ON SUBFINITENESS OF GRADED LINEAR SERIES

Huayi Chen & Hideaki Ikoma

Abstract. — Hilbert's 14th problem studies the finite generation property of the intersection of an integral algebra of finite type with a subfield of the field of fractions of the algebra. It has a negative answer due to the counterexample of Nagata. We show that a subfinite version of Hilbert's 14th problem has a confirmative answer. We then establish a graded analogue of this result, which permits to show that the subfiniteness of graded linear series does not depend on the function field in which we consider it. Finally, we apply the subfiniteness result to the study of geometric and arithmetic graded linear series.

Contents

1. Introduction

Let k be a field and X be an integral projective scheme over $\operatorname{Spec} k$. If D is a Cartier divisor on X, as a *graded linear series* of D, one refers to a graded

Huayi Chen is partially supported by ANR-14-CE25-0015, Hideaki Ikoma is supported by JSPS Grant-in-Aid for Young Scientists (B) No. 16K17559 and partially by JSPS Grant-in-Aid (S) No. 16H06335.

sub-k-algebra of $\bigoplus_{n\in\mathbb{N}} H^0(X, nD)$. The graded linear series are closely related to the positivity of the divisor and are objects of central interest in the study of the geometry of the underlying polarised scheme (X, D). Classically the asymptotic behaviour of graded linear series of finite type is well understood through the theory of Hilbert polynomials. Several results in birational algebraic geometry, such as Fujita's approximation theorem [**9, 24**], show that certain graded linear series, even though not of finite type, still have a similar asymptotic behaviour as in the finite generation case. More recently, Lazarsfeld-Mustaţă [**16**] and Kaveh-Khovanskii [**14, 13**] have proposed, after ideas of Okounkov [**22, 23**], a method to encode the asymptotic behaviour of dimensions of the homogeneous components of a given graded linear series into a convex body (called the *Newton-Okounkov body*) in an Euclidean space.

Note that a graded linear series of a Cartier divisor is always a graded subalgebra of a graded algebra of finite type. It is then quite natural to ask if there is a nice birational geometry for algebras of subfinite type (namely subalgebras of an algebra of finite type) over a field.

From the point of view of birational geometry, it is more convenient to consider graded linear series of a finitely generated field extension K/k without specifying a polarised model of K. In this framework, as a *graded linear series* of K/k, we refer to a graded sub-k-algebra $V_\bullet$ of the polynomial algebra $K[T]$ such that $V_0 = k$ and that V_n is a finite dimensional vector space over k for any $n \in \mathbb{N}$. In [**6**], a new construction of Newton-Okounkov bodies has been proposed by using ideas from Arakelov geometry, which only depends on a choice of a tower of successive field extensions $k = K_0 \subset K_1 \subset \cdots \subset K_d = K$ such that each extension K_{i+1}/K_i is transcendental and of transcendence degree 1. The construction is valid for graded linear series of subfinite type (namely contained in a graded linear series of finite type of K/k) whose field of rational functions $k(V_\bullet)$ coincides with K (see Definition 3.1). One may expect that the same method applies to general graded linear series of subfinite type $V_\bullet$ by considering $V_\bullet$ as a graded linear series of $k(V_\bullet)/k$. However, the main obstruction to this strategy is that *a priori* the condition of subfiniteness depends on the extension K/k with respect to which we consider the graded linear series. This leads to the following subfiniteness problem: given a graded linear series $V_\bullet$ of K/k of subfinite type, does there exist a graded linear series $W_\bullet$ of finite type of the extension $k(V_\bullet)/k$ which contains $V_\bullet$?

Note that the above problem is closely related to Hilbert's fourteenth problem [(1)]. In fact, given a graded linear series $V_\bullet$ of K/k which is contained in a graded linear series of finite type $V'_\bullet$. The intersection of $V'_\bullet$ with $k(V_\bullet)[T]$

1. Let k be a field and $k(x_1, \ldots, x_n)$ be the field of rational functions of n variables. Hilbert's fourteenth problem asked whether the intersection of a subfield of $k(x_1, \ldots, x_n)$ and the polynomial algebra $k[x_1, \ldots, x_n]$ is finitely generated over k (as a k-algebra).

gives a graded linear series of $k(V_\bullet)/k$ containing $V_\bullet$, where $k(V_\bullet)$ is the field of rational functions of $V_\bullet$. Unfortunately the intersection is not necessarily a k-algebra of finite type, as is shown by Nagata's counterexamples [**21, 20**] to Hilbert's fourteenth problem.

Note that the above subfiniteness problem actually asks for a weaker condition than the finite generation of the intersection of $V'_\bullet$ with $k(V_\bullet)[T]$. It suffices that the intersection is contained in a graded linear series of finite type of $k(V_\bullet)$. Similarly, we can consider the following subfinite version of Hilbert's fourteenth problem, which actually has a positive answer (see Theorem 2.6 and Corollary 2.7 *infra*).

Theorem 1.1. — *Let k be a field, R be an integral k-algebra of finite type and K be the field of fractions of R. Let K' be an extension of k which is contained in K. Then there exists a finitely generated sub-k-algebra R' of K' containing $R \cap K'$, such that* $\mathrm{Frac}(R') = \mathrm{Frac}(R \cap K')$.

The method of proof consists of an induction argument with respect to the field extension K/k which permits to reduce the problem to the case where the extension K/k is monogenerated. Similar method can be applied to the graded case (but with more subtleties because of the grading structure), which leads to the following result and gives a confirmative answer to the subfiniteness problem of graded linear series. It shows that the subfiniteness of graded linear series is an absolute condition, which does not depend on the choice of field extension with respect to which the graded linear series is considered (see Theorem 3.7 and Corollary 4.11 *infra*).

Theorem 1.2. — *Let k be a field and K/k be a finitely generated field extension. Let $V_\bullet$ be a graded linear series of K/k which is of subfinite type. Then there exists a graded linear series of finite type $W_\bullet$ of K/k such that $V_\bullet \subset W_\bullet$ and $k(V_\bullet) = k(W_\bullet)$.*

Recall that Hilbert's fourteenth problem is reformulated in a geometric setting by Zariski [**25**], see also [**19**] and the survey article [**18**]. Note that Theorem 1.1 can be compared with the following result in [**25**].

***Theorem 1.3* (Zariski)**. — *Let k be a field, A an integrally closed k-algebra of finite type, $K := \mathrm{Frac}(A)$, and K'/k a subextension of K/k. There then exist an integrally closed k-algebra B of finite type and an ideal I of B such that the fraction field of B is k-isomorphic to the fraction field of $A \cap K'$ and that*

$$A \cap K' = \bigcup_{n \in \mathbb{N}} (B : I^n),$$

where $(B : I^n) := \{x \in \mathrm{Frac}(B) \,:\, xI^n \subset B\}$ denotes the ideal quotient.

Inspired by this result, we establish the following projective version of Zariski's theorem and deduce an alternative proof for Theorem 1.2 (see Corollary 4.11 *infra*).

Theorem 1.4. — *Let K/k be a finitely generated field extension and $W_\bullet$ a graded linear series of K/k that is generated over k by the homogeneous elements of degree 1. We assume that W_1 contains $1 \in K$ and that the projective spectrum $P := \mathrm{Proj}(W_\bullet)$ is a normal scheme. Let K'/k be a subextension of K/k and $W_\bullet \cap K' := \bigoplus_{n\in\mathbb{N}}(W_n \cap K')$. Then there exist a rational fibration $\pi : P \dashrightarrow X$ of P over an integral normal projective k-scheme X and an effective $\mathbb{Q}$-Weil divisor D on X having the following properties.*

(i) $\mathrm{Rat}(X) = k(W_\bullet \cap K')$.
(ii) $W_n \cap K' \subset H^0(X, nD)$ *for every sufficiently positive integer n.*
(iii) For every $\xi' \in X$ *with* $\dim \mathcal{O}_{X,\xi'} = 1$*, there exists a* $\xi \in P$ *with* $\dim \mathcal{O}_{P,\xi} = 1$ *such that* $\xi' = c_X(\mathrm{ord}_\xi |_{\mathrm{Rat}(X)})$.

Moreover, if the transcendence degree of K' over k is 1, then we can replace the property (ii) by (ii') below.

(ii') $W_n \cap K' = H^0(X, nD)$ *for every sufficiently positive integer n.*

As an application of the above subfiniteness results, we establish a Fujita approximation theorem for general graded linear series of subfinite type (see Theorem 5.2 *infra*) and an upper bound for the Hilbert-Samuel function of such graded linear series (see Theorem 5.3 *infra*). More precisely, we obtain the following results.

Theorem 1.5. — *Let K/k be a finitely generated field extension. For any graded linear series $V_\bullet$ of K/k of subfinite type, whose Kodaira-Iitaka dimension d is nonnegative, the limit*

$$\mathrm{vol}(V_\bullet) = \lim_{n\in\mathbb{N},\, V_n\neq\{0\},\, n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!}$$

exists in $(0, +\infty)$. Moreover, $\mathrm{vol}(V_\bullet)$ is equal to the supremum of $\mathrm{vol}(W_\bullet)$, where $W_\bullet$ runs over the set of all graded linear series contained in $V_\bullet$ having d as the Kodaira-Iitaka dimension. Finally, there exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that

$$f(n) = \mathrm{vol}(V_\bullet)\frac{n^d}{d!} + O(n^{d-1})$$

and that $\dim_k(V_n) \leqslant f(n)$ for any $n \in \mathbb{N}$.

We also apply the above results to the study of graded linear series in the arithmetic setting (see Theorem 5.6 *infra*).

The article is organised as follows. In the second section, we prove a weaker form of Hilbert's 14th problem as the subfiniteness result stated in

Theorem 1.1. In the third section, we prove a graded analogue of Theorem 1.1 in the setting of graded linear series. In the fourth section we consider the subfiniteness problem in the geometric setting as a projective analogue of Zariski's result and establish Theorem 1.4. Finally in the fifth section, we develop various applications.

Notation and conventions. —

1. The field of fractions of an integral domain A is denoted by $\mathrm{Frac}(A)$.

2. Let K/k be an extension of fields. We denote by $\mathrm{tr.deg}_k(K)$ the transcendence degree of K over k.

3. Let S be a scheme. For any $i \in \mathbb{N}$, we denote by $S^{(i)}$ the set of points x of S such that the local ring $\mathcal{O}_{S,x}$ has i as its Krull dimension. If S is an integral scheme, we denote by $\mathrm{Rat}(S)$ the field of rational functions on S.

4. Let k be a field and S be a projective normal scheme over $\operatorname{Spec} k$. As *Weil divisor* (resp. $\mathbb{Q}$-*Weil divisor*) on S one refers to an element

$$D = \sum_{V \in S^{(1)}} n_V V$$

in $\mathbb{Z}^{\oplus S^{(1)}}$ (resp. $\mathbb{Q}^{\oplus S^{(1)}}$). If all coefficients n_V are nonnegative, we say that D is *effective*, denoted by $D \geqslant 0$. If ϕ is a nonzero rational function on S, we denote by (ϕ) the principal Weil divisor associated with ϕ, namely

$$(\phi) := \sum_{V \in S^{(1)}} \mathrm{ord}_V(\phi) V.$$

The map $(\cdot) : \mathrm{Rat}(X)^\times \to \mathbb{Z}^{\oplus S^{(1)}}$ is a group homomorphism and induces a $\mathbb{Q}$-linear map from $\mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathbb{Q}^{\oplus S^{(1)}}$ which we denote by $(\cdot)_{\mathbb{Q}}$. If D is a $\mathbb{Q}$-Weil divisor on S, we define

$$H^0(X, D) := \big\{\phi \in \mathrm{Rat(S)}^\times \;:\; D + (\phi \otimes 1)_{\mathbb{Q}} \geqslant 0\big\} \cup \{0\} \tag{1.1}$$

and

$$R(D)_\bullet := \bigoplus_{n \geqslant 0} H^0(X, nD) T^n. \tag{1.2}$$

Note that $R(D)_\bullet$ is a graded sub-k-algebra of the polynomial algebra $\mathrm{Rat}(S)[T]$.

5. Let K/k be a field extension. As *discrete valuation* of K over k, we refer to a valuation $\nu : K \to \mathbb{Z} \cup \{+\infty\}$ such that $\nu(a) = 0$ for any $a \in k^\times$. Given such a valuation ν, we denote by $O_\nu := \{f \in K \;:\; \nu(f) \geqslant 0\}$ its valuation ring, $\mathfrak{m}_\nu$ the maximal ideal of O_ν and $\kappa(\nu) := O_\nu/\mathfrak{m}_\nu$ the residue field. Note that $\kappa(\nu)$ is an extension of k and O_ν is a k-algebra. If O_ν is equal to K, we say that the valuation ν is *trivial* (note that in this case $\nu(a) = 0$ for any $a \in K^\times$). Two discrete valuations ν_1 and ν_2 of K over

k are said to be *equivalent* if there exists an order-preserving isomorphism $\iota : \nu_1(K^\times) \to \nu_2(K^\times)$ such that $\nu_2 = \iota \circ \nu_1$.

Let K/k be a finitely generated extension of fields. We say that a discrete valuation ν of K over k is *divisorial* if

$$\mathrm{tr.deg}_k(K) = \mathrm{tr.deg}_k(\kappa(\nu)) + \mathrm{rk}_{\mathbb{Z}}(\nu(K^\times)). \tag{1.3}$$

In the case where the valuation is not trivial, the divisorial condition above is just

$$\mathrm{tr.deg}_k(K) = \mathrm{tr.deg}_k(\kappa(\nu)) + 1$$

Let K'/k be a subextension of K/k and let ν be a discrete valuation of K over k which is nontrivial. Then the restriction of ν to K' is a discrete valuation of K' over k. We define the *ramification index* of ν with respect to K' as the unique integer $e(K',\nu) \in \mathbb{N}$ satisfying

$$\nu(K'^\times) = e(K',\nu)\nu(K^\times). \tag{1.4}$$

6. Let k be a field and S be an integral separated k-scheme. Given a discrete valuation ν of $\mathrm{Rat}(S)$ over k, we say that a point x of S is the *centre* of ν in S if

$$\mathcal{O}_{S,x} \subset O_\nu \quad \text{and} \quad \mathfrak{m}_x = \mathfrak{m}_\nu \cap \mathcal{O}_{S,x}, \tag{1.5}$$

where $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{S,x}$. By the valuative criterion of separation, if the centre of ν in S exists, then it is unique. In the case where the centre of ν in S exists, we denote it by $c_S(\nu)$. If S is proper over k, then by the valuative criterion of properness every discrete valuation of $\mathrm{Rat}(S)$ over k has a centre in S.

A discrete valuation ν is trivial if and only if the centre of ν in S is the generic point. Moreover, each regular point $\xi \in S^{(1)}$ defines a discrete valuation $\mathrm{ord}_\xi : \mathrm{Rat}(S) \to \mathbb{Z} \cup \{+\infty\}$ whose centre is ξ.

7. Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded ring. We denote by $\mathrm{Proj}(R_\bullet)$ the projective spectrum of $R_\bullet$. If $M_\bullet$ is a graded $R_\bullet$-module, we denote by $\widetilde{M_\bullet}$ the quasi-coherent $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}$-module associated with $M_\bullet$ (see [**10**, §II.2.5]). For any $m \in \mathbb{N}$, we let $M(m)_\bullet$ be the $\mathbb{N}$-graded $R_\bullet$-module such that $M(m)_n = M_{n+m}$ for any $n \in \mathbb{N}$, and let $M_{\geqslant m}$ be the $\mathbb{N}$-graded sub-$R_\bullet$-module of $M_\bullet$ such that $(M_{\geqslant m})_n = \{0\}$ if $n < m$ and $(M_{\geqslant m})_n = M_n$ if $n \geqslant m$. In particular, one has $M(m)_\bullet = M_{\geqslant m}(m)_\bullet$. The quasi-coherent sheaf $\widetilde{R(m)_\bullet}$ is denoted by $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m)$. Note that if $R_\bullet$ is generated as an R_0-algebra by R_1, then $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m)$ are invertible $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}$-modules for all $m \in \mathbb{N}$, and one has canonical isomorphisms

$$\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m) \otimes_{\mathcal{O}_{\mathrm{Proj}(R_\bullet)}} \mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m') \cong \mathcal{O}_{\mathrm{Proj}(R_\bullet)}(m+m')$$

for all $(m, m') \in \mathbb{N}^2$.

8. Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded ring. We say that $R_\bullet$ is *essentially integral* if the ideal $R_{\geqslant 1}$ of $R_\bullet$ does not vanish and if the product of two nonzero homogeneous elements of positive degree of $R_\bullet$ is nonzero. Note that if $R_\bullet$ is essentially integral then the scheme $\mathrm{Proj}(R_\bullet)$ is integral (see [**10**, Proposition II.2.4.4])

2. A weak form of Hilbert's fourteenth problem

Let k be a field, R be a finitely generated integral k-algebra and K be the field of fractions of R. Clearly K is a finitely generated extension of k. Let K' be a subextension of K/k, which is also a finitely generated extension (see [**4**, Chapitre V, §14, n°7, Corollaire 3]). We consider the intersection $R\cap K'$ and ask the following question which could be considered as a weaker form of Hilbert's fourteenth problem: *does there exist a finitely generated sub-k-algebra R' of K' containing $R\cap K'$ such that* $\mathrm{Frac}(R') = \mathrm{Frac}(R\cap K')$. In this section, we give a confirmative answer to this question.

Definition 2.1. — Let k be a field and A be a k-algebra. We say that A is of *subfinite type* if it is a sub-k-algebra of a k-algebra of finite type.

Lemma 2.2. — *An injective homomorphism of rings $A\to B$ yields a dominant morphism* $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$.

Proof. — Let $\mathfrak{p}$ be a minimal prime ideal of A and $S := A\setminus\mathfrak{p}$. Since the homomorphism of rings $A\to B$ is injective, also is the localised homomorphism $A_{\mathfrak{p}}\to S^{-1}B$. Hence $S^{-1}B$ is nonzero. In particular, there exists a prime ideal $\mathfrak{P}$ of B such that $\mathfrak{P}\cap S=\emptyset$, or equivalently, $\mathfrak{P}\cap A\subset\mathfrak{p}$. Since $\mathfrak{P}\cap A$ is a prime ideal of A and $\mathfrak{p}$ is a minimal prime ideal of A, one has $\mathfrak{P}\cap A=\mathfrak{p}$. □

Proposition 2.3. — *Let k be a field and A be a k-algebra of subfinite type. We assume that A is an integral domain. Then there exists a k-algebra of finite type containing A, which is also an integral domain.*

Proof. — Let B be a k-algebra of finite type such that $A\subset B$. By Lemma 2.2, one can find a prime ideal $\mathfrak{p}$ of B such that $\mathfrak{p}\cap A=\{0\}$. Since B is a k-algebra of finite type, also is $B/\mathfrak{p}$. The proposition is thus proved. □

Lemma 2.4. — *Let A be a k-algebra which is an integral domain, and K the field of fractions of A. Let K'/K be a finite extension of K generated by one element α and B' a sub-k-algebra of finite type of K' which contains A. Then there exists a sub-k-algebra of finite type B of K which contains A.*

Proof. — Let $f\in K[T]$ be the minimal polynomial of α over K, which is assume to be monic. Let $F_1,\ldots,F_n$ be polynomials in $K[T]$ such that

$B' = k[F_1(\alpha), \ldots, F_n(\alpha)]$. Let $S \subset K$ be the (finite) set of the coefficients of the polynomials f, $F_1, \ldots, F_n$. We claim that A is contained in $k[S]$. In fact, suppose that an element u of A is written in the form $\varphi(F_1(\alpha), \ldots, F_n(\alpha))$, where $\varphi \in k[X_1, \ldots, X_n]$, then by Euclidean division the polynomial $\varphi(F_1, \ldots, F_n) \in k[S][T]$ can be written as $fg + u$, where g is a polynomial in $k[S][T]$. Therefore, one has $u \in k[S]$. □

Lemma 2.5. — *Let A be a k-algebra which is an integral domain, and K the field of fractions of A. Let K'/K be a purely transcendental extension of transcendence degree 1 and B' a sub-k-algebra of finite type of K' which contains A. Then there exists a sub-k-algebra of finite type B of K which contains A.*

Proof. — Let $\alpha \in K'$ be a transcendental element over K such that $K' = K(\alpha)$. Assume that $B' = k[\varphi_1(\alpha), \ldots, \varphi_n(\alpha)]$, where each φ_i is a rational function of the form F_i/G_i, where F_i and G_i are polynomials of one variable with coefficients in K and $G_i \neq 0$. Let β be an element in the algebraic closure of the field K such that $G_i(\beta) \neq 0$ in $K'(\beta)$ for any $i \in \{1, \ldots, n\}$. Then one has $A \subset \widetilde{B} := k[\varphi_1(\beta), \ldots, \varphi_n(\beta)] \subset K(\beta)$. In fact, if an element u of A can be written as $P(\varphi_1(\alpha), \ldots, \varphi_n(\alpha))$, where P is a polynomial with coefficients in k, then, since α is transcendental over $K(\beta)$, by considering α as the variable of rational functions and by specifying its value by β, we obtain that $u = P(\varphi_1(\beta), \ldots, \varphi_n(\beta))$. Finally, by applying Lemma 2.4 to $A \subset \widetilde{B}$ and the finite extension $K(\beta)/K$, we obtain that there exists a k-algebra of finite type $B \subset K$ such that $A \subset B$. □

Theorem 2.6. — *Let k be a field and A be a k-algebra of subfinite type. We assume in addition that A is an integral domain and we denote by K the field of fractions of A. Then there exists a sub-k-algebra of finite type B of K such that $A \subset B$.*

Proof. — By Proposition 2.3, there exists a k-algebra of finite type B' which is an integral domain containing A. Let K' be the field of fractions of B', it is a finitely generated extension of K. Therefore there exists a sequence of extensions
$$K = K_0 \subsetneq K_1 \subsetneq \ldots \subsetneq K_n = K'$$
such that each extension K_i/K_{i-1} is generated by one element, $i \in \{1, \ldots, n\}$. The extension K_i/K_{i-1} is either generated by an algebraic element over K_{i-1} or is purely transcendental of transcendence degree 1. By induction we obtain that, for any $i \in \{0, \ldots, n-1\}$, there exists a sub-k-algebra of finite type B_i of K_i such that $B_i \supset A$. The theorem is thus proved. □

Corollary 2.7. — *Let k be a field, R be an integral k-algebra of finite type and K be the field of fractions of R. Let K' be an extension of k which is* *contained in K. Then there exists a finitely generated sub-k-algebra R' of K' containing $R \cap K'$, such that* $\mathrm{Frac}(R') = \mathrm{Frac}(R \cap K')$.

Proof. — By definition, $R \cap K'$ is an integral k-algebra of subfinite type. By Theorem 2.6, there exists a sub-k-algebra of finite type R' of $\mathrm{Frac}(R \cap K')$ such that $R \cap K' \subset R'$. Clearly one has $\mathrm{Frac}(R') = \mathrm{Frac}(R \cap K')$ since $R \cap K' \subset R' \subset \mathrm{Frac}(R \cap K')$. The assertion is thus proved. □

3. Graded linear series and subfiniteness

Let k be a field and K/k be a finitely generated field extension. Let

$$K[T] = \bigoplus_{n \in \mathbb{N}} KT^n$$

be the graded ring of polynomials of one variable with coefficients in K.

Definition 3.1. — As a *graded linear series* of K/k we refer to a graded sub-k-algebra

$$V_\bullet = \bigoplus_{n \in \mathbb{N}} V_n T^n$$

of $K[T]$ such that $V_0 = k$ and that V_n is a finite dimensional k-vector subspace of K for any $n \in \mathbb{N}_{\geqslant 1}$.

Let $V_\bullet$ and $V'_\bullet$ be two graded linear series of K/k. If $V_n \subset V'_n$ for any $n \in \mathbb{N}$, we say that $V_\bullet$ *is contained in* $V'_\bullet$, or $V_\bullet$ *contains* $V'_\bullet$, and denote it by $V_\bullet \subset V'_\bullet$.

Let $V_\bullet$ be a graded linear series of K/k. If $V_\bullet$ is finitely generated as a k-algebra, we say that $V_\bullet$ is *of finite type*. If $V_\bullet$ is contained in a graded linear series of finite type, we say that it is *of subfinite type*.

Let $V_\bullet$ be a graded linear series of K/k. We denote by $k(V_\bullet)$ the subextension of K/k generated by elements of the form f/g, where f and g are nonzero elements of K such that there exists $n \in \mathbb{N}_{\geqslant 1}$ with $\{f, g\} \subset V_n$. The field $k(V_\bullet)$ is called *the field of rational functions of* $V_\bullet$.

Lemma 3.2. — *Given any graded linear series $V_\bullet$ of K/k, one has*

$$k(V_n) = k(V_\bullet)$$

for every sufficiently positive integer n with $V_n \neq \{0\}$, where $k(V_n)$ denotes the subextension of K/k generated by the elements of the form f/g with $\{f, g\} \subset V_n$, $g \neq 0$.

Proof. — First, we note that if $\ell \in \mathbb{N}_{\geqslant 1}$ is an index such that V_ℓ contains a nonzero element h, then $k(V_m) \subset k(V_{m+\ell n})$ for any $m, n \in \mathbb{N}_{\geqslant 1}$. In fact, if $\{f, g\} \subset V_m$ and $g \neq 0$, then

Misuse of k as the index since k is referred to the base field.

$$\frac{f}{g} = \frac{fh^n}{gh^n} \text{ and } \{fh^n, gh^n\} \subset V_{m+\ell n}$$

for any $n \in \mathbb{N}_{\geqslant 1}$.

By changing the grading of $V_\bullet$, we may assume without loss of generality that $\{n \in \mathbb{N} : V_n \neq \{0\}\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module. There exist integers $\{n_1, \ldots, n_r\} \subset \mathbb{N}_{\geqslant 1}$ and nonzero elements $\{f_1, \ldots, f_r, g_1, \ldots, g_r\} \subset K$ such that $\{f_i, g_i\} \subset V_{n_i}$ for any $i \in \{1, \ldots, r\}$ and that $k(V_\bullet) = k(f_1/g_1, \ldots, f_r/g_r)$. Set $p := \mathrm{lcm}(n_1, \ldots, n_r)$. By the above observation, we can assume $\{f_i, g_i\} \subset V_p$ for any i, and one has

$$k(V_\bullet) = k(f_1/g_1, \ldots, f_r/g_r) = k(V_p).$$

Moreover, by the hypothesis that $\{n \in \mathbb{N} : V_n \neq \{0\}\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module, we can find a positive integer q such that p and q are coprime and that $k(V_p) = k(V_q) = k(V_\bullet)$.

To conclude the proof, it suffices to show that $\{pm + qn : m, n \in \mathbb{N}\}$ contains every sufficiently positive integer. Since p and q are coprime, we can fix $x, y \in \mathbb{Z}$ such that $px - qy = 1$. Moreover, we can assume that both x and y are positive. For any r with $0 \leqslant r < q$ and any n with $n \geqslant (q-1)y$,

$$qn + r = prx + q(n - ry) \in \{pm + qn : m, n \in \mathbb{N}\}.$$

Hence $\{pm+qn : m, n \in \mathbb{N}\}$ contains every integer not less than $q(q-1)y$. □

Remark 3.3. — Let $V_\bullet$ be a graded linear series of K/k and f be a nonzero element of K. We denote by $V_\bullet(f)$ the graded linear series $\bigoplus_{n\in\mathbb{N}} f^n V_n T^n$, where $f^n V_n := \{f^n g : g \in V_n\}$, called the *twist of* $V_\bullet$ *by* f. Note that the twist does not change the field of rational functions: one has $k(V_\bullet(f)) = k(V_\bullet)$ for any $f \in K \setminus \{0\}$.

Proposition 3.4. — *Let $W_\bullet$ be a graded linear series of finite type of K/k. Let n_0 be an integer such that $n_0 \geqslant 1$. There exist an integer $r \geqslant 1$ and a family $(f_i T^{n_i})_{i=1}^r$ of homogeneous elements in $W_\bullet$ such that the following conditions are fulfilled:*

(1) for any $i \in \{1, \ldots, r\}$, one has $n_i \geqslant n_0$;
(2) for any integer $n \geqslant n_0$, the vector space W_n is generated by elements of the form $f_1^{a_1} \cdots f_r^{a_r}$, where $a_1, \ldots, a_r$ are natural numbers such that $a_1 n_1 + \cdots + a_r n_r = n$.

Proof. — Suppose that $W_\bullet$ is generated by $W_1 T \oplus \cdots \oplus W_d T^d$. We claim that the graded linear series

$$k \oplus \bigoplus_{n \geqslant n_0} W_n T^n$$

is generated by $W_{n_0} T^{n_0} \oplus \cdots \oplus W_{2n_0+d-2} T^{2n_0+d-2}$. Let n be an integer such that $n \geqslant 2n_0 + d - 2$. Since $W_\bullet$ is generated by $W_1 T \oplus \cdots \oplus W_d T^d$, we obtain that

$$W_n=\sum_{\substack{(a_1,\ldots,a_d)\in\mathbb{N}^d\\ a_1+2a_2+\cdots+da_d=n}} W_1^{a_1}\cdots W_d^{a_d}.$$

Let $(a_1,\ldots,a_d)$ be an element in $\mathbb{N}^d$ such that $a_1+2a_2+\cdots+da_d=n$. Since $n\geqslant 2n_0+d-2$, there exist an integer $m\geqslant 1$ and a family

$$\big\{(a_1^{(i)},\ldots,a_d^{(i)})\,:\,i\in\{1,\ldots,m\}\big\}$$

of elements in $\mathbb{N}^d$ such that

$$\forall\, j\in\{1,\ldots,d\},\quad a_j^{(1)}+\cdots+a_j^{(m)}=a_j,$$

$$\forall\, i\in\{1,\ldots,m-1\},\quad n_0\leqslant a_1^{(i)}+2a_2^{(i)}+\cdots+da_d^{(i)}\leqslant n_0+d-1,$$

and

$$n_0\leqslant a_1^{(m)}+2a_2^{(m)}+\cdots+da_d^{(m)}\leqslant 2n_0+d-2.$$

Therefore

$$W_n=\sum_{\substack{(b_{n_0},\ldots,b_{2n_0+d-2})\in\mathbb{N}^{n_0+d-1}\\ n_0b_{n_0}+\cdots+(2n_0+d-2)b_{2n_0+d-2}=n}} W_{n_0}^{b_{n_0}}\cdots W_{2n_0+d-2}^{b_{2n_0+d-2}},$$

which concludes the claim (b_j corresponds to the number of $i\in\{1,\ldots,m\}$ such that $a_1^{(i)}+2a_2^{(i)}+\cdots+da_d^{(i)}=j$). Finally it suffices to choose a family of homogeneous elements in $W_\bullet$ which forms a basis of $W_{n_0}T^{n_0}\oplus\cdots\oplus W_{2n_0+d-2}T^{2n_0+d-2}$. □

Lemma 3.5. — *Let $K/k'/k$ be extensions of fields. We assume that the extension K/k is finitely generated and the extension k'/k is finite. Let $W'_\bullet$ be a graded linear series of finite type of K/k' and let*

$$W_\bullet=k\oplus\bigoplus_{n\in\mathbb{N}_{\geqslant 1}} W'_nT^n.$$

Then $W_\bullet$ is a graded linear series of finite type of K/k.

Proof. — Let $(f_iT^{n_i})_{i=1}^r$ be a system of generators of $W'_\bullet$. Let $(\theta_j)_{j=1}^m$ be a basis of k' over k. We claim that $W_\bullet$ is generated by

$$(\theta_jf_iT^{n_i})_{(i,j)\in\{1,\ldots,r\}\times\{1,\ldots,m\}}. \tag{3.1}$$

In fact, if φ is an element of W'_n, then it can be written as

$$\sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r\\ a_1n_1+\cdots+a_rn_r=n}}\lambda_{\boldsymbol{a}}f_1^{a_1}\cdots f_r^{a_r},$$

where the coefficients $\lambda_{\boldsymbol{a}}$ belong to k'. By writing $\lambda_{\boldsymbol{a}}$ as a linear combination of $(\theta_j)_{j=1}^m$, we obtain that φ lies in the graded linear series of K/k generated by (3.1). The lemma is thus proved. $\square$

Definition 3.6. — Let $V_\bullet$ be a graded linear series of K/k. We assume that there exists $n \in \mathbb{N}_{\geqslant 1}$ such that $V_n \neq \{0\}$. We define the *Kodaira-Iitaka dimension* of $V_\bullet$ as the transcendence degree of $k(V_\bullet)$ over k. We refer the readers to [**13**, §3] and [**7**, §2] for the definition of Kodaira-Iitaka dimension in the setting of graded linear series of Cartier divisors or line bundles. If $V_n = \{0\}$ for any $n \in \mathbb{N}_{\geqslant 1}$, then by convention the *Kodaira-Iitaka dimension* of $V_\bullet$ is defined to be $-\infty$.

Theorem 3.7. — *Let $V_\bullet$ be a graded linear series of K/k. Assume that there exists a graded linear series of finite type $V'_\bullet$ of K/k which contains $V_\bullet$. Then there exists a graded linear series of finite type $W_\bullet$ of K/k such that $V_\bullet \subset W_\bullet$ and $k(V_\bullet) = k(W_\bullet)$.*

Proof. — *Step 1: reduction to the case where* $1 \in V_1$ *and* $k(V'_1) = k(V'_\bullet)$. Let $\Theta := \{n \in \mathbb{N}_{\geqslant 1} : V_n \neq \{0\}\}$. The assertion of the theorem is trivial when $\Theta = \varnothing$. In the following, we assume that Θ is not empty, and hence it is a subsemigroup of $\mathbb{N}_{\geqslant 1}$. Let $a \in \mathbb{N}_{\geqslant 1}$ be a generator of the subgroup of $\mathbb{Z}$ generated by Θ. As $\bigoplus_{n\in\mathbb{N}} V'_{an}T^{an}$ is a k-algebra of finite type (see for example [**10**, Lemme II.2.1.6.(iv)]), by changing the grading we can reduce the problem to the case where $a = 1$. In particular, there exists an $m \in \mathbb{N}_{\geqslant 1}$ such that the vector spaces V_m and V_{m+1} are both nonzero. We pick $x \in V_m \setminus \{0\}$ and $y \in V_{m+1} \setminus \{0\}$. By replacing $V_\bullet$ by the graded linear series generated by $V_\bullet$ and $(y/x)T$ and replacing $V'_\bullet$ by the graded linear series generated by $V'_\bullet$ and $(y/x)T$ (this procedure does not change the fields of rational functions), we reduce the problem to the case where $V_1 \neq \{0\}$. Finally, by replacing $V_\bullet$ by $V_\bullet(f^{-1})$ and $V'_\bullet$ by $V'_\bullet(f^{-1})$ (see Remark 3.3 for the notation), where f is a nonzero element of V_1 (again this procedure does not change the fields of fractions, see Remark 3.3), we reduce the problem to the case where $1 \in V_1$. Moreover, by replacing $V'_\bullet$ by the graded linear series generated by $V'_\bullet$ and $\alpha_1 T, \ldots, \alpha_m T$, where $\{\alpha_1, \ldots, \alpha_m\}$ is a system of generators of $k(V'_\bullet)$ over k, we may assume that $k(V'_1) = k(V'_\bullet)$.

Step 2: reduction to the simple extension case by induction. As explained in the previous step, we can assume $1 \in V_1$ and $k(V'_1) = k(V'_\bullet)$. Since $k(V'_\bullet)/k(V_\bullet)$ is a finitely generated extension of fields (where V_1 is assumed to contain 1), there exist successive extensions of fields

$$k(V_\bullet) = K_0 \subsetneq K_1 \subsetneq \ldots \subsetneq K_b = k(V'_\bullet)$$

such that each extension K_i/K_{i-1} is generated by one element of V'_1.

Assume that the theorem has been proved for the case where $k(V'_\bullet)/k(V_\bullet)$ is generated by one element in V'_1. Then by induction we can show that, for any $i \in \{0, \ldots, b\}$, there exists a graded linear series of finite type $W^{(i)}_\bullet$, which contains $V_\bullet$ and such that $k(W^{(i)}_\bullet) = K_i$. In fact, we can choose $W^{(r)}_\bullet = V'_\bullet$. Assume that we have chosen a graded linear series of finite type $W^{(i+1)}_\bullet$ such that $W^{(i+1)}_\bullet \supset V_\bullet$ and $k(W^{(i+1)}_\bullet) = K_{i+1}$, where $i \in \{0, \ldots, b-1\}$. Let $V^{(i)}_\bullet$ be the graded linear series generated by $V_\bullet$ and a finite system of generators of K_i/k in V'_1. The graded linear series $V^{(i)}_\bullet$ contains $V_\bullet$ and $K_i = k(V^{(i)}_1)$. Without loss of generality we may assume that $V^{(i)}_\bullet \subset W^{(i+1)}_\bullet$ and that the extension K_{i+1}/K_i is generated by one element α in $W^{(i+1)}_1$, otherwise we just replace $W^{(i+1)}_\bullet$ by the graded linear series generated by $W^{(i+1)}_\bullet$, $V^{(i)}_1$ and a generator of the extension K_{i+1}/K_i in V'_1. It is a graded linear series of finite type which contains $V_\bullet$ and has K_{i+1} as its field of rational functions. If the theorem has been proved for the simple extension case, then we obtain the existence of a graded linear series of finite type $W^{(i)}_\bullet$ such that $V_\bullet \subset W_\bullet$ and $k(W^{(i)}_\bullet) = K_i$.

Note that the graded linear series $W_\bullet = W^{(0)}_\bullet$ satisfies the conditions $V_\bullet \subset W_\bullet$ and $k(V_\bullet) = k(W_\bullet)$. Therefore, to prove the theorem it suffices to prove the particular case where the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element in V'_1. Similarly, to prove the theorem under the supplementary condition that the extension $k(V'_\bullet)/k(V_\bullet)$ is algebraic, it suffices to prove the particular case where the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element in V'_1 which is algebraic over $k(V_\bullet)$.

Step 3: algebraic extension case. In this step, we prove the theorem under the assumption that the extension $k(V'_\bullet)/k(V_\bullet)$ is algebraic. As explained in the previous two steps, we may suppose without loss of generality that $1 \in V_1$, $k(V'_1) = k(V'_\bullet)$ and the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element α in V'_1 which is algebraic over $k(V_\bullet)$.

Let

$$G(X) := X^\delta + \xi_1 X^{\delta-1} + \cdots + \xi_\delta \in k(V_\bullet)[X]$$

be the minimal polynomial of α over $k(V_\bullet)$. By Proposition 3.4, there exist an integer $r \in \mathbb{N}_{\geqslant 1}$ and homogeneous elements $(f_i T^{n_i})_{i=1}^r$ with $n_i \geqslant \delta$ for any $i \in \{1, \ldots, r\}$, which generates the graded linear series

$$k \oplus \bigoplus_{n \geqslant \delta} V'_n T^n.$$

Since $1 \in V_n \subset V'_n$ for any $n \in \mathbb{N}_{\geqslant 1}$, for any $i \in \{1, \ldots, r\}$, one has $f_i \in k(V'_\bullet)$. Moreover, since the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by α (which is of degree δ over $k(V_\bullet)$), there exist polynomials

$$F_i(X) := \eta_{i,1} X^{\delta-1} + \cdots + \eta_{i,\delta} \in k(V_\bullet)[X], \quad i \in \{1, \ldots, r\}$$

such that $f_i = F_i(\alpha)$ for any $i \in \{1, \ldots, r\}$. We introduce the following polynomials in $k(V_\bullet)[T, Y]$

$$\widetilde{G}(T, Y) = Y^\delta + (\xi_1 T)Y^{\delta-1} + \cdots + \xi_\delta T^\delta,$$
$$\widetilde{F}_i(T, Y) = (\eta_{i,1} T^{n_i-\delta+1})Y^{\delta-1} + \cdots + \eta_{i,\delta} T^{n_i}.$$

Note that one has $\widetilde{G}(T, TX) = G(X)T^\delta$ and $\widetilde{F}(T, TX) = F_i(X)T^{n_i}$.

We let $W_\bullet$ be the graded linear series generated by $V_1 T \oplus \cdots \oplus V_{\delta-1} T^{\delta-1}$ and the elements $\xi_1 T, \ldots, \xi_\delta T^\delta$, $\eta_{i,1} T^{n_i-\delta+1}, \ldots, \eta_{i,\delta} T^{n_i}$ $(i \in \{1, \ldots, r\})$. It is a graded linear series of finite type of K/k such that $k(W_\bullet) \subset k(V_\bullet)$. It remains to prove that $W_\bullet$ contains $V_\bullet$. Clearly $V_n \subset W_n$ for $n \in \{1, \ldots, \delta - 1\}$. Let $n \in \mathbb{N}_{\geqslant \delta}$ and φ be an element in $V_n \subset V'_n$. By definition φ can be written in the form

$$\sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1 + \cdots + a_r n_r = n}} \lambda_{\boldsymbol{a}} f_1^{a_1} \cdots f_r^{a_r} = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1 + \cdots + a_r n_r = n}} \lambda_{\boldsymbol{a}} F_1(\alpha)^{a_1} \cdots F_r(\alpha)^{a_r},$$

where $\lambda_{\boldsymbol{a}} \in k$. We consider the element

$$\widetilde{F}(T, Y) = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1 + \cdots + a_r n_r = n}} \lambda_{\boldsymbol{a}} \widetilde{F}_1(T, Y)^{a_1} \cdots \widetilde{F}_r(T, Y)^{a_r} \in k(V_\bullet)[T, Y].$$

Viewed as a polynomial on Y with coefficients in $k(V_\bullet)[T]$, the coefficients of $\widetilde{F}(T, Y)$ can be written as the values of certain polynomials on $\eta_{i,1} T^{n_i-\delta+1}, \ldots, \eta_{i,\delta} T^{n_i}$ $(i \in \{1, \ldots, r\})$. Note that one has

$$\widetilde{F}(T, TX) = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1 + \cdots + a_r n_r = n}} \lambda_{\boldsymbol{a}} F_1(X)^{a_1} \cdots F_r(X)^{a_r} T^n.$$

Therefore $\widetilde{F}(T, T\alpha) - \varphi T^n = 0$ in $k(V'_\bullet)[T]$. Since G is the minimal polynomial of α, an Euclidean division argument shows that φT^n can be written as a polynomial of $\xi_1 T, \ldots, \xi_\delta T^\delta$, $\eta_{i,1} T^{n_i-\delta+1}, \ldots, \eta_{i,\delta} T^{n_i}$ $(i \in \{1, \ldots, r\})$. The theorem is thus proved in the particular case where $k(V'_\bullet)/k(V_\bullet)$ is an algebraic extension.

Step 4: general case. In this step, we prove the theorem in the general case. As explained in steps 1 and 2, we may assume that $1 \in V_1$, $k(V'_1) = k(V'_\bullet)$ and that the extension $k(V'_\bullet)/k(V_\bullet)$ is generated by one element α in V'_1 which is transcendental over $k(V_\bullet)$ (the algebraic case has already been treated in Step 3).

Since $V'_\bullet$ is of finite type, there exist an integer $r \geqslant 1$ and homogeneous elements $(f_i T^{n_i})_{i=1}^r$ which generate $V'_\bullet$ as a k-algebra. As $k(V'_\bullet)/k(V_\bullet)$ is generated by α, there exists rational functions P_i/Q_i $(i \in \{1, \ldots, r\})$, where $\{P_i, Q_i\} \subset k(V_\bullet)[X]$, $Q_i \neq 0$, such that $f_i = P_i(\alpha)/Q_i(\alpha)$.

Let θ be an element in the algebraic closure $\overline{k}$ of k, such that $Q_i(\theta) \neq 0$ for any $i \in \{1, \ldots, r\}$. Let $\widehat{k} = k(\theta)$ and $\widehat{K} = K(\theta)$. Then $\widehat{K}/K$ is a finite extension of field, and $\widehat{K}/\widehat{k}$ is a purely transcendental extension generated by α. Let $\widehat{V}_\bullet$ and $\widehat{V}'_\bullet$ be the graded sub-$\widehat{k}$-algebra of $\widehat{K}[T]$ generated by $V_\bullet$ and $V'_\bullet$ respectively. Then $\widehat{V}'_\bullet$ is generated as a $\widehat{k}$-algebra by $(f_i T^{n_i})_{i=1}^r$. We let $\widehat{W}_\bullet$ be the graded linear series of $\widehat{K}/\widehat{k}$ generated by T and elements of the form $(P_i(\theta)/Q_i(\theta))T^{n_i}$, where $i \in \{1, \ldots, r\}$. This is a graded linear series of finite type. Note that $P_i(\theta)/Q_i(\theta) \in \widehat{k}(\widehat{V}_\bullet)$ for any $i \in \{1, \ldots, r\}$. Therefore $\widehat{k}(\widehat{W}_\bullet) \subset \widehat{k}(\widehat{V}_\bullet)$.

Let $n \in \mathbb{N}_{\geqslant 1}$ and φ be an element of $\widehat{V}_n \subset \widehat{V}'_n$. By definition φ can be written in the form

$$\varphi = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1+\cdots+a_r n_r = n}} \lambda_{\boldsymbol{a}} f_1^{a_1} \cdots f_r^{a_r},$$

where the coefficients $\lambda_{\boldsymbol{a}}$ belong to $\widehat{k}$. As α is transcendental over $\widehat{k}(\widehat{V}_\bullet)$, we obtain that

$$\varphi = \sum_{\substack{\boldsymbol{a}=(a_1,\ldots,a_r)\in\mathbb{N}^r \\ a_1 n_1+\cdots+a_r n_r = n}} \lambda_{\boldsymbol{a}} \prod_{i=1}^r \Big(\frac{P_i(\theta)}{Q_i(\theta)}\Big)^{a_i},$$

which shows that $\varphi \in W_n$. Therefore one has $\widehat{V}_\bullet \subset \widehat{W}_\bullet$, which implies that $\widehat{k}(\widehat{V}_\bullet) = \widehat{k}(\widehat{W}_\bullet)$ since we have already seen that $\widehat{k}(\widehat{W}_\bullet) \subset \widehat{k}(\widehat{V}_\bullet)$.

Let

$$W'_\bullet := k \oplus \bigoplus_{n\in\mathbb{N}_{\geqslant 1}} \widehat{W}_n T^n.$$

Since $\widehat{W}_\bullet$ is a graded linear series of finite type of $\widehat{K}/\widehat{k}$, by Lemma 3.5 we obtain that $W'_\bullet$ is a graded linear series of $\widehat{K}/k$ of finite type. Moreover, one has $V_\bullet \subset W'_\bullet$ and $k(W'_\bullet) \subset \widehat{k}(\widehat{W}_\bullet) = \widehat{k}(\widehat{V}_\bullet)$ is a finite extension of $k(V_\bullet)$. Therefore, by the algebraic extension case of the theorem proved in Step 3 we obtain the existence of a graded linear series of finite type $W_\bullet$ of $\widehat{K}/k$ such that $V_\bullet \subset W_\bullet$ and that $k(V_\bullet) = k(W_\bullet)$. Moreover, the equality $k(V_\bullet) = k(W_\bullet)$ and the assumption $1 \in V_1 \subset W_1$ imply that $W_\bullet$ is a graded linear series of $k(V_\bullet)/k$ (and hence a graded linear series of K/k). The theorem is thus proved. □

4. A subfinite version of Zariski's theorem

4.1. Preliminaries. — In this section, we collect several basic facts on the valuations and on the graded rings, which we use to show Theorem 1.4.

4.1.1. — Let K/k be a finitely generated field extension. We set

$$\begin{aligned}&\mathsf{D}(K/k)\\&\quad=\{\nu\,:\,\text{divisorial valuations of } K \text{ over } k\}\\&\quad=\left\{\nu\,:\,\begin{array}{l}\nu \text{ is a discrete valuation of } K \text{ over } k \text{ which either is trivial}\\ \text{or satisfies } \nu(K^{\times})\cong\mathbb{Z} \text{ and } \mathrm{tr.deg}_k(\kappa(\nu))=\mathrm{tr.deg}_k(K)-1\end{array}\right\}\end{aligned}$$

(see Notation and conventions **5**).

Lemma 4.1. — *Let K/k be a finitely generated field extension and K'/k be a subextension of K/k. If $\nu\in\mathsf{D}(K/k)$, then the restriction $\nu':=\nu|_{K'}$ belongs to $\mathsf{D}(K'/k)$.*

Proof. — Let $\nu\in\mathsf{D}(K/k)$. The assertion is trivially true if the restriction of ν to K' is a trivial valuation. In the rest of the proof, we assume that ν' is not trivial (and consequently ν is not trivial). By applying [**3**, Chapitre VI, §10, n°3, Corollaire 1] to K'/k and K/K', one has

$$\mathrm{tr.deg}_k(\kappa(\nu'))+1=\mathrm{tr.deg}_k(\kappa(\nu'))+\mathrm{rk}(\nu(K'^{\times}))\leqslant\mathrm{tr.deg}_k(K') \tag{4.1}$$

and

$$\mathrm{tr.deg}_{\kappa(\nu')}(\kappa(\nu))=\mathrm{tr.deg}_{\kappa(\nu')}(\kappa(\nu))+\mathrm{rk}(\nu(K^{\times})/\nu(K'^{\times}))\leqslant\mathrm{tr.deg}_{K'}(K), \tag{4.2}$$

respectively. By taking the sum of these two inequalities, one obtains

$$\begin{aligned}\mathrm{tr.deg}_k(\kappa(\nu'))+\mathrm{tr.deg}_{\kappa(\nu')}(\kappa(\nu))&\leqslant\mathrm{tr.deg}_k(K')+\mathrm{tr.deg}_{K'}(K)-1\\&=\mathrm{tr.deg}_k(K)-1.\end{aligned} \tag{4.3}$$

Since ν belongs to $\mathsf{D}(K/k)$, one has

$$\mathrm{tr.deg}_k(\kappa(\nu))=\mathrm{tr.deg}_k(\kappa(\nu'))+\mathrm{tr.deg}_{\kappa(\nu')}(\kappa(\nu))=\mathrm{tr.deg}_k(K)-1,$$

which means that the inequality in (4.3) is actually an equality. Hence both inequalities (4.1) and (4.2) are equalities. In particular, one has $\mathrm{tr.deg}_k(\kappa(\nu'))=\mathrm{tr.deg}_k(K')-1$. □

Lemma 4.2. — *Let $\pi:X\to X'$ be a dominant morphism of integral separated k-schemes, $K:=\mathrm{Rat}(X)$, $K':=\mathrm{Rat}(X')$, and ν a discrete valuation of K/k. If the centre $c_X(\nu)$ of ν in X exists, then $\pi(c_X(\nu))$ is the center of $\nu|_{K'}$ in X', namely $\pi(c_X(\nu))=c_{X'}(\nu|_{K'})$.*

Proof. — Since the morphism π is dominant, it induces an injective homomorphism of fields $\mathrm{Rat}(X')\to\mathrm{Rat}(X)$, which allows to consider K' as a subfield of K. Recall that the centre $c_X(\nu)$ is the unique point $x\in X$ satisfying $\mathcal{O}_{X,x}\subset O_\nu$ and $\mathfrak{m}_x=\mathfrak{m}_\nu\cap\mathcal{O}_{X,x}$ (see Notation and conventions **6**). Note that

$$O_{\nu|_{K'}}=\big\{f\in K'\,:\,\nu(f)\geqslant 0\big\}=O_\nu\cap K',\quad\text{and}\quad\mathfrak{m}_{\nu|_{K'}}=\mathfrak{m}_\nu\cap K'.$$

Hence $O_{X',\pi(c_X(\nu))} \subset O_{\nu|_{K'}}$ and $\mathfrak{m}_{\pi(c_X(\nu))} \subset \mathfrak{m}_{\nu|_{K'}}$ (which implies $\mathfrak{m}_{\pi(c_X(\nu))} = \mathfrak{m}_{v|_{K'}} \cap \mathcal{O}_{X',\pi(c_X(\nu))}$ since $\mathfrak{m}_{\pi(c_X(\nu))}$ is a maximal ideal). □

4.1.2. — Let $R_\bullet$ be a graded ring and let $P := \mathrm{Proj}(R_\bullet)$. For each homogeneous element $a \in R_{\geqslant 1}$, let

$$(R_\bullet)_{(a)} := \left\{ \frac{f}{a^p} \ : \ \deg f = p \deg a \right\} \tag{4.4}$$

be the degree 0 component of the localisation $R_\bullet[1/a]$, and let $D_{\mathrm{Proj}(R_\bullet)+}(a) := \mathrm{Spec}((R_\bullet)_{(a)})$ denote the affine open subscheme of $\mathrm{Proj}(R_\bullet)$.

Set $\mathcal{O}_P(n) := \widetilde{R(n)_\bullet}$ (see Notation and conventions **7**). Given an $s \in R_n$, the local sections $s/1 \in H^0(D_{P+}(a), \mathcal{O}_P(n)) = (R(n)_\bullet)_{(a)}$ for $a \in R_1$ glue up to a global section $\alpha_n(s) \in H^0(P, \mathcal{O}_P(n))$. The following lemmas are well-known.

Lemma 4.3 **(**[**10**, Proposition II.2.7.3]**).** — *Suppose that the irrelevant ideal $R_{\geqslant 1}$ is finitely generated, and let $M_\bullet$ be a finitely generated graded $R_\bullet$-module. If $\widetilde{M_\bullet} = 0$, then $M_n = \{0\}$ for any sufficiently positive integer n.*

Lemma 4.4. — *Let $R_\bullet$ be a graded ring and $P = \mathrm{Proj}(R_\bullet)$. If $R_\bullet$ is essentially integral and is generated as an R_0-algebra by finitely many homogeneous elements in R_1, then the canonical homomorphism $\alpha_\bullet : R_\bullet \to R(\mathcal{O}_P(1))_\bullet := \bigoplus_{n\in\mathbb{N}} H^0(P, \mathcal{O}_P(n))$ is injective and any element of $R(\mathcal{O}_P(1))_\bullet$ is integral over $R_\bullet$.*

Proof. — Suppose that $R_\bullet$ is generated as an R_0-algebra by $\{a_1, \ldots, a_r\} \subset R_1\backslash\{0\}$, where $a_1, \ldots, a_r$ are all non zerodivisors in $R_{\geqslant 1}$ since $R_\bullet$ is essentially integral (see Notation and conventions **8**). Given any $\mathfrak{p} \in P$, one can find an a_i such that $a_i \notin \mathfrak{p}$; hence $(D_{P+}(a_i))_{i\in\{1,\ldots,r\}}$ covers P. Thus, a section in $R(\mathcal{O}_P(1))_\bullet$ can naturally be identified with an element in

$$\bigcap_{i=1}^{r} R_\bullet[1/a_i], \tag{4.5}$$

where the intersection is taken in $R_\bullet[1/(a_1 \ldots a_r)]$. In particular, $\alpha_\bullet$ is injective.

Given any homogeneous element $u \in R(\mathcal{O}_P(1))_\bullet$, one can find an $e \geqslant 1$ such that $a_i^e u \in R_\bullet$ for every i by (4.5). Since $a_1, \ldots, a_r$ generates $R_{\geqslant 1}$, one obtains $R_{\geqslant re} u \subset R_{\geqslant re}$. Moreover, by induction,

$$R_{\geqslant re} u^n \subset R_{\geqslant re} u^{n-1} \subset \cdots \subset R_{\geqslant re} u \subset R_{\geqslant re}$$

for every $n \geqslant 1$. It implies that $R_\bullet[u] \subset (1/a_1)^{re} R_\bullet$; hence u is integral over $R_\bullet$ (see for example [**17**, Theorem 9.1]). □

Lemma 4.5. — *We keep the notation of Lemma 4.4. Suppose that $R_\bullet$ is a Noetherian integral domain and is generated as an R_0-algebra by finitely many homogeneous elements in R_1.*

(1) If $R_\bullet$ is an N-1 ring, then there exists an $n_0 \geqslant 0$ such that α_n is isomorphic for every $n \geqslant n_0$.
(2) If $R_\bullet$ is an integrally closed domain, then α_n is isomorphic for every $n \geqslant 0$.

Proof. — (1) Recall that an integral domain is called an N-1 ring if its integral closure in its fraction field is a finite generated module over itself. Note that the graded rings $R_\bullet$ and $R'_\bullet := R(\mathcal{O}_P(1))_\bullet$ have the same homogeneous fraction field, which is the field of rational functions of the scheme $\mathrm{Proj}(R_\bullet)$. In particular, any homogeneous element of $R'_\bullet$ belongs to the homogeneous fraction field of $R_\bullet$, which is contained in the fraction field of $R_\bullet$. By Lemma 4.4 we obtain that $R'_\bullet$ is contained in the integral closure of $R_\bullet$ and hence is a module of finite type over $R_\bullet$ by the Noetherian and N-1 hypotheses.

We consider the exact sequence of $\mathcal{O}_{\mathrm{Proj}(R_\bullet)}$-modules:

$$0 \longrightarrow \widetilde{\mathrm{Ker}(\alpha_\bullet)} \longrightarrow \widetilde{R_\bullet} \xrightarrow{\widetilde{\alpha_\bullet}} \widetilde{R'_\bullet} \longrightarrow \widetilde{\mathrm{Coker}(\alpha_\bullet)} \longrightarrow 0.$$

Since $\widetilde{\alpha_\bullet}$ is isomorphic by [**10**, Proposition II.2.7.11], we have $\widetilde{\mathrm{Ker}(\alpha_\bullet)} = \widetilde{\mathrm{Coker}(\alpha_\bullet)} = 0$. Hence, by Lemma 4.3, we conclude.

(2) If $R_\bullet$ is integrally closed, the above argument actually leads to $R_\bullet = R'_\bullet$ since $R'_\bullet$ is contained in the integral closure of $R_\bullet$. □

Given a graded ring $R_\bullet$ and a positive integer d, the *Veronese subring* $R_\bullet^{(d)}$ of $R_\bullet$ is defined as

$$R_n^{(d)} := \begin{cases} R_n & \text{if } d \text{ divides } n, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

for any $n \in \mathbb{N}$.

Lemma 4.6 **([10**, Proposition II.2.4.7**]).** — *For any $d \in \mathbb{N}_{\geqslant 1}$, the natural inclusion $u_\bullet : R_\bullet^{(d)} \to R_\bullet$ induces an isomorphism $\mathrm{Proj}(u_\bullet) : \mathrm{Proj}(R_\bullet) \to \mathrm{Proj}(R_\bullet^{(d)})$ sending $\mathfrak{p}$ to $\mathfrak{p} \cap R_\bullet^{(d)}$. Moreover, the open subscheme $D_{\mathrm{Proj}(R_\bullet)+}(a)$ is isomorphic to $D_{\mathrm{Proj}(R_\bullet^{(d)})+}(a^d)$ via $\mathrm{Proj}(u_\bullet)$ for each homogeneous element $a \in R_{\geqslant 1}$.*

Lemma 4.7. — *If $R_\bullet$ is of finite type over R_0, then $\mathrm{Proj}(R_\bullet)$ is projective over $\mathrm{Spec}(R_0)$.*

Proof. — By [**2**, Chapitre III, §1, n°3, Proposition 3] and Lemma 4.6, there exists a positive integer n_0 such that $R_\bullet^{(n_0)} = R_0[R_{n_0}]$ and that the natural morphism $\mathrm{Proj}(R_\bullet) \to \mathrm{Proj}(R_\bullet^{(n_0)})$ is isomorphic over $\mathrm{Spec}(R_0)$. Since the graded ring $R'_\bullet := \bigoplus_{n\in\mathbb{N}} R_{n_0 n}^{(n_0)}$ is finitely generated by R'_1 as an R_0-algebra, $R'_\bullet$ can be written as a quotient of a suitable polynomial algebra $R_0[T_1, \ldots, T_r]$ with $\deg T_i = 1$ for each i. Then $\mathrm{Proj}(R'_\bullet) \to \mathbb{P}^r_{R_0}$ is a closed embedding and

$\mathrm{Proj}(R'_\bullet)$ is projective over $\mathrm{Spec}(R_0)$ in the sense of Hartshorne [**12**, page 103]. The isomorphisms $R'_n \to R^{(n_0)}_{n_0 n}$ yield an isomorphism $(R'_\bullet)_{(a)} \to (R^{(n_0)}_\bullet)_{(a)}$ for each homogeneous element $a \in R'_1$. Hence, $\mathrm{Proj}(R^{(n_0)}_\bullet) \to \mathrm{Proj}(R'_\bullet)$ is an isomorphism and $\mathrm{Proj}(R^{(n_0)}_\bullet)$ is also projective over $\mathrm{Spec}(R_0)$. □

4.2. Proof of Theorem 1.4. — We begin with a reminder on the notation and the hypotheses of the theorem. Let K/k be a finitely generated field extension and $W_\bullet$ a graded linear series of K/k (see Definition 3.1). We assume that it is generated as a k-algebra by (a finite number of) homogeneous elements of degree 1. Moreover, we assume that W_1 contains $1 \in K$ and that the projective spectrum $P = \mathrm{Proj}(W_\bullet)$ is a normal scheme. Let K'/k be a subextension of K/k.

Our purpose is to construct a rational fibration $\pi : P \dashrightarrow X$ of integral normal projective k-schemes and an effective $\mathbb{Q}$-Weil divisor D on X satisfying the properties (i)–(iii) predicted in the theorem.

We set $W'_\bullet := W_\bullet \cap K' = \bigoplus_{n\in\mathbb{N}}(W_n \cap K')$ and $Q := \mathrm{Proj}(W'_\bullet)$. Note that Q may not be of finite type over k; hence, may not be proper over k. Since $W_\bullet \cap K' = W_\bullet \cap k(W_\bullet \cap K')$, we can assume without loss of generality that $k(W_\bullet \cap K') = K'$ and $k(W_\bullet) = K$. We divide the rest of the proof into six steps.

Step 1: In this step, we give a valuation theoretic interpretation of the required statement. Let H be the effective Cartier divisor on P defined by the image of 1 via $W_1 \to H^0(P, \mathcal{O}_P(1))$. By Lemma 4.5(1), one has

$$
\begin{aligned}
W_n &= \{\phi \in K \,:\, nH + (\phi) \geqslant 0\} \\
&= \left\{\phi \in K \,:\, \mathrm{ord}_\xi(\phi) \geqslant -n\,\mathrm{ord}_\xi(H),\ \forall\, \xi \in P^{(1)}\right\}
\end{aligned}
\tag{4.7}
$$

for every sufficiently positive integer n. Therefore,

$$
W'_n = \left\{\phi \in K' \,:\, \mathrm{ord}_\xi|_{K'}(\phi) \geqslant -n\,\mathrm{ord}_\xi(H),\ \forall\, \xi \in P^{(1)}\right\}
\tag{4.8}
$$

for every sufficiently positive integer n.

Step 2: In this step, we show the following.

Claim 4.8. — *The scheme Q is normal. Moreover, for any $\xi' \in Q^{(1)}$, there exists a $\xi \in P^{(1)}$ such that $\mathrm{ord}_\xi|_{K'}$ is equivalent to $\mathrm{ord}_{\xi'}$ (see Notation and conventions* **5***).*

It is a consequence of the following.

Claim 4.9. — *Let a be a homogeneous element of degree $d \geqslant 1$ in $W'_\bullet$, $R := (W_\bullet)_{(a)}$ and $R' := (W'_\bullet)_{(a)}$. Then $R' = R \cap K'$.*

Proof of Claim 4.9. — The inclusion $R' \subset R \cap K'$ is obvious. If b is an element of W_{dn} such that $b/a^n \in K'$, then $b \in W_{dn} \cap K'$. Hence we obtain $b/a^n \in R'$. $\square$

For each homogeneous element a of positive degree in $W'_\bullet$, we consider the morphism

$$\operatorname{Spec}\left((W_\bullet)_{(a)}\right) \longrightarrow \operatorname{Spec}\left((W'_\bullet)_{(a)}\right). \tag{4.9}$$

Since P is normal, $R := (W_\bullet)_{(a)}$ is integrally closed; hence we have

$$R = \bigcap_{\mathfrak{P} \in \operatorname{Spec}(R)^{(1)}} R_{\mathfrak{P}}.$$

By Claim 4.9, we know that

$$R' = R \cap K' = \bigcap_{\mathfrak{P} \in \operatorname{Spec}(R)^{(1)}} R_{\mathfrak{P}} \cap K' \tag{4.10}$$

is a Krull ring; thus in particular, an integrally closed domain. Given any $\xi' = \mathfrak{p} \in \operatorname{Spec}(R')^{(1)}$, one can find a $\mathfrak{P} \in \operatorname{Spec}(R)^{(1)}$ such that

$$R'_{\mathfrak{p}} = R_{\mathfrak{P}} \cap K' \tag{4.11}$$

(see [**17**, Theorem 12.3]). To show Claim 4.8, it suffices to take $\xi := \mathfrak{P}$. In fact, if $x \in R' \setminus \mathfrak{p}$, then by (4.11)

$$\frac{1}{x} = \frac{a}{y} \in R_{\mathfrak{P}} \cap K' \quad (a \in R,\, y \in R \setminus \mathfrak{P}).$$

Thus, if $x \in \mathfrak{P}$, then $y = ax \in \mathfrak{P}$, which is a contradiction. Hence, we know $\mathfrak{p} \supset \mathfrak{P} \cap K'$. Since $\mathfrak{p}$ has height 1 and $\mathfrak{P} \cap K' \neq \{0\}$, we have $\mathfrak{p} = \mathfrak{P} \cap K'$.

Step 3: Fix an $n_0 \geqslant 1$ such that

$$k\left(\left\{\frac{f}{g} \,:\, f, g \in W_n \cap K',\, g \neq 0\right\}\right) = K'$$

for every n divisible by n_0 (see Lemma 3.2). Let $V^0_\bullet$ be the graded sub-k-algebra of $W'_\bullet$ generated by W'_{n_0} and $v_\bullet : V^0_\bullet \to W'_\bullet$ the natural inclusion map.

Let $\nu_0 : X_0 \to \operatorname{Proj}(V^0_\bullet)$ be a normalisation. Then X_0 is an integral normal projective k-scheme with $\operatorname{Rat}(X_0) = k(V^0_\bullet) = K'$ (Lemma 4.7). Set

$$G(v_\bullet) := \left\{\mathfrak{p} \in Q \,:\, \mathfrak{p} \not\supset v_\bullet(V^0_{\geqslant 1})\right\}$$

(see Notation and conventions **7**) and let $\operatorname{Proj}(v_\bullet) : G(v_\bullet) \to \operatorname{Proj}(V^0_\bullet)$ be the morphism defined by $v_\bullet$. Since Q is normal, $\operatorname{Proj}(v_\bullet)$ induces $\mu : G(v_\bullet) \to X_0$. Since

$$\mu^{-1}\left(\nu_0^{-1}(D_{\operatorname{Proj}(V^0_\bullet)+}(a))\right) = \operatorname{Spec}((W'_\bullet)_{(a)})$$

for each homogeneous element a in $V^0_{\geqslant 1}$, μ is an affine morphism ([**10**, Definition II.1.2.1]).

Step 4: In this step, we show that, given any $\eta \in X_0^{(1)}$, one can find an $\eta' \in Q^{(1)}$ such that $\eta' \in G(v_\bullet)$ and $\mu(\eta') = \eta$. We suppose that one finds an $\eta \in X_0^{(1)}$ such that there is no $\eta' \in G(v_\bullet)^{(1)}$ with $\mu(\eta') = \eta$, from which we are going deduce a contradiction.

Since X_0 is normal and

$$\mathrm{Proj}(V^0_\bullet) = \bigcup_{a:\ \text{homogeneous}} D_{\mathrm{Proj}(V^0_\bullet)_+}(a),$$

one can fix a homogeneous element $a \in V^0_{\geqslant 1}$ and an l belonging to the integral closure $\widehat{A}$ of $A := (V^0_\bullet)_{(a)}$ in K' such that $\overline{\{\eta\}}$ is defined by a single nonzero equation $f \in \widehat{A}[1/l]$ on the affine open subscheme

$$U := \mathrm{Spec}(\widehat{A}[1/l]) = \Big\{x \in \nu_0^{-1}(D_{\mathrm{Proj}(V^0_\bullet)_+}(a)) \ :\ l(x) \neq 0\Big\}.$$

The hypothesis implies that $\mu^* f$ generates the unit ideal on $\mu^{-1}(U) = \mathrm{Spec}((W'_\bullet)_{(a)}[1/l])$. Thus, there exist $n_1, p \in \mathbb{N}_{\geqslant 1}$, $q \in \mathbb{N}$, and $g \in W'_{n_1}$ such that

$$f \cdot \frac{g}{a^p} = l^q \quad (n_1 = p \deg a). \tag{4.12}$$

Let $V^1_\bullet$ be the graded sub-k-algebra of $W'_\bullet$ generated by W'_{n_1} over k and $\nu_1 : X_1 \to \mathrm{Proj}(V^1_\bullet)$ a normalisation of $\mathrm{Proj}(V^1_\bullet)$. Then X_1 is also an integral normal projective k-scheme with $\mathrm{Rat}(X_1) = K'$ (see Step 3). Let $V^{0(n_1)}_\bullet$ denote the Veronese subalgebra of $V^0_\bullet$ (see (4.6)). We consider the natural inclusion homomorphism $w_\bullet : V^{0(n_1)}_\bullet \to V^1_\bullet$ and the commutative diagram

$$\begin{tikzcd}
X_0 \arrow[d, "\nu_0"'] & & X_1 \arrow[ll, dashed, "u"'] \arrow[d, "\nu_1"] \\
\mathrm{Proj}(V^0_\bullet) \arrow[r, "\sim"] & \mathrm{Proj}(V^{0(n_1)}_\bullet) & \mathrm{Proj}(V^1_\bullet) \arrow[l, dashed, "\mathrm{Proj}(w_\bullet)"'] \\
\mathrm{Spec}(A) \arrow[u, hook] \arrow[r, equal] & D_{\mathrm{Proj}(V^{0(n_1)}_\bullet)_+}(a^{n_1}) \arrow[u, hook] & \mathrm{Spec}(B), \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

where we set $B := (V^1_\bullet)_{(a^{n_1})}$ (see also Lemma 4.6). Let $\widehat{B}$ denote the integral closure of B in K'. On the open subscheme $u^{-1}(U) = \mathrm{Spec}(\widehat{B}[1/l])$, the equation $u^* f$ is invertible since

$$f^{n_1} \cdot \frac{g^{n_1}}{a^{n_1 p} l^{n_1 q}} = 1$$

by (4.12) and $g^{n_1}/(a^{n_1 p} l^{n_1 q}) \in \widehat{B}[1/l]$.

However, on the other hand, the valuation ord_η has a centre η_1 on X_1, and $\mathcal{O}_{X_1,\eta_1}$ is a discrete valuation ring. Thus, by the valuative criterion of properness [**10**, Corollaire II.7.3.6], η_1 is contained in the maximal domain of definition for u and is mapped to $\eta \in U$ by u. Therefore, $\eta_1 \in \nu_1^{-1}(G(w_\bullet))$; hence $\eta_1 \in u^{-1}(U)$ and $u^* f$ vanishes at η_1, which leads to a contradiction.

Step 5: By Claim 4.8 and Step 4, given any $\eta \in X_0^{(1)}$, there is a $\xi \in P^{(1)}$ such that ord_η is equivalent to $\mathrm{ord}_\xi|_{K'}$ (see Notation and conventions **5**).

Claim 4.10. — *The set*

$$\Sigma := \left\{ \xi \in P^{(1)} \,:\, c_{X_0}(\mathrm{ord}_\xi|_{K'}) \notin X_0^{(0)} \cup X_0^{(1)} \right\}$$

is finite. In particular, we obtain a surjective map

$$(P^{(1)} \cup P^{(0)}) \setminus \Sigma \to X_0^{(1)} \cup X_0^{(0)}, \quad \xi \mapsto c_{X_0}(\mathrm{ord}_\xi|_{K'}).$$

Proof of Claim 4.10. — The inclusion $K' \subset K$ yields a morphism $\pi : U' \to X_0$ from a nonempty open subscheme U' of P. By the theorem of generic flatness [**11**, Théorème IV.6.9.1], there is a nonempty open subscheme $U'' \subset X_0$ such that $\pi : \pi^{-1}(U'') \to U''$ is flat. If $\xi \in \pi^{-1}(U'')^{(1)}$, then

$$\dim \mathcal{O}_{X_0,\pi(\xi)} = \dim \mathcal{O}_{U',\xi} - \dim_\xi \left(\pi^{-1}(\pi(\xi))\right) = 0 \text{ or } 1$$

[**12**, Proposition III.9.5]. Thus π maps $\pi^{-1}(U'')^{(1)}$ into $X_0^{(1)} \cup X_0^{(0)}$. □

Let e_ξ denote the ramification index of ord_ξ (see (1.4)) and set

$$D := \sum_{\xi' \in X_0^{(1)}} \min\left\{ \frac{\mathrm{ord}_\xi(H)}{e_\xi} \,:\, \xi \in P^{(1)},\, \xi \mapsto \xi',\, e_\xi \neq 0 \right\} \overline{\{\xi'\}}. \tag{4.13}$$

By (4.8) and Claim 4.10, we have

$$\begin{aligned} W'_n &= \left\{ \phi \in K' \,:\, \mathrm{ord}_\xi|_{K'}(\phi) \geqslant -n\,\mathrm{ord}_\xi(H),\ \forall\, \xi \in P^{(1)} \right\} \\ &\subset \left\{ \phi \in K' \,:\, \mathrm{ord}_\xi|_{K'}(\phi) \geqslant -n\,\mathrm{ord}_\xi(H),\ \forall\, \xi \in P^{(1)} \setminus \Sigma \right\} \\ &= \left\{ \phi \in K' \,:\, e_\xi\,\mathrm{ord}_{\xi'}(\phi) \geqslant -n\,\mathrm{ord}_\xi(H),\ \forall\, \xi' \in X_0^{(1)} \right\} = H^0(X_0, nD) \end{aligned}$$

for every $n \gg 0$.

Step 6: Finally, we consider the case where $\mathrm{tr.deg}_k K' = 1$. In this case, X is a regular projective curve over $\mathrm{Spec}\, k$, which (as a set) is canonically in bijection with $\mathsf{D}(K'/k)$. Moreover, we have a surjective map

$$P^{(1)} \to X^{(1)} \cup X^{(0)}, \quad \xi \mapsto c_X(\mathrm{ord}_\xi|_{\mathrm{Rat}(X)}).$$

Hence, if we set

$$D := \sum_{\xi' \in X^{(1)}} \min\left\{ \frac{\mathrm{ord}_\xi(H)}{e_\xi} \ : \ \xi \in P^{(1)},\, \xi \mapsto \xi',\, e_\xi \neq 0 \right\} \xi', \tag{4.14}$$

then, by (4.8), we have $W_n \cap K' = H^0(X, nD)$ for every sufficiently positive integer n.

4.3. Rational fibrations associated with graded linear series. — In the following, we give an alternative proof for Theorem 1.2 by using the projective version of Zariski's result (Theorem 1.4).

Corollary 4.11. — *Let K/k be a finitely generated field extension and K'/k a subextension of K/k. Let $V_\bullet$ be a graded linear series of K'/k. If $V_\bullet$ is contained in a graded linear series $W_\bullet$ of K/k and of finite type over k, then $V_\bullet$ is contained a graded linear series $W'_\bullet$ of K'/k and of finite type over k.*

Proof. — We divide the proof into three steps.

Step 1: In this step, we make several reductions of the theorem. By the same arguments as in the step 1 of Theorem 3.7, we can assume that V_1 contains 1.

Claim 4.12. — *By enlarging K if necessary, we can assume that $W_\bullet$ is generated by W_1 over k.*

Proof of Claim 4.12. — Let $f_1T^{d_1}, \ldots, f_rT^{d_r} \in W_{\geqslant 1}$ be homogeneous generators of $W_\bullet$ over k. Let $T_1, \ldots, T_r$ be variables with $\deg T_i = 1$ for every i. One can find a homogeneous prime ideal $\mathfrak{p}$ of $W_\bullet[T_1, \ldots, T_r]$ such that $\mathfrak{p}$ contains

$$I := (T_1^{d_1} - f_1T^{d_1}, \ldots, T_r^{d_r} - f_rT^{d_r})$$

and such that $\mathfrak{p} \cap V_\bullet = \{0\}$. In fact, let $W'_\bullet := W_\bullet[T_1, \ldots, T_r]/I$ and let a be a homogeneous element of degree $\geqslant 1$. Since the morphism $\mathrm{Spec}\big((W'_\bullet)_{(a)}\big) \to \mathrm{Spec}\big((V_\bullet)_{(a)}\big)$ is dominant (Lemma 2.2), there exists a homogeneous prime ideal $\mathfrak{p} \in \mathrm{Proj}(W'_\bullet)$ such that $\mathfrak{p} \cap V_\bullet = \{0\}$. We set $U_\bullet := W'_\bullet/\mathfrak{p}$. Then $U_\bullet$ is a graded linear series of $k(U_\bullet)/k$, $W_\bullet \to U_\bullet$ is injective, and $U_\bullet$ is generated by $U_1 = W_1 + W_0T_1 + \cdots + W_0T_r$. □

In particular, we can assume that $P := \mathrm{Proj}(W_\bullet)$ is a projective scheme over k and that $\mathcal{O}_P(1) := \widetilde{W(1)_\bullet}$ is an invertible sheaf on P.

Step 2: Let $u : \widehat{P} \to P$ be a normalisation and H the Cartier divisor defined by the image of 1 via $V_1 \to H^0(\widehat{P}, u^*\mathcal{O}_P(1))$. We choose a very ample divisor $\widehat{H}$ such that $\widehat{H} - H$ is effective and such that $R(\widehat{H})_\bullet$ is generated by $R(\widehat{H})_1T$ over $R(\widehat{H})_0$.

Note that the graded k-algebra

$$\widehat{W}_\bullet := k \oplus \bigoplus_{n\geqslant 1} H^0(\widehat{P}, n\widehat{H})T^n$$

is a graded linear series of K/k and of finite type over k (Lemma 3.5) and that $\mathrm{Proj}(\widehat{W}_\bullet)$ is isomorphic to $\widehat{P}$ over k.

Applying Theorem 1.4 to $\widehat{W}_\bullet$ and K'/k, we can find an integral normal projective k-scheme X, an effective $\mathbb{Q}$-divisor D on X, and an integer $n_0 \geqslant 1$ such that $\mathrm{Rat}(X) \subset K'$ and such that $V_n \subset R(\widehat{H})_n \cap K' \subset H^0(X, nD)$ for every n with $n \geqslant n_0$.

Step 3: Let $\widehat{D}$ be a very ample divisor on X such that $\widehat{D} - D$ is effective and such that $R(\widehat{D})_\bullet$ is finitely generated over k. Let $W'_\bullet$ be the graded linear series generated by a basis of

$$\bigoplus_{n<n_0} V_n T^n$$

over k and by finite number of generators of $R(\widehat{D})_\bullet$ over k. Then W'_n contains V_n for every $n \geqslant 0$ and $W'_\bullet$ is finitely generated over k. □

Theorem 1.4 is comparable with the existence theorem of Iitaka fibrations for line bundles on normal projective varieties (see for example [**15**, Theorem 2.1.33]). As a consequence of Theorem 1.4, we can give an estimate of the following type for graded linear series of subfinite type (see also [**15**, Corollary 2.1.38] and Theorem 5.2 *infra*).

Corollary 4.13. — *Let K/k be a finitely generated field extension and $V_\bullet$ a graded linear series of K/k and of subfinite type. Let d be the Kodaira-Iitaka dimension of $V_\bullet$. If d is nonnegative, then there exist an integral normal projective k-scheme X and $\mathbb{Q}$-Cartier divisors D, D' on X such that the rational function field of X is k-isomorphic to $k(V_\bullet)$, that both D and D' have Kodaira-Iitaka dimension d, and that*

$$H^0(X, nD') \subset V_n \subset H^0(X, nD) \subset k(V_\bullet)$$

for every sufficiently positive n with $V_n \neq \{0\}$.

Proof. — The existence of D results from the same arguments as in Corollary 4.11. Thus, it suffices to show the existence of D' having the prescribed properties. By changing the grading of $V_\bullet$, we may assume that $\{n \in \mathbb{N} : V_n \neq \{0\}\}$ generates $\mathbb{Z}$ as a $\mathbb{Z}$-module. Choose any sufficiently positive integer p_0 such that $k(V_{p_0}) = k(V_\bullet)$ (see Lemma 3.2). Let $W_\bullet$ be the sub-k-algebra of $V_\bullet$ generated by V_{p_0}, and set

$$W'_\bullet := \bigoplus_{n\in\mathbb{N}} W_{p_0 n}.$$

Let $P := \mathrm{Proj}(W'_\bullet)$ and $\mathcal{O}_P(1) := \widetilde{W'_\bullet(1)}$. By Lemma 4.5, $W'_n = H^0(P, \mathcal{O}_P(n)) \subset V_{p_0 n}$ for every $n \gg 1$. Let $\nu : \widehat{P} \to P$ be a normalisation. Let p be any sufficiently positive integer divisible by p_0. Then one can find an ample divisor A on $\widehat{P}$ such that

$$H^0(\widehat{P}, nA) = H^0(P, \nu_*(\mathcal{O}_{\widehat{P}}(nA))) \subset H^0(P, \mathcal{O}_P(pn/p_0)) \subset V_{pn}$$

for every positive integer n (see [**5**, Démonstration de Proposition 3.6]).

Repeating the same arguments, one can choose an integral normal projective k-scheme X, two big Cartier divisors A, A' on X, and two coprime positive integers p, p' such that

$$H^0(X, nA) \subset V_{pn} \text{ and } H^0(X, nA') \subset V_{p'n}$$

for any positive integer n. Moreover, one can choose an ample $\mathbb{Q}$-Cartier divisor D' on X and two coprime positive integers q, q' such that $qq'D'$ is integral, that q (resp. q') is divisible by p (resp. p'), and that

$$H^0(X, qnD') \subset H^0(X, (qn/p)A) \subset V_{qn}$$

and

$$H^0(X, q'nD') \subset H^0(X, (q'n/p')A) \subset V_{q'n}$$

hold for every integer $n \in \mathbb{N}_{\geq 1}$.

Since

$$H^0(X, qnD') \otimes_k H^0(X, q'n'D') \to H^0(X, (qn + q'n')D')$$

is surjective for any sufficiently positive integers n, n' (see for example [**15**, Example 1.2.22], which is valid over fields of arbitrary characteristics), we have $H^0(X, nD') \subset V_n$ for every sufficiently positive n (recall the arguments in Lemma 3.2). □

Theorem 4.14 **(Fujita** [**8**, Appendix]**).** — *Let X be an integral normal projective k-scheme and D an effective Cartier divisor on X. If the Kodaira-Iitaka dimension of D is 1, then the section ring $R(D)_\bullet$ is finitely generated.*

Proof. — Let $K := \mathrm{Rat}(X)$ and let $C = \mathsf{D}(k(R(D)_\bullet)/k)$ be the smooth projective k-curve with rational function field k-isomorphic to $K' := k(R(D)_\bullet)$. The inclusion $K' \subset K$ defines a rational map $X \dashrightarrow C$ and, by taking a suitable blow-up $\mu : \widehat{X} \to X$, one obtains a flat morphism $\pi : \widehat{X} \to C$. If we set

$$E := \sum_{\xi' \in C^{(1)}} \min\left\{ \frac{\mathrm{ord}_\xi(\mu^* D)}{e_\xi} \,:\, \xi \in \widehat{X}^{(1)},\, \xi \mapsto \xi',\, e_\xi \neq 0 \right\} \xi',$$

then $H^0(C, nE) = H^0(\widehat{X}, n\mu^*D) = H^0(X, nD)$ for every positive integer n. Hence the result is reduced to the classic case of curves. □

Remark 4.15. — 1. If X is a surface, Zariski [**26**] completely classified the cases where $R(D)_\bullet$ is finitely generated [**26**, Theorem 10.6 and Proposition 11.5]. Later, Fujita [**8**] generalised the case where the Kodaira-Iitaka dimension is one to the form of Theorem 4.14 by using the Iitaka fibrations.

2. For a nef and big Cartier divisor D on X, $R(D)_\bullet$ is finitely generated if and only if D is semiample (see [**15**, Theorem 2.3.15]).

5. Applications

In this section, we apply the subfinite criterion (Theorem 1.2) to the study of Fujita approximation for general subfinite graded linear series. Throughout the section, we let k be a field and K/k be a finitely generated k-algebra.

Definition 5.1. — Let $V_\bullet$ be a graded linear series of K/k and d be its Kodaira-Iitaka dimension (see Definition 3.6). If $d \neq -\infty$, we define the *volume* of $V_\bullet$ as

$$\mathrm{vol}(V_\bullet) := \limsup_{n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!}. \tag{5.1}$$

A priori this invariant takes value in $[0, +\infty]$. We will see below that, if in addition the graded linear series $V_\bullet$ is of subfinite type, then its volume is always a positive real number.

We say that a graded linear series $V_\bullet$ *satisfies the Fujita approximation property* if

$$\sup_{\substack{W_\bullet \subset V_\bullet \\ W_\bullet \text{ of finite type} \\ \dim(W_\bullet)=\dim(V_\bullet)}} \mathrm{vol}(W_\bullet) = \mathrm{vol}(V_\bullet),$$

where $W_\bullet$ runs over the set of all graded linear series of finite type which are contained in $V_\bullet$ and such that $W_\bullet$ has the same Kodaira-Iitaka dimension as $V_\bullet$.

The purpose of the section is to establish the following approximation result.

Theorem 5.2. — *Any graded linear series $V_\bullet$ of K/k which is of subfinite type and has nonnegative Kodaira-Iitaka dimension d satisfies the Fujita approximation property. Moreover, one has*

$$\mathrm{vol}(V_\bullet) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!} \in (0, +\infty),$$

where $\mathbb{N}(V_\bullet) = \{n \in \mathbb{N} : V_n \neq \{0\}\}$.

Proof. — By changing the grading we may assume without loss of generality that $V_n \neq \{0\}$ for sufficiently positive integer n. Let K' be the homogeneous fraction field $k(V_\bullet)$. Note that K'/k is a subextension of K/k and hence is finitely generated. Moreover, by Theorem 1.2, we obtain that $V_\bullet$ viewed as a graded linear series of K'/k is of subfinite type. Therefore, the assertions follow from [**6**, Theorem 1.1] (by definition $V_\bullet$ is birational if we consider it as a graded linear series of K'). □

By combining the results of [**5**] and the subfiniteness result (Theorem 1.2), we obtain the following upper bound for the Hilbert-Samuel function of general graded linear series of subfinite type.

Theorem 5.3. — *Let $V_\bullet$ be a graded linear series of K/k and d its Kodaira-Iitaka dimension. There then exists a function $f : \mathbb{N} \to \mathbb{R}_+$ such that*

$$f(n) = \mathrm{vol}(V_\bullet)\frac{n^d}{d!} + O(n^{d-1}), \quad n \to +\infty$$

and

$$\forall\, n \in \mathbb{N}, \quad \dim_k(V_n) \leqslant f(n).$$

Remark 5.4. — The result [**6**, Theorem 1.1] actually provides more geometric information about the graded linear series of subfinite type. Let K/k be a finitely generated transcendental field extension and let d be the transcendence degree of K/k. We fix a flag

$$k = K_0 \subset K_1 \subset \ldots \subset K_d = K$$

of subfields of K containing k such that each extension K_i/K_{i-1} is transcendental and has transcendence degree 1. Let $\mathcal{A}(K/k)$ be the set of all graded linear series of subfinite type $V_\bullet$ of K/k such that $k(V_\bullet) = k$. Then there has been constructed in [**6**] a map Δ from $\mathcal{A}(K/k)$ to the set of convex bodies in $\mathbb{R}^d$ which satisfies the following conditions.

(a) If $V_\bullet$ and $V'_\bullet$ are two graded linear series in $\mathcal{A}(K/k)$ such that $V_\bullet \subset V'_\bullet$, then one has $\Delta(V_\bullet) \subset \Delta(V'_\bullet)$.
(b) If $V_\bullet$ and $W_\bullet$ are two graded linear series in $\mathcal{A}(K/k)$, then
$$\Delta(V_\bullet \cdot W_\bullet) \supset \Delta(V_\bullet) + \Delta(W_\bullet) := \{x + y \,:\, x \in \Delta(V_\bullet),\, y \in \Delta(W_\bullet)\},$$
where $V_\bullet \cdot W_\bullet$ denotes the graded linear series whose n-th homogeneous component is the k-vector space generated by $\{fg \,:\, f \in V_n,\ g \in W_n\}$.
(c) For any graded linear series $V_\bullet$ in $\mathcal{A}(K/k)$, the volume of $V_\bullet$ identifies with the Lebesgue measure of $\Delta(V_\bullet)$ multiplied by $d!$.

This allows us to construct the arithmetic analogue of Newton-Okounkov bodies for general arithmetic graded linear series of subfinite type, using the ideas of [**1**].

In what follows, we assume that k is a number field. We denote by M_k the set of all places of k. For each $v \in M_k$, let $|\cdot|_v$ be an absolute value on k which extends either the usual absolute value or certain p-adic absolute value (so that $|p|_v = p^{-1}$) on $\mathbb{Q}$.

As *adelic vector bundle* on $\operatorname{Spec} k$, we refer to the data $\overline{V} = (V, (\|\cdot\|_v)_{v\in M_k})$ of a finite dimensional vector space V over k and a family of norms $\|\cdot\|_v$ over $V \otimes_k k_v$ such that there exists a basis $(e_i)_{i=1}^r$ of V over k and a finite subset S of M_k satisfying the following condition:

$$\forall\, v \in M_k \setminus S, \quad \forall\, (\lambda_1, \ldots, \lambda_r) \in k_v^r, \quad \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_v = \max_{i\in\{1,\ldots,r\}} |\lambda_i|_v.$$

Given an adelic vector bundle $\overline{V}$ on $\operatorname{Spec} k$, for any nonzero element $s \in V$, we define the *Arakelov degree of* s as

$$\widehat{\deg}(s) := -\sum_{v\in M_k} [k_v : \mathbb{Q}_v] \ln \|s\|_v.$$

By the product formula

$$\forall\, a \in k^\times, \quad \sum_{v\in M_k} [k_v : \mathbb{Q}_v] \ln |a|_v = 0$$

we obtain that

$$\forall\, a \in k^\times, \quad \widehat{\deg}(as) = \widehat{\deg}(s).$$

Moreover, the *Arakelov degree of* $\overline{V}$ is defined as

$$-\sum_{v\in M_k} \ln \|\eta\|_{v,\det},$$

where η is a nonzero element of $\det(V)$, and

$$\|\eta\|_{v,\det} = \inf\{\|x_1\|_v \cdots \|x_r\|_v \,:\, \eta = x_1 \wedge \cdots \wedge x_r\}.$$

Again by the product formula we obtain that the definition does not depend on the choice of $\eta \in \det(V) \setminus \{0\}$.

Let $\overline{V}$ be an adelic vector bundle of rank r on $\operatorname{Spec} k$. For any $t \in \mathbb{R}$, let

$$\mathcal{F}^t(V) = \operatorname{Vect}_k(\{s \in V \setminus \{0\} \,:\, \widehat{\deg}(s) \geqslant t\}).$$

This is a decreasing $\mathbb{R}$-filtration on V, called the $\mathbb{R}$-*filtration by minima.* Note that for any $i \in \{1, \ldots, r\}$, the number

$$\lambda_i(\overline{V}) = \sup\{t \in \mathbb{R} \,:\, \operatorname{rk}_k(\mathcal{F}^t(V)) \geqslant i\}$$

coincides with the minus logarithmic version of the i-th minima in the sense of Roy and Thunder. For any $s \in V$, we let

$$\lambda(s) := \sup\{t \in \mathbb{R} \,:\, s \in \mathcal{F}^t(V)\}.$$

In the following, we let K/k be a finitely generated field extension of the number field k. Let $V_\bullet$ be a graded linear series of subfinite type of

K/k. For each $n \in \mathbb{N}$, we equip V_n with a structure of adelic vector bundle $(V_n, (\|\cdot\|_{n,v})_{v\in M_k})$ on $\operatorname{Spec} k$ such that, for any $v \in M_k$,

$$\forall\,(n,m) \in \mathbb{N}^2,\ \forall\,(s_n, s_m) \in V_n \times V_m, \quad \|s_n \cdot s_m\|_v \leqslant \|s_n\|_v \cdot \|s_m\|_v. \tag{5.2}$$

We assume in addition that

$$\lambda_{\max}(\overline{V}_\bullet) := \limsup_{n\to+\infty} \frac{\lambda_1(\overline{V}_n)}{n} < +\infty.$$

This condition implies that $V_\bullet$ has a nonnegative Kodaira-Iitaka dimension. For any $t \in \mathbb{R}$, let

$$V_\bullet^t := \bigoplus_{n\in\mathbb{N}} \mathcal{F}^{nt}(V_n).$$

It is a graded linear series of K/k. By definition one has $V_n^t = \{0\}$ if $n \in \mathbb{N}_{\geqslant 1}$ and $t > \lambda_{\max}(\overline{V}_\bullet)$.

Proposition 5.5. — *For any $t < \lambda_{\max}(\overline{V}_\bullet)$, one has $k(V_\bullet) = k(V_\bullet^t)$.*

Proof. — Clearly one has $k(V_\bullet) \supset k(V_\bullet^t)$. It suffices to prove the converse inclusion. Let $n \geqslant 1$ be an integer and f, g be nonzero elements in V_n. Since $t < \lambda_{\max}(V_\bullet)$ there exist $m \in \mathbb{N}_{\geqslant 1}$ and $s \in V_m$ such that $\lambda(s) > mt$. Thus for sufficiently positive integer ℓ one has $\lambda(s^\ell f) > (\ell m+n)t$ and $\lambda(s^\ell g) > (\ell m+n)t$. Therefore $\{s^\ell f, s^\ell g\} \subset V_{\ell m+n}^t$, which implies $f/g \in k(V_\bullet^t)$. □

The above proposition allows us to consider $V_\bullet^t$ as a birational graded linear series of $k(V_\bullet)/k$ and to construct its Newton-Okounkov body as reminded in Remark 5.4. We define the *concave transform* of $\overline{V}_\bullet$ as the function $G_{\overline{V}_\bullet}$ on $\Delta(V_\bullet)$ sending $x \in \Delta(V_\bullet)$ to

$$\sup\{t < \lambda_{\max}(\overline{V}_\bullet) \,:\, x \in \Delta(V_\bullet^t)\}.$$

By the condition (b) in Remark 5.4, the function $G_{\overline{V}_\bullet}$ is concave.

The following result generalises [**1**, Theorem 2.8] to the case of subfinite adelically normed graded linear series.

Theorem 5.6. — *Let K/k be a finitely generated extension of a number field k, and $\overline{V}_\bullet = \bigoplus_{n\in\mathbb{N}} \overline{V}_n$ a graded linear series of subfinite type of K/k of Kodaira-Iitaka dimension $d \geqslant 0$, equipped with structures of adelic vector bundles on* $\operatorname{Spec} k$*, which satisfy the submultiplicativity condition* (5.2) *and the condition $\lambda_{\max}(\overline{V}_\bullet) < +\infty$. Then the sequence of measures*

$$\frac{1}{\operatorname{rk}_k(V_n)} \sum_{i=1}^{\operatorname{rk}_k(V_n)} \delta_{\lambda_i(V_n)/n}, \quad n \in \mathbb{N}(V_\bullet) = \{m \in \mathbb{N} \,:\, V_m \neq \{0\}\}$$

converges weakly to a Boreal probability measure on $\mathbb{R}$*, which is the image of the uniform measure*

$$\frac{1}{\mathrm{vol}(\Delta(V_\bullet))} \mathbb{1}_{\Delta(V_\bullet)}(x)\,\mathrm{d}x$$

by the concave transform $G_{\overline{V}_\bullet}$.

Proof. — For any $t < \lambda_{\max}(\overline{V}_\bullet)$, the graded linear series $V_\bullet^t$ has the same homogeneous fraction field as $V_\bullet$ (see Proposition 5.5). Hence we can construct a decreasing family $(\Delta(V_\bullet^t))_{t<\lambda_{\max}(\overline{V}_\bullet)}$ of convex bodies contained in $\Delta(V_\bullet)$, as described in Remark 5.4. Moreover, if t_1 and t_2 are two real numbers which are $< \lambda_{\max}(\overline{V}_\bullet)$. Then by the same method as in [**1**, §1.3], we obtain the desired result. $\square$

References

[1] S. Boucksom & H. Chen – "Okounkov bodies of filtered linear series", *Compositio Mathematica* **147** (2011), no. 4, p. 1205–1229.

[2] N. Bourbaki – *éléments de mathématique. Fascicule XXVIII. Algèbre commutative. Chapitre 3: Graduations, filtra- tions et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire*, Actualités Scientifiques et Industrielles, No. 1293, Hermann, Paris, 1961.

[3] ———, *Éléments de mathématique*, Hermann, Paris, 1964, Fasc. XXX. Algèbre commutative. Chapitres 5 et 6.

[4] N. Bourbaki – *Éléments de mathématique*, Masson, Paris, 1981, Algèbre. Chapitres 4 à 7.

[5] H. Chen – "Majorations explicites des fonctions de Hilbert-Samuel géométrique et arithmétique", *Mathematische Zeitschrift* **279** (2015), no. 1-2, p. 99–137.

[6] ———, "Okounkov bodies: an approach of function field arithmetic", preprint available at `https://webusers.imj-prg.fr/~huayi.chen/Recherche/okounkov_cdf.pdf`, 2017.

[7] S. D. Cutkosky – "Asymptotic multiplicities of graded families of ideals and linear series", *Advances in Mathematics* **264** (2014), p. 55–113.

[8] T. Fujita – "On L-dimension of coherent sheaves", *Journal of the Faculty of Science. University of Tokyo. Section IA. Mathematics* **28** (1981), no. 2, p. 215–236.

[9] ———, "Approximating Zariski decomposition of big line bundles", *Kodai Mathematical Journal* **17** (1994), no. 1, p. 1–3.

[10] A. Grothendieck & J. Dieudonné – "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes", *Institut des Hautes Études Scientifiques. Publications Mathématiques* **8** (1961), p. 222.

[11] ———, "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II", *Institut des Hautes Études Scientifiques. Publications Mathématiques* (1965), no. 24, p. 231.

[12] R. Hartshorne – *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[13] K. Kaveh & A. G. Khovanskii – "Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory", *Annals of Mathematics. Second Series* **176** (2012), no. 2, p. 925–978.

[14] K. Kaveh & A. Khovanskii – "Algebraic equations and convex bodies", in *Perspectives in analysis, geometry, and topology*, Progr. Math., vol. 296, Birkhäuser/Springer, New York, 2012, p. 263–282.

[15] R. Lazarsfeld – *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[16] R. Lazarsfeld & M. Mustaţă – "Convex bodies associated to linear series", *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série* **42** (2009), no. 5, p. 783–835.

[17] H. Matsumura – *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[18] D. Mumford – "Hilbert's fourteenth problem — the finite generation of subrings such as rings of invariants", in *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974)*, Amer. Math. Soc., Providence, R. I., 1976, p. 431–444.

[19] M. Nagata – "Addition and corrections to my paper "A treatise on the 14-th problem of Hilbert"", *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **30** (1957), p. 197–200.

[20] ———, "On the 14-th problem of Hilbert", *American Journal of Mathematics* **81** (1959), p. 766–772.

[21] ———, "On the fourteenth problem of Hilbert", in *Proc. Internat. Congress Math. 1958*, Cambridge Univ. Press, New York, 1960, p. 459–462.

[22] A. Okounkov – "Brunn-Minkowski inequality for multiplicities", *Inventiones Mathematicae* **125** (1996), no. 3, p. 405–411.

[23] ———, "Why would multiplicities be log-concave?", in *The orbit method in geometry and physics (Marseille, 2000)*, Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, p. 329–347.

[24] S. Takagi – "Fujita's approximation theorem in positive characteristics", *Journal of Mathematics of Kyoto University* **47** (2007), no. 1, p. 179–202.

[25] O. Zariski – "Interprétations algébrico-géométriques du quatorzième problème de Hilbert", *Bulletin des Sciences Mathématiques. 2e Série* **78** (1954), p. 155–168.

[26] O. Zariski – "The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface", *Annals of Mathematics. Second Series* **76** (1962), p. 560–615.

August 2, 2017

HUAYI CHEN, Université Paris Diderot, Institut de Mathématiques de Jussieu - Paris Rive Gauche, Bâtiment Sophie Germain, Boîte Courrier 7012, 75205 Paris Cedex 13, France. • Beijing International Center for Mathematical Research, Peking University, Beijing, 100871, China. • *E-mail :* huayi.chen@imj-prg.fr
Url : webusers.imj-prg.fr/~huayi.chen

HIDEAKI IKOMA, Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan
E-mail : ikoma@math.kyoto-u.ac.jp