Jet schemes of complex plane branches and equisingularity

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Abstract

For \( m \in \mathbb{N} \), we determine the irreducible components of the \( m \)-th Jet Scheme of a complex branch \( C \) and we give formulas for their number \( N(m) \) and for their codimensions, in terms of \( m \) and the generators of the semigroup of \( C \). This structure of the Jet Schemes determines and is determined by the topological type of \( C \).

1 Introduction

Let \( \mathbb{K} \) be an algebraically closed field. The space of arcs \( X_\infty \) of an algebraic \( \mathbb{K} \)-variety \( X \) is a non-noetherian scheme in general. It has been introduced by Nash in [N]. Nash has initiated its study by looking at its image by the truncation maps \( X_\infty \to X_m \) in the jet schemes of \( X \). The \( m^{th} \)-jet scheme \( X_m \) of \( X \) is a \( \mathbb{K} \)-scheme of finite type which parametrizes morphisms \( \text{Spec} \mathbb{K}[t]/(t)^{m+1} \to X \). From now on, we assume \( \text{char} \mathbb{K} = 0 \).

In [N], Nash has derived from the existence of a resolution of singularities of \( X \), that the number of irreducible components of the Zariski closure of the set of the \( m \)-truncations of arcs on \( X \) that send \( 0 \) into the singular locus of \( X \) is constant for \( m \) large enough.

Besides a theorem of Kolchin asserts that if \( X \) is irreducible, then \( X_\infty \) is also irreducible. More recently, the jet schemes have attracted attention from various viewpoints. In [Mus], Mustata has characterized the locally complete intersection varieties having irreducible \( X_m \) for \( m \geq 0 \). In [ELM], a formula comparing the codimensions of \( Y_m \) in \( X_m \) with the log canonical threshold of a pair \((X,Y)\) is given. In this work, we consider a curve \( C \) in the complex plane \( \mathbb{C}^2 \) with a singularity at \( 0 \) at which it is analytically irreducible (i.e. the formal neighborhood \((C,0)\) of \( C \) at \( 0 \) is a branch). We determine the irreducible components of the space \( C_m^0 := \pi_m^{-1}(0) \) where \( \pi_m : C_m \to C \) is the canonical projection, and we show that their number is not bounded as \( m \) grows. More precisely, let \( x \) be a transversal parameter in the local ring \( O_{\mathbb{C}^2,0} \), i.e. the line \( x = 0 \) is transversal to \( C \) at \( 0 \) and following [ELM], for \( e \in \mathbb{N} \), let

\[
\text{Cont}^e(x)_m (\text{resp. } \text{Cont}^>e(x)_m) := \{ \gamma \in C_m \mid \text{ord}_x \circ \gamma = e (\text{resp. } > e) \},
\]

where \( \text{Cont} \) stands for contact locus. Let \( \Gamma(C) = \langle \beta_0, \cdots, \beta_g \rangle \) be the semigroup of the branch \((C,0)\) and let \( e_i = \gcd(\beta_0, \cdots, \beta_i) \), \( 0 \leq i \leq g \). Recall that \( \Gamma(C) \) and the
topological type of $C$ near 0 are equivalent data and characterize the equisingularity class of $(C, 0)$ as defined by Zariski in [Z2]. We show in theorem 4.9 that the irreducible components of $C_m$ are

$$C_{mn1} = \text{Cont}^{\kappa \bar{\beta}_0}(x)_m,$$

for $1 \leq \kappa$ and $\kappa \bar{\beta}_0 + e_1 \leq m$,

$$C^j_{mnv} = \text{Cont}^{\kappa \bar{\beta}_0}(x)_m$$

for $2 \leq j \leq g, 1 \leq \kappa, \kappa \not\equiv 0 \mod \frac{\bar{e}_{j-1}}{e_j}$ and $\kappa \bar{\beta}_j + e_1 \leq m < \kappa \bar{\beta}_j$,

$$B_m = \text{Cont}^{\kappa \bar{\beta}_0}(x)_m,$$

if $q \bar{\beta}_j + e_1 \leq m < (q + 1)n_1 \bar{\beta}_1 + e_1$.

These irreducible components give rise to infinite and finite inverse systems represented by a tree. We recover $(\bar{\beta}_0, \ldots, \bar{\beta}_g)$ from the tree and the multiplicity $\bar{\beta}_0$ in corollary 4.13, and we give formulas for the number of irreducible components of $C_m^0$ and their codimensions in terms of $m$ and $(\bar{\beta}_0, \ldots, \bar{\beta}_g)$ in proposition 4.7 and corollary 4.10. We recover the fact coming from [ELM] and [I] that

$$\min_m \frac{\text{codim}(C_m^0, C_m^2)}{m + 1} = \frac{1}{\bar{\beta}_0} + \frac{1}{\bar{\beta}_1}.$$

The structure of the paper is as follows: The basics about Jet schemes and the results that we will need are presented in section 2. In section 3 we present the definitions and the results we will need about branches. The last section is devoted to the proof of the main result and corollaries.

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## 2 Jet schemes

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic. Let $X$ be a $\mathbb{K}$-scheme of finite type over $\mathbb{K}$ and let $m \in \mathbb{N}$. The functor $F_m : \mathbb{K} - \text{Schemes} \rightarrow \text{Sets}$ which to an affine scheme defined by a $\mathbb{K}$-algebra $A$ associates

$$F_m(\text{Spec}(A)) = \text{Hom}_\mathbb{K}(\text{Spec}(A[t]/(t^{m+1})), X)$$

is representable by a $\mathbb{K}$-scheme $X_m [V]$. $X_m$ is the $m$-th jet scheme of $X$, and $F_m$ is isomorphic to its functor of points. In particular the closed points of $X_m$ are in bijection
with the $\mathbb{K}[t]/(t^{m+1})$ points of $X$.

For $m, p \in \mathbb{N}, m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \to A[t]/(p^{m+1})$ induces a canonical projection $\pi_{m,p} : X_m \to X_p$. These morphisms clearly verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$.

Note that $X_0 = X$. We denote the canonical projection $\pi_{m,0} : X_m \to X_0$ by $\pi_m$.

**Example 1.** Let $X = \text{Spec} \frac{\mathbb{K}[x_0, \ldots, x_n]}{(f_1, \ldots, f_r)}$ be an affine $\mathbb{K}$–scheme. For a $\mathbb{K}$-algebra $A$, to give a $A$-point of $X_m$ is equivalent to give a $k$-algebra homomorphism

$$\varphi : \frac{\mathbb{K}[x_0, \ldots, x_n]}{(f_1, \ldots, f_r)} \to A[t]/(t^{m+1}).$$

The map $\varphi$ is completely determined by the image of $x_i, i = 0, \ldots, n$

$$x_i \mapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)} t + \cdots + x_i^{(m)} t^m$$

such that $f_i(\phi(x_0), \ldots, \phi(x_n)) \in (t^{m+1}), i = 1, \ldots, r$.

If we write

$$f_t(\phi(x_0), \ldots, \phi(x_n)) = \sum_{j=0}^{m} F_t^{(j)}(x_0, \ldots, x^{(j)}) t^j \mod (t^{m+1})$$

where $x^{(j)} = (x^{(j)}_0, \ldots, x^{(j)}_n)$, then

$$X_m = \text{Spec} \frac{\mathbb{K}[x_0, \ldots, x_n]}{(f_t^{(j)_{j=0,\ldots,m}})}$$

**Example 2.** From the above example, we see that the $m$-th jet scheme of the affine space $\mathbb{A}^n_k$ is isomorphic to $\mathbb{A}_k^{(m+1)n}$ and that the projection $\pi_{m,m-1} : (\mathbb{A}^n_k)_m \to (\mathbb{A}^n_k)_{m-1}$ is the map that forgets the last $n$ coordinates.

Let $\text{char}(\mathbb{K}) = 0$, $S = \mathbb{K}[x_0, \ldots, x_n]$ and $S_m = \mathbb{K}[x_0^{(0)}, \ldots, x^{(m)}]$. Let $D$ be the $k$-derivation on $S_m$ defined by $D(x^{(j)}_i) = (j+1)x^{(j+1)}_i$ if $0 \leq j < m$, and $D(x^{(m)}_i) = 0$. For $f \in S$ let $f^{(1)} := D(f)$ and we recursively define $f^{(m)} = D(f^{(m-1)})$.

**Proposition 2.1.** Let $X = \text{Spec}(S/(f_1, \ldots, f_r)) = \text{Spec}(R)$ and $R_m = \Gamma(X_m)$. Then

$$R_{m} = \text{Spec}(\frac{\mathbb{K}[x_0, \ldots, x_n]}{(f_t^{(j)_{j=0,\ldots,m}})}).$$

**Proof:** For a $\mathbb{K}$-algebra $A$, to give an $A$-point of $X_m$ is equivalent to give an homomorphism

$$\phi : \mathbb{K}[x_0, \ldots, x_n] \to A[t]/(t^{m+1})$$

which can be given by

$$x_i \mapsto \frac{x_i^{(0)}}{0!} + \frac{x_i^{(1)}}{1!} t + \cdots + \frac{x_i^{(m)}}{m!} t^m.$$
Then for a polynomial \( f \in S \), we have
\[
\phi(f) = \sum_{j=0}^{m} \frac{f^{(j)}(x^{(0)}, \ldots, x^{(j)})}{j!} t^j.
\]
To see this, it is sufficient to remark that it is true for \( f = x_i \), and that both sides of the equality are additive and multiplicative in \( f \), and the proposition follows.

**Remark 2.2.** Note that the proposition shows the linearity of the equations \( F_i^j(x^{(0)}, \ldots, x^{(j)}) \) defining \( X_m \) with respect to the new variables i.e \( x^{(j)} \). We can deduce from this that if \( X \) is a nonsingular \( k \)-variety of dimension \( n \), then the projections \( \pi_{m,m-1} : X_m \rightarrow X_{m-1} \) are locally trivial fibrations with fiber \( \mathbb{A}_k^n \). In particular, \( X_m \) is a non singular variety of dimension \( (m+1)n \).

### 3 Semigroup of complex branches

The main references for this section are [Z],[Me],[A],[Sp],[GP],[GT],[LR]. Let \( f \in \mathbb{C}[[x,y]] \) be an irreducible power series, which is \( y \)-regular (i.e \( f(0,y) = y^{\beta_0}u(y) \) where \( u \) is invertible in \( \mathbb{C}[[y]] \)) and such that \( \text{mult}_0 f = \beta_0 \) and let \( C \) be the analytically irreducible plane curve(branch for short) defined by \( f \) in \( \text{Spec} \ \mathbb{C}[[x,y]] \). By the Newton-Puiseux theorem, the roots of \( f \) are
\[
y = \sum_{i=0}^{\infty} a_i w^i x^{\frac{i}{e_0}} \quad (1)
\]
where \( w \) runs over the \( \beta_0 \)-th-roots of unity in \( \mathbb{C} \).This is equivalent to the existence of a parametrization of \( C \) of the form
\[
x(t) = t^{\beta_0} \quad y(t) = \sum_{i \geq \beta_0} a_i t^i.
\]
We recursively define \( \beta_i = \min\{i, a_i \neq 0, \ gcd(\beta_0, \ldots, \beta_{i-1}) \text{ is not a divisor of } i\} \).
Let \( e_0 = \beta_0 \) and \( e_i = \gcd(e_{i-1}, \beta_i), i \geq 1 \). Since the sequence of positive integers
\[
e_0 > e_1 > \cdots > e_i > \cdots
\]
is strictly decreasing, there exists \( g \in \mathbb{N} \), sucht that \( e_g = 1 \). The sequence \( (\beta_1, \ldots, \beta_g) \) is the sequence of Puiseux exponents of \( C \). We set
\[
n_i := \frac{e_{i-1}}{e_i}, m_i := \frac{\beta_i}{e_i}, i = 1, \ldots, g
\]
and by convention, we set \( \beta_{g+1} = +\infty \) and \( n_{g+1} = 1 \).

On the other hand, for \( h \in \mathbb{C}[[x,y]] \), we define the intersection number
\[
(f, h)_0 = (C, C_h)_0 := \dim_{\mathbb{C}[[x,y]]} \frac{\mathbb{C}[[x,y]]}{(f, h)} = \text{ord}_t h(x(t), y(t))
\]
where $C_h$ is the Cartier divisor defined by $h$ and \{x(t), y(t)\} is as above. The mapping $v_f: \frac{C[x,y]}{(f)} \rightarrow \mathbb{N}$, $h \rightarrow (f, h)_0$ defines a divisorial valuation. We define the semigroup of $C$ to be the semigroup of $v_f$ i.e $\Gamma(C) = \Gamma(v_f) = \{(f, h)_0 \in \mathbb{N}, h \not\equiv 0 \text{ mod}(f)\}$. The following propositions and theorem from [Z] characterize the structure of $\Gamma(C)$.

**Proposition 3.1.** There exists a unique sequence of $g + 1$ positive integers $(\beta_0, \cdots, \beta_g)$ such that:

i) $\beta_0 = \beta_0$,

ii) $\beta_i = \min\{\Gamma(C) \mid 1 \leq \beta_i < \beta_i + 1\}$,

iii) $\Gamma(C) = \langle \beta_0, \cdots, \beta_g \rangle$.

where for $i = 1, \cdots, g + 1, < \beta_0, \cdots, \beta_i \rangle$ is the semigroup generated by $\beta_0, \cdots, \beta_i$. By convention, we set $\beta_{g+1} = +\infty$.

**Proposition 3.2.** The sequence $(\beta_0, \cdots, \beta_g)$ verifies:

i) $e_i = \gcd(\beta_0, \cdots, \beta_i)$, $0 \leq i \leq g$,

ii) $\beta_0 = \beta_0, \beta_1 = \beta_1$ and $\beta_i = n_{i-1} \beta_{i-1} + \beta_i - \beta_i$. In particular $n_i \beta_i < \beta_{i+1}$, for $i = 2, \cdots, g$.

**Theorem 3.3.** The sequence $(\beta_0, \cdots, \beta_g)$ and the sequence $(\beta_0, \cdots, \beta_g)$ are equivalent data. They determine and are determined by the topological type of $C$.

Then from the appendix of [Z], [A] or [Sp], we can choose a system of approximate roots (or a minimal generating sequence) \{x_0, \cdots, x_{g+1}\} of the divisorial valuation $v_f$. We set $x = x_0, y = x_1$; for $i = 2, \cdots, g + 1, x_i \in C[[x, y]]$ is irreducible; for $1 \leq i \leq g$, the analytically irreducible curve $C_i = \{x_i = 0\}$ has $i - 1$ Puiseux exponents and $C_{g+1} = C$. This sequence also verifies

i) $v_f(x_i) = \beta_i$, $0 \leq i \leq g$,

ii) $\Gamma(C_i) = \langle \beta_0, \cdots, \beta_i \rangle$ and the Puiseux sequence of $C_i$ is $(\beta_i, \beta_{i-1})$, $2 \leq i \leq g + 1$.

iii) for $1 \leq i \leq g$, there exists a unique system of nonnegative integers $b_{ij}$, $0 \leq j < i$ such that for $1 \leq j < i$, $b_{ij} = n_j \beta_i = \Sigma_{0 \leq j < i} b_{ij} \beta_j$. Furthermore, for $1 \leq i \leq g$, one can choose $x_i$ such that they satisfy identities of the form

$$x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i-1}} - \sum_{\gamma = (\gamma_0, \cdots, \gamma_i)} c_{i, \gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i}, (\ast)$$

with, $0 \leq \gamma_j < n_j$, for $1 \leq j \leq i$, and $\sum_{j} \gamma_j \beta_j > n_i \beta_i$ and with $c_{i, \gamma}, c_i \in \mathbb{C}$ and $c_i \neq 0$. These last equations $(\ast)$ let us realize $C$ as a complete intersection in $\mathbb{C}^{g+1} = Spec \mathbb{C}[[x_0, \cdots, x_g]]$ defined by the equations

$$f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i-1}} - \sum_{\gamma = (\gamma_0, \cdots, \gamma_i)} c_{i, \gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i})$$

for $1 \leq i \leq g$, with $x_{g+1} = 0$ by convention.
Proposition 4.1. Let \( h \in \mathbb{C}[[x, y]] \) be a \( g \)-regular irreducible power series with multiplicity \( p = \text{ord}_y h(0, y) \). Let \( y(x^{\frac{1}{\beta_0}}) \) and \( z(x^{\frac{1}{\beta}}) \) be respectively roots of \( f \) and \( h \) as in (1). We call contact order of \( f \) and \( h \) in their Puiseux series the following rational number
\[
\alpha_f(h) := \max\{\text{ord}_x (y(x^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{\beta}})); w^{\beta_0} = 1, \lambda^p = 1\} = \\
\max\{\text{ord}_x (y(x^{\frac{1}{\beta}}) - z(x^{\frac{1}{\beta}})); w^{\beta} = 1\} = \\
\max\{\text{ord}_x (y(x^{\frac{1}{\beta}}) - \lambda x^{\frac{1}{\beta}}); \lambda^p = 1\} = \alpha_h(f).
\]
The following formula is from [Me], see also [GP]. Let \( \beta \in \mathbb{C} \) \((-\infty,0)\) be an integer. Then
\[
\frac{(f,h)_0}{p} = \sum_{k=1}^{i-1} \frac{e_k - e_k \beta + e_i \alpha_f(h)}{\beta_0} = (\beta_i + e_i - 1 + (\beta_0 \alpha_f(h) - \beta_i - 1) e_i) \frac{1}{\beta_0}.
\]

Corollary 3.5. [A][GP] Let \( i > 0 \) be an integer. Then \( \alpha_f(h) \leq \frac{\beta_i}{\beta_0} \) iff \( \frac{(f,h)_0}{p} \leq e_i - \frac{\beta_i}{\beta_0} \). Moreover \( \alpha_f(h) = \frac{\beta_i}{\beta_0} \) iff \( \frac{(f,h)_0}{p} = e_i - \frac{\beta_i}{\beta_0} \). In particular \( \alpha_f(x_i) = \frac{\beta_i}{\beta_0}, 1 \leq i \leq g \). We say that \( C_i x_i = 0 \) has maximal contact with \( C \).

4 Jet schemes of complex branches

We keep the notations of sections 2 and 3. We consider a curve \( C \subset \mathbb{C}^2 \) with a branch of multiplicity \( \beta_0 > 1 \) at 0, defined by \( f \). Note that in suitable coordinates we can write
\[
f(x_0, x_1) = (x_1^{n_1} - c x_0^{m_1})^{e_1} + \sum_{a^\beta + b^\beta > \beta_0^\beta} c_{ab} x_0^a x_1^b; c \in \mathbb{C}^* \text{ and } c_{ab} \in \mathbb{C}.
\]
We look for the irreducible components of \( C_m^0 := (\pi_m^{-1}(0)) \) for every \( m \in \mathbb{N} \), where \( \pi_m : C_m \rightarrow C \) is the canonical projection. Let \( j_m^0 \) be the radical of the ideal defining \( (\pi_m^{-1}(0)) \) in \( \mathbb{C}_m^2 \). In the sequel, we will denote the integral part of a rational number \( r \) by \( \lceil r \rceil \).

Proposition 4.1. For \( 0 < m < n_1 \beta_1 \), we have that
\[
(C_m^0)_{\text{red}} = (\pi_m^{-1}(0))_{\text{red}} = \text{Spec} \frac{\mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)}]}{(x_0^{(0)}, \ldots, x_0^{(\lceil \frac{m}{\beta_1} \rceil 0)}, x_1^{(0)}, \ldots, x_1^{(\lceil \frac{m}{\beta_1} \rceil 0)})},
\]
and
\[
(C_{n_1 \beta_1}^0)_{\text{red}} = (\pi_{n_1 \beta_1}^{-1}(0))_{\text{red}} = \text{Spec} \frac{\mathbb{C}[x_0^{(0)}, \ldots, x_0^{(n_1 \beta_1)}, x_1^{(0)}, \ldots, x_1^{(n_1 \beta_1)}]}{(x_0^{(0)}, \ldots, x_0^{(n_1 \beta_1)}, x_1^{(0)}, \ldots, x_1^{(n_1 \beta_1)})} - c_{\beta_1}^{(m_1 \beta_1)}.
\]
Proof: We write $f = \sum (a,b) c_{ab} f_{ab}$ where $(a,b) \in \mathbb{N}^2$, $f_{ab} = x_0^a x_1^b$, $c_{ab} \in \mathbb{C}$ and $a \beta_0 + b \beta_1 = \beta_0 \beta_1$ (the segment $\{ (0, \beta_0)(\beta_1, 0) \}$ is the Newton Polygon of $f$). Let $\text{supp}(f) = \{(a,b) \in \mathbb{N}^2; c_{ab} \neq 0 \}$.

For $0 < m < n_1 \beta_1$, the proof is by induction on $m$. For $m = 1$ we have that

$$F^{(1)} = \sum_{(a,b) \in \text{supp}(f)} c_{ab} F^{(1)}_{ab}$$

where $(F^{(0)}, \ldots, F^{(i)})$ (resp. $F_{ab}^{(0)}, \ldots, F_{ab}^{(i)}$) is the ideal defining the $i$-th jet scheme of $C^{(i)}$ of $C$ resp. $C_{ab}^{(i)}$ the $i$-th jet scheme of $C_{ab} = \{f_{ab} = 0\}$ in $\mathbb{C}_i^2$. Then we have

$$F_{ab}^{(1)} = \sum_{i_k = 1} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_{a+1})} \cdots x_1^{(i_{a+b})}$$

where $\beta_1 (a + b) \geq a \beta_0 + b \beta_1 \geq \beta_0 \beta_1$ so $a + b \geq \beta_0 > 1$. Then for every $(a,b) \in \text{supp}(f)$ and every $(i_1, \ldots, i_a, \ldots, i_{a+b}) \in \mathbb{N}^{a+b}$ such that $\sum_{k=1}^{i_k = 1} i_k = 1$ there exists $1 \leq k \leq a + b$ such that $i_k \neq 0$, this means that $F_{ab}^{(1)} \in (x_0^{(0)}, x_1^{(0)})$ and since we are looking over the origin, we have that $(x_0^{(0)}, x_1^{(0)}) \subseteq J_1^0$ therefore $(\pi_1^{-1}(0))_{red} = \text{Spec}(\mathbb{C}^{x_0^{(m)}, x_1^{(0)}, x_1^{(m)}})/(x_0^{(0)}, x_1^{(0)})$. (In fact this is nothing but the Zariski tangent space of $C$ at $0$).

Suppose that the lemma holds until $m - 1$ i.e.

$$(\pi_{m-1}^{-1}(0))_{red} = \text{Spec}(\mathbb{C}^{x_0^{(m)}, \cdots, x_0^{(m-1)}, x_1^{(0)}, \cdots, x_1^{(m-1)}})/(x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m-1)})).$$

First case: If $[\frac{m-1}{\beta_1}] = [\frac{m}{\beta_1}]$ and $[\frac{m-1}{\beta_0}] = [\frac{m}{\beta_0}]$. We have

$$F^{(m)} = \sum_{(a,b) \in \text{supp}(f)} c_{ab} \sum_{i_k = m} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_{a+1})} \cdots x_1^{(i_{a+b})}$$

Let $(a,b) \in \text{supp}(f)$; if for every $k = 1, \ldots, a$, we had $i_k \geq [\frac{m}{\beta_1}] + 1$, and for every $k = a + 1, \ldots, a + b$, we had $i_k \geq [\frac{m}{\beta_0}] + 1$, then

$$m \geq a([\frac{m}{\beta_1}] + 1) + b([\frac{m}{\beta_0}] + 1) > \frac{m}{\beta_1} a + \frac{m}{\beta_0} b = m \frac{a \beta_0 + b \beta_1}{\beta_0 \beta_1} \geq m.$$ 

The contradiction means that there exists $1 \leq k \leq a$ such that $i_k \leq [\frac{m-1}{\beta_1}]$ or there exists $a + 1 \leq k \leq a + b$ such that $i_k \leq [\frac{m}{\beta_0}]$. So $F^{(m)}$ lies in the ideal generated by $J_{m-1}^0$ in $\mathbb{C}^{x_0^{(m)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}}$ and $J_0^m = J_{m-1}^0 \cap \mathbb{C}^{x_0^{(m)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}}$.

Second case: If $[\frac{m-1}{\beta_1}] = [\frac{m}{\beta_1}]$ and $[\frac{m-1}{\beta_0}] + 1 = [\frac{m}{\beta_0}]$ (i.e. $\beta_0$ divides $m$). We have that

$$F^{(m)} = F^{(m)}_{0,\beta_0} + \sum_{(a,b) \in \text{supp}(f); (a,b) \neq (0,\beta_0)} F^{(m)}_{ab}, \quad (**)$$

where

$$F^{(m)}_{0,\beta_0} = \sum_{i_k = m} x_0^{(i_1)} \cdots x_0^{(i_m)} = x_1^{(m)} \beta_0 + \sum_{\sum i_k = m; (i_1, \ldots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \ldots, \frac{m}{\beta_0})} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})},$$
The third case i.e. if \( i_k = m \) and \( (i_1, \ldots, i_{\beta_0}) \neq (m, \ldots, m) \) implies that there exists \( 1 \leq k \leq \beta_0 \) such that \( i_k < \frac{m}{\beta_0} \), so
\[
\sum_{i_k = m; (i_1, \ldots, i_{\beta_0}) \neq (m, \ldots, m)} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})} \in J^0_{m-1, \mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)}]}.
\]
For the same reason as above, we have that
\[
\sum_{(a, b) \in \text{supp}(f): (a, b) \neq (0, \beta_0)} F_{ab}^{(m)} \in J^0_{m-1, \mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)}]}.
\]
From (**) we deduce that \( x_1^{(\frac{m}{\beta_0})} \in J^0_m \) and \( F^{(m)} \in (x_0^{(0)}, \ldots, x_0^{(\frac{m}{\beta_0})}, x_1^{(0)}, \ldots, x_1^{(\frac{m}{\beta_0})}) \). Then \( J^0_m = (x_0^{(0)}, \ldots, x_0^{(\frac{m}{\beta_0})}, x_1^{(0)}, \ldots, x_1^{(\frac{m}{\beta_0})}) \).
The third case i.e. if \( \lceil \frac{m-1}{\beta_1} \rceil + 1 = \lfloor \frac{m}{\beta_0} \rfloor \) and \( \lceil \frac{m-1}{\beta_1} \rfloor = \lfloor \frac{m}{\beta_0} \rfloor \) is discussed as the second one.
Note that these are the only three possible cases since \( m < n_1 \beta_1 = \text{lcm}(\beta_0, \beta_1) \) (here \( \text{lcm} \) stands for the least common multiple).

For \( m = n_1 \beta_1 \), we have that \( F^{(m)} \) is the coefficient of \( t^m \) in the expansion of
\[ f(x_0^{(0)} + x_0^{(1)} t + \cdots + x_0^{(m)} t^m, x_1^{(0)} + x_1^{(1)} t + \cdots + x_1^{(m)} t^m). \]
But since we are interested in the radical of the ideal defining the \( m \)-th jet scheme, and we have found that \( x_1^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)} \in J^0_{m-1} \subseteq J^0_m \), we can annihilate \( x_0^{(0)}, \ldots, x_0^{(n_1-1)}, x_1^{(0)}, \ldots, x_1^{(m-1)} \) in the above expansion. Using (s), we see that the coefficient of \( t^m \) is \( (x_1^{(m_1)} - c x_0^{(m_1)})^{\nu_1} \).

In the sequel if \( A \) is a ring, \( f \subseteq A \) an ideal and \( f \in A \), we denote by \( V(I) \) the subvariety of \( \text{Spec} A \) defined by \( I \) and by \( D(f) \) the open set in \( \text{Spec} A \), \( D(f) := \text{Spec} A_f \).

The proof of the following corollary is analogous to that of proposition 4.1.

**Corollary 4.2.** Let \( m \in \mathbb{N}; \) let \( k \geq 1 \) be such that \( m = kn_1 \beta_1 + i; 1 \leq i \leq n_1 \beta_1 \). Then if \( i < n_1 \beta_1 \), we have that
\[
\text{Cont} > km_1(x_0^{(0)}) = \left\langle \pi_{m, kn_1 \beta_1}^{-1}(V(x_0^{(0)}, \ldots, x_0^{(kn_1)})) \right\rangle_{\text{red}} =
\]
\[
\text{Spec} 
\frac{\mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)}]}{(x_0^{(0)}, \ldots, x_0^{(kn_1)}, \ldots, x_0^{(kn_1+\frac{m}{\beta_1})}, x_1^{(0)}, \ldots, x_1^{(kn_1)}, \ldots, x_1^{(kn_1+\frac{m}{\beta_1})})}
\]
and if \( i = n_1 \beta_1 \)
\[
\text{Cont} > km_1(x_0^{(0)}) = \left\langle \pi_{m, kn_1 \beta_1}^{-1}(V(x_0^{(0)}, \ldots, x_0^{(kn_1)})) \right\rangle_{\text{red}} =
\]
\[
\text{Spec} 
\frac{\mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)}]}{(x_0^{(0)}, \ldots, x_0^{(k+1)n_1-1}, x_1^{(0)}, \ldots, x_1^{((k+1)n_1-1)}, x_0^{((k+1)n_1)^m} - c x_0^{((k+1)n_1)^m})}
\]
We now consider the case of a plane branch with one Puiseux exponent.
Lemma 4.3. Let $C$ be a plane branch with one Puiseux exponent. Let $m, k \in \mathbb{N}$, such that $k \neq 0$ and $m \geq kn_1 \bar{\beta}_1 + 1$, and let $\pi_{m, kn_1 \bar{\beta}_1} : C_m \to C_{kn_1 \bar{\beta}_1}$ be the canonical projection. Then

$$C^k_m := \pi_{m, kn_1 \bar{\beta}_1}^{-1}(V(x_0^{(0)}, \ldots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{red}$$

is irreducible of codimension $k(m_1 + n_1) + 1 + (m - kn_1 \bar{\beta}_1)$ in $\mathbb{C}^2_m$.

Proof: First note that since $e_1 = 1$, we have $m_1 = \bar{\beta}_1 = \bar{\beta}_1$. Let $I^0_m$ be the ideal defining $C^k_m$ in $\mathbb{C}^2_m \cap D(x^{(kn_1)}_0)$. Since $m \geq kn_1 \bar{\beta}_1$, by corollary 4.2, $x^{(0)}_1, \ldots, x^{(kn_1-1)}_1 \in I^0_m$. So $I^0_m$ is the radical of the ideal $I^{0k}_m := (x^{(0)}_0, \ldots, x^{(kn_1-1)}_0, x^{(0)}_1, \ldots, x^{(kn_1-1)}_1, F^{(0)}, \ldots, F^{(m)})$.

Now it follows from $\diamond$ and proposition 2.3 that

$$F(l) \in (x^{(0)}_0, \ldots, x^{(kn_1-1)}_0, x^{(0)}_1, \ldots, x^{(kn_1-1)}_1) \text{ for } 0 \leq l < kn_1 m_1,$$

$$F^{(kn_1 m_1)} \equiv x_1^{(kn_1) m_1} - cx_0^{(kn_1) m_1} \mod (x^{(0)}_0, \ldots, x^{(kn_1-1)}_0, x^{(0)}_1, \ldots, x^{(kn_1-1)}_1),$$

$$F^{(kn_1 m_1 + l)} \equiv n_1 x_1^{(kn_1) m_1 - 1} x^{(kn_1 + l)}_1 - m_1 cx_0^{(kn_1) m_1 - 1} x^{(kn_1 + l)}_0,$$

$$+ H_l(x^{(0)}_0, \ldots, x^{(kn_1 + l-1)}_0, x^{(0)}_1, \ldots, x^{(kn_1 + l-1)}_1) \mod (x^{(0)}_0, \ldots, x^{(kn_1-1)}_0, x^{(0)}_1, \ldots, x^{(kn_1-1)}_1),$$

for $1 \leq l \leq m - kn_1 m_1$.

This implies that $I^{0k}_m : (x^{(0)}_0, \ldots, x^{(kn_1-1)}_0, x^{(0)}_1, \ldots, x^{(kn_1-1)}_1, F^{(kn_1 m_1)}$, $\ldots$, $F^{(m)})$. Moreover the subscheme of $\mathbb{C}^2_m \cap D(x^{(kn_1)}_0)$ defined by $I^{0k}_m$ is isomorphic to the product of $\mathbb{C}^s(\mathbb{C}^*)$ isomorphic to the regular locus of $x^{(kn_1) m_1 - 1} - cx_0^{(kn_1) m_1}$ by an affine space and its codimension is $k(m_1 + n_1) + 1 + (m - kn_1 m_1)$; so it is reduced and irreducible, and it is nothing but $C^k_m$, or equivalently $I^0_m = I^{0k}_m$.

\[ \square \]

Corollary 4.4. Let $C$ be a plane branch with one Puiseux exponent. Let $m \in \mathbb{N}, m \neq 0$. let $q \in \mathbb{N}$ be such that $m = qn_1 \bar{\beta}_1 + i; 0 < i \leq n_1 \bar{\beta}_1$. Then $C^0_m = \pi_{m, qn_1 \bar{\beta}_1}^{-1}(0)$ has $q + 1$ irreducible components which are:

$$C_{mkI} = \mathbb{C}^k_m, 1 \leq k \leq q,$$

and $\mathbf{B}_m = \text{Cont}^{>q n_1}(x)_m = \pi_{m, qn_1 \bar{\beta}_1}^{-1}(V(x_0^{(0)}, \ldots, x_0^{(qn_1)}))$.

We have that

$$\text{codim}(C_{mkI}, \mathbb{C}^2_m) = k(m_1 + n_1) + 1 + (m - kn_1 m_1)$$

and

$$\text{codim}(\mathbf{B}_m, \mathbb{C}^2_m) = q(m_1 + n_1) + \left[ \frac{i}{\beta_0} \right] + \left[ \frac{i}{\beta_1} \right] + 2 = \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] + 2 \text{ if } i < n_1 \bar{\beta}_1$$

$$\text{codim}(\mathbf{B}_m, \mathbb{C}^2_m) = (q + 1)(m_1 + n_1) + 1 \text{ if } i = n_1 \bar{\beta}_1.$$

\textbf{Proof : } The codimensions and the irreducibility of $B_m$ and $C_{mkI}$ follow from corollary 4.2 and lemma 4.3. This shows that if $1 \leq k < k' \leq q$, we have $\text{codim}(C_{mkI}, C^2_m) < \text{codim}(C_{mkI}, \mathbb{C}^2_m)$, then $C_{mkI} \not\subseteq C_{mkI}$. On the other hand, since $C_{mkI} \subseteq V(x_0^{(kn)})$ and $C_{mkI} \not\subseteq V(x_0^{(km)})$, we have that $C_{mkI} \not\subseteq C_{mkI}$. This also shows that $\dim B_m = \dim C_{mkI}$ for $1 \leq k \leq q$, therefore $B_m \not\subseteq C_{mkI}, 1 \leq k \leq q$. But $C_{mkI} \not\subseteq B_m$ because $B_m \subseteq V(x_0^{(qm)})$ and $C_{mkI} \not\subseteq V(x_0^{(km)})$ for $1 \leq k \leq q$. We thus have that $C_{mkI} \not\subseteq B^m$ and $B^m \not\subseteq C_{mkI}$. We conclude the corollary from the fact that by construction $C^0_m = \cup_{k=1}^q C_{mkI} \cup B_m$. \qed

To understand the general case, i.e. to find the irreducible components of $C^0_m$ where $C$ has a branch with $g$ Puiseux exponents at 0, since for $kn_1 \beta_1 < m \leq (k+1)n_1 \beta_1, m, k \in \mathbb{N}$ we know by corollary 4.2 the structure of the $m$-jets that project to $V(x_0^{(0)}, \ldots, x_0^{(km)}) \cap C^0_{kn_1 \beta_1}$, we have to understand for $m > kn_1 \beta_1$ the $m$-jets that project to $V(x_0^{(0)}, \ldots, x_0^{(kn)}) \cap \mathbb{C}_{\beta}^{\operatorname{dim}C}$. Let $m, k \in \mathbb{N}$ be such that $m \geq kn_1 \beta_1$. Let $j = \max\{l, n_2 \cdots n_l \} \; \; \text{divides} \; k\}$ (we set $j = 2$ if the greatest common divisor $(k, n_2) = 1 \; \; \text{or} \; \; (g = 1)$. Set $\kappa$ such that $k = \kappa n_2 \cdots n_l - 1$, then we have $kn_1 = \kappa n_2 \cdots n_l$.

\textbf{Proposition 4.5.} Let $2 \leq j \leq g + 1$; for $i = 2, \ldots, g$, and $kn_1 \beta_1 < m < \kappa e_i - \beta_i$, we have that

$$C^k_m = \pi^{-1}_m (C^2_{\beta_i} (C^k_{\beta_i})),$$

where $\pi_m : \mathbb{C}^2_m \rightarrow \mathbb{C}^2_{\beta_i}$ is the canonical map. For $j < g + 1$ and $m \geq \kappa \beta_j$, we have that

$$C^k_m = \emptyset.$$

\textbf{Proof : } Let $\phi \in C^k_m$. Let $\tilde{\phi} : \text{Spec } \mathbb{C}[t] \rightarrow (\mathbb{C}^2, 0)$ be such that $\phi = \tilde{\phi} \mod t^{m+1}$. Let $\tilde{f} \in \mathbb{C}[x, y]$ be a function that defines the branch $\tilde{C}$ image of $\tilde{\phi}$. We may assume that the map $\text{Spec } \mathbb{C}[t] \rightarrow \tilde{C}$ be the normalization of $\tilde{C}$. Since $\operatorname{ord}_x x_0 \circ \tilde{\phi} = kn_1, \operatorname{ord}_y x_0 \circ \tilde{\phi} = km_1$ the multiplicity $m(\tilde{f})$ of $\tilde{C}$ at the origin is $\operatorname{ord}_x \tilde{f}(0, x_1) = kn_1 = \kappa n_2 \cdots n_l - 1$.

\textbf{Claim :} If $(f, \tilde{f})_0 < \kappa e_i - \beta_i$, then $(f, \tilde{f})_0 = n_i \cdots n_l (x_i, \tilde{f})_0$.

Indeed, we have that $\frac{(f, \tilde{f})_0}{\operatorname{ord}_x f(0, y)} < e_i - \frac{\beta_i}{\beta_0}$, therefore by corollary 3.5 we have that

$$o_f (\tilde{f}) < \frac{\beta_i}{\beta_0} = o_f(x_i).$$

We will prove that $o_f (\tilde{f}) = o_x (\tilde{f})$. (It was pointed by the referee that this follows from [A]. For the convenience of the reader we give a detailed proof below.)

Let $y(x^{\frac{1}{n}}), z(x^{\frac{1}{n_1 - n_i - 1}})$ and $u(x^{m(\beta)})$ be respectively Puiseux-roots of $f, x_i$ and $\tilde{f}$. There exist $w, \lambda \in \mathbb{C}$ such that $w^{\frac{1}{n_2 \cdots n_l}} = 1, \lambda^{m(\beta)} = 1$ and

$$o_f (\tilde{f}) = \operatorname{ord}_x (u(\lambda x^{m(\beta)}) - y(x^{\frac{1}{n}}))).$$
and
\[ o_f(x_i) = \text{ord}_x(y(x^\frac{1}{n_i}) - z(wx^{\frac{1}{n_1-n_i-1}})) . \]

Since \( o_f(\tilde{f}) < o_f(x_i) \), we have that
\[
\begin{align*}
o_f(\tilde{f}) &= \text{ord}_x(u(\lambda x^\frac{1}{m(\ell)}) - y(x^\frac{1}{n_0}) + y(x^\frac{1}{n_0}) - z(wx^{\frac{1}{n_1-n_i-1}})) \\
&= \text{ord}_x(u(\lambda x^\frac{1}{m(\ell)}) - z(wx^{\frac{1}{n_1-n_i-1}})) \leq o_{x_i}(\tilde{f}) .
\end{align*}
\]

On the other hand, there exist \( \lambda \) and \( \delta \in \mathbb{C} \), such that \( \lambda^m(\tilde{f}) = 1, \delta^{n_0} = 1 \) and such that
\[
o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^\frac{1}{m(\ell)}) - z(x^{\frac{1}{n_1-n_i-1}}))
\]
and
\[
o_f(x_i) = \text{ord}_x(y(x^\frac{1}{n_0}) - z(x^{\frac{1}{n_1-n_i-1}})).
\]

We have then that
\[
o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^\frac{1}{m(\ell)}) - y(x^\frac{1}{n_0}) + y(x^\frac{1}{n_0}) - z(wx^{\frac{1}{n_1-n_i-1}})).
\]

Now
\[
\text{ord}_x(u(\lambda x^\frac{1}{m(\ell)}) - y(x^\frac{1}{n_0})) \leq o_f(\tilde{f}) < o_f(x_i) = \text{ord}_x(y(x^\frac{1}{n_0}) - z(wx^{\frac{1}{n_1-n_i-1}})).
\]

So
\[
o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^\frac{1}{m(\ell)}) - y(x^\frac{1}{n_0})) \leq o_f(\tilde{f}).
\]

We conclude that \( o_f(\tilde{f}) = o_{x_i}(\tilde{f}) \), and since the sequence of Puiseux exponents of \( C_i \) is \( (\frac{\beta_0}{n_1-n_0}, \cdots, \frac{\beta_{i-1}}{n_1-n_0}) \), applying proposition 3.4 to \( C \) and \( C_i \), we find that \( (f, \tilde{f})_0 = (n_1-n_0)_{x_i, \tilde{f}}(i, f) \) and claim follows.

On the other hand by the corollary 3.5 applied to \( f \) and \( \tilde{f} \), \((f, \tilde{f})_0 \geq \kappa e_{i-1} \frac{\beta_i}{\epsilon_j} \) if and only if \( o_f(\tilde{f}) \geq \frac{\beta_i}{\epsilon_0} = o_{x_i}(f) = o_f(x_i) \) so \( o_f(\tilde{f}) \geq \frac{\beta_i}{\epsilon_0} \) if and only if \( o_{x_i}(\tilde{f}) \geq \frac{\beta_i}{\epsilon_0} \), therefore \((x_i, \tilde{f})_0 \geq \kappa \frac{\beta_i}{\epsilon_j} \). This proves the first assertion.

The second assertion is a direct consequence of lemma 5.1 in [GP].

To further analyse the \( C^k \)'s, we realize, as in section 3, \( C \) as a complete intersection in \( \mathbb{C}^{g+1} = \text{Spec} \mathbb{C}[x_0, \cdots, x_g] \) defined by the ideal \((f_1, \cdots, f_g)\) where
\[
f_i = x_{i+1} - (x_{i}^{n_i} - c_i x_0^b_0 \cdots x_{i-1}^{b_{i-1}}) - \sum_{\gamma=(\gamma_0, \cdots, \gamma_i)} c_{\gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i}
\]
for \( 1 \leq i \leq g \) and \( x_{g+1} = 0 \). This will let us see the \( C^k \)'s as fibrations over some reduced scheme that we understand well.

We keep the notations above and let \( I_0^m \) be the radical of the ideal defining \( C_0^m \) in \( \mathbb{C}^{g+1} \) and let \( I_0^m \) be the ideal defining \( C_0^m = (V(I_0^m, x_0^{k_0}, \cdots, x_0^{k_{n_1-1}}) \cap D(x_0^{k_{n_1}})) \) in \( D(x_0^{k_{n_1}}) \).
Lemma 4.6. Let \( k \neq 0 \), \( j \) and \( \kappa \) as above. For \( 1 \leq i < j \leq g \) (resp. \( 1 \leq i < j - 1 = g \)) and for \( \kappa n_i \cdots n_{j-1} \hat{\beta}_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{j+1} \), we have

\[
\{m\}^0 = \left( x_0^{(0)}, \ldots, x_0^{(\kappa \hat{\beta}_0 - 1)} \right),
\]

\[
x_l^{(0)}, \ldots, x_l^{(\kappa \hat{\beta}_0 - 1)}, F_l^{(\kappa n_i \hat{\beta}_i)}, \ldots, F_l^{(m)}, 1 \leq l \leq i,
\]

\[
\{l+1\}^0, \ldots, \{l+1\}^{(\kappa \hat{\beta}_0 - 1)}.
\]

Moreover for \( 1 \leq l \leq i \),

\[
F_l^{(\kappa n_i \hat{\beta}_i)} = \left( x_l^{(\kappa \hat{\beta}_0) n_0} - c_l x_0^{(\kappa \hat{\beta}_0)}, \ldots, x_l^{(0)} \right)
\]

\[
\mod ((x_l^{(0)}, \ldots, x_l^{(\kappa \hat{\beta}_0 - 1)})_{0 \leq l \leq i}, x_l^{(0)}, \ldots, x_l^{(0)}),
\]

for \( 1 \leq l < i \) and \( \kappa \frac{n_i \hat{\beta}_i}{n_j \cdots n_g} \leq n < \kappa \frac{n_j + 1}{n_j \cdots n_g} \) (resp. \( l = i \) and \( \kappa \frac{n_i \hat{\beta}_i}{n_j \cdots n_g} \leq n \leq \frac{m}{n_{i+1} \cdots n_g} \))

\[
F_l^{(n)} = -n_i x_l^{(\kappa \hat{\beta}_0) n_0} x_l^{(\kappa \hat{\beta}_0)} x_l^{(\kappa \hat{\beta}_0)} \cdots x_l^{(\kappa \hat{\beta}_0)} +
\]

\[
\sum_{0 \leq h \leq l-1} c_h x_l^{(\kappa \hat{\beta}_0) n_0} \cdots x_l^{(\kappa \hat{\beta}_0)}
\]

\[
F_l^{(n)} = x_l^{(n)} + H_l(x_0^{(0)}, \ldots, x_0^{(0)}),
\]

for \( 1 \leq l < i \) and \( \frac{n_i + 1}{n_j \cdots n_g} \leq n \leq m \) (resp. \( l = i \) and \( \frac{m}{n_{i+1} \cdots n_g} \leq n \leq m \), or \( i + 1 \leq l \leq g - 1 \) and \( 0 \leq n \leq m \),

\[
F_l^{(n)} = x_l^{(n)} + H_l(x_0^{(0)}, \ldots, x_0^{(n)}),
\]

For \( i = j - 1 = g \) and \( m \geq \kappa n_g \bar{\beta}_g \),

\[
\{m\}^0 = \left( x_g^{(0)}, \ldots, x_g^{(\kappa \hat{\beta}_0 - 1)} \right),
\]

\[
x_l^{(0)}, \ldots, x_l^{(\kappa \hat{\beta}_0 - 1)}, F_l^{(\kappa n_i \hat{\beta}_i)}, \ldots, F_l^{(m)}, 1 \leq l \leq g,
\]

where for \( 1 \leq l \leq g \) and \( \kappa n_i \hat{\beta}_i \leq n \leq m \), the above formula for \( F_l^{(n)} \) remains valid,

\[
F_g^{(\kappa n_i \hat{\beta}_i)} = -(x_g^{(\kappa \hat{\beta}_0) n_0} - c_g x_0^{(\kappa \hat{\beta}_0)} \cdots, x_g^{(\kappa \hat{\beta}_0) n_0} \cdots, x_g^{(\kappa \hat{\beta}_0) n_0} - c_g x_0^{(\kappa \hat{\beta}_0)} \cdots, x_g^{(\kappa \hat{\beta}_0) n_0} \cdots, x_g^{(\kappa \hat{\beta}_0) n_0})
\]
Proof: First assume that \( \kappa n_i \leq \kappa n_i + n - \kappa n_i \beta_i \) for \( 1 \leq i < j \leq g \) and for \( \kappa n_i \beta_i < n \leq m \),

\[
F_g(n) \equiv - (n_g x_g (\kappa \beta_g)_g - 1 \ x_g (\kappa \beta_g + n - \kappa n_i \beta_i)) - \\
c_g \sum_{0 \leq h \leq g - 1} b_g x_0 (\kappa \beta_0)_h \cdots x_h (\kappa \beta_h)_h - 1 \ x_h (\kappa \beta_h + n - \kappa n_i \beta_i) \cdots x_{g - 1} (\kappa \beta_{g - 1})_{g - 1}^{b(g - 1)} + \\
H_g (\cdots, x_h (\kappa \beta_h + n - \kappa n_i \beta_i), \cdots)
\]

mod \((x_0^{(0)}, \cdots, x_l^{(\kappa \beta_l - 1)}))_{0 \leq l \leq g}

and for \( \kappa n_i \beta_i < n \leq m \),

\[
F_g(n) \equiv - (n_g x_g (\kappa \beta_g)_g - 1 \ x_g (\kappa \beta_g + n - \kappa n_i \beta_i)) - \\
c_g \sum_{0 \leq h \leq g - 1} b_g x_0 (\kappa \beta_0)_h \cdots x_h (\kappa \beta_h)_h - 1 \ x_h (\kappa \beta_h + n - \kappa n_i \beta_i) \cdots x_{g - 1} (\kappa \beta_{g - 1})_{g - 1}^{b(g - 1)} + \\
H_g (\cdots, x_h (\kappa \beta_h + n - \kappa n_i \beta_i), \cdots)
\]

mod \((x_0^{(0)}, \cdots, x_l^{(\kappa \beta_l - 1)}))_{0 \leq l \leq g}

Proof: First assume that \( \kappa n_i \leq \kappa n_i + n - \kappa n_i \beta_i \) for \( 1 \leq i < j \leq g \) (resp. \( 1 \leq i < j - 1 = g \)). By proposition 4.5, we have that 

\[
\pi^{-1} \left( \frac{\kappa}{n_i + 1 - n} \right) \left( \frac{\kappa}{m} \right) \left( \frac{\kappa I}{n_i + 1 - n} \right)
\]

where \( \pi : C^2 \to C^2 \left( \frac{m}{n_i + 1 - n} \right) \) is the canonical map. Now \( C^2 = \text{Spec} \mathbb{C}[x_0, x_1] \) (resp. \( C_{i + 1} = V(x_{i + 1}) \)) is realized as the complete intersection in \( C_{i + 1} = \text{Spec} \mathbb{C}[x_0, \cdots, x_g] \) defined by the ideal \((f_1, \cdots, f_{g - 1})(\text{resp. } (f_1, \cdots, f_{g - 1}, x_{i + 1})) \). So since \( m \geq \kappa n_i \beta_i, \ x_m \) is the radical of the ideal \( I_m \)

\[
(0) \cdots, x_0^{(k_1 - 1)}, x_1^{(0)}, \cdots, x_{k_1 - 1}^{(0)}, F_{g - 1}^{(0)}, \cdots, x_{i + 1}^{(0)}, F_{g + 1 - 1}^{(0)}, \cdots, x_{n_i + 1 - m}^{(0)}
\]

We first observe that \( F_{g}(n) \equiv x_2^{(n)} \mod (x_0^{(0)}, \cdots, x_0^{(k_1 - 1)}, x_1^{(0)}, \cdots, x_{k_1 - 1}^{(0)} - c_1 x_0^{(m_1)}) \) for \( 0 \leq n < k_1 \beta_1 \). Now since \( \frac{m}{n_2 - n_2} \geq \frac{m}{n_2 - n_2} \geq k_1 m_1 \), we have

\[
F_1^{(k_1 - 1)} \equiv - (x_1^{(k_1 - 1)} - c_1 x_0^{(k_1 - 1)} - c_1 x_0^{(k_1 - 1)})
\]

and

\[
F_1^{(n)} \equiv - (n_1 x_1^{(k_1 - 1)} - c_1 x_0^{(k_1 - 1)} - c_1 x_0^{(k_1 - 1)})
\]

mod \((x_0^{(0)}, \cdots, x_0^{(k_1 - 1)}, x_1^{(0)}, \cdots, x_{k_1 - 1}^{(0)} - c_1 x_0^{(m_1 - 1)}) \)

for \( k_1 \beta_1 < n \leq \frac{m}{n_2 - n_2} \). Finally, for \( l = 1 \) and \( \frac{m}{n_2 - n_2} \leq n < m \), or \( 2 \leq l \leq g - 1 \) and \( 0 \leq n \leq m \), we have

\[
F_l^{(n)} = x_{l + 1}^{(n)} + H_l(x_0^{(0)}, \cdots, x_l^{(n)}), \cdots, x_l^{(0)}, \cdots, x_l^{(n)}
\]

As a consequence for \( i = 1 \), the subscheme of \( \mathbb{C}^{n+1} \cap D(x_0^{(k_1 - 1)}) \) defined by \( I_m \) is isomorphic to the product of \( \mathbb{C}^* \) by an affine space, so it is reduced and irreducible and \( I_m \) is a
prime ideal in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, \cdots, x_g^{(0)}, \cdots, x_g^{(m)}]_{x_0^{(kn_1)}}$, generated by a regular sequence, i.e. the proposition holds for $i = 1$.
Assume that it holds for $i < j - 1 < g$ (resp. $i < j - 2 = g - 1$). For $\kappa n_{i+1} \cdots n_{j-1} \beta_{i+1} \leq m < \kappa n_{i+2} \cdots n_{j-1} \beta_{i+2}$, the ideal in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, \cdots, x_g^{(0)}, \cdots, x_g^{(m)}]_{x_0^{(kn_1)}}$ generated by $I_{\kappa n_{i+1} \cdots n_{j-1} \beta_{i+1} - 1}^m$ is contained in $I_{\kappa n_{i+1} \cdots n_{j-1} \beta_{i+1} - 1}^m$. By the inductive hypothesis, $x_l^{(0)}, \cdots, x_l^{(\frac{\kappa \beta_l - 1}{n_j - n_g})} \in I_{\kappa n_{i+1} \cdots n_{j-1} \beta_{i+1} - 1}^m$ for $l = 1, \cdots, i + 1$. So $I_{\kappa n_{i+1} \cdots n_{j-1} \beta_{i+1} - 1}^m$ is the radical of

$$I_{\kappa n_{i+1} \cdots n_{j-1} \beta_{i+1} - 1}^m = \left( x_0^{(0)}, \cdots, x_0^{(\frac{\kappa \beta_l - 1}{n_j - n_g})}, x_l^{(0)}, \cdots, x_l^{(\frac{\kappa \beta_l - 1}{n_j - n_g})}, F_l^{(0)}, \cdots, F_l^{(m)}, 1 \leq l \leq i + 1, x_{i+2}, \cdots, x_{i+2}^{(\frac{m}{n_{i+2} - n_g})}, F_{i+2}^{(0)}, \cdots, F_{i+2}^{(m)}, i + 2 \leq l \leq g - 1 \right).$$

Now for $0 \leq n < \frac{\kappa n_i \beta_i}{n_j \cdots n_g}$ we have

$$F_l^{(n)} \equiv x_{l+1}^{(n)} \mod \left( x_0^{(0)}, \cdots, x_l^{(\frac{\kappa \beta_l - 1}{n_j - n_g})}, x_l^{(0)}, \cdots, x_l^{(\frac{\kappa \beta_l - 1}{n_j - n_g})} \right), 1 \leq l \leq i + 1.$$

Here since $\beta_{i+1} > n_i \beta_i$, for $1 \leq l \leq i$ and $\frac{m}{n_{i+2} - n_g} \geq \left[ \frac{m}{n_{i+2} - n_g} \right] \geq \frac{\kappa n_{i+1} \beta_{i+1}}{n_j \cdots n_g}$, we can delete $F_l^{(n)}$, $1 \leq l \leq i + 1$, $0 \leq n < \frac{\kappa n_i \beta_i}{n_j \cdots n_g}$ from the above generators of $I_{\kappa n \beta - 1}^m$. The identities relative to the $F_l^{(n)}$ for $1 \leq l \leq i + 1$, $\frac{\kappa n_i \beta_i}{n_j \cdots n_g} \leq n \leq m$ or $i + 2 \leq l \leq g - 1$ and $0 \leq n \leq m$ follow immediately from (6). Hence the subscheme of $\mathbb{C}^{g+1} \cap D(x_0^{(kn_1)})$ defined by $I_{\kappa n \beta - 1}^m$ is isomorphic to the product of $\mathbb{C}^*$ by an affine space, so it is reduced and irreducible and $I_{\kappa n \beta - 1}^m$ is a prime ideal in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, \cdots, x_g^{(0)}, \cdots, x_g^{(m)}]_{x_0^{(kn_1)}}$, generated by a regular sequence, i.e. the proposition holds for $i + 1$.

The case $i = j - 1 = g$ and $m \geq \kappa n_g \beta_g$ follows by similar arguments.

As an immediate consequence we get

**Proposition 4.7.** Let $C$ be a plane branch with $g$ Puiseux exponents. Let $k \neq 0, j$ and $\kappa$ as above. For $m \geq \kappa n_1 \beta_1$, let $\pi_{m,kn_1 \beta_1} : C_m \to C_{kn_1 \beta_1}$ be the canonical projection and let $C_{m}^k := \pi_{m,kn_1 \beta_1}^{-1} (D(x_0^{(kn_1)}) \cap V(x_0^{(0)}, \cdots, x_0^{(kn_1-1)}))_{\text{red}}$. Then for $1 \leq i < j \leq g$ (resp. $1 \leq i < j - 1 = g$) and $\kappa n_i \cdots n_{j-1} \beta_{i+1} \leq m < \kappa n_i \cdots n_{j-1} \beta_{i+1}$, $C_{m}^k$ is irreducible of codimension

$$\frac{\kappa}{n_j \cdots n_g} (\beta_0 + \beta_1 + \sum_{l=1}^{i-1} (\beta_{l+1} - n_l \beta_l)) + \left( \frac{m}{n_{i+1} \cdots n_g} \right) - \frac{\kappa n_i \beta_i}{n_j \cdots n_g} + 1.$$
in $C^2_m$. (We suppose that the sum in the formula is equal to 0 when $i = 1$.) For $j \leq g$ and $m \geq \kappa \bar{\beta}_i$ (resp. $j = g + 1$ and $m \geq \kappa n_i \bar{\beta}_i$),

$$C^k_m = \emptyset$$

(resp. $C^k_m$ is of codimension

$$\kappa(\bar{\beta}_0 + \bar{\beta}_1 + \sum_{l=1}^{g-1}(\bar{\beta}_{l+1} - n_l \bar{\beta}_1)) + m - \kappa n_i \bar{\beta}_i + 1$$

in $C^2_m$.

The referee kindly pointed out that for $m \in \mathbb{N}$ such that $\kappa n_i \cdots n_{j-1} \bar{\beta}_i \leq m < \kappa n_i+1 \cdots n_{j-1} \bar{\beta}_{i+1}$, the codimension of $C^k_m$ can also be written as:

$$\frac{k}{e_{j-1}}(\bar{\beta}_0 + \bar{\beta}_1 - \bar{\beta}_i + 1) + \frac{m}{e_i} + 1.$$ 

For $k' \geq k$ and $m \geq k' n_1 \bar{\beta}_1$, we now compare $\text{codim}(C^k_m, C^2_m)$ and $\text{codim}(C^k_m, C^2_m)$.

**Corollary 4.8.** For $k' \geq k \geq 1$ and $m \geq k' n_1 \bar{\beta}_1$, if $C^k_m$ and $C^k_{m'}$ are nonempty, we have

$$\text{codim}(C^k_m, C^2_m) \leq \text{codim}(C^k_m, C^2_m).$$

**Proof:** Let $\gamma^k : [k n_1 \bar{\beta}_1, \infty[ \rightarrow [k(n_1 + m_1), \infty]$ be the piecewise linear function given by

$$\gamma^k(m) = \frac{k}{e_1}(\bar{\beta}_0 + \bar{\beta}_1 + \sum_{l=1}^{i-1}(\bar{\beta}_{l+1} - n_l \bar{\beta}_1)) + \left(\frac{m}{e_i} - \frac{k n_i \bar{\beta}_i}{e_1}\right) + 1$$

for $1 \leq i \leq g$ and $\frac{k n_i \bar{\beta}_i}{e_2 \cdots n_{i-1}} \leq m < \frac{k n_i \bar{\beta}_i}{e_2 \cdots n_{i-1}}$. (Recall that by convention $\bar{\beta}_{g+1} = \infty$.)

In view of proposition 4.7, we have that $\text{codim}(C^k_m, C^2_m) = [\gamma^k(m)]$ for $k \equiv 0 \mod n_2 \cdots n_{j-1}$ and $k' \not\equiv 0 \mod n_2 \cdots n_{j}$ with $2 \leq j \leq g$ and any integer $m \in [k n_1 \bar{\beta}_1, \frac{k n_i \bar{\beta}_i}{e_2 \cdots n_{j-1}}]$ or for $k \equiv 0 \mod n_2 \cdots n_{g}$ and any integer $m \geq k n_1 \bar{\beta}_1$. Similarly we define $\gamma^{k'} : [k' n_1 \bar{\beta}_1, \infty[ \rightarrow [k'(n_1 + m_1), \infty]$ by changing $k$ to $k'$.

Let $\Gamma^k$ (resp. $\Gamma^{k'}$) be the graph of $\gamma^k$ (resp. $\gamma^{k'}$) in $\mathbb{R}^2$. Now let $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\tau(a, b) = (a, b - 1)$ and let $\lambda^{k'/k} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\lambda^{k'/k}(a, b) = \frac{k'}{k}(a, b)$. We note that $\tau(\Gamma^k) = \lambda^{k'/k}(\tau(\Gamma^k))$; we also note that the endpoints of $\tau(\Gamma^k)$ and $\tau(\Gamma^{k'})$ lie on the line through 0 with slope $\frac{\bar{\beta}_0 + \bar{\beta}_1}{e_1 n_1 \bar{\beta}_1} = \frac{1}{e_1} \frac{n_1 + m_1}{n_1 m_1} < \frac{1}{e_1}$. Since $\frac{k}{\kappa} \geq 1$, the image of $\tau(\Gamma^k)$ by $\lambda^{k'/k}$ lies in the interior subset of $\mathbb{R}^2_0$ with boundary the union of $\tau(\Gamma^k)$, of the segment joining its endpoint $(k n_1 \bar{\beta}_1, \frac{k}{k'}(\bar{\beta}_0 + \bar{\beta}_1))$ to $(k n_1 \bar{\beta}_1, 0)$ and of $[k n_1 \bar{\beta}_1, \infty[ \times 0$. This implies that $\gamma^{k'}(m) \leq \gamma^k(m)$ for $m \geq k' n_1 \bar{\beta}_1$, hence $[\gamma^{k'}(m)] \leq [\gamma^k(m)]$ and the claim.

$\square$
Theorem 4.9. Let $C$ be a plane branch with $g \geq 2$ Puiseux exponents. Let $m \in \mathbb{N}$. For $1 \leq m < n_1 \beta_1 + e_1, C_{m}^0 = \text{Cont}^{>0}(x_0)_m$ is irreducible. For $qn_1 \beta_1 + e_1 \leq m < (q + 1)n_1 \beta_1 + e_1$, with $q \geq 1$ in $\mathbb{N}$, the irreducible components of $C_{m}^0$ are:

$$C_{mn} = \text{Cont}^{e \beta_0}(x_0)_m$$

for $1 \leq k$ and $\kappa \beta_0 \beta_1 + e_1 \leq m$,

$$C_{m}^{\kappa n} = \text{Cont}^{e \beta_0 g}(x_0)_m$$

for $j = 2, \cdots, g, 1 \leq k$ and $k \neq 0 \mod n_j$ and such that $\kappa n_1 \cdots n_{j-1} \beta_1 + e_1 \leq m < \kappa \beta_j$.

$$B_m = \text{Cont}^{>n_1 q}(x_0)_m.$$  

Proof: We first observe that for any integer $k \neq 0$ and any $m \geq n_1 \beta_1$,

$$(C_{m}^0)^{\text{red}} = \cup_{1 \leq k \leq k} C_m^k \cup \text{Cont}^{kn_1}(x_0)_m$$

where $C_m^k := \text{Cont}^{km_1}(x_0)_m$. Indeed, for $k = 1$, we have that $(C_{m}^0)^{\text{red}} \subset V(x_0^0, \cdots, x_0^{(n_1-1)})$ by proposition 4.1. Arguing by induction on $k$, we may assume that the claim holds for $m \geq (k-1)n_1 \beta_1$. Now by corollary 4.2, we know that for $m \geq kn_1 \beta_1$, $\text{Cont}^{(k-1)n_1}(x_0)_m \subset V(x_0^0, \cdots, x_0^{(kn_1-1)})$, hence the claim for $m \geq kn_1 \beta_1$.

We thus get that for $qn_1 \beta_1 + e_1 \leq m < (q + 1)n_1 \beta_1 + e_1$,

$$(C_{m}^0)^{\text{red}} = \cup_{1 \leq k \leq q} C_m^k \cup \text{Cont}^{>n_1 q}(x_0)_m.$$  

By proposition 4.7, for $1 \leq k \leq q, C_m^k$ is either irreducible or empty. We first note that if $C_m^k \neq \emptyset$, then $\overline{C_m^k} \subset \text{Cont}^{>n_1 q}(x_0)_m$. Similarly, if $1 \leq k \leq 2$ and $C_m^k$ and $C_m^{k'}$ are nonempty, then $\overline{C_m^k} \subset \overline{C_m^{k'}}$. On the other hand by corollary 4.8, we have that $\text{codim}(C_m^k, C_m^{k'}) \leq \text{codim}(C_m^k, C_m^{k'})$. So $\overline{C_m^k} \subset \overline{C_m^{k'}}$. Finally we will show that $\text{Cont}^{>n_1 q}(x_0)_m \subset \overline{C_m^k}$ if $C_m^k \neq \emptyset$ for $1 \leq k \leq q$. To do so, it is enough to check that $\text{codim}(C_m^k, C_m^{k'}) \leq \text{codim}(\text{Cont}^{>n_1 q}(x_0)_m, C_m^{k'})$. For $m \neq [q_1 \beta_1 + e_1, (q + 1)n_1 \beta_1]$, we have

$$\delta(m) := \text{codim}(\text{Cont}^{>n_1 q}(x_0)_m, C_m^{k}) = 2 + q(n_1 + m_1) + [\frac{m - qn_1 \beta_1}{\beta_0}] + [\frac{m - qn_1 \beta_1}{\beta_0}]$$

by corollary 4.2. Let $\lambda : [q_1 \beta_1 + e_1] \rightarrow [q(n_1 + m_1), \infty] = \frac{q(n_1 + m_1)}{\kappa} + \frac{m - qn_1 \beta_1}{\beta_1} + 1$. For simplicity, set $i = m - qn_1 \beta_1$. For any integer $i$ such that $e_1 \leq i < n_1 \beta_1 = n_1 m_1 e_1$, we have $1 + [\frac{i}{n_1 e_1}] + [\frac{i}{n_1 e_1}] \leq [\frac{i}{\beta_1}]$. Indeed this is true for $i = e_1$ and it follows by induction on $i$ from the fact that for any pair of integers $(b, a)$, we have $[\frac{b+1}{a}] = [\frac{b}{a}]$ if and only if $b + 1 \equiv 0 \mod a$ and $[\frac{b+1}{a}] = [\frac{b}{a}] + 1$ otherwise, since $i < n_1 m_1 e_1$. So $\delta(m) \leq \lambda(m)$.

But in the proof of corollary 4.8, we have checked that if $C_m^k \neq \emptyset$, then $\text{codim}(C_m^k, C_m^{k'}) = [\gamma^k(m)]$. We have also checked that for $q \geq k$ and $m \geq qn_1 \beta_1$, $\gamma^k(m) \geq \gamma^q(m)$. Finally in
view of the definitions of $\gamma^q$ and $\lambda^q$, we have $\gamma^q(m) \geq \lambda^q(m)$, so $[\gamma^q(m)] \geq [\lambda^q(m)] \geq \delta^q(m)$. For $m = (q + 1)n_1\beta_1$, we have $\delta^q(m) = (q + 1)(n_1 + m_1) + 1$ by corollary 4.2. For $m \in [(q+1)n_1\beta_1, (q+1)n_1\beta_1 + 1]$, we have $\text{Cont}^{\geq qn_1}(x_0)_m = C_m^{q+1} \cup \text{Cont}^{>(q+1)n_1}(x_0)_m$ and $\text{Cont}^{>(q+1)n_1}(x_0)_m = V(x_0^{(0)}, \ldots, x_0^{((q+1)n_1)}, x_1^{(0)}, \ldots, x_1^{((q+1)n_1 - 1)}, x_1^{((q+1)n_1)}, x_1^{((q+1)n_1 + 1)}, x_1^{(0)}, \ldots, x_1^{((q+1)n_1 - 1)}, x_1^{((q+1)n_1 - 1)}, x_1^{((q+1)n_1)}) \cap D(x_0^{((q+1)n_1)})$, again by corollary 4.2. If in addition we have $m < (q+1)\beta_2$, then by proposition 4.5 $C_m^{q+1} = V(x_0^{(0)}, \ldots, x_0^{((q+1)n_1 - 1)}, x_1^{(0)}, \ldots, x_1^{((q+1)n_1 - 1)}, x_1^{((q+1)n_1)})$, thus we have $\text{Cont}^{\geq qn_1}(x_0)_m = C_m^{q+1}$ and $\delta^q(m) = (q + 1)(n_1 + m_1) + 1$. We have $(q + 1)n_1\beta_1 + e_1 \leq (q + 1)\beta_2$ if $q + 1 \geq n_2$, because $\beta_2 - n_1\beta_1 \equiv 0 \mod (e_2)$. If not, we may have $(q+1)\beta_2 < (q+1)n_1\beta_1 + e_1$, so for $(q+1)\beta_2 \leq m < (q+1)n_1\beta_1 + e_1$, we have $C_m^{q+1} = \emptyset$, $\text{Cont}^{\geq qn_1}(x_0)_m = \text{Cont}^{(q+1)n_1}(x_0)_m$ and $\delta^q(m) = (q + 1)(n_1 + m_1) + 2$.

In both cases, for $m \in [(q+1)n_1\beta_1, (q+1)n_1\beta_1 + 1]$, we have $\delta^q(m) \leq (q + 1)(n_1 + m_1) + 2$.

Since $[\lambda^q(m)] = q(n_1 + m_1) + n_1m_1 + 1$, we conclude that $[\lambda^q(m)] \subseteq \delta^q(m)$, so for $1 \leq k \leq q$, if $C_m^{k} \neq \emptyset$, we have $[\gamma^k(m)] \geq \delta^q(m)$. This proves that the irreducible components of $C_m^{0}$ are the $C_m^{k}$ for $1 \leq k \leq q$ and $C_m^{q+1} \neq \emptyset$, and $\text{Cont}^{\geq qn_1}(x_0)_m$, hence the claim in view of the characterization of the nonempty $C_m^{k}$'s given in proposition 4.5.

□

Corollary 4.10. Under the assumption of theorem 4.9, let $q_0 + 1 = \min\{\alpha \in \mathbb{N}; \alpha(\beta_2 - n_1\beta_1) \geq e_1\}$. Then $0 \leq q_0 \leq n_2$. For $1 \leq m < (q_0 + 1)n_1\beta_1 + e_1$, $C_m^{0}$ is irreducible and we have $\text{codim}(C_m^{0}, C_m^{2}) = 2 + \frac{m}{\beta_0} + \frac{m}{\beta_1}$ for $0 \leq q \leq q_0$ and $qn_1\beta_1 + e_1 \leq m < (q + 1)n_1\beta_1$ or $0 \leq q < q_0$ and $(q + 1)\beta_2 \leq m < (q + 1)n_1\beta_1 + e_1$. For $q \geq q_0 + 1$ in $\mathbb{N}$ and $qn_1\beta_1 + e_1 \leq m < (q + 1)n_1\beta_1 + e_1$, the number of irreducible components of $C_m^{0}$ is:

$$N(m) = q + 1 - \sum_{j=2}^{q} \left(\frac{m}{\beta_j} - \frac{m}{n_j\beta_j}\right)$$

and $\text{codim}(C_m^{0}, C_m^{2}) = 2 + \frac{m}{\beta_0} + \frac{m}{\beta_1}$ for $qn_1\beta_1 + e_1 \leq m < (q + 1)n_1\beta_1$ or $(q + 1)n_1\beta_1 \leq m < (q + 1)n_1\beta_1 + e_1$.

Proof: We have already observed that $n_2(\beta_2 - n_1\beta_1) \geq e_1$ because $\beta_2 - n_1\beta_1 \equiv 0 \mod (e_2)$, so $1 \leq q_0 + 1 \leq n_2$.

For $qn_1\beta_1 + e_1 \leq m < (q + 1)n_1\beta_1 + e_1$, with $q \geq 1$, we have seen in the proof of theorem
Corollary 4.11. If the plane curve $C$ has a branch at 0, with multiplicity $\beta_0$, and first Puiseux exponent $\bar{\beta}_1$, then

$$\min m \frac{\text{codim}(C^0_m, C^2_m)}{m + 1} = \frac{1}{\beta_0} + \frac{1}{\bar{\beta}_1}.$$
Proof: For any $m, p \neq 0$ in $\mathbb{N}$, we have $m - p\lfloor \frac{m}{p} \rfloor \leq p - 1$ and $m - p\lfloor \frac{m}{p} \rfloor = p - 1$ if and only if $m + 1 \equiv 0 \pmod{p}$; so for any $m \in \mathbb{N}, 2 + \lfloor \frac{m}{q_0} \rfloor + \lfloor \frac{m}{q_1} \rfloor \geq (m + 1)(\frac{1}{q_0} + \frac{1}{q_1})$ and we have equality if and only if $m + 1 \equiv 0 \pmod{(\beta_0)}$ and mod $(\beta_1)$ or equivalently $m + 1 \equiv 0 \pmod{(n_1\beta_1)}$ since $n_1\beta_1$ is the least common multiple of $\beta_0$ and $\beta_1$. If not we have $1 + \lfloor \frac{m}{q_0} \rfloor + \lfloor \frac{m}{q_1} \rfloor \geq (m + 1)(\frac{1}{q_0} + \frac{1}{q_1})$. Now if $(q + 1)n_1\beta_1 < m < (q + 1)n_1\beta_1 + 1$ with $q \in \mathbb{N}$, we have $(q + 1)n_1\beta_1 < m + 1 \leq (q + 1)n_1\beta_1 + 1 < (q + 2)n_1\beta_1$, so $m + 1 \not\equiv 0 \pmod{(n_1\beta_1)}$. If $(q + 1)n_1\beta_1 \leq m < (q + 1)\beta_2$ with $q \in \mathbb{N}$ and $q < q_0$, then $(q + 1)n_1\beta_1 < m + 1 \leq (q + 1)n_1\beta_1 + 1 < (q + 2)n_1\beta_1$, so $m + 1 \not\equiv 0 \pmod{(n_1\beta_1)}$. So in both cases, we have $1 + \lfloor \frac{m}{q_0} \rfloor + \lfloor \frac{m}{q_1} \rfloor \geq (m + 1)(\frac{1}{q_0} + \frac{1}{q_1})$. The claim follows from corollary 4.10.

It also follows immediately from corollary 4.10.

Corollary 4.12. Let $q_0 \in \mathbb{N}$ as in corollary 4.10. There exists $n_1\beta_1$ linear functions, $L_0, \cdots, L_{n_1\beta_1}$ such that $\dim(C^0_m) = L_i(m)$ for any $m \equiv i \pmod{n_1\beta_1}$ such that $m \geq q_0n_1\beta_1 + 1$.

The canonical projections $\pi_{m+1,m}: C^0_{m+1} \rightarrow C^0_m, m \geq 1$, induce infinite inverse systems

$$B_{m+1} \rightarrow B_m \rightarrow \cdots \rightarrow B_1$$

and finite inverse systems

$$C_{(m+1)\kappa I} \rightarrow C_{m\kappa I} \rightarrow \cdots \rightarrow C_{(\kappa\beta_0\beta_1+e_1)\kappa I} \rightarrow B_{\kappa\beta_0\beta_1+e_1-1}$$

for $2 \leq j \leq g$, and $\kappa \not\equiv 0 \pmod{(n_j)}$.

We get a tree $T_{C,0}$ by representing each irreducible component of $C^0_m, m \geq 1$, by a vertex $v_{i,m}, 1 \leq i \leq N(m)$, and by joining the vertices $v_{i,m+1}$ and $v_{i+1,m}$ if $\pi_{m+1,m}$ induces one of the above maps between the corresponding irreducible components.

This tree only depends on the semigroup $\Gamma$.

Conversely, we recover $\beta_0, \cdots, \beta_g$ from this tree and $\max\{m, \text{codim}(B_{m}, C^0_m) = 2\} = \beta_0 - 1$. Indeed the number of edges joining two vertices from which an infinite branch of the tree starts is $\beta_0\beta_1$. We thus recover $\beta_1$ and $e_1$. We recover $\beta_2, \cdots, \beta_g - n_1\beta_1, \cdots, \beta_j - n_1 \cdots n_{j-1}\beta_1, \cdots, \beta_g - n_1 \cdots n_{g-1}\beta_1$, hence $\beta_2, \cdots, \beta_g$ from the number of edges in the finite branches.

Corollary 4.13. Let $C$ be a plane branch with $g \geq 1$ Puiseux exponents. The tree $T_{C,0}$ described above and $\max\{m, \dim C^0_m = 2m\}$ determines the sequence $\beta_0, \cdots, \beta_g$ or equivalently the equisingularity class of $C$ and conversely.

We represent below the tree for the branch defined by $f(x, y) = (y^2 - x^3)^2 - 4x^6y - x^9 = 0$, whose semigroup is $< \beta_0 = 4, \beta_1 = 6, \beta_2 = 15 >$, and for which we have $e_1 = 2, e_2 = 1$ and $n_1 = n_2 = 2$. 
References


