φ of $\mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ such that

$$\mathrm{ord}_x(\varphi)\geqslant\begin{cases}-\mu_{\mathrm{inf},x}(g), & \text{if } x\in S,\\ 0, & \text{if } x\in X^{(1)}\setminus S.\end{cases}\tag{6.5}$$

Note that $\mu_{\mathrm{inf},x}(g)\leqslant\mathrm{ord}_x(D)$ for any $x\in X^{(1)}$. Hence, $\varphi\in\Gamma(D)_{\mathbb{R}}^{\times}$. Moreover, by (6.5) one has

$$g+g_{(\varphi)}\geqslant 0.$$

Hence, $(D,g)+\widehat{(\varphi)}$ is effective.

(4) Note that $\mu_{\mathrm{inf},x}(g)\leqslant\mathrm{ord}_x(D)=0$ for any $x\in X^{(1)}\setminus\mathrm{Supp}(D)$, we obtain that $\mu_{\mathrm{inf},x}(g)=0$ for all but finitely many $x\in X^{(1)}$. Therefore, $\sum_{x\in X^{(1)}}\mu_{\mathrm{inf},x}(g)x$ is well defined as an $\mathbb{R}$-divisor on X.

Assume that the $\mathbb{R}$-divisor $\sum_{x\in S}\mu_{\mathrm{inf},x}(g)x$ is principal, namely of the form (φ) for some $\varphi\in\mathrm{Rat}(X)_{\mathbb{R}}^{\times}$. Then the metrised $\mathbb{R}$-divisor

$$(D,g)-\widehat{(\varphi)}$$

is effective. Conversely, if ϕ is an element of $\mathrm{Rat}(X)_{\mathbb{R}}^{\times}$, which is different from $-\sum_{x\in X^{(1)}}\mu_{\mathrm{inf},x}(g)x$, then there exists $x\in X^{(1)}$ such that $\mathrm{ord}_x(\phi)<-\mu_{\mathrm{inf},x}(g)$ since

$$\sum_{x\in X^{(1)}}\mathrm{ord}_x(\phi)[\kappa(x):k]=-\sum_{x\in X^{(1)}}\mu_{\mathrm{inf},x}(g)[\kappa(x):k]=0.$$

Therefore, the function $g+g_{(\phi)}$ can not be non-negative. ■

Combining Propositions 6.7 and 6.8, we obtain the following criterion of effectivity up to $\mathbb{R}$-linear equivalence for metrised $\mathbb{R}$-divisors.

Theorem 6.9. Let (D, g) be a metrised $\mathbb{R}$-divisor on X. Then (D, g) is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor if and only if $\mu_{\mathrm{inf},x}(g) = 0$ for all but finitely many $x \in X^{(1)}$ and one of the following conditions holds:

(a) $\mu_{\mathrm{inf}}(g) > 0$,

(b) $\sum_{x\in X^{(1)}} \mu_{\mathrm{inf},x}(g)x$ is a principal $\mathbb{R}$-divisor on X.

6.3 Criterion of pseudo-effectivity

By using the criteria of effectivity up to $\mathbb{R}$-linear equivalence in the previous subsection, we prove a numerical criterion of pseudo-effectivity in terms of the invariant $\mu_{\mathrm{inf}}(\cdot)$.

Lemma 6.10. Let (D, g) be a metrised $\mathbb{R}$-divisor. Assume that $(D, g + \varepsilon)$ is pseudo-effective for any $\varepsilon > 0$. Then (D, g) is also pseudo-effective.

Proof. Let (D_1, g_1) be a big metrised $\mathbb{R}$-divisor. By Proposition 6.2, one has $\deg(D_1) > 0$ and $\lambda_{\mathrm{ess}}(D_1, g_1) > 0$. Let ε be a positive number such that $\varepsilon < \lambda_{\mathrm{ess}}(D_1, g_1)$. By (5.12) one has

$$\lambda_{\mathrm{ess}}(D_1, g_1 - \varepsilon) = \lambda_{\mathrm{ess}}(D_1, g_1) - \varepsilon > 0.$$

Hence, $(D_1, g_1 - \varepsilon)$ is big (by Proposition 6.2). Therefore,

$$(D, g) + (D_1, g_1) = (D + D_1, g + g_1) = (D, g + \varepsilon) + (D_1, g_1 - \varepsilon)$$

is big. ■

Proposition 6.11. A metrised $\mathbb{R}$-divisor (D, g) on X is pseudo-effective if and only if $\mu_{\mathrm{inf}}(g) \geqslant 0$.

Proof. "$\Longleftarrow$": For any $\varepsilon > 0$, one has $\mu_{\mathrm{inf}}(g + \varepsilon) > 0$. By Theorem 6.9, $(D, g + \varepsilon)$ is $\mathbb{R}$-linearly equivalent to an effective metrised $\mathbb{R}$-divisor, and hence is pseudo-effective (see Proposition 6.8 (2)). By Lemma 6.10, we obtain that (D, g) is pseudo-effective.

"$\Longrightarrow$": We begin with the case where $\deg(D) > 0$. If (D, g) is pseudo-effective, then by Corollary 6.4, one has $\lambda_{\mathrm{ess}}(D, g) \geqslant 0$. Hence, $(D, g + \varepsilon)$ is big for any $\varepsilon > 0$ (by

(5.12) and Proposition 6.2). In particular, one has $\mu_{\inf}(g+\varepsilon)\geqslant 0$ for any $\varepsilon>0$. For each $x\in X^{(1)}$, the function $(\varepsilon>0)\mapsto\mu_{\inf,x}(g+\varepsilon)$ is decreasing and bounded from below by $\mu_{\inf,x}(g)$. Moreover, for any $\xi\in\,]\eta_0,x_0[$ one has

$$\inf_{\varepsilon>0}\frac{g(\xi)+\varepsilon}{t(\xi)}=\frac{g(\xi)}{t(\xi)}$$

and hence

$$\inf_{\varepsilon>0}\mu_{\inf,x}(g+\varepsilon)\leqslant\frac{g(\xi)}{t(\xi)}.$$

Therefore, we obtain

$$\inf_{\varepsilon>0}\mu_{\inf,x}(g+\varepsilon)=\mu_{\inf,x}(g).$$

By the monotone convergence theorem we deduce that

$$\mu_{\inf}(g)=\inf_{\varepsilon>0}\mu_{\inf}(g+\varepsilon)\geqslant 0.$$

We now treat the general case. Let y be a closed point of X. We consider y as an $\mathbb{R}$-divisor on X and denote it by D_y. Let g_y be the canonical Green function associated with D_y. As D_y is effective and $g_y\geqslant 0$, we obtain that (D_y,g_y) is effective and hence pseudo-effective. Therefore, for any $\delta>0$,

$$(D,g)+\delta(D_y,g_y)=(D+\delta D_y,g+\delta g_y)$$

is pseudo-effective. Moreover, one has $\deg(D+\delta D_y)>0$. Therefore, by what we have shown above, one has

$$\mu_{\inf}(g+\delta g_y)=\mu_{\inf}(g)+\delta[\kappa(y):k]\geqslant 0.$$

Since $\delta>0$ is arbitrary, one obtains $\mu_{\inf}(g)\geqslant 0$. ■

6.4 Positivity of Green functions

Let D be an $\mathbb{R}$-divisor on X such that $\Gamma(D)^{\times}_{\mathbb{R}}$ is not empty. For any Green function g of D, we define a map

$$\widetilde{g}:X^{\mathrm{an}}\setminus\{x_0\,:\,x\in X^{(1)}\}\longrightarrow\mathbb{R}$$

as follows. For any $\xi\in X^{\mathrm{an}}\setminus\{x_0\,:\,x\in X^{(1)}\}$, let

$$\widetilde{g}(\xi):=\sup_{s\in\Gamma(D)^{\times}_{\mathbb{R}}}\big(\ln|s|(\xi)-\ln\|s\|_g\big).\tag{6.6}$$

Proposition 6.12. Let D be an $\mathbb{R}$-divisor on X such that $\Gamma(D)^{\times}_{\mathbb{Q}}$ is not empty. For any $\xi \in X^{\mathrm{an}} \setminus \{x_0 \,:\, x \in X^{(1)}\}$ one has

$$\widetilde{g}(\xi) = \sup_{s \in \Gamma(D)^{\times}_{\mathbb{Q}}} \big(\ln|s|(\xi) - \ln\|s\|_g\big). \tag{6.7}$$

Proof. Without loss of generality, we may assume that D is effective. For clarifying the presentation, we denote temporarily by

$$\widetilde{g}_0(\xi) := \sup_{s \in \Gamma(D)^{\times}_{\mathbb{Q}}} \big(\ln|s|(\xi) - \ln\|s\|_g\big).$$

Let s be an element of $\Gamma(D)^{\times}_{\mathbb{R}}$, which is written in the form $s_1^{a_1} \cdots s_r^{a_r}$, where $s_1, \ldots, s_r$ are elements of $\mathrm{Rat}(X)^{\times}_{\mathbb{Q}}$ and $a_1, \ldots, a_r$ are positive real numbers, which are linearly independent over $\mathbb{Q}$. Let $\{x_1, \ldots, x_n\}$ be the support of (s). By Lemma 5.9, for any $i \in \{1, \ldots, r\}$, the support of (s_i) is contained in $\{x_1, \ldots, x_n\}$. Since s belongs to $\Gamma(D)^{\times}_{\mathbb{Q}}$, for $j \in \{1, \ldots, n\}$, one has

$$a_1 \operatorname{ord}_{x_j}(s_1) + \cdots + a_r \operatorname{ord}_{x_j}(s_r) + \operatorname{ord}_{x_j}(D) \geqslant 0.$$

By Lemma 5.10 and Remark 5.11, there exist a sequence $(\varepsilon^{(m)})_{m \in \mathbb{N}}$ in $\mathbb{Q}_{>0}$ and a sequence

$$\boldsymbol{\delta}^{(m)} = (\delta_1^{(m)}, \ldots, \delta_r^{(m)}), \quad m \in \mathbb{N}$$

of elements of $\mathbb{R}^r_{>0}$, which satisfy the following conditions:

(1) the sequence $(\varepsilon^{(m)})_{m \in \mathbb{N}}$ converges to 0,

(2) the sequence $(\boldsymbol{\delta}^{(m)})_{m \in \mathbb{N}}$ converges to $(0, \ldots, 0)$, and

(3) if we denote by $u^{(m)}$ the element

$$s_1^{\delta_1^{(m)}} \cdots s_r^{\delta_r^{(m)}}$$

in $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$, one has $u^{(m)} \in \Gamma(\varepsilon^{(m)} D)^{\times}_{\mathbb{R}}$ and

$$s^{(m)} := (s u^{(m)})^{(1+\varepsilon^{(m)})^{-1}} \in \mathrm{Rat}(X)^{\times}_{\mathbb{Q}},$$

and hence it belongs to $\Gamma(D)^{\times}_{\mathbb{Q}}$.

Note that one has

$$\|su^{(m)}\|_{(1+\varepsilon^{(m)})g} \leqslant \|s\|_g \cdot \|u^{(m)}\|_{\varepsilon^{(m)}g}.$$

Since $u^{(m)} \in \Gamma(\varepsilon^{(m)}D)_{\mathbb{R}}^{\times}$, one has

$$-\ln\|u^{(m)}\|_{\varepsilon^{(m)}} = \inf\Big(\varepsilon^{(m)}g + \sum_{i=1}^{r} \delta_i^{(m)} g_{(s_i)}\Big) \geqslant \varepsilon^{(m)} \inf \varphi_g.$$

Therefore,

$$-\ln\|s\|_g \leqslant -(1+\varepsilon^{(m)}) \ln\|s^{(m)}\|_g - \varepsilon^{(m)} \inf \varphi_g.$$

Thus,

$$\begin{aligned}\ln|s|(\xi) - \ln\|s\|_g &= (1+\varepsilon^{(m)}) \ln|s^{(m)}|(\xi) - \sum_{i=1}^{r} \delta_i^{(m)} \ln|s_i|(\xi) - \ln\|s\|_g \\ &\leqslant (1+\varepsilon^{(m)})\widetilde{g}_0(\xi) - \sum_{i=1}^{r} \delta_i^{(m)} \ln|s_i|(\xi) - \varepsilon^{(m)} \inf \varphi_g.\end{aligned}$$

Taking the limit when $m \to +\infty$, we obtain

$$\ln|s|(\xi) - \ln\|s\|_g \leqslant \widetilde{g}_0(\xi).$$

The proposition is thus proved. ■

Proposition 6.13. Let D be an $\mathbb{R}$-divisor on X such that $\Gamma(D)_{\mathbb{R}}^{\times}$ is not empty. For any Green function g of D, the function $\widetilde{g}$ extends on X^{an} to a convex Green function of D, which is bounded from above by g.

Proof. We first show that $\widetilde{g}$ is bounded from above by g. For any $s \in \Gamma(D)_{\mathbb{R}}^{\times}$ one has

$$\forall\, \xi \in X^{\mathrm{an}}, \quad -\ln\|s\|_g = \inf(g_{(s)} + g) \leqslant g(\xi) - \ln|s|(\xi),$$

so that

$$\forall\, \xi \in X^{\mathrm{an}}, \quad \ln|s|(\xi) - \ln\|s\|_g \leqslant g(\xi).$$

It remains to check that $\widetilde{g}$ extends by continuity to a convex Green function of D.

We first treat the case where $\deg(D) = 0$. By Remark 4.11 we obtain that $\Gamma(D)^{\times}_{\mathbb{R}}$ contains a unique element s and one has $D = -(s)$. Therefore,

$$\widetilde{g} = \ln|s| - \ln\|s\|_g = g_D - \ln\|s\|_g,$$

which clearly extends to a convex Green function of D.

In the following, we assume that $\deg(D) > 0$. Let x be an element of $X^{(1)}$. The function $\widetilde{g} \circ \xi_x|_{\mathbb{R}_{>0}}$ (see Section 3.1) can be written as

$$(t \in \mathbb{R}_{>0}) \longmapsto \sup_{s \in \Gamma(D)^{\times}_{\mathbb{R}}} -t\,\mathrm{ord}_x(s) - \ln\|s\|_g,$$

which is the supremum of a family of affine functions on $t > 0$. Therefore, $\widetilde{g} \circ \xi_x|_{\mathbb{R}_{>0}}$ is a convex function on $\mathbb{R}_{>0}$. This expression also shows that, for any $s \in \Gamma(D)^{\times}_{\mathbb{R}}$, one has

$$\liminf_{\xi \to x_0} \frac{\widetilde{g}(\xi)}{t(\xi)} \geqslant \mathrm{ord}_x(s^{-1}).$$

By Proposition 4.10 (see also Remark 4.11), one has

$$\liminf_{\xi \to x_0} \frac{\widetilde{g}(\xi)}{t(\xi)} \geqslant \sup_{s \in \Gamma(D)^{\times}_{\mathbb{R}}} \mathrm{ord}_x(s^{-1}) = \mathrm{ord}_x(D).$$

Moreover, since $\widetilde{g} \leqslant g$ and since g is a Green function of D, one has

$$\limsup_{\xi \to x_0} \frac{\widetilde{g}(\xi)}{t(\xi)} \leqslant \lim_{\xi \to x_0} \frac{g(\xi)}{t(\xi)} = \mathrm{ord}_x(D).$$

Therefore, one has

$$\lim_{\xi \to x_0} \frac{\widetilde{g}(\xi)}{t(\xi)} = \mathrm{ord}_x(D).$$

The proposition is thus proved. ∎

Definition 6.14. Let (D, g) be a metrised $\mathbb{R}$-divisor on X such that $\Gamma(D)^{\times}_{\mathbb{R}}$ is not empty. We call $\widetilde{g}$ the *plurisubharmonic envelope* of the Green function g. In the case where the equality $g = \widetilde{g}$ holds, we say that the Green function g is *plurisubharmonic*. Note that $\widetilde{g}$ is bounded from above by the convex envelope $\breve{g}$ of g.

Remark 6.15. If we set $\varphi = g - \widetilde{g}$, then φ is a non-negative continuous function on X^{an}, so that, in some sense, the decomposition $(D, g) = (D, \widetilde{g}) + (0, \varphi)$ gives rise to a Zariski decomposition of (D, g) on X.

Theorem 6.16. Let (D, g) be a metrised $\mathbb{R}$-Cartier divisor on X such that $\Gamma(D)_{\mathbb{R}}^{\times}$ is not empty. Then $\widetilde{g}(\eta_0) = g(\eta_0)$ if and only if $\mu_{\inf}(g - g(\eta_0)) \geqslant 0$. Moreover, in the case where these equivalent conditions are satisfied, $\widetilde{g}$ identifies with the convex envelope $\breve{g}$ of g.

Proof. **Step 1:** We first treat the case where $\deg(D) = 0$. In this case $\Gamma(D)_{\mathbb{R}}^{\times}$ contains a unique element s (with $D = -(s)$) and one has (see the proof of Proposition 6.13)

$$\widetilde{g} = g_D - \ln\|s\|_g.$$

Hence,

$$\widetilde{g}(\eta_0) = -\ln\|s\|_g = \inf(g_{(s)} + g) = \inf \varphi_g.$$

Note that $g(\eta_0) = \varphi_g(\eta_0)$. Therefore, the equality $\widetilde{g}(\eta_0) = g(\eta_0)$ holds if and only if φ_g attains its minimal value at η_0, or equivalently

$$\forall\, x \in X^{(1)}, \quad \mu_{\inf,x}(g - g(\eta_0)) = \mathrm{ord}_x(g).$$

In particular, if $\widetilde{g}(\eta_0) = g(\eta_0)$, then

$$\mu_{\inf}(g - g(\eta_0)) = \sum_{x \in X^{(1)}} \mathrm{ord}_x(g)[\kappa(x) : k] = 0.$$

Conversely, if $\mu_{\inf}(g - g(\eta_0)) \geqslant 0$, then by (6.2) one obtains that

$$\mu_{\inf}(g - g(\eta_0)) = 0$$

and the equality $\mu_{\inf,x}(g - g(\eta_0)) = \mathrm{ord}_x(g)$ holds for any $x \in X^{(1)}$. Hence, $\widetilde{g}(\eta_0) = g(\eta_0)$.

If φ is a bounded Green function on X^{an}, which is bounded from above by φ_g, by Proposition 2.2 one has

$$\varphi(\xi) \leqslant \varphi(\eta_0) \leqslant \varphi_g(\eta_0) = g(\eta_0)$$

for any $\xi \in X^{\mathrm{an}}$. In the case where the inequality $\widetilde{g}(\eta_0) = g(\eta_0)$ holds, the function $\widetilde{g} = g_D + g(\eta_0)$ is the largest convex Green function of D, which is bounded from above by g, namely the equality $\widetilde{g} = \breve{g}$ holds.

Step 2: In the following, we assume that $\deg(D) > 0$. By replacing g by $g - g(\eta_0)$ it suffices to check that, in the case where $g(\eta_0) = 0$, the equality $\widetilde{g}(\eta_0) = 0$ holds if and only if $\mu_{\inf}(g) \geqslant 0$. By definition one has

$$\widetilde{g}(\eta_0) = \sup_{s \in \Gamma(D)^{\times}_{\mathbb{R}}} (-\ln\|s\|_g).$$

Step 2.1: We first assume that $\widetilde{g}(\eta_0) = 0$ and show that $\mu_{\inf}(g) \geqslant 0$. Let s be an element of $\Gamma(D)^{\times}_{\mathbb{R}}$. By definition one has

$$-\ln\|s\|_g = \inf_{\xi \in X^{\mathrm{an}}} (g + g_{(s)})(\xi).$$

Let (D_1, g_1) be a big metrised $\mathbb{R}$-divisor. We fix $s_1 \in \Gamma(D_1)^{\times}_{\mathbb{R}}$ such that $\|s_1\|_{g_1} < 1$ (see Proposition 6.2 for the existence of s_1). Since $\widetilde{g}(\eta_0) = 0$, there exists $s \in \Gamma(D)^{\times}_{\mathbb{R}}$ such that

$$\|ss_1\|_{g+g_1} \leqslant \|s\|_g \cdot \|s_1\|_{g_1} < 1.$$

Therefore, $\lambda_{\mathrm{ess}}(D + D_1, g + g_1) > 0$ and hence $(D + D_1, g + g_1)$ is big (see Proposition 6.2). We then obtain that (D, g) is pseudo-effective and hence $\mu_{\inf}(g) \geqslant 0$ (see Proposition 6.5).
Step 2.2: We now show that $\mu_{\inf}(g) > 0$ implies $\widetilde{g}(\eta_0) = 0$. For $\varepsilon > 0$, let

$$U_\varepsilon := \{\xi \in X^{\mathrm{an}} \,:\, g(\xi) > -\varepsilon\}.$$

This is an open subset of X^{an}, which contains η_0. Hence, there exists a finite set $X^{(1)}_\varepsilon$ of closed points of X, which contains the support of D and such that, for any closed point x of X lying outside of $X^{(1)}_\varepsilon$, one has $g|_{[\eta_0, x_0]} > -\varepsilon$. Moreover, for any $x \in X^{(1)} \setminus \mathrm{Supp}(D)$ one has $\mu_{\inf,x}(g) \leqslant 0$ since g is bounded on $[\eta_0, x_0]$. Therefore, the condition $\mu_{\inf}(g) > 0$ implies that

$$\sum_{x \in X^{(1)}_\varepsilon} \mu_{\inf,x}(g)[\kappa(x) : k] > 0. \tag{6.8}$$

We let s_ε be an element of $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that $\mathrm{ord}_x(s_\varepsilon) \geqslant -\mu_{\inf,x}(g)$ for any $x \in X^{(1)}_\varepsilon$ and that $\mathrm{ord}_x(s_\varepsilon) \geqslant 0$ for any $x \in X^{(1)} \setminus X^{(1)}_\varepsilon$. This is possible by the inequality (6.8). In fact, the $\mathbb{R}$-divisor

$$E = \sum_{x \in X_\varepsilon^{(1)}} \mu_{\mathrm{inf},x}(g) \cdot x$$

has a positive degree, and hence $\Gamma(E)_{\mathbb{R}}^{\times}$ is not empty. Note that $\mu_{\mathrm{inf},x}(g) \leqslant \mathrm{ord}_x(D)$ for any $x \in X^{(1)}$. Therefore, $D + (s_\varepsilon)$ is effective. Moreover, for any $x \in X^{(1)} \setminus X_\varepsilon^{(1)}$ and $\xi \in [\eta_0, x_0[$ one has

$$(g - \ln|s_\varepsilon|)(\xi) \geqslant g(\xi) \geqslant -\varepsilon.$$

Therefore, we obtain $\|s_\varepsilon\| \leqslant \mathrm{e}^\varepsilon$ since $g - \ln|s_\varepsilon| \geqslant 0$ on $[\eta_0, x_0[$ for any $x \in X_\varepsilon^{(1)}$. This leads to $\widetilde{g}(\eta_0) = 0$ since ε is arbitrary.

Step 2.3: We assume that $\mu_{\mathrm{inf}}(g) > 0$ and show that $\breve{g} = \widetilde{g}$. By definition, for any $x \in X^{(1)}$, the function $\widetilde{g} \circ \xi_x|_{\mathbb{R}_{>0}}$ can be written as the supremum of a family of affine functions, hence it is a convex function on $\mathbb{R}_{>0}$ bounded from above by g. In the following, we fix a closed point x of X.

Without loss of generality, we may assume that x belongs to $X_\varepsilon^{(1)}$ for any $\varepsilon > 0$. Note that for any $\xi \in [\eta_0, x_0]$ one has

$$\widetilde{g}(\xi) \geqslant \ln|s_\varepsilon|(\xi) - \ln\|s_\varepsilon\|_g \geqslant \mu_{\mathrm{inf},x}(g)t(\xi) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, one has $\widetilde{g}(\xi) \geqslant \mu_{\mathrm{inf},x}(g)t(\xi)$.

Let a and b be real numbers such that $at(\xi) + b \leqslant g(\xi)$ for any $\xi \in [\eta_0, x_0[$. Then one has $b \leqslant 0$ since $g(\eta_0) = 0$. Moreover, one has

$$a = \lim_{\xi \to x_0} \frac{at(\xi) + b}{t(\xi)} \leqslant \lim_{\xi \to x_0} \frac{g(\xi)}{t(\xi)} = \mathrm{ord}_x(D).$$

We will show that $at(\xi) + b \leqslant \widetilde{g}(\xi)$ for any $\xi \in [\eta_0, x_0[$. This inequality is trivial when $a \leqslant \mu_x(g)$ since $\widetilde{g}(\xi) \geqslant \mu_{\mathrm{inf},x}(g)t(\xi)$ and $b \leqslant 0$. In the following, we assume that $a > \mu_x(g)$.

For any $\varepsilon > 0$, we let $s_\varepsilon^{a,b}$ be an element of $\mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ such that

$$\mathrm{ord}_y(s_\varepsilon^{a,b}) \geqslant \begin{cases} -a & \text{if } y = x, \\ -\mu_{\mathrm{inf},y}(g) & \text{if } y \in X_\varepsilon^{(1)},\ y \neq x, \\ 0 & \text{if } y \in X^{(1)} \setminus X_\varepsilon^{(1)}. \end{cases}$$

This is possible since $\mu_{\inf}(g) > 0$ and $a > \mu_{\inf,x}(g)$. Note that $s_\varepsilon^{a,b}$ belongs to $\Gamma(D)_{\mathbb{R}}^{\times}$. Moreover, for $\xi \in [\eta_0, x_0[$, one has

$$g(\xi) - \ln|s_\varepsilon^{a,b}|(\xi) \geqslant g(\xi) - at(\xi) \geqslant b;$$

for any $y \in X_\varepsilon^{(1)} \setminus \{x\}$, one has

$$g(\xi) - \ln|s_\varepsilon^{a,b}|(\xi) = g(\xi) - \mu_{\inf,y}(g)t(\xi) \geqslant 0;$$

for any $y \in X^{(1)} \setminus X_\varepsilon$, one has $g(\xi) - \ln|s_\varepsilon^{a,b}|(\xi) \geqslant g(\xi) \geqslant -\varepsilon$. Therefore, we obtain

$$-\ln\|s_\varepsilon^{a,b}\| \geqslant \min\{-\varepsilon, b\}.$$

As a consequence, for any $\xi \in [\eta_0, x_0[$, one has

$$\widetilde{g}(\xi) \geqslant \ln|s_\varepsilon^{a,b}|(\xi) - \ln\|s\|_g = at(\xi) + \min\{-\varepsilon, b\}.$$

Since $b \leqslant 0$ and since $\varepsilon > 0$ is arbitrary, we obtain $\widetilde{g}(\xi) \geqslant at(\xi) + b$.

Step 3: In this step, we assume that $\deg(D) > 0$ and $\mu_{\inf}(g - g(\eta_0)) = 0$. We show that and $\breve{g} = \widetilde{g}$. Without loss of generality, we assume that $g(\eta_0) = 0$. Since

$$\deg(D) = \sum_{x \in X^{(1)}} \mu_x(g)[\kappa(x) : k] > 0,$$

there exists $y \in X^{(1)}$ such that

$$\mu_{\inf,y}(g) < \mathrm{ord}_x(D) = \mu_y(g).$$

We let g_0 be the bounded Green function on $\mathcal{T}(X^{(1)})$ such that $g_0(\xi) = 0$ for

$$\xi \in \bigcup_{x \in X^{(1)},\, x \neq y} [\xi_0, x_0],$$

and

$$g_0(\xi) = \min\{t(\xi), 1\}, \quad \text{for } \xi \in [\eta_0, y_0].$$

One has $g_0 \geqslant 0$, and

$$\sup_{\xi \in X^{(1)}} g_0(\xi) \leqslant 1.$$

For any $\varepsilon > 0$, we denote by g_ε the Green function $g + \varepsilon g_0$. One has

$$\mu_{\mathrm{inf},x}(g_\varepsilon) > \mu_{\mathrm{inf},x}(g) \geqslant 0.$$

Moreover, by definition $g_\varepsilon(\eta_0) = 0$. Therefore, by what we have shown in Step 2.2, one has

$$\widetilde{g}_\varepsilon(\eta_0) = \sup_{s \in \Gamma(D)^\times_{\mathbb{R}}} \big(- \ln\|s\|_{g_\varepsilon} \big) = 0.$$

Note that for any $s \in \Gamma(D)^\times_{\mathbb{R}}$ one has

$$\mathrm{e}^{\varepsilon} \|s\|_{g_\varepsilon} \geqslant \|s\|_g \geqslant \|s\|_{g_\varepsilon}.$$

Hence, we obtain

$$\widetilde{g}_\varepsilon - \varepsilon \leqslant \widetilde{g} \leqslant \widetilde{g}_\varepsilon.$$

Since $\widetilde{g}_\varepsilon(\eta_0) = 0$ for any $\varepsilon > 0$, we obtain $\widetilde{g}(\eta_0) = 0$. Finally, the inequalities

$$g_\varepsilon - \varepsilon \leqslant g \leqslant g_\varepsilon$$

lead to

$$\breve{g}_\varepsilon - \varepsilon \leqslant \breve{g} \leqslant \breve{g}_\varepsilon.$$

By what we have shown in Step 2.3, one has $\widetilde{g}_\varepsilon = \breve{g}_\varepsilon$ for any $\varepsilon > 0$. Therefore, the equality $\widetilde{g} = \breve{g}$ holds. ■

Corollary 6.17. Let (D, g) be a metrised $\mathbb{R}$-divisor on X such that $\Gamma(D)^\times_{\mathbb{R}} \neq \emptyset$. Then g is plurisubharmonic if and only if it is convex and $\mu_{\mathrm{inf}}(g - g(\eta_0)) \geqslant 0$.

6.5 Global positivity conditions under metric positivity

Let X be a regular projective curve over $\operatorname{Spec} k$ and (D, g) be a metrised $\mathbb{R}$-divisor. In this section, we consider global positivity conditions under the hypothesis that g is plurisubharmonic.

Proposition 6.18. Let (D, g) be a metrised $\mathbb{R}$-divisor such that $\Gamma(D)^\times_{\mathbb{R}}$ is not empty and that the Green function g is plurisubharmonic.

(1) (D, g) is pseudo-effective if and only if $g(\eta_0) \geqslant 0$.

(2) One has $\lambda_{\mathrm{ess}}(D, g) = g(\eta_0)$.

(3) The metrised $\mathbb{R}$-divisor (D, g) is big if and only if $\deg(D) > 0$ and $g(\eta_0) > 0$.

Proof. (1) We have already seen in Proposition 6.5 that, if (D, g) is pseudo-effective, then $g(\eta_0) \geqslant 0$. It suffices to prove that $g(\eta_0) \geqslant 0$ implies that (D, g) is pseudo-effective. Since g is plurisubharmonic, by Theorem 6.16 one has

$$\mu_{\inf}(g) \geqslant \mu_{\inf}(g - g(\eta_0)) \geqslant 0.$$

By Proposition 6.11, one obtains that (D, g) is pseudo-effective

(2) By (5.6) and Proposition 5.8, it suffices to prove that $g(\eta_0) \leqslant \lambda_{\mathrm{ess}}(D, g)$. In the case where $\deg(D) = 0$, the hypotheses that $\Gamma(D)^{\times}_{\mathbb{R}}$ is not empty and g is plurisubharmonic imply that D is a principal $\mathbb{R}$-divisor, $\Gamma(D)^{\times}_{\mathbb{R}}$ contains a unique element s with $D = -(s)$, and $g - g(\eta_0)$ is the canonical Green function of D (see the 1st step of the proof of Theorem 6.16). Therefore, one has

$$\lambda_{\mathrm{ess}}(D, g) = -\ln\|s\|_g = g(\eta_0).$$

In the following we treat the case where $\deg(D) > 0$. Since g is plurisubharmonic, by Theorem 6.16 one has $\mu_{\inf}(g - g(\eta_0)) \geqslant 0$, so that $(D, g - g(\eta_0))$ is pseudo-effective (see Proposition 6.11). As $\deg(D) > 0$, by Corollary 6.4 and (5.12), one has

$$\lambda_{\mathrm{ess}}(g - g(\eta_0)) = \lambda_{\mathrm{ess}}(g) - g(\eta_0) \geqslant 0.$$

(3) Follows from (2) and Proposition 6.2. ■

7 Hilbert–Samuel Formula on Curves

Let k be a field equipped with the trivial valuation. Let X be a regular and irreducible projective curve over k. The purpose of this section is to prove a Hilbert–Samuel formula for metrised $\mathbb{R}$-divisors on X.

Definition 7.1. We identify X^{an} with the infinite tree $\mathcal{T}(X^{(1)})$ and consider the weight function $w : X^{(1)} \to \,]0, +\infty[$ defined as $w(x) = [\kappa(x) : k]$. If $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ are metrised $\mathbb{R}$-divisors on X such that g_1 and g_2 are both pairable (see Definition 3.8) we define $(\overline{D}_1 \cdot \overline{D}_2)$ as the pairing $\langle g_1, g_2\rangle_w$, namely

$$\begin{aligned}(\overline{D}_1 \cdot \overline{D}_2) &= g_2(\eta_0)\deg(D_1) + g_1(\eta_0)\deg(D_2) \\ &\quad - \sum_{x\in X^{(1)}} [\kappa(x):k] \int_0^{+\infty} \varphi'_{g_1\circ\xi_x}(t)\varphi'_{g_2\circ\xi_x}(t)\,\mathrm{d}t.\end{aligned} \tag{7.1}$$

Remark 7.2. We compare the above construction to the arithmetic intersection product in the classic form. We assume that g_1 is plurisubharmonic. Denote by

$$h_{\overline{D}_1}(D_2) := \sum_{x\in X^{(1)}} [\kappa(x):k]\varphi_{g_1}(x_0)\,\mathrm{ord}_x(D_2).$$

This term should be considered as the logarithmic height of the $\mathbb{R}$-divisor with respect to $\overline{D}_1$ (see Remark 5.7). By Proposition 2.3, we can rewrite $(\overline{D}_1 \cdot \overline{D}_2)$ as (see Convention 3.9 for the notation of $\varphi'_{g_1}(\eta_0; x)$)

$$\begin{aligned}&g_2(\eta_0)\deg(D_1) + g_1(\eta_0)\deg(D_2) + \sum_{x\in X^{(1)}} [\kappa(x):k] g_2(\eta_0)\varphi'_{g_1}(\eta_0;x) \\ &\quad + \sum_{x\in X^{(1)}} [\kappa(x):k] \int_0^{+\infty} \varphi_{g_2\circ\xi_x}(t)\,\mathrm{d}\varphi'_{g_1\circ\xi_x}(t) \\ &= h_{\overline{D}_1}(D_2) + \sum_{x\in X^{(1)}} [\kappa(x):k]\,\mathrm{ord}_x(D_2)(\varphi_{g_1}(\eta_0) - \varphi_{g_1}(x_0)) \\ &\quad + \sum_{x\in X^{(1)}} [\kappa(x):k]\bigg(\int_0^{+\infty} \varphi_{g_2}\circ\xi_x(t)\,\mathrm{d}\varphi'_{g_1\circ\xi_x}(t) + \int_0^{+\infty} g_2(\eta_0)\varphi'_{g_1}(\eta_0;x)\bigg) \\ &= h_{\overline{D}_1}(D_2) + \int_{X^{\mathrm{an}}} g_2(\xi)\bigg(\gamma_{g_1}(\mathrm{d}\xi) + \sum_{x\in X^{(1)}} [\kappa(x):k]\varphi'_{g_1}(\eta_0;x)\delta_{\eta_0}\bigg),\end{aligned}$$

where on $]\eta_0, x_0]$ the measure γ_{g_1} identifies with the push-forward of $\mathrm{d}\varphi'_{g_1\circ\xi_x}$ by ξ_x, and in the 2nd equality we have applied the equality (see Proposition 2.4)

$$\varphi_{g_1}(\eta_0) - \varphi_{g_1}(x_0) = \int_0^{+\infty} t\,\mathrm{d}\varphi'_{g_1\circ\xi_x}(t).$$

The measure

$$\gamma_{g_1}(\mathrm{d}\xi) + \sum_{x\in X^{(1)}} [\kappa(x):k]\varphi'_{g_1}(\eta_0;x)\delta_{\eta_0}$$

should be considered as the Monge–Ampère measure of (D_1, g_1).

Remark 7.3. Assume that s is an element of $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that

$$\overline{D}_2 = \widehat{(s)} = ((s), g_{(s)}).$$

One has (see Definition 3.8)

$$(\overline{D}_1, \overline{D}_2) = \langle g_1, g_{(s)} \rangle_W = g_1(\eta_0) \deg((s)) = 0.$$

Theorem 7.4. Let $\overline{D} = (D, g)$ be a metrised $\mathbb{R}$-divisor on X such that $\Gamma(D)^{\times}_{\mathbb{R}} \neq \emptyset$ and g is plurisubharmonic. Then $\widehat{\mathrm{vol}}_{\chi}(\overline{D}) = (\overline{D} \cdot \overline{D})$.

Remark 7.5. Let g_D be the canonical Green function of D and $\varphi_g := g - g_D$ (considered as a continuous function on X^{an}). Note that a plurisubharmonic Green function is convex (see Proposition 6.13). Therefore, by Proposition 3.12, one has

$$\mu_{\mathrm{inf},x}(g - g(\eta_0)) = \mathrm{ord}_x(D) + \varphi_g'(\eta_0; x).$$

Theorem 6.16 shows that

$$\mu_{\mathrm{inf}}(g - g(\eta_0)) = \deg(D) + \sum_{x \in X^{(1)}} \varphi_g'(\eta_0; x)[\kappa(x) : k] \geqslant 0. \tag{7.2}$$

In the case where $\deg(D) = 0$, one has $g = g(\eta_0) + g_D$ (see Step 1 in the proof of Theorem 6.16). Therefore, one has

$$(\overline{D} \cdot \overline{D}) = 2g(\eta_0) \deg(D) = 0 = \widehat{\mathrm{vol}}_{\chi}(\overline{D}),$$

where the last equality comes from (3) of Proposition 5.23. Therefore, to prove Theorem 7.4, it suffices to treat the case where $\deg(D) > 0$.

Assumption 7.6. Let Σ be the set consisting of closed points x of X such that φ_g is not a constant function on $[\eta_0, x_0]$. Note that Σ is countable by Proposition 3.6. Here we consider additional assumptions (i)–(iv):

- (i) D is a divisor.
- (ii) Σ is finite.
- (iii) $\varphi_g(\eta_0) = 0$.
- (iv) $\mu_{\mathrm{inf}}(g - g(\eta_0)) \geqslant 0$.

These assumptions actually describe a special case of the setting of the above theorem, but it is an essential case because the theorem in general is a consequence of its assertion under these assumptions by using the continuity of $\widehat{\mathrm{vol}}_\chi(\cdot)$. Before starting the proof of Theorem 7.4 under the above assumptions, we need to prepare several facts. For a moment, we proceed with arguments under Assumption 7.6. Let $L = \mathcal{O}_X(D)$ and h be the continuous metric of L given by $\exp(-\varphi_g)$. For $x \in \Sigma$, let

$$a_x := \varphi_g'(\eta_0; x) \quad \text{and} \quad \varphi_x' := \varphi_{g\circ\xi_x}'.$$

For $x \in \Sigma$ and $n \in \mathbb{N}_{\geqslant 1}$, we set $a_{x,n} = \lfloor -na_x \rfloor$. One has

$$a_{x,n} \leqslant -na_x < a_{x,n} + 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{a_{x,n}}{n} = -a_x.$$

Moreover, as

$$\sum_{x\in\Sigma} a_x[\kappa(x) : k] + \deg(L) > 0$$

by our assumptions, there exists $n_0 \in \mathbb{N}_{\geqslant 1}$ such that

$$\begin{aligned}&\frac{2(\mathrm{genus}(X) - 1) + \sum_{x\in\Sigma} a_{x,n}[\kappa(x) : k] + \sum_{x\in\Sigma}[\kappa(x) : k]}{n} \\ &\leqslant \frac{2(\mathrm{genus}(X) - 1) + \sum_{x\in\Sigma}[\kappa(x) : k]}{n} - \sum_{x\in\Sigma} a_x[\kappa(x) : k] < \deg(L)\end{aligned}$$

holds for any integer $n \geqslant n_0$, that is,

$$\forall\, n \in \mathbb{N}_{\geqslant n_0}, \quad 2(\mathrm{genus}(X) - 1) + \sum_{x\in\Sigma}(a_{x,n} + 1)[\kappa(x) : k] < n \deg(L). \tag{7.3}$$

We set

$$D_n = \sum_{x\in\Sigma}(a_{x,n} + 1)x \quad \text{and} \quad D_{x,n} = D_n - (a_{x,n} + 1)x.$$

Note that

$$\begin{cases} H^0(X, nL \otimes \mathcal{O}_X(-D_n)) = \{s \in H^0(X, nL) \ : \ \mathrm{ord}_x(s) \geqslant a_{x,n} + 1 \ (\forall x \in \Sigma)\}, \\ H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n} - ix)) \\ \qquad = \left\{s \in H^0(X, nL) \ : \ \mathrm{ord}_y(s) \geqslant a_{y,n} + 1 \ (\forall y \in \Sigma \setminus \{x\}) \text{ and } \mathrm{ord}_x(s) \geqslant i\right\} \end{cases}$$

Lemma 7.7. For any integer n such that $n \geqslant 0$, the following assertions hold.

(1) $\sum_{x \in \Sigma} H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n})) = H^0(X, nL)$.

(2) One has

$$H^0(X, nL)/H^0(X, nL \otimes \mathcal{O}_X(-D_n)) \\ = \bigoplus_{x \in \Sigma} H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n}))/H^0(X, nL \otimes \mathcal{O}_X(-D_n))$$

Proof. (1) Let us consider a natural homomorphism

$$\bigoplus_{x \in \Sigma} nL \otimes \mathcal{O}_X(-D_{x,n}) \to nL.$$

Note that the above homomorphism is surjective and the kernel is isomorphic to $(nL \otimes \mathcal{O}_X(-D_n))^{\oplus \operatorname{card}(\Sigma) - 1}$. Moreover, by Serre's duality,

$$H^1(X, nL \otimes \mathcal{O}_X(-D_n)) \simeq H^0(X, \omega_X \otimes -nL \otimes \mathcal{O}_X(D_n))^\vee$$

and

$$\deg(\omega_X \otimes -nL \otimes \mathcal{O}_X(D_n)) \\ = 2(\mathrm{genus}(X) - 1) - n\deg(L) + \sum_{x \in \Sigma} (a_{x,n} + 1)[\kappa(x) : k] < 0,$$

so that $H^1(X, nL \otimes \mathcal{O}_X(-D_n)) = 0$. Therefore, one has (1).

(2) By (1), it is sufficient to see that if

$$\sum_{x \in \Sigma} s_x \in H^0(X, nL \otimes \mathcal{O}_X(-D_n))$$

and

$$\forall x \in \Sigma, \quad s_x \in H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n})),$$

then

$$s_x \in H^0(X, nL \otimes \mathcal{O}_X(-D_n))$$

for all $x \in \Sigma$. Indeed, as

$$\forall y \in \Sigma \setminus \{x\}, \quad s_y \in H^0(X, \mathcal{O}_X(-(a_{x,n}+1)x))$$

and

$$\sum_{y \in \Sigma} s_y \in H^0(X, \mathcal{O}_X(-(a_{x,n}+1)x)),$$

we obtain

$$s_x \in H^0(X, \mathcal{O}_X(-(a_{x,n}+1)x)),$$

so that $s_x \in H^0(X, \mathcal{O}_X(-D_n))$, as required. ■

Lemma 7.8. For $x \in \Sigma$ and $i \in \{0, \ldots, a_{x,n}\}$,

$$\dim_k \left(H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n} - ix))/H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n} - (i+1)x))\right) = [\kappa(x) : k].$$

Proof. Let us consider an exact sequence

$$0 \to nL \otimes \mathcal{O}_X(-D_{x,n} - (i+1)x) \to nL \otimes \mathcal{O}_X(-D_{x,n} - ix) \to \kappa(x) \to 0,$$

so that, since

$$\begin{aligned}
&\deg(\omega_X \otimes -nL \otimes \mathcal{O}_X(D_{x,n} + (i+1)x)) \\
&= 2(\mathrm{genus}(X) - 1) - n\deg(L) + \big((i+1) - (a_{x,n}+1)\big)[\kappa(x) : k] \\
&\quad + \sum_{y \in \Sigma} (a_{y,n}+1)[\kappa(y) : k] \\
&\leqslant 2(\mathrm{genus}(X) - 1) - n\deg(L) + \sum_{y \in \Sigma} (a_{y,n}+1)[\kappa(y) : k] < 0,
\end{aligned}$$

one has the assertion as before. ■

By Lemma 7.8, for each $x \in \Sigma$, there are

$$s^{(\ell)}_{x,0}, \dots, s^{(\ell)}_{x,a_{x,n}} \in H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n})), \quad \ell \in \{1, \dots, [\kappa(x) : k]\}$$

such that the classes of $s^{(\ell)}_{x,0}, \dots, s^{(\ell)}_{x,a_{x,n}}$ form a basis of

$$H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n}))/H^0(X, nL \otimes \mathcal{O}_X(-D_n))$$

and

$$s^{(\ell)}_{x,i} \in H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n} - ix)) \setminus H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n} - (i+1)x))$$

whose classes form a basis of

$$H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n} - ix))/H^0(X, nL \otimes \mathcal{O}_X(-D_{x,n} - (i+1)x))$$

for $i = 0, \dots, a_{x,n}$. Moreover, we choose a basis $\{t_1, \dots, t_{e_n}\}$ of $H^0(X, nL \otimes \mathcal{O}_X(-D_n))$. Then, by Lemma 7.7,

$$\Delta_n := \{t_1, \dots, t_{e_n}\} \cup \bigcup_{x \in \Sigma} \left\{ s^{(\ell)}_{x,0}, \dots, s^{(\ell)}_{x,a_{x,n}} \ : \ \ell \in \{1, \dots, [\kappa(x) : k]\} \right\}$$

forms a basis of $H^0(X, nL)$.

Lemma 7.9.

(1) The equality

$$\|s^{(\ell)}_{x,i}\|_{nh} = \exp(-n\varphi^*_x(i/n))$$

holds for $x \in \Sigma$, $\ell \in \{1, \dots, [\kappa(x) : k]\}$ and $i \in \{0, \dots, a_{x,n}\}$. Moreover, $\|t_j\|_{nh} = 1$ for all $j \in \{1, \dots, e_n\}$.

(2) The basis Δ_n of $H^0(X, nL)$ is orthogonal with respect to $\|.\|_{nh}$.

Proof. First of all, note that, for $s \in H^0(X, nL) \setminus \{0\}$ and $\xi \in X^{\mathrm{an}}$,

$$-\ln |s|_{nh}(\xi) = \begin{cases} t(\xi)\,\mathrm{ord}_x(s) \geq 0 & \text{if } \xi \in [\eta_0, x_0] \text{ and } x \notin \Sigma, \\ n\big(t(\xi)(\mathrm{ord}_x(s)/n) + \varphi_x(t(\xi))\big) & \text{if } \xi \in [\eta_0, x_0] \text{ and } x \in \Sigma, \end{cases}$$

so that

$$\|s\|_{nh} = \max\left\{1,\ \max_{x\in\Sigma}\{\exp(-n\varphi_x^*(\mathrm{ord}_x(s)/n))\}\right\}. \tag{7.4}$$

(1) The assertion follows from (7.4) because $\varphi_x^*(\lambda) = 0$ if $\lambda \geqslant -a_x$.

(2) Fix $s \in H^0(X, nL) \setminus \{0\}$. We set

$$s = b_1 t_1 + \cdots + b_{e_n} t_{e_n} + \sum_{x\in\Sigma}\sum_{i=0}^{a_{x,n}}\sum_{\ell=1}^{[\kappa(x):k]} c_{x,i}^{(\ell)} s_{x,i}^{(\ell)}.$$

If $s \in H^0(X, nL \otimes \mathcal{O}_X(-D_n))$, then $c_{x,i}^{(\ell)} = 0$ for all x, i and ℓ. Thus,

$$1 = \max_{j\in\{1,\ldots,e_n\}}\{|b_j| \cdot \|t_j\|_{nj}\} = \|s\|_{nh}.$$

Next we assume that $s \notin H^0(X, nL \otimes \mathcal{O}_X(-D_n))$. If we set

$$T = \{x \in \Sigma \ :\ \mathrm{ord}_x(s) \leqslant a_{x,n}\},$$

then $T \neq \emptyset$ and, for $x \in \Sigma$ and $\ell \in \{1, \ldots, [\kappa(x) : k]\}$,

$$\begin{cases} c_{x,0}^{(\ell)} = \cdots = c_{x,a_{x,n}}^{(\ell)} = 0 & \text{if } x \notin T, \\ c_{x,0}^{(\ell)} = \cdots = c_{x,\mathrm{ord}_x(s)-1}^{(\ell)} = 0, \quad (c_{x,\mathrm{ord}_x(s)}^{(\ell)})_{\ell=1}^{[\kappa(x):k]} \neq (0, \ldots, 0) & \text{if } x \in T. \end{cases}$$

Therefore, by (7.4),

$$\begin{aligned} \max\left\{\max_{j=1,\ldots,e_n}\{|b_j| \cdot \|t_j\|_{nh}\},\ \max_{\substack{x\in\Sigma,\\ i=0,\ldots,a_{x,n}}}\{|c_{x,i}| \cdot \|s_{x,i}\|_{nh}\}\right\} &= \max_{x\in T,\,\ell}\{\|s_{x,\mathrm{ord}_x(s)}^{(\ell)}\|_{nh}\} \\ = \max_{x\in\Sigma,\,\ell}\{\|s_{x,\mathrm{ord}_x(s)}^{(\ell)}\|_{nh}\} &= \max_{x\in\Sigma}\{\exp(-n\varphi_x^*(\mathrm{ord}_x(s)/n))\} = \|s\|_{nh}, \end{aligned}$$

as required. ∎

Let us begin the proof of Theorem 7.4 under Assumption 7.6. By Lemma 7.9 together with Definition 3.8 and Proposition 2.6,

$$\lim_{n\to\infty}\frac{\widehat{\deg}\left(H^0(X,nL),\|.\|_{nh}\right)}{n^2/2}=2\sum_{x\in\Sigma}\lim_{n\to\infty}[\kappa(x):k]\sum_{i=0}^{a_{x,n}}\frac{1}{n}\varphi_x^*(i/n)$$

$$=2\sum_{x\in\Sigma}[\kappa(x):k]\int_0^{-a_x}\varphi_x^*(\lambda)\mathrm{d}\lambda=-\sum_{x\in\Sigma}[\kappa(x):k]\int_0^{\infty}(\varphi_x')^2\mathrm{d}t=(\overline{D}\cdot\overline{D}),$$

as required.

Proof of Theorem 7.4 without additional assumptions. First of all, note that Σ is a countable set (cf. Proposition 3.6).

Step 1: (the case where D is Cartier divisor, Σ is finite and $\varphi_g'(\eta_0)+\deg(D)>0$). By the previous observation,

$$\widehat{\mathrm{vol}}_\chi(D,g-g(\eta_0))=\big((D,g-g(\eta_0))\cdot(D,g-g(\eta_0))\big).$$

On the other hand, by Proposition 5.14, one has

$$\widehat{\mathrm{vol}}_\chi(D,g)=\widehat{\mathrm{vol}}_\chi(D,g-g(\eta_0))+2\deg(D)g(\eta_0).$$

Moreover, by the bilinearity of the arithmetic intersection pairing, one has

$$\big(\overline{D}\cdot\overline{D}\big)=\big((D,g-g(\eta_0))\cdot(D,g-g(\eta_0))\big)+2\deg(D)g(\eta_0).$$

Thus, the assertion follows.

Step 2: (the case where D is Cartier divisor and Σ is finite). For $0<\varepsilon<1$, we set $g_\varepsilon=g_D^{\mathrm{can}}+\varepsilon\varphi_g$. If $\varphi_g'(\eta_0)=0$, then $\Sigma=\emptyset$, so that the assertion is obvious. Thus, we may assume that $\varphi_g'(\eta_0)<0$. As $\varphi_g'(\eta_0)+\deg(D)\geqslant 0$, we have $\varepsilon\varphi_g'(\eta_0)+\deg(D)>0$. Therefore, by Step 1,

$$\widehat{\mathrm{vol}}_\chi(D,g_\varepsilon)=\big((D,g_\varepsilon)\cdot(D,g_\varepsilon)\big).$$

Thus, the assertion follows by Proposition 5.24.

Step 3: (the case where D is Cartier divisor and Σ is infinite). We write Σ in the form of a sequence $\{x_1,\ldots,x_n,\ldots,\}$. For any $n\in\mathbb{Z}_{\geqslant 1}$, let g_n be the Green function defined as follows:

$$\forall\, \xi \in X^{\mathrm{an}}, \quad g_n(\xi) = g_D(\xi) + \begin{cases} \varphi_g(\xi) & \text{if } \xi \in \bigcup_{i=1}^{n} [\eta_0, x_{i,0}], \\ g(\eta_0) & \text{otherwise.} \end{cases}$$

Note that

$$\lim_{n\to\infty} \sup_{\xi \in X^{\mathrm{an}}} |\varphi_{g_n}(\xi) - \varphi_g(\xi)| = 0.$$

Indeed, as φ_g is continuous at η_0, for any $\varepsilon > 0$, there is an open set U of X^{an} such that $\eta_0 \in U$ and $|\varphi_g(\xi) - \varphi_g(\eta_0)| \leqslant \varepsilon$ for any $\xi \in U$. Since $\eta_0 \in U$, one can find N such that $[\eta_0, x_{n,0}] \subseteq U$ for all $n \geqslant N$. Then, for $n \geqslant N$,

$$|\varphi_g(\xi) - \varphi_{g_n}(\xi)| \begin{cases} \leqslant \varepsilon & \text{if } \xi \in [\eta_0, x_{i,0}] \text{ for some } i > n, \\ = 0 & \text{otherwise,} \end{cases}$$

as required. Thus, by (2) in Proposition 5.23, the assertion is a consequence of Step 2.

Step 4: (the case where D is $\mathbb{Q}$-Cartier divisor). Choose a positive integer a such that aD is Cartier divisor. Then, by Step 3,

$$\widehat{\mathrm{vol}}_\chi(a\overline{D}) = (a\overline{D} \cdot a\overline{D}) = a^2(\overline{D} \cdot \overline{D}).$$

By Corollary 5.21, one has $\widehat{\mathrm{vol}}_\chi(a\overline{D}) = a^2\widehat{\mathrm{vol}}_\chi(\overline{D})$. Hence, the equality

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = (\overline{D} \cdot \overline{D})$$

holds.

Step 5: (general case). By our assumption, there are metrised $\mathbb{Q}$-Cartier divisors $(E_1, h_1), \ldots, (E_r, h_r)$ and $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ such that $E_1, \ldots, E_r$ are semiample, $h_1, \ldots, h_r$ are plurisubharmonic, and $(D, g) = a_1(E_1, h_1) + \cdots + a_r(E_r, h_r)$. We choose sequences $\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{r,n}\}_{n=1}^{\infty}$ of positive rational numbers such that $\lim_{n\to\infty} a_{i,n} = a_i$ for $i = 1, \ldots, r$. We set $(D_n, g_n) = a_{1,n}(E_1, h_1) + \cdots + a_{r,n}(E_r, h_r)$. Then we may assume that $\deg(D_n) > 0$. By Step 4, then $widehat\mathrm{vol}_\chi(D_n, g_n) = (D_n, g_n)^2$. On the other hand, by Proposition 5.24, $\widehat{\mathrm{vol}}_\chi(D, g) =$

$\lim_{n\to\infty} \widehat{\mathrm{vol}}_\chi(D_n, g_n)$. Moreover,

$$\big((D,g)\cdot(D,g)\big) = \lim_{n\to\infty} \big((D_n,g_n)\cdot(D_n,g_n)\big).$$

Thus, the assertion follows. ■

Remark 7.10. Let $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ be metrised $\mathbb{R}$-divisors such that $\deg(D_1) > 0$ and $\deg(D_2) > 0$. Let $\overline{D} = (D_1 + D_2, g_1 + g_2)$. If g_1 and g_2 are plurisubharmoic, then Theorems 5.22 and 7.4 lead to the following inequality:

$$\frac{(\overline{D}\cdot\overline{D})}{\deg(D)} \geqslant \frac{(\overline{D}_1\cdot\overline{D}_1)}{\deg(D_1)} + \frac{(\overline{D}_2\cdot\overline{D}_2)}{\deg(D_2)}. \tag{7.5}$$

This inequality actually holds without plurisubharmonic condition (namely it suffices that g_1 and g_2 are pairable). In fact, by (7.1) one has

$$\frac{(\overline{D}_i\cdot\overline{D}_i)}{\deg(D_i)} = 2g_i(\eta_0) - \sum_{x\in X^{(1)}} \frac{[\kappa(x):k]}{\deg(D_i)} \int_0^{+\infty} \varphi'_{g_i\circ\xi_x}(t)^2\,\mathrm{d}t$$

for $i \in \{1,2\}$, and

$$\begin{aligned}\frac{(\overline{D}\cdot\overline{D})}{\deg(D)} &= 2(g_1(\eta_0) + g_2(\eta_0)) \\ &\quad - \sum_{x\in X^{(1)}} \frac{[\kappa(x):k]}{\deg(D_1)+\deg(D_2)} \int_0^{+\infty} (\varphi'_{g_1\circ\xi_x}(t) + \varphi'_{g_2\circ\xi_x}(t))^2\,\mathrm{d}t,\end{aligned}$$

which leads to

$$\begin{aligned}&(\deg(D_1)+\deg(D_2))\bigg(\frac{(\overline{D}\cdot\overline{D})}{\deg(D)} - \frac{(\overline{D}_1\cdot\overline{D}_1)}{\deg(D_1)} - \frac{(\overline{D}_2\cdot\overline{D}_2)}{\deg(D_2)}\bigg) \\ &= \sum_{x\in X^{(1)}} [\kappa(x):k]\bigg(\frac{\deg(D_2)}{\deg(D_1)}\int_0^{+\infty} \varphi'_{g_1\circ\xi_x}(t)^2\,\mathrm{d}t + \frac{\deg(D_1)}{\deg(D_2)}\int_0^{+\infty} \varphi'_{g_2\circ\xi_x}(t)^2\,\mathrm{d}t \\ &\quad - 2\int_0^{+\infty} \varphi'_{g_1\circ\xi_x}(t)\varphi'_{g_2\circ\xi_x}(t)\,\mathrm{d}t\bigg) \geqslant 0,\end{aligned}$$

by using Cauchy–Schwarz inequality and the inequality of arithmetic and geometric means.

The inequality (7.5) leads to

$$2(\overline{D}_1 \cdot \overline{D}_2) \geqslant \frac{\deg(D_2)}{\deg(D_1)}(\overline{D}_1 \cdot \overline{D}_1) + \frac{\deg(D_1)}{\deg(D_2)}(\overline{D}_2 \cdot \overline{D}_2).$$

In the case where $(\overline{D}_1 \cdot \overline{D}_2)$ and $(\overline{D}_2 \cdot \overline{D}_2)$ are non-negative, by the inequality of arithmetic and geometric means, we obtain that

$$(\overline{D}_1 \cdot \overline{D}_2) \geqslant \sqrt{(\overline{D}_1 \cdot \overline{D}_1)(\overline{D}_2 \cdot \overline{D}_2)},$$

where the equality holds if and only if $\overline{D}_1$ and $\overline{D}_2$ are proportional up to $\mathbb{R}$-linear equivalence. This could be considered as an analogue of the arithmetic Hodge index inequality of Faltings [18, Theorem 4] and Hriljac [21, Theorem 3.4], see also [30, Theorem 7.1] and [4, §5.5].

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