Jet schemes of normal toric surfaces

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Abstract

For \( m \in \mathbb{N}, m \geq 1 \), we determine the irreducible components of the \( m - th \) jet scheme of a normal toric surface \( S \). We give formulas for the number of these components and their dimensions. This permits to determine the log canonical threshold of a toric surface embedded in an affine space. When \( m \) varies, these components give rise to projective systems, with which we associate a weighted oriented graph. We prove that the data of this graph is equivalent to the data of the analytical type of \( S \). Besides, we classify these irreducible components by an integer invariant that we call index of speciality. We prove that for \( m \) large enough, the set of components with index of speciality \( 1 \), is in \( 1 - 1 \) correspondance with the set of exceptional divisors that appear on the minimal resolution of \( S \).

1 Introduction

Nash has introduced the arc space of a variety \( X \) in order to investigate the intrinsic data of the various resolutions of singularities of \( X \). The analogy with \( p \)-adic numbers has led Kontsevich [K], Denef and Loeser [DL1] to invent motivic integration and to introduce several rational series that generalize analogous series in the \( p \)-adic context [DL2]. The geometric counterpart of the theory of motivic integration has been used by Ein, Mustata and others to obtain formulas controlling discrepancies in terms of invariant of jet schemes -these are finite dimensional approximations of the arc space-[Mus2],[ELM],[EM],[dFEI]. Roughly speaking, while we can extract informations about abstract resolutions of singularities from the arc space and vice versa, we can extract informations about embedded resolutions of singularities from the jet schemes and vice versa. This partly explains why the arc space of a toric variety -which has been intensively studied [KKMS],[L],[B-GS],[I],[IK]- is well understood. Indeed, we know an equivariant abstract resolution of a toric variety, what permits to understand the action of the arc space of the torus on its arc space [I], but an equivariant embedded resolution is less accessible.

Note that despite that jet schemes were the subject of numerous articles in the last decade, few is known about their geometry for specific class of singularities, except for the
following three classes: monomial ideals [GS], determinantal varieties [D],[SS],[Y], plane
branches [Mo1]. It seems that it is a challenge to understand the structure of Jet schemes
of toric varieties.

In this article, we study the jets schemes of normal toric surface singularities. Beside
being the simplest toric singularities, this class of singularities is interesting from the fol-
lowing points of view:

In [Ni], Nicaise has computed the Igusa motivic Poincaré series for toric surface sin-
gularities and has proved that we can not extract the analytical type of the surface from
this series. We will prove that the data of the number of irreducible components and their
dimensions is equivalent to the data of the motivic Poincaré series. On the other hand, we
will assign to the jet schemes of a toric surface an oriented weighted graph that contains
informations about how their irreducible components behave under the transition mor-
phisms, and we will prove in corollary 4.18 that the data of this graph is equivalent to the
analytical type of the surface. This is a first instance where one can extract from the Jet
schemes an invariant (the oriented graph) which is a complete invariant of the analytical
type of a singularity.

The Nash map for a toric surface $S$ which assigns to every irreducible component of the
space of arcs centered in the singular locus an exceptional divisor on the minimal resolution
of $S$ is bijective [IK]. In general it is a difficult task to relate the irreducible components of
the jet schemes to the irreducible components of the arc space. For a given $m$, we classify
these irreducible components by an integer invariant that we call index of speciality (4.14).
We prove that for $m$ large enough, the components with index of speciality 1, are in $1 - 1$
correspondance with the exceptional divisors that appear on the minimal resolution of $S$.
This is to compare with a result that we have obtained in [Mo2] for rational double point
singularities.

Despite that these singularities are not complete intersections and therefore we do not
have a definition of non-degeneracy with respect to their Newton polyhedra in the sense of
Kouchnirenko [Ko], they heuristically are non-degenerate because they are desingularized
with one toric morphism. The number of irreducible components of their $m$–th jet scheme
is increasing when $m$ grows (corllary 4.15). For plane branches, this property only holds
for branches with at most one Puiseux pair ([Mo1]). This is one of the expected features
of Newton non degenracy. This might be an approach towards defining Newton polyhedra
without coordinates.

These surfaces are examples of varieties having rational singularities, but which need
not be locally complete intersection, therefore we cannot characterize their rationality by
[Mus1] via their jet schemes. We will prove that these latter have special properties, for
example: for a given $m \in \mathbb{N}$, we will prove that the irreducible components of the $m$–th
jet scheme of a toric surface, which have the same index of speciality (see 4.14 for a defi-
nition) are equidimensional. It would be interesting to figure out if this remains true for
all rational singularities. Note that apart from the case of the $A_n$ singularities, these jet schemes are never irreducible, in contrary with the case of jet schemes of rational complete intersection singularities, which are always irreducible [Mus1].

We determine the irreducible components of the jet schemes of a toric surface as the closure of certain contact loci, and we give formulas for their number and dimensions. As a byproduct, we will deduce using Mustata’s formula from [Mus2] the log canonical threshold of the pair $S \subset \mathbb{A}^e$, where $e$ is the embedding dimension of $S$.

One of our motivations in the study of irreducible components of jet schemes is Teissier’s approach to resolution of singularities. Teissier conjectured that one can re-embed a singular variety in such a way that it can be desingularized by one toric morphism (a "local" version of this conjecture can be found in [T]). A jet schemes viewpoint on this re-embedding in the case of irreducible plane curves was given in [LMR]. A toric surface is naturally well embedded, but jet schemes may also determine special toric morphisms which resolve the singularities of the surface. In a work in progress, we use the description of the irreducible components of this paper to construct an embedded resolution of a toric surface singularity. This can be thought as an inverse embedded Nash problem [LMR],[ELM].

Some of the results of this paper were announced in [Mo3].

The structure of the paper is as follows: in section two we present a reminder on jet schemes and on toric surfaces. In section three we study the jet schemes of the $A_n$ singularities. The last section is the heart of the paper and is devoted to jet schemes of toric surfaces of embedding dimension bigger than or equal to four.

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2 Jet schemes and toric surfaces

2.1 Jet schemes

Let $\mathbb{K}$ be a field. Let $X$ be a $\mathbb{K}$-scheme of finite type over $\mathbb{K}$. For $m \in \mathbb{N}$, the functor $F_m : \mathbb{K}-\text{Schemes} \to \text{Sets}$ which to an affine scheme defined by a $\mathbb{K}$-algebra $A$ associates

$$F_m(\text{Spec}(A)) = \text{Hom}_\mathbb{K}(\text{Spec}A[t]/(t^{m+1}), X)$$

is representable by a $\mathbb{K}$-scheme $X_m$ [V]. We call $X_m$ the $m$-th jet scheme of $X$ and we have that $F_m$ is isomorphic to its functor of points. In particular the $\mathbb{K}$-points of $X_m$ are
in bijection with the \( \mathbb{K}[t]/(t^{m+1}) \)-points of \( X \).

For \( m, p \in \mathbb{N}, m > p \), the truncation homomorphism \( A[t]/(t^{m+1}) \to A[t]/(t^{p+1}) \) induces a canonical projection \( \pi_{m,p} : X_m \to X_p \). These morphisms are affine and for \( p < m < q \) they clearly verify \( \pi_{m,p} \circ \pi_{q,m} = \pi_{q,p} \). This yields an inverse system whose limit \( X_\infty \) is a scheme called the arc space of \( X \). Note that \( X_0 = X \). We denote the canonical projections \( X_m \to X_0 \) by \( \pi_m \) and \( X_\infty \to X_m \) by \( \Psi_m \). See [EM] for more about jet schemes.

**Example 1.** Let \( X = \text{Spec} \ \mathbb{K}[x_1, \ldots, x_n] / (f_1, \ldots, f_r) \) be an affine \( \mathbb{K} \)-scheme. For a \( \mathbb{K} \)-algebra \( A \), an \( A \)-point of \( X_m \) is a \( \mathbb{K} \)-algebra homomorphism

\[
\varphi : \frac{\mathbb{K}[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)} \to A[t]/(t^{m+1}).
\]

This homomorphism is completely determined by the image of \( x_i, i = 1, \ldots, n \)

\[
x_i \mapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)} t + \cdots + x_i^{(m)} t^m
\]

and it should verify that \( \varphi(f_i) = f_i(\phi(x_1), \ldots, \phi(x_n)) \in (t^{m+1}), \ i = 1, \ldots, r \).

Therefore if we set

\[
f_l(\phi(x_1), \ldots, \phi(x_n)) = \sum_{j=0}^{m} F_l^{(j)}(x_1^{(j)}, \ldots, x_n^{(j)}) t^j \mod (t^{m+1})
\]

where \( x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)}) \), then we have that

\[
X_m = \text{Spec} \ \frac{\mathbb{K}[x^{(0)}, \ldots, x^{(m)}]}{(F_l^{(j)})_{l=1, \ldots, r}}
\]

**Example 2.** From the above example, we see that the \( m \)-th jet scheme of the affine space \( \mathbb{A}^n \) is isomorphic to \( \mathbb{A}^{(m+1)n} \) and that the projection \( \pi_{m,m-1} : \mathbb{A}^n_m \to \mathbb{A}^n_{m-1} \) is the map that forgets the last \( n \) coordinates.

**Remark 2.1.** Note that in general, if \( X \) is a nonsingular variety of dimension \( n \), then all the projections \( \pi_{m,m-1} : X_m \to X_{m-1} \) are locally trivial fibrations with fiber \( \mathbb{A}^n \). In particular \( X_m \) is of dimension \( n(m+1) \) ([EM]).

### 2.2 Toric surfaces

Let \( S \) be a singular affine normal toric surface defined over the field \( \mathbb{K} \). There exist two coprime integers \( p \) and \( q \) such that \( S \) is defined by the cone \( \sigma \subset N = \mathbb{Z}^2 \) generated by \( (1,0) \) and \( (p,q) \) and \( 0 < p < q \), i.e. \( S = \text{Spec} \mathbb{K}[x^u, \ u \in \sigma^\vee \cap M] \) where \( \sigma^\vee \) is the dual cone of \( \sigma \) and \( M \) is the dual lattice of \( N \) ([O]). We have the Hirzebruch-Jung continued fraction expansion in terms of \( c_j \geq 2 \) :
\[ \frac{q}{p} = c_2 - \frac{1}{c_3 - \frac{1}{\cdots - \frac{1}{c_{e-1}}}} \]

which we denote by \([c_2, \ldots, c_{e-1}]\). Let \(\theta^\vee\) be the convex hull of \((\sigma^\vee \cap M) \setminus 0\) and let \(\partial \theta^\vee\) be its boundary polygon. Let \(u_1, u_2, \ldots, u_h\) be the points of \(M\) lying in this order on the compact edges of \(\partial \theta^\vee\), with \(u_1 = (0, 1)\) and \(u_h = (q, -p)\). Then from \([O]\), proposition 1.21, we have that \(h = e\) is the embedding dimension of \(S\) and the \(u_i\) form a minimal system of generators of the semigroup \(\sigma^\vee \cap M\). Moreover we have

\[ u_{i-1} + u_{i+1} = c_i u_i, \text{ for } i = 2, \ldots, e - 1. \]

For \(i = 1, \ldots, e\), we will denote by \(x_i\) the regular function on \(S\) defined by \(x_i\). Riemenschneider has exhibited the generators of the ideal defining \(S\) in \(\mathbb{A}^e = \text{Spec} \mathbb{K}[x_1, \ldots, x_e]\). They can be given in a quasi-determinantal format (\([R]\), \([St]\)):

\[
\begin{pmatrix}
x_1 & x_2 & \cdots & x_{e-2} & x_{e-1}
x_2 & x_2^{c_2-2} & \cdots & x_{e-1}^{c_{e-1}-2} & x_e
\end{pmatrix}
\]

where the generalised minors of a quasi-determinant

\[
\begin{pmatrix}
f_1 & f_2 & \cdots & f_{k-1} & f_k \\
h_{1,2} & \cdots & h_{k-1,k} & f_k \\
g_1 & g_2 & \cdots & g_{k-1} & g_k
\end{pmatrix}
\]

are \(f_i g_j - g_i (\prod_{n=1}^{j-1} h_{n,n+1}) f_j\).

They can be written as follows:

\[ E_{ij} = x_i x_j - x_{i+1} x_{i+1}^{-c_{i+1}-2} x_{i+2}^{-c_{i+2}-2} \cdots x_{j-2}^{-c_{j-2}-2} x_{j-1}^{-c_{j-1}-2} x_{j-1}, \]

where \(1 \leq i < j - 1 \leq e - 1\).

Let \(b_i \in \mathbb{N}, b_i \geq 2\), be such that \(q/(q - p) = [b_1, \ldots, b_r]\). Let \(l_0 = (1, 0), \ldots, l_{e+1} = (p, q)\) in this order be the elements of \(N\) lying on the compact edges of the boundary \(\partial \theta\) of the convex hull \(\theta\) of \((\sigma \cap N) \setminus 0\).

**Proposition 2.2.** We have that \(r = s\) and is equal to the number of irreducible components of the exceptional curve for the minimal resolution of singularities of \(S\). Moreover we have that

\[ c_2 + \cdots + c_{e-1} - 2(e - 2) + 1 = s. \]

See lemma 1.22 and corollary 1.23 in \([O]\) for a proof.
3 Jet schemes of toric surfaces of embedding dimension \( e = 3 \)

If \( R \) is a ring, \( I \subseteq R \) an ideal, we denote by \( V(I) \) the subvariety of \( \text{Spec} \, R \) defined by \( I \).

Let \( S \) be the variety defined in \( \mathbb{A}^3 \) by the equation \( f(x,y,z) = xy - z^{e+1} = 0 \). \( S \) has an \( A_n \) singularity at the origin 0 and is nonsingular elsewhere. Note that an affine toric surface of embedding dimension 3 has this type of singularities (see section 2.1). If we set

\[
f \left( \sum_{i=0}^{m} x^{(i)} t^i, \sum_{i=0}^{m} y^{(i)} t^i, \sum_{i=0}^{m} z^{(i)} t^i \right) = \sum_{i=0}^{m} F^{(i)} t^i \mod t^{m+1},
\]

then \( S_m \) is defined in \( \mathbb{A}_m^{3(m+1)} = \mathbb{A}_m^3 \) by the ideal \( I_m = (F^{(0)}, F^{(1)}, ..., F^{(m)}) \). By remark 2.1, the morphism \( \pi^{-1}_m(S\{0\}) \to S\{0\} \) is a locally trivial fibration, therefore we have that \( \pi^{-1}_m(S\{0\}) \) is an irreducible component of \( S_m \) of codimension \( m + 1 \) in \( \mathbb{A}_m^3 \). On the other hand, we will prove in the coming lines that the dimension of \( S_m^0 := \pi^{-1}_m(0) \) in \( \mathbb{A}_m^3 \) is \( m + 2 \), which means that \( S_m \) is irreducible for every \( m \in \mathbb{N} \); indeed, since \( I_m \) is generated by \( m + 1 \) equations, any irreducible component of \( S_m \) could have codimension at most \( m + 1 \). (Note that the irreducibility of \( S_m \) follows from [Mus1] because \( S \) is locally a complete intersection with a rational singularity, but we give here a direct proof in this simple case.)

We claim that for \( m \leq n \), we have \( S^0_m = Z_m^0 \), where \( Z \subset \mathbb{A}^3 \) is the hypersurface defined by \( xy = 0 \). Indeed, a \( m \)-jet \( \gamma_m = (x = \sum_{i=0}^{m} x^{(i)} t^i, y = \sum_{i=0}^{m} y^{(i)} t^i, z = \sum_{i=0}^{m} z^{(i)} t^i) \in (\mathbb{A}^3)_m \) centered at the origin (i.e. \( x^{(0)} = y^{(0)} = z^{(0)} = 0 \)) is in \( S^0_m \) if and only if \( xy - z^{m+1} \equiv 0 \mod t^{m+1} \), but since \( z_0 = 0 \) and \( m \leq n \), we have that \( \text{ord}_t \, x y \geq n + 1 \geq m + 1 \), therefore this is equivalent to \( \text{ord}_t \, xy \geq m + 1 \) and therefore to \( \gamma \in Z^0_m \).

But clearly for \( m \leq n \), the irreducible components of \( Z^0_m \) are the subvarieties defined by the ideals

\[
I_m^l = (x^{(0)}, ..., x^{(l-1)}, y^{(0)}, ..., y^{(m-l)}, z^{(0)}), l = 1, ..., m.
\]

Notice that the codimension of \( C_m^l := V(I_m^l) \) in \( \mathbb{A}_m^3 \) is equal to \( m + 2 \) for \( l = 1, ..., m \). We deduce that for \( m \leq n \), \( S_m \) is irreducible of codimension \( m + 1 \). On the other hand, for \( m \geq n + 1 \) we have that \( C_m^{l} := \pi^{-1}_{m,n}(V(I_m^l)) \) is defined in \( (\mathbb{A}^3)_m \) by the ideal \( I_m^l = (I_m^{l}, J_{m-(n+1)}) \) where \( J_{m-(n+1)} \) is the ideal obtained from the ideal defining \( X_{m-(n+1)} \) in \( \mathbb{A}_m^{3-(n+1)} \) by changing variables. Indeed if we set

\[
f \left( \sum_{i=0}^{m} x^{(i)} t^i, \sum_{i=0}^{m} y^{(i)} t^i, \sum_{i=0}^{m} z^{(i)} t^i \right) =
\]

\[
f \left( t^l \sum_{i=0}^{m-l} x^{(l+i)} t^i, t^{m-l+1} - \sum_{i=0}^{m-l+1} y^{(m-l+1+i)} t^i, t(t^{m-l+1} - \sum_{i=0}^{m-l+1} z^{(m-l+1+i)} t^i) \right) =
\]

\[
t^{n+1} f \left( \sum_{i=0}^{m-l} x^{(l+i)} t^i, \sum_{i=0}^{m-l+1} y^{(m-l+1+i)} t^i, \sum_{i=0}^{m-l+1} z^{(m-l+1+i)} t^i \right)
\]
\[ t^{m+1} \left( \sum_{i=0}^{m-(n+1)} G_l^{(i)} i^i \right) \mod t^{m+1}, \quad (\diamond) \]

then \( J_{m-(n+1)}^l \) is generated by \( G_l^{(i)} \), \( i = 0, \ldots, m-(n+1) \), and by comparing \((\diamond)\) with \((\diamond)\), we get that

\[ G_l^{(i)} = F^{(i)}(x^{(l)}, \ldots, x^{(l+i)}, y^{(n-l+1)}, \ldots, y^{(n-l+1+i)}, z^{(1)}, \ldots, z^{(1+i)}). \]

We deduce that for \( l = 1, \ldots, n \),

\[ \text{Codim } (\pi_{m,n}^{-1}(V(I_n^l)), \mathbb{A}_m^3) = n + 2 + \text{Codim } (S_{m-(n+1)}^l, \mathbb{A}_m^3). \]

This implies by a simple induction on \( m \) that for \( l = 1, \ldots, n \),

\[ \text{Codim } \pi_{m,n}^{-1}(V(I_n^l)) = m + 2. \]

Therefore \( \text{Codim } (S_m^0, \mathbb{A}_m^3) = m + 2 \), so \( S_m \) is irreducible. It follows that \( \pi_{m,n}^{-1}(V(I_n^l)) \) which is isomorphic to \( S_{m-(n+1)} \times \mathbb{A}_m^{2n+1} \) is irreducible. On the other hand, the ideal defining \( S_m^0 \) in \( \mathbb{A}_m^3 \) is generated by the \( m + 2 \) functions \( x^{(0)}, y^{(0)}, z^{(0)}, F^{(i)}, i = 2, \ldots, m \).

We conclude:

**Theorem 3.1.** For \( m \in \mathbb{N}, n \geq 1 \), the scheme \( S_m^0 \) of \( m \)-jets centered in the singular locus of an \( A_n \) singularity is a complete intersection scheme. For \( m \leq n \) this scheme has \( m \) irreducible components each of codimension \( m + 2 \). For \( m \geq n + 1 \), it has \( n \) irreducible components each of codimension \( m + 2 \). The scheme \( S_m \) is irreducible.

We obtain a graph \( \Gamma \) by representing every irreducible component of \( S_m^0 \), \( m \geq 1 \), by a vertex \( v_{i,m} \) and by joining the vertices \( v_{i,m+1} \) and \( v_{i,m} \) if the morphism \( \pi_{m+1,m} \) induces a morphism between the corresponding irreducible components. From the computations above we deduce that the graph \( \Gamma \) for the singularity \( A_4 \) is the following:

\[ \text{\vdots} \]
4 Jet schemes of toric surfaces of embedding dimension $e \geq 4$

We keep the notations introduced in section 2 and we begin by introducing some more notations. Let $f \in \mathbb{K}[x_1, \ldots, x_e]$; for $m, p \in \mathbb{N}$ such that $p \leq m$, we set:

$$\text{Cont}^p(f)_m(\text{resp.} \text{Cont}^{>p}(f)_m) := \{ \gamma \in S_m | \text{ord}_\gamma(f) = p(\text{resp.} > p) \},$$

$$\text{Cont}^p(f) = \{ \gamma \in S_\infty | \text{ord}_\gamma(f) = p \},$$

where $\text{ord}_\gamma(f)$ is the $t$-order of $f \circ \gamma$.

For $a, b \in \mathbb{N}, b \neq 0$, we denote by $\lceil \frac{a}{b} \rceil$ the round-up of $\frac{a}{b}$. For $i = 2, \cdots, e - 1, s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$ (i.e. $m \geq 2s - 1 \geq 1$) and $l \in \{s, \ldots, m_s^i\}$, where

$$m_s^i := \min\{(c_i - 1)s, (m + 1) - s\},$$

we set

$$D_{i,m}^{s,l} := \text{Cont}^s(x_i)_m \cap \text{Cont}^l(x_{i+1})_m,$$

and

$$C_{i,m}^{s,l} := D_{i,m}^{s,l}.$$

As in the previous section, if $R$ is a ring, $I \subseteq R$ an ideal, we denote by $V(I)$ the subvariety of $\text{Spec} R$ defined by $I$. For $f \in R$, we denote by $D(f)$ the open set $D(f) := \text{Spec} R_f$.

We will prove that the irreducible components of $S_m^0 := \pi_m^{-1}(0), m \geq 1$, are among the closed sets $C_{i,m}^{s,l}$. We begin by giving an overview of the strategy of the proof.
The first remark is that $S_1^0$, which is the Zariski tangent space of $S$ at 0, is isomorphic to an affine space (lemma 4.1)

$$S_1^0 = \text{Spec} \left( \mathbb{K} \left[ x_1^{(0)}, \ldots, x_e^{(0)}, x_1^{(1)}, \ldots, x_e^{(1)} \right] \right).$$

A key idea is to stratify it as follows

$$S_1^0 = \left( S_1^0 \cap D(x_1^{(1)}) \right) \cup \ldots \cup \left( S_1^0 \cap D(x_e^{(1)}) \right) \cup \left( S_1^0 \cap V(x_1^{(1)}, \ldots, x_e^{(1)}) \right).$$

First we study $\pi_{m,1}^{-1}(S_1^0 \cap D(x_i^{(1)}))$, for $i = 2, \ldots, e-1$ and $m \geq 2$. By using syzygies between the equations defining $S$ (lemma 4.3), we construct in proposition 4.8 a trivial fibration from $\pi_{m,1}^{-1}(S_1^0 \cap D(x_i^{(1)}))$ to a constructible subset of the jet schemes of an $A_{e_i}$ singularity. This latter constructible subset is introduced and studied in lemma 4.7, what permits to us to determine the irreducible components of the Zariski closure of $\pi_{m,1}^{-1}(S_1^0 \cap D(x_i^{(1)}))$, for $i = 2, \ldots, e-1$, namely the $C_{i,m}^{1,2}$'s.

The constructibles $\pi_{m,1}^{-1}(S_1^0 \cap D(x_i^{(1)}))$ for $i = 1, e$ are irreducible (proposition 4.8) and included in the Zariski closure of $\pi_{m,1}^{-1}(S_1^0 \cap D(x_i^{(1)}))$ for $i = 1, e-1$ (proposition 4.10). The proof of the latter statement in the case where the embedding dimension $e = 4$ is based on dimension arguments, then we use induction on $e$. For this purpose, we approximate $S$ by toric surfaces which are of less embedding dimension.

For $m = 2$, we prove that $\pi_{2,1}^{-1}(S_1^0 \cap V(x_1^{(1)}, \ldots, x_e^{(1)}))$ is included in the Zariski closure of $\pi_{m,1}^{-1}(S_1^0 \cap D(x_i^{(1)}))$ for any $i = 2, \ldots, e-1$ (proposition 4.10). The proof of the latter statement in the case where the embedding dimension $e = 4$ is based on dimension arguments, then we use induction on $e$. For this purpose, we approximate $S$ by toric surfaces which are of less embedding dimension.

For $m = 3$, $\pi_{3,1}^{-1}(S_1^0 \cap V(x_1^{(1)}, \ldots, x_e^{(1)}))$, is an irreducible component of $S_1^0$, and is an affine space that we stratify in a similar way to (∗) (see the case $m = 2n + 1$ in proposition 4.10).

We then, as above, consider the inverse image by $\pi_{m,3}$, $m \geq 4$ of each strata. The inverse images by $\pi_{m,3}$ of the open stratas will be understood again by comparison with some subsets of the jet schemes of $A_{e_i}$ singularities and they will give rise to a new generation of irreducible components, namely the $C_{i,m}^{2,2}$’s. Then we study the inverse image by $\pi_{4,3}$ and $\pi_{5,3}$ of the closed strata. This phenomena is understood by an induction on $m$, (more precisely on $n$), which permits us to cover $S_m^0$ by irreducible subsets. In theorem 4.12, we prove that there are no inclusions between these subsets.

Our first aim is to prove the irreducibility of the $C_{i,m}^{s,1}$’s and to compute their codimensions in $A_{e_i}^n$, this is the subject of proposition 4.5. We begin by some preparatory lemmas.

**Lemma 4.1.** For $i = 2, \ldots, e-1$, $s \geq 1$, the ideal defining $C_{i,2s-1}^{s,1}$ in $A_{2s-1}^e$ is

$$I_{i,2s-1}^{s,1} = (x_j^{(b)}, 1 \leq j \leq e, 0 \leq b < s).$$
Note that \( C_{i,2s-1}^{s,s} \) does not depend on \( i \). For \( j = 1, e, \) we set
\[
C_{j,2s-1}^{s,s} := C_{i,2s-1}^{s,s}, \quad i = 2, \cdots, e - 1.
\]

Proof: Let us prove that \( D_{i,2s-1}^{s,s} = V(I_{i,2s-1}^{s,s}) \cap D(x_i^{(s)}x_{i+1}^{(s)}) \). Let \( \gamma \in \mathbb{A}_{2s-1}^s \) such that \( \text{ord}_{s,x_i} = \text{ord}_{s,x_{i+1}} = s \). So, we have \( \text{ord}_{s,x_i} = c_is > 2s - 1 \) because \( c_i \geq 2 \). If moreover \( \gamma \) lies in \( S_{2s-1} \), then it satisfies \( E_{i-1,i+1} \equiv t^{2s} \), which is equivalent to \( \text{ord}_{s,x_{i+1}} \geq s \), because \( x_i \circ \gamma \equiv 0 \mod t^{2s} \) and \( \text{ord}_{s,x_{i+1}} = s \). The same argument, using \( E_{i-2,i} \), \( E_{i,i+2} \) and so on by induction, using the other \( E_{ji} \)'s and \( E_{ij} \)'s, gives that \( \text{ord}_{s,x_j} \geq s \). We deduce
\[
D_{i,2s-1}^{s,s} \subset V(I_{i,2s-1}^{s,s}) \cap D(x_i^{(s)}x_{i+1}^{(s)}).
\]
The opposite inclusion comes from the fact that a jet in \( V(I_{i,2s-1}^{s,s}) \cap D(x_i^{(s)}x_{i+1}^{(s)}) \subset \mathbb{A}_{2s-1}^s \) satisfies all the equations of \( S \) modulo \( t^{2s} \). Since \( V(I_{i,2s-1}^{s,s}) \subset \mathbb{A}_{2s-1}^s \) is irreducible, the lemma follows.

Lemma 4.2. For \( i = 2, \cdots, e - 1, \ m \in \mathbb{N}, \ s \in \{1, \ldots, [m/2]\} \) and \( l \in \{s, \ldots, m_i\} \), we have that
\[
C_{i,m}^{s,l} \subset \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}).
\]

Proof: For \( \gamma \in D_{i,m}^{s,l} \), we have that
\[
E_{i-1,i+1} \circ \gamma = (x_{i-1} \circ \gamma)(x_{i+1} \circ \gamma) - (x_i \circ \gamma)^{c_i} \in (t)^{m+1}.
\]
If \( c_is \geq m+1 \), then \( \text{ord}_{s,x_{i+1}} \geq m+1-l \geq s \), and if \( c_is < m+1 \) then \( \text{ord}_{s,x_{i-1}} = c_is-l \geq s \). Moreover, since for \( i < j-1 \leq e-1 \) (resp. \( 1 \leq j < i - 1 \)), we have
\[
E_{ij} \circ \gamma = (x_i \circ \gamma)(x_j \circ \gamma) - (x_{i+1} \circ \gamma)(x_{i+1}^{(c_{i+1}-2)} \circ \gamma)(x_{i+1} \circ \gamma)^{(c_{i+1}-2)} \circ \gamma)(x_{i-1} \circ \gamma)^{(c_{i-1}-2)} \circ \gamma)(x_{i-1} \circ \gamma) \in (t)^{m+1},
\]
(resp. \( E_{ji} \circ \gamma = (x_j \circ \gamma)(x_i \circ \gamma) - (x_{j+1} \circ \gamma)(x_{j+1}^{(c_{j+1}-2)} \circ \gamma)(x_{j+1} \circ \gamma)^{(c_{j+1}-2)} \circ \gamma)(x_{j-1} \circ \gamma)^{(c_{j-1}-2)} \circ \gamma)(x_{j-1} \circ \gamma) \in (t)^{m+1}, \)
\[
\text{ord}_{s,x_i} = s, \text{ord}_{s,x_{i+1}} \geq s (\text{resp. ord}_{s,x_{i-1}} \geq s),
\]
\[
c_i+1, (\text{resp. } c_i-1) \geq 2 \text{ and } m + 1 \geq 2s,
\]
we get by ascending (resp. descending) induction on \( j \) that \( \text{ord}_{s,x_j} \geq s \), and therefore \( D_{i,m}^{s,l} \subset \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}) \). The lemma follows since \( \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}) \) is closed.

Lemma 4.3. For \( i = 2, \ldots, e - 1, \ m \in \mathbb{N}, \ s \in \{1, \ldots, [m/2]\} \),
\[
\pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s} \cap D(x_i^{(s)})) = \{ \gamma \in \mathbb{A}_m^e : \text{ord}_{s,x_j} \geq s, \ j = 1, \cdots, e, \ \text{ord}_{s,x_i} = s,
\]
\[
\text{ord}_{s,x_{i-1,i+1}} \geq m + 1, \ \text{ord}_{s,x_{ij}} (\text{resp. ord}_{s,x_{ji}}) \geq m + 1, \text{ for } 1 \leq j < i - 1 \}
\]
(resp. \( i < j - 1 \leq e - 1 \)).
Proof: The inclusion \( \mathcal{S} \subset \mathcal{S} \) is an immediate consequence of lemma 4.1. To get the other inclusion, it is enough to check that for every \( \gamma \in \mathbb{A}_m^e \) enjoying the conditions listed above, we also have \( \text{ord}_\gamma E_jh \geq m + 1 \) for \( 1 \leq j < h - 1 \leq e - 1 \).

If \( i < j \), the syzygy

\[
x_iE_{jh} - x_jE_{ih} + x_{j+1}^{c_{j+1}-2} \cdots x_h^{c_h-2} x_{h-1}E_{i,j+1} = 0
\]

implies that \( \text{ord}_\gamma E_{jh} \geq m + 1 \), because \( \text{ord}_\gamma x_j \) and \( \text{ord}_\gamma x_{h-1} \geq s \) and \( \text{ord}_\gamma x_i = s \). Similarly if \( h < i \), the syzygy

\[
x_iE_{jh} - x_hE_{ji} + x_{j+1}^{c_{j+1}-2} \cdots x_{h-1}^{c_{h-1}-2} E_{h-1,i} = 0
\]

implies that \( \text{ord}_\gamma E_{jh} \geq m + 1 \), because \( \text{ord}_\gamma x_h \) and \( \text{ord}_\gamma x_{j+1} \geq s \) and \( \text{ord}_\gamma x_i = s \).

Assume now that \( 1 \leq j < i - 1 \) and \( h = i + 1 \); the syzygy

\[
x_{i+1}E_{ji} - x_iE_{j,i+1} + x_{j+1}^{c_{j+1}-2} \cdots x_{i-1}^{c_{i-1}-2} E_{i-1,i+1} = 0
\]

implies that \( \text{ord}_\gamma E_{j,i+1} \geq m + 1 \).

Similarly if \( j = i - 1 \) and \( i + 1 < h \leq e \), the syzygy

\[
x_{i-1}E_{ih} - x_iE_{i-1,h} + x_{i+1}^{c_{i+1}-2} \cdots x_h^{c_h-2} x_{h-1}E_{i-1,i+1} = 0
\]

implies that \( \text{ord}_\gamma E_{i-1,h} \geq m + 1 \).

Finally, if \( 1 \leq j < i - 1 \) and \( i + 1 < h \leq e \), the syzygy

\[
x_jE_{ih} - x_jE_{jh} + x_{i+1}^{c_{i+1}-2} \cdots x_{h-1}^{c_{h-1}-2} x_{h-1}E_{j,i+1} = 0
\]

implies that \( \text{ord}_\gamma E_{j,h} \geq m + 1 \), taking into account that we have shown above that \( \text{ord}_\gamma E_{j,i+1} \geq m + 1 \).

\[\square\]

Remark 4.4. Note that the syzygies (4.1), . . . , (4.5) are syzygies in the ring of polynomials and not in the ring of regular functions on \( S \). This is essential for the conclusion in the above lemma.

Proposition 4.5. For \( i = 2, \ldots , e - 1, \ m \in \mathbb{N}, \ s \in \{1, \ldots , \lfloor \frac{m}{2} \rfloor \} \) and \( l \in \{s, \ldots , m_s^i\} \), \( C_{i,m}^{s,l} \) is irreducible, and its codimension in \( \mathbb{A}_m^e \) is equal to

\[se + (m - (2s - 1))(e - 2).\]

Proof: First, since the ideal defining \( S \) in \( \mathbb{A}_m^e \) is generated by \( E_{jh}, 1 \leq j < h - 1 \leq e - 1 \), we have that

\[D_{i,m}^{s,l} \subset U_{i,m}^{s,l} := \{\gamma \in \mathbb{A}_m^e; \ \text{ord}_\gamma E_{ij}(\text{resp.} \text{ord}_\gamma E_{ji}) \geq m + 1 \text{ for } i < j - 1 \leq e - 1\]
For $\gamma \in U_{i,m}^l$, we have by the proof of 4.2 that for $j = 1, \ldots, e$, $ord_\gamma x_j \geq s$. It follows from lemma 4.3 that $D_{i,m}^s = U_{i,m}^s$.

The irreducibility of $C_{i,m}^s$ follows from the fact that $D_{i,m}^s = U_{i,m}^s$ is isomorphic to the product of a two dimensional torus by an affine space. Indeed, set $x_j \circ \gamma = \sum_{0 \leq \nu \leq m} x_j^{(\nu)} t^\nu$, $1 \leq j \leq e$. If $ord_\gamma x_i = s$ and $ord_\gamma x_{i+1} = l$, we have $ord_\gamma E_{i-1,l} \geq m + 1$, if and only if $x^{(\nu)}_{i-1} = 0$ for $0 \leq \nu \leq m - l$ if $c_i s \geq m + 1$ (resp. $x^{(\nu)}_{i-1} = 0$ for $0 \leq \nu < c_i s - l$ and is a polynomial function of $x_i^{(s)}$, $\ldots$, $x_i^{(m-c_i s + s)}$, $1/x_{i+1}^{(l)} x_{i+1}^{(l)}$ for $c_i s - l \leq \nu \leq m - l$ if $c_i s < m + 1$). Similarly, $ord_\gamma E_{ij}$ (resp. $ord_\gamma E_{ji}$) $\geq m + 1$ for $i + 1 < j \leq e$ (resp. $1 \leq j < i - 1$) if and only if $x_j^{(\nu)} = 0$ for $0 \leq \nu < s$ and is a polynomial function of $1/x_i^{(s)} x_i^{(s)}$, $\ldots$, $x_i^{(m-s)}$, $x_i^{(l)} x_{i+1}^{(l)}$, $\ldots$, $x_i^{(m-l)}$ for $s \leq \nu \leq m - s$ (resp. $x_j^{(\nu)} = 0$ for $0 \leq \nu < s$ and is a polynomial function of $1/x_i^{(s)} x_i^{(s)}$, $\ldots$, $x_i^{(m-s)}$, $x_{i-1}^{(s)} x_{i-1}^{(s)}$, $\ldots$, $x_{i-1}^{(m-s)}$ for $s \leq \nu \leq m - s$ since $ord_\gamma x_{i-1} \geq s$ as soon as $ord_\gamma E_{i-1,l} \geq m + 1$). As a consequence, the codimension of $D_{i,m}^s$, hence of its closure $C_{i,m}^s$, in $k^e$, is

$$m + s + 1 + (e - i - 1)(m - s + 1) + (i - 2)(m - s + 1) = (e - 2)(m + 1) - (e - 4)s = se + (m - (2s - 1))(e - 2).$$

The next propositions are preparatory for the proof of proposition 4.10, which states that $S_m^0 : = \pi_m^{-1}(O)$ is the union of the $C_{i,m}^s$.

**Proposition 4.6.**

1. For $i = 2, \ldots, e - 1$ and $s \in \mathbb{N}$ such that $1 \leq s \leq l \leq (c_i - 1)s$, we have $\text{Cont}^\sigma(x_i) \cap \text{Cont}^\sigma(x_{i+1}) \neq \emptyset$, hence for $s \in \{1, \ldots, [\frac{m}{2}]\}$ and $l \in \{s, \ldots, m^2\}$, we have $\Psi_m^{-1}(D_{i,m}^s) \neq \emptyset$.

2. For $s \in \mathbb{N}$, $s \geq 1$, $\text{Cont}^\sigma(x_1) \cap \text{Cont}^\sigma(x_2) \neq \emptyset$ and is included in

$$\Psi_{2s-1}^{-1}(C_{i,2s-1}^s \cap D(x_1^{(s)}) \cap D(x_2^{(s)})).$$

**Proof:** (1) We will prove that there exists an arc $h$ on $S$, whose generic point lies in the torus, and such that $h \in Cont^\sigma(x_i) \cap Cont^\sigma(x_{i+1})$. Note that the data of an arc $h$ on $S$ meeting the torus is equivalent to the data of a vector $v_h = (a, b) \in \sigma \cap N$; moreover for any $u \in M \cap \sigma^\vee$, we have that $h \in Cont^{\sigma, u}(x^u)$, where we denote by $v_h, u$ the scalar product of $v_h$ and $u$, and by $x^u$ the regular function defined by $u$ on $S([LR], \text{proposition 3.3})$. Let $u_i, i = 1, \ldots, e$, be the system of minimal generators of $\sigma^\vee \cap M$, defined in 2.2 such that $x^{u_i} = x_i$. Therefore to prove that there exists an arc $h$ as above, it is sufficient to prove that there exists $(a, b) \in \sigma \cap N$ such that $(a, b), u_i = s$ and $(a, b), u_{i+1} = l$. Since $u_i$ and $u_{i+1}$ determine a $\mathbb{Z}$–basis of $M$, there exists a unique $(a, b) \in N$ such that $(a, b), u_i = s$ and $(a, b), u_{i+1} = l$. Let’s prove that $(a, b)$ lies in the interior of $\sigma$, i.e. that for $j = 1, \ldots, e$, $(a, b), u_j > 0$. Since $u_{i-1} = c_i u_i - u_{i+1}$, we have that
(a, b).u_{i-1} = c_i s - l which is greater than or equal to s because by hypothesis, we have 
\( s \leq l \leq s(c_i - 1) \). Similarly we have that 
\((a, b).u_{i+2} = c_{i+1} l - s \) which is greater than or equal to l. Since 
\( c_i \geq 2 \), for \( i = 1, \ldots, e \), by descending (respectively ascending) induction we find that 
\((a, b).u_{j-1} \geq (a, b).u_j \), for \( j = 2, \ldots, i \) (respectively \( (a, b).u_{j-1} \leq (a, b).u_j \), for \( j = i + 2, \ldots, e \)) and the proposition follows.

(2) We have that \( u_1 = (0, 1), u_2 = (1, 0) \). We need to prove that the unique vector \( v = (a, b) \in N \) such that \((a, b).(0, 1) = b = s \) and \((a, b).(1, 0) = a = s \), also belongs to \( \sigma \); in fact it is clear that \( (s, s) \) belongs to the interior of \( \sigma \). We also need to prove that for \( j = 3, \ldots, e \), we have that \( (s, s).u_j \geq s \) since \( u_j \in \sigma^V \) and \((1, 1) \) lies in the interior of \( \sigma \), we have that \((1, 1).u_j > 0 \), moreover \( u_j \in M \) and \((1, 1) \) \in \( N \), so \((1, 1).u_j \in \mathbb{Z} \) and \((1, 1).u_j \geq 1 \).

\[ \square \]

**Lemma 4.7.** For \( i = 2, \ldots, e - 1 \), let \( X^i = \text{Spec} \mathbb{K}[x_{i-1}, x_i, x_{i+1}]/(x_{i-1}x_{i+1} - x_i^c) \). For \( s \in \{1, \ldots, \lceil \frac{m}{2} \rceil \} \), let
\[ V_{s,m}^i := \{ \gamma \in X^i_m : \text{ord}_\gamma(x_j) \geq s, \ j = i - 1, i + 1, \ \text{ord}_\gamma(x_i) = s \} \]
and for \( l \in \{s, \ldots, m_i^s\} \), let
\[ \Delta_{s,m}^{i,l} := \{ \gamma \in X^i_m : \text{ord}_\gamma(x_i) = s, \ \text{ord}_\gamma(x_{i+1}) = l \}. \]
Then, the irreducible components of \( \overline{V_{s,m}^i} \) are the \( \Delta_{s,m}^{i,l} \), \( l \in \{s, \ldots, m_i^s\} \).

**Proof:** First, assume that \( m + 1 \leq c_i s \), so that \( m_i^s = m + 1 - s \). We have that
\[ V_{s,m}^i = \{ \gamma \in K^3_m : \text{ord}_\gamma(x_j) \geq s, j = i - 1, i + 1, \text{ord}_\gamma(x_i) = s \}
\]
and for \( l \in \{s, \ldots, m + 1 - s\} \),
\[ \Delta_{s,m}^{i,l} = \{ \gamma \in K^3_m : \text{ord}_\gamma(x_i) = s, \text{ord}_\gamma(x_{i+1}) = l, \text{ord}_\gamma(x_{i-1}) \geq m + 1 - l \} = \]
\[ V(x_{i-1}^{(0)}, x_i^{(m-l)}, x_{i+1}^{(0)}, \ldots, x_i^{(s-1)}, x_{i+1}^{(0)}, \ldots, x_i^{(l-1)}) \cap D(x_i^{(s)}, x_i^{(l)}) \].

Since \( s \leq l \leq m + 1 - s \), we have that \( \Delta_{s,m}^{i,l} \subset V_{s,m}^i \), so \( \cup_{s \leq l \leq m+1-s} \Delta_{s,m}^{i,l} \subset \overline{V_{s,m}^i} \). Now for \( \gamma \in V_{s,m}^i \), we have that \( \text{ord}_\gamma(x_i) = s, l := \text{ord}_\gamma(x_{i+1}) \geq s \) and \( \text{ord}_\gamma(x_{i-1}) \geq m + 1 - l \). If \( l \leq m + 1 - s \), we thus have that \( \gamma \in \Delta_{s,m}^{i,m} \); if \( l > m + 1 - s \), we have that \( \text{ord}_\gamma(x_{i-1}) \geq s \), hence \( \gamma \in \Delta_{s,m}^{i,l} \subset V_{s,m}^i \), and \( \cup_{s \leq l \leq c_i-1-s} \Delta_{s,m}^{i,l} \subset \overline{V_{s,m}^i} \). Hence the claim. Now assume that \( c_i s < m + 1 \), so that \( m_i^s = (c_i - 1)s \). For \( l \in \{s, \ldots, (c_i - 1)s\} \) and \( \gamma \in \Delta_{s,m}^{i,l} \), we thus have that \( \text{ord}_\gamma(x_i) = s, \text{ord}_\gamma(x_{i+1}) = l \geq s \), and \( \text{ord}_\gamma(x_{i-1}) \geq c_i s \), hence \( \text{ord}_\gamma(x_{i-1}) = c_i s - l \geq s \), therefore \( \Delta_{s,m}^{i,l} \subset V_{s,m}^i \), and \( \cup_{s \leq l \leq (c_i - 1)s} \Delta_{s,m}^{i,l} \subset \overline{V_{s,m}^i} \).

On the other hand \( V_{s,m}^i = (\pi_{m,c_i,s-1}^i)^{-1}(\overline{V_{s,m}^i} \cap \overline{V_{s,m}^i}) \) where \( \pi_{m,c_i,s-1}^i : X_m \rightarrow X_{c_i,s-1} \) is the
natural map and for \( s \leq l \leq (c_i - 1)s \) we have that 
\[ \Delta_{i,m}^{s,l} = (\pi_{m,c_i,s-1})^{-1}(\Delta_{i,c_i,s}^{s,l}). \]
Now we have just seen that 
\[ \overline{V}_{i,c_i,s-1}^{s,l} = \bigcup_{s \leq l \leq (c_i - 1)s} \Delta_{i,m}^{s,l} \] 
and that 
\[ \Delta_{i,c_i,s}^{s,l} = V(x_i(0), \ldots, x_i^{(c_i - 1)s - 1}, x_i^{(0)} \ldots, x_i^{(s-1)} \ldots, x_i^{(l-1)}). \]

As a consequence \((\pi_{m,c_i,s-1})^{-1}(\Delta_{i,m}^{s,l})\) is isomorphic to the product of an affine space by the space of \((m - c_i)s\)-jets of the surface \( \text{Spec} \mathbb{K}[x_i^{(c_i - 1)s - 1}, x_i^{(s)} \ldots, x_i^{(l-1)}] \), and this latter is irreducible by section 3, hence coincides with \( \Delta_{i,m}^{s,l} \). So \( \overline{V}_{i,m} \subset \bigcup_{s \leq l \leq (c_i - 1)s} \Delta_{i,m}^{s,l} \), hence the claim.

\[ \square \]

**Proposition 4.8.** Let \( m, s \in \mathbb{N} \) such that \( s \in \{1, \ldots, \lceil \frac{m}{2} \rceil \} \).

1. For \( i = 2, \ldots, e - 1 \), the irreducible components of \( \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}(x_i^s)) \) are the \( C_{i,m}^{s,l}, l \in \{s, \ldots, m_i^s\} \).

2. For \( i = 1, e \), we have that \( \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}(x_i^s)) \) is irreducible of codimension 
\[ se + (m - (2s - 1))(e - 2) \]
in \( \mathcal{A}_m^e \).

**Proof:** (1) By the lemmas 4.2 and 4.3, we have that 
\[ D_{i,m}^{s,l} \subset \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}(x_i^s)) = \{ \gamma \in \mathcal{A}_m^e : \text{ord}_x x_j \geq s, j = 1, \ldots, e, \text{ord}_x x_i = s, \text{ord}_x E_{i-1,i+1} \geq m + 1, \text{ord}_x E_{j,i}(\text{resp. \text{ord}_x E_{i,j}) \geq m + 1, \text{for} 1 \leq j < i - 1 (\text{resp.} i < j - 1 \leq e - 1) \}. \]

The projection \( \mathcal{A}_m^e \rightarrow \mathcal{A}_3 \) which sends \( (x_1, \ldots, x_e) \) to \( (x_{i-1}, x_i, x_{i+1}) \) induces a natural map \( p^i : S \rightarrow X^i \) and the induced map \( p^i_m : S_m \rightarrow X_m \) sends \( \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}(x_i^s)) \) (resp. \( D_{i,m}^{s,l} \)) into \( V_i^{s,l} := \{ \gamma \in X_m^i, \text{ord}_x(x_j) \geq s, j = i - 1, i + 1, \text{ord}_x(x_i) = s \} \) (resp. \( \Delta_{i,m}^{s,l} := \{ \gamma \in X_m^i, \text{ord}_x(x_i) = s, \text{ord}_x(x_{i+1}) = l \} \). Now in view of lemma 4.3, the maps 
\[ \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}(x_i^s)) \rightarrow V_i^{s,l} \text{ and } D_{i,m}^{s,l} \rightarrow \Delta_{i,m}^{s,l} \]
are isomorphic to a trivial fibration of rank \( s(e - 3) \). By lemma 4.7, the irreducible components of \( V_i^{s,l} \) are the \( \Delta_{i,m}^{s,l}, l \in \{s, \ldots, m_i^s\} \). Since \( V_i^{s,l} = \overline{V}_{i,m}^{s,l} \cap D(x_i^s) \), we thus have 
\[ V_i^{s,l} = \bigcup_l (\Delta_{i,m}^{s,l} \cap D(x_i^s)) \text{; so } \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s}(x_i^s)) \simeq \bigcup_l \Omega_{i,m}^{s,l} \] 
where 
\[ \Omega_{i,m}^{s,l} = (\Delta_{i,m}^{s,l} \cap D(x_i^s)) \times \mathcal{A}_m^e. \] As a consequence \( \Omega_{i,m}^{s,l} \) is irreducible and we have that \( D_{i,m}^{s,l} \subset \Omega_{i,m}^{s,l} \). Moreover 
\[ \text{Codim}(\Omega_{i,m}^{s,l}, \mathcal{A}_m^e) = (e - 3)(m + 1) + (m + s + 1) - s(e - 3) = \]
(m + 1)(e - 2) - s(e - 4) = \text{Codim}(C_{s,m}^{i,l}, \mathcal{K}_m^e),

hence \( C_{i,m}^{s,l} = \Omega_{i,m}^{s,l} \) and the claim follows since \( C_{i,m}^{s,l} \neq C_{i,m}^{s,l'} \) for \( l \neq l' \).

(2) Assume \( i = 1 \), the case \( i = e \) follows in the same way. We first check that

\[
\pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s} \cap D(x_1^{(s)})) = \\
\{ \gamma \in \mathbb{A}^{s,m}_m, \text{ord}_\gamma(x_j) \geq s, j = 1, \ldots, e, \text{ord}_\gamma(x_1) = s, \\
\text{ord}_\gamma(E_{ij}) \geq m + 1 \text{ for } 3 \leq j \leq e \}.
\]

The inclusion \( " \subset " \) is clear. To get the opposite inclusion we have to prove that the conditions just listed imply that \( \text{ord}_\gamma(E_{jh}) \geq m + 1 \) for \( 2 \leq j < h - 1 \leq e - 1 \). This is an immediate consequence of the syzygy

\[ x_1E_{jh} - x_jE_{1h} + x_{j+1}^{c_j+1-2} \cdots x_{h-1}^{c_h-1-2} x_{h-1}E_{1,j+1} = 0. \]

Therefore, \( \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s} \cap D(x_1^{(s)})) \) is isomorphic to the product of \( \mathbb{K}^* \) by an affine space of dimension \((m - s) + (m - s + 1) + s(e - 2)\) and its Zariski closure is irreducible of codimension \((m + 1)(e - 2) - s(e - 4)\) in \( \mathbb{A}^{s,m}_m \).

\( \square \)

**Lemma 4.9.** For \( i = 2, \ldots, e - 2 \), we have that

\[
C_{i,m}^{s,s} = C_{i+1,m}^{s,\ell_{i+1}}.
\]

**Proof:** If \( m + 1 \leq c_{i+1}s \), by definition \( m_{i+1}^s = m + 1 - s \), and in view of lemma 4.1 and lemma 4.2, we have that \( D_{i,m}^{s,s} \subset \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s} \cap D(x_1^{(s)})) \). Now by proposition 4.3.2, the irreducible components of \( \pi_{m,2s-1}^{-1}(C_{i,2s-1}^{s,s} \cap D(x_1^{(s)})) \) are the \( C_{i,m}^{s,l} \) for \( l \in \{s, \ldots, m_{i+1}^s\} \). Since \( C_{i,m}^{s,m_{i+1}^s} = D_{i,m}^{s,s} \) is irreducible, and its codimension in \( \mathbb{A}^{s,m}_m \) coincides with the codimension of any of the \( C_{i,m}^{s,l} \), there exists \( l \) such that \( C_{i,m}^{s,l} = C_{i+1,m}^{s,\ell_{i+1}} \), with \( s \leq l \leq m + 1 - s \). So \( D_{i,m}^{s,s} \) and \( D_{i+1,m}^{s,l} \) are dense open subsets of \( C_{i,m}^{s,s} \) and there exists \( \gamma \in D_{i,m}^{s,s} \cap D_{i+1,m}^{s,l} \). We thus have \( \text{ord}_\gamma x_i = \text{ord}_\gamma x_{i+1} = s \), and \( \text{ord}_\gamma x_{i+2} = l \).

But \( E_{i,i+2} = x_i x_{i+2} - x_{i+1}^{c_{i+1}} \) and \( \text{ord}_\gamma E_{i,i+2} \geq m + 1 \). Since \( m + 1 \leq c_{i+1}s \), this implies \( \text{ord}_\gamma x_{i+2} = l \geq m + 1 - s \), so \( l = m + 1 - s \), i.e. \( C_{i,m}^{s,s} = C_{i+1,m}^{s,m_{i+1}^s} \).

Assume now that \( m + 1 > c_{i+1}s \); for any \( \gamma \in D_{i,m}^{s,s} \), we have that \( \text{ord}_\gamma x_i = \text{ord}_\gamma x_{i+1} = s \) and \( \text{ord}_\gamma E_{i,i+2} \geq m + 1 \), hence \( \text{ord}_\gamma x_{i+2} = (c_{i+1} - 1)s = m_{i+1}^s \), which implies that \( D_{i,m}^{s,s} \subset D_{i+1,m}^{s,m_{i+1}^s} \). Since both are irreducible and have the same dimension, we deduce by passing to the closure that \( C_{i,m}^{s,s} = C_{i+1,m}^{s,m_{i+1}^s} \).

\( \square \)

Let \( S_m^0 := \pi_{m-1}^{-1}(O) \), where \( O \) is the singular point of \( S \). Note that \( \pi_{m-1}^{-1}(S - \{0\}) \) is an irreducible component of \( S_m^0 \) of codimension \((m + 1)(e - 2)\) in \( \mathbb{A}^{s,m}_m \); we will see that the irreducible components of \( S_m^0 \) have codimension less than or equal to \((m + 1)(e - 2)\), therefore they are irreducible components of \( S_m^0 \).
Proposition 4.10.

\[ S^0_m = \bigcup_{i \in \{2, e\}, s \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor \}, j \in \{s, \ldots, m\}} C^{s, l}_{i, m}. \]

Proof: We first look at the case \( m = 2n + 1, \; n \geq 0 \). We claim that

\[ S^0_{2n+1} = \bigcup_{i \in \{1, \ldots, e\}, s \in \{1, \ldots, n\}} \pi^{-1}_{2n+1, 2s-1}(C^{s, s}_{i, 2s-1} \cap D(x^{(s)}_{i}) ) \cup C^{n+1, n+1}_{i, 2n+1}. \] (\( \diamond \))

The proof of the claim is by induction on \( n \). By lemma 4.1, we have that \( S^0_i = C^{i, i}_{i, i} \) for any \( i = 1, \ldots, e \), hence the case \( n = 0 \). Using the inductive hypothesis for \( n - 1 \), and the fact that for \( s \in \{1, \ldots, n - 1\} \) we have that \( \pi_{2n-1, 2s-1} \cap \pi_{2n+1, 2n-1} = \pi_{2n-1, 2s-1} \), we obtain:

\[ S^0_{2n+1} = \pi^{-1}_{2n+1, 2n-1}(S^0_{2n-1}) = \bigcup_{i \in \{1, \ldots, e\}, s \in \{1, \ldots, n-1\}} \pi^{-1}_{2n+1, 2s-1}(C^{s, s}_{i, 2s-1} \cap D(x^{(s)}_{i}) ) \cup \pi^{-1}_{2n+1, 2n-1}(C^{n, n}_{i, 2n-1}). \]

The claim follows from the stratification

\[ C^{n, n}_{i, 2n-1} = \bigcup_{j=1, \ldots, e}(C^{n, n}_{i, 2n-1} \cap D(x^{(n)}_{j})) \cup (C^{n, n}_{i, 2n-1} \cap V(x^{(1)}_{i}, \ldots, x^{(e)}_{i})), \]

and from the fact that by lemma 4.1, \( \pi^{-1}_{2n+1, 2n-1}(C^{n, n}_{i, 2n-1} \cap D(x^{(n)}_{i})) = C^{n, n}_{i, 2n+1} \). We then conclude the proof of the proposition for \( m = 2n + 1 \) in two steps: First by using proposition 4.8 (1). Second, by proposition 4.6 we have that \( \pi^{-1}_{2n+1, 2s-1}(C^{s, s}_{i, 2s-1} \cap D(x^{(s)}_{i}) \cap D(x^{(s)}_{j}) \neq \emptyset \), hence \( \pi^{-1}_{2n+1, 2s-1}(C^{s, s}_{i, 2s-1} \cap D(x^{(s)}_{i}) \cap \pi^{-1}_{2n+1, 2s-1}(C^{s, s}_{i, 2s-1} \cap D(x^{(s)}_{j}) \neq \emptyset \); since by 4.8 (2) this latter is irreducible as any irreducible component of the former, its generic point coincides with the generic point of one of the irreducible components of \( \pi^{-1}_{2n+1, 2s-1}(C^{s, s}_{i, 2s-1} \cap D(x^{(s)}_{j}) \)).

The case \( m = 2(n + 1), \; n \geq 0 \): by (\( \diamond \)) we just need to prove that for \( n \geq 0 \), and \( i = 1, \ldots, e \) we have that

\[ \pi^{-1}_{2(n+1), 2n+1}(C^{n+1, n+1}_{i, 2n+1}) = \bigcup_{l=2n+2, (2(n+1))^{n+1}} C^{n+1, l}_{i, 2(n+1)}. \]

First note that by lemma 4.1 and 4.2, we have the inclusion

\[ \pi^{-1}_{2(n+1), 2n+1}(C^{n+1, n+1}_{i, 2n+1}) \supseteq \bigcup_{l=2(n+1), (2(n+1))^{n+1}} C^{n+1, l}_{i, 2(n+1)}. \] (\( \diamond \diamond \))

The proof of the opposite inclusion is by induction on the embedding dimension \( e \) of \( S \). First assume that \( e = 4 \); the equations defining \( S \) in \( A^4 \) are \( E_{13}, E_{14}, E_{24} \). So the ideal defining \( \pi^{-1}_{2(n+1), 2n+1}(C^{n+1, n+1}_{i, 2n+1}) \) in \( A^4_{2(n+1)} \) is generated by

\[ (x^{(0)}_{j}, \ldots, x^{(n)}_{j}); \; j = 1, \ldots, 4; \; E_{13}^{(2n+2)}, E_{14}^{(2n+2)}, E_{24}^{(2n+2)}). \]
hence every irreducible component of \(\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1})\) has codimension in \(\mathbb{A}^4_{2(n+1)}\) less than or equal to \(4(n+1) + 3 = 4n + 7\).

Now we have that
\[
\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1}) = \bigcup_{j=1,\ldots,4} \pi^{-1}_{2(n+1),2n+1}((C_{i,2n+1}^{n+1,n+1} \cap D(x_j^{(n+1)})))
\]
\[\bigcup \pi^{-1}_{2(n+1),2n+1}((C_{i,2n+1}^{n+1,n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)})))
\]
\[= \bigcup_{j=1,\ldots,4} \pi^{-1}_{2(n+1),2n+1}((C_{i,2n+1}^{n+1,n+1} \cap D(x_j^{(n+1)})))
\]
\[\bigcup \pi^{-1}_{2(n+1),2n+1}((C_{i,2n+1}^{n+1,n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)}))).
\]

Moreover by proposition 4.6, we have that
\[
\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_1^{(n+1)})) \cap \pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_2^{(n+1)})) \neq \emptyset.
\]

By proposition 4.8 (2), \(\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_1^{(n+1)}))\) is irreducible, and it coincides with an irreducible component of \(\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_2^{(n+1)}))\).

Similarly \(\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_4^{(n+1)}))\) coincides with an irreducible component of \(\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_3^{(n+1)}))\).

In addition by lemma 4.1 and proposition 4.8. 1), we have that for \(j = 2, 3,\)
\[
\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_j^{(n+1)})) = \bigcup_{l=n+1,\ldots,(2n+1)} C_{j,l,2(n+1)}^{n+1,l}
\]

Hence \(\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1}) = \)
\[
\bigcup_{l=n+1,\ldots,(2n+1)} C_{j,l,2(n+1)}^{n+1,l} \cup \pi^{-1}_{2(n+1),2n+1}((C_{i,2n+1}^{n+1,n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)}))).
\]

Finally we have that \(\pi^{-1}_{2(n+1),2n+1}((C_{i,2n+1}^{n+1,n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)}))) =\)
\[
\{\gamma \in S_{2(n+1)}, \text{ord}_x x_j \geq n + 2, j = 1, \ldots, 4\} = \{\gamma \in \mathbb{A}^4_{2(n+1)}, \text{ord}_x x_j \geq n + 2, j = 1, \ldots, 4\}
\]
\[= V(x_j^{(0)}, \ldots, x_j^{(n+1)}, j = 1, \ldots, 4)
\]
is irreducible of codimension \(4(n+2)\) in \(\mathbb{A}^4_{2(n+1)}\). Since \(4(n+2) > 4n + 7\), it is not an irreducible component of \(\pi^{-1}_{2(n+1),2n+1}(C_{i,2n+1}^{n+1,n+1})\), hence the claim.
We now assume the lemma to be true for toric surfaces $\tilde{S}$ of embedding dimension $\tilde{e}$ with $4 \leq \tilde{e} \leq e - 1$. We have that 

$$\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_e^{(n+1)})) \cup \pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap V(x_e^{(n+1)})).$$

Again by proposition 4.6 and proposition 4.8, $\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_e^{(n+1)}))$ coincides with one of the irreducible components of $\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_e^{(n+1)}))$, namely the $C_{e-1,2(n+1)}^{n+1,n+1}$ for $l \in \{n+1, \ldots, (2(n+1))^{n+1} \}$.

So it remains to determine $\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap V(x_e^{(n+1)}))$. The discussion splits into two cases:

i) There exists $h \in \{3, \ldots, e\}$ such that $c_{h-1} > 2$ and $c_h = \cdots = c_{e-1} = 2$.

By lemma 4.1, we have that $\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap V(x_e^{(n+1)})) = \{\gamma \in S_{2(n+1)}; \text{ord}_\gamma x_j \geq n + 1, 1 \leq j \leq e - 1, \text{ord}_\gamma x_e \geq n + 2\} = \{\gamma \in A^{e}_{2(n+1)}; \text{ord}_\gamma x_j \geq n + 1, 1 \leq j \leq e - 1, \text{ord}_\gamma x_e \geq n + 2, \text{ord}_\gamma E_{jk} \geq 2n + 3, 1 \leq j < k - 1 \leq e - 1\}.$

Now recall that $E_{e-2,e} = x_{e-2} - x_{e-1}^e$. If $h < e$, we have that $c_h = 2$, so for $\gamma \in A^{e}_{2(n+1)}$ such that $\text{ord}_\gamma x_{e-2} \geq n + 1, \text{ord}_\gamma x_e \geq n + 2$ and $\text{ord}_\gamma E_{e-2,e} \geq 2n + 3$, we thus have that $2\text{ord}_\gamma x_{e-1} \geq 2n + 3$ hence $\text{ord}_\gamma x_{e-1} \geq n + 2$. Similarly, if $i \geq h$, for $\gamma \in A^{e}_{2(n+1)}$ such that $\text{ord}_\gamma x_{i-1} \geq n + 1, \text{ord}_\gamma x_{i} \geq n + 2$ and $\text{ord}_\gamma E_{i-1,i} \geq 2n + 3$, we get that $\text{ord}_\gamma x_{i} \geq n + 2$.

By descending induction on $i$, this shows that $\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap V(x_e^{(n+1)})) \subset V(x_{h}^{(n+1)}, \ldots, x_{n+1}^{(n+1)}).$

Note that this inclusion is verified by definition when $h = e$. Moreover, for $\gamma \in A^{e}_{2(n+1)}$ such that $\text{ord}_\gamma x_j \geq n + 1$ (resp. $n + 2$) for $1 \leq j < h$ (resp. $h \leq j \leq e$), we have that $\text{ord}_\gamma E_{jk} \geq 2n + 3$ if $h \leq k \leq e$, indeed we have that $\quad \text{ord}_\gamma x_j x_k \geq n + 1 + n + 2 = 2n + 3$, and $\text{ord}_\gamma x_{j+1} x_{j+1}^{e_j+1} - 2 \ldots x_{k-2}^{e_j+1} - 2 x_{k-1} \geq 3(n + 1)$ (resp. $n + 1 + n + 2$) for $k = h$ (resp. $k > h$). Therefore we have that $\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap V(x_e^{(n+1)})) = \{\gamma \in A^{e}_{2(n+1)}; \text{ord}_\gamma x_j \geq n + 1, 1 \leq j \leq h - 1, \text{ord}_\gamma x_j \geq n + 2, h \leq j \leq e, \text{ord}_\gamma E_{jk} \geq 2n + 3, 1 \leq j < k - 1 \leq h - 2\}.$

If $h \geq 5$, this can be interpreted geometrically as follows: Let $\tilde{S}$ be the toric surface in $A^{h-1} = \text{Spec}[x_i, \ldots, x_{h-1}]$ defined by the ideal generated by $(E_{jk}, 1 \leq j < k - 1 \leq h - 2)$ and for $i = 2, \ldots, h - 2, m \in \mathbb{N}, s \in \{1, \ldots, \left\lceil \frac{m}{2} \right\rceil\}, l \in \{s, \ldots, m_s\}$, let $\tilde{D}_{i,m}^{n,l} = \{\gamma \in \tilde{S}_m; \text{ord}_\gamma x_i = s, \text{ord}_\gamma x_{i+1} = l\}$.
and \( \tilde{C}_{i,m}^{s,l} = \tilde{D}_{i,m}^{s,l} \); finally for \( m > p \), let \( \pi_{m,p} : \tilde{S}_m \longrightarrow \tilde{S}_p \) be the canonical projection. By lemma 4.1 again, we have that
\[
\tilde{\pi}_{2(n+1),2n+1}^{-1}(\tilde{C}_{i,2n+1}^{n+1}) = \{ \gamma \in \mathbb{A}^{h-1}_{2(n+1)}; \ ord_x x_j \geq n + 1, 1 \leq j \leq h - 1, \ord_x E_{jk} \geq 2n + 3, 1 \leq j < k - 1 \leq h - 2 \}.
\]

Therefore we deduce that \( \tilde{\pi}_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1} \cap V(x_{e}^{n+1})) = \tilde{\pi}_{2(n+1),2n+1}^{-1}(\tilde{C}_{i,2n+1}^{n+1}) \times \text{Spec} \mathbb{K}[x_{j}^{(n+2)}, \ldots, x_{j}^{(2(n+1))}, j = h, \ldots, e] \)
which by the inductive hypothesis is equal to
\[
\bigcup_{i=2, \ldots, h-2; \ l=n+1, \ldots, (2(n+1))^{n+1}} C_{i,2(n+1)}^{n+1,l} \times \text{Spec} \mathbb{K}[x_{j}^{(n+2)}, \ldots, x_{j}^{(2(n+1))}, j = h, \ldots, e].
\]

Now we claim that
\[
\bigcup_{i=2, \ldots, h-2; \ l=n+1, \ldots, (2(n+1))^{n+1}} C_{i,2(n+1)}^{n+1,l} \subset V(x_{h}^{(n+1)}, \ldots, x_{e}^{(n+1)}).
\]

Indeed, let \( \gamma \in D_{i,2(n+1)}^{n+1,l} \) for some \( i \) and \( l \) in the above union. We have that \( \gamma \in \pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1}) \), so \( \ord_x x_j \geq n + 1 \) for \( 1 \leq j \leq e \), \( \ord_x x_i = n + 1 \) and \( \ord_x E_{ie} \geq 2n + 3 \). Since \( i \leq h - 2 \) and \( c_{h-1} \geq 2 \), this implies that
\[
\ord_x x_{i+1} x_{e+1}^{-2} \ldots x_{e-1}^{-2} x_{e-1} - 2 \geq 2n + 3,
\]
therefore \( \ord_x x_i x_e \geq 2n + 3 \), thus \( \ord_x x_e \geq n + 2 \), and since we have proved that
\[
\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1} \cap V(x_{e}^{n+1})) \subset V(x_{h}^{(n+1)}, \ldots, x_{e}^{(n+1)}),
\]
we deduce that \( C_{i,2(n+1)}^{n+1,l} = D_{i,2(n+1)}^{n+1,l} \subset V(x_{h}^{(n+1)}, \ldots, x_{e}^{(n+1)}). \)

Finally by proposition 4.5, \( C_{i,2(n+1)}^{n+1,l} \) (resp. \( C_{i,2(n+1)}^{n+1,l} \)) is irreducible of codimension \( (n+1)e + e - 2(\text{resp. } (n + 1)(h - 1) + h - 3) \) in \( \mathbb{A}^{e}_{2(n+1)} \) (resp. \( \mathbb{A}^{h-1}_{2(n+1)} \)), therefore
\[
\dim C_{i,2(n+1)}^{n+1,l} = \dim \tilde{C}_{i,2(n+1)}^{n+1,l'} \times \text{Spec} \mathbb{K}[x_{j}^{(n+2)}, \ldots, x_{j}^{(2(n+1))}, j = h, \ldots, e]
\]
for any \( i' \in \{ 2, \ldots, h-2 \}, l' \in \{ n+1, \ldots, (2(n+1))^{n+1} \} \), and we deduce from the inclusion (\( \infty \)) that \( C_{i,2(n+1)}^{n+1,l} \) coincides with \( \tilde{C}_{i',2(n+1)}^{n+1,l'} \times \text{Spec} \mathbb{K}[x_{j}^{(n+2)}, \ldots, x_{j}^{(2(n+1))}, j = h, \ldots, e] \) for some \( i' \in \{ 2, \ldots, h-2 \}, l' \in \{ n+1, \ldots, (2(n+1))^{n+1} \} \).

But we have that \( \ord_x x_i = n + 1 \), \( \ord_x (x_{i+1}) = l \) for the generic point of \( C_{i,2(n+1)}^{n+1,l} \); therefore since \( i + 1 \leq h - 1 \), we have that \( \ord_x x_i = n + 1 \) and \( \ord_x x_{i+1} = l \) for \( \gamma \) the generic point of \( \tilde{C}_{i,2(n+1)}^{n+1,l'} \). Therefore \( \tilde{C}_{i,2(n+1)}^{n+1,l} \) and \( \tilde{C}_{i,2(n+1)}^{n+1,l'} \subset \tilde{C}_{i,2(n+1)}^{n+1,l} \). But
since they are irreducible of the same codimension in $\mathbb{A}^{h-1}_{2n+1}$ they are equal, so we have that
\[ C^{n+1,l}_{i,2(n+1)} = C^{n+1,l}_{i,2(n+1)} \times \text{Spec}\mathbb{k}[x_j^{(n+2)}, \ldots, x_j^{(2n+1)}], j = h, \ldots, e]. \]

We thus have that
\[ \pi_{2(n+1),2n+1}^{-1}(C^{n+1,n+1}_{i,2n+1} \cap V(x_e^{(n+1)})) = \bigcup_{i=2,\ldots,h-2;l=n+1;\ldots,(2(n+1))^{n+1}} C^{n+1,l}_{i,2(n+1)}, \]

and the claim follows. (Note that we get that
\[ \tilde{\pi}_2(2(n+1),2n+1)(C^{n+1,n+1}_{i,2n+1} \cap V(x_e^{(n+1)})) = \bigcup_{i=2,\ldots,h-2;l=n+1;\ldots,(2(n+1))^{n+1}} C^{n+1,l}_{i,2(n+1)}, \]

as an immediate consequence of lemma 4.1 and lemma 4.9.)

If $h = 4$, let $\tilde{S}$ be the toric surface in $\mathbb{A}^3 = \text{Spec}\mathbb{k}[x_1, x_2, x_3]$ defined by the ideal $(E_{1,3})$ and let $\tilde{C}^{n+1}_{2(n+1)} = \{ \gamma \in \tilde{S}_{2(n+1)}; \text{ord}_\gamma x_j \geq n + 1, j = 1, 2, 3 \}$. The equality $(\circ \circ \circ)$ reduces to
\[ \pi_{2(n+1),2n+1}^{-1}(C^{n+1,n+1}_{i,2n+1} \cap V(x_e^{(n+1)})) = C^{n+1,n+1}_{i,2(n+1)} \times \text{Spec}\mathbb{k}[x_j^{(n+2)}, \ldots, x_j^{(2n+1)}], j = 4, \ldots, e]. \]

Since $E_{13} = x_1x_3 - x_2^2$, if $c_2 > 2$, $\tilde{C}^{n+1}_{2(n+1)} \subset \text{Spec}\mathbb{k}[x_j^{(n+1)}, \ldots, x_j^{(2n+1)}], j = 1, \ldots, 3$ is defined by the ideal $(x_1^{(n+1)}, x_3^{(n+1)})$, so $\tilde{C}^{n+1}_{2(n+1)} = V(x_1^{(n+1)}) \cup V(x_3^{(n+1)})$ while it is irreducible if $c_2 = 2$.

We check as above that
\[ \bigcup_{l=n+1;\ldots,(2(n+1))^{n+1}} C^{n+1,l}_{2,2(n+1)} \subset V(x_4^{(n+1)}, \ldots, x_e^{(n+1)}) \]

and that dim $C^{n+1,l}_{2,2(n+1)}$ coincides with the dimension of any irreducible components of $\tilde{C}^{n+1}_{2(n+1)} \times \text{Spec}\mathbb{k}[x_j^{(n+2)}, \ldots, x_j^{(2n+1)}], j = 4, \ldots, e].$ Again in view of $(\circ \circ)$, each $C^{n+1,l}_{2,2(n+1)}$ is an irreducible component of $\tilde{C}^{n+1}_{2(n+1)} \times \text{Spec}\mathbb{k}[x_j^{(n+2)}, \ldots, x_j^{(2n+1)}], j = 4, \ldots, e].$

If $c_2 = 2$, then $(2(n+1))^{n+1} = n + 1$ and we thus have
\[ \pi_{2(n+1),2n+1}^{-1}(C^{n+1,n+1}_{i,2n+1} \cap V(x_e^{(n+1)})) = C^{n+1,n+1}_{2,2(n+1)}. \]

If $c_2 > 2$, we have that $(2(n+1))^{n+1} = n + 2$, and the same argument as above shows that
\[ C^{n+1,n+1}_{2,2(n+1)} = V(x_1^{(n+1)}) \times \text{Spec}\mathbb{k}[x_j^{(n+2)}, \ldots, x_j^{(2n+1)}], j = 4, \ldots, e] \]
\[ C^{n+1,n+2}_{2,2(n+1)} = V(x_3^{(n+1)}) \times \text{Spec}\mathbb{k}[x_j^{(n+2)}, \ldots, x_j^{(2n+1)}], j = 4, \ldots, e]. \]

We thus have
\[ \pi_{2(n+1),2n+1}^{-1}(C^{n+1,n+1}_{i,2n+1} \cap V(x_e^{(n+1)})) = \bigcup_{l=n+1;\ldots,(2(n+1))^{n+1}} C^{n+1,l}_{2,2(n+1)} \]
hence the claim.

Finally if \( h = 3 \), by \( (\circ \circ \circ) \) we have that

\[
\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap V(x^{(n+1)})) = \Spec \mathbb{K}[x_j^{(n+1)}, \ldots, x_j^{(2(n+1))}, j = 1, 2] \times \Spec \mathbb{K}[x_j^{(n+2)}, \ldots, x_j^{(2(n+1))}, j = 3, \ldots, e].
\]

Now we have that \( C_{2,2(n+1)}^{n+1,n+2} \subset V(x_3^{(n+1)}, \ldots, x_e^{(n+1)}) \). Indeed, for \( \gamma \in D_{2,2(n+1)}^{n+1,n+2} \), we have that \( \text{ord}_x x_2 = n+1, \text{ord}_x x_3 = n+2, \text{ord}_x x_j \geq n+1, j = 4, \ldots, e \) and \( \text{ord}_x E_{2j} \geq 2n+3 \) for \( j = 4, \ldots, e \). Since \( c_3 = \ldots = c_{e-1} = 2 \), this implies that \( \text{ord}_x x_j \geq n+2 \) for \( j = 4, \ldots, e \), so \( \gamma \in V(x_3^{(n+1)}, \ldots, x_e^{(n+1)}) \). We conclude that \( \pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap V(x^{(n+1)})) = C_{2,2(n+1)}^{n+1,n+2} \) because both sets are irreducible and have the same dimension, and the claim follows in this case.

ii) If \( c_2 = \cdots = c_{e-1} = 2 \) then

\[
\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1}) = V(x_i^{(0)}, \ldots, x_i^{(n)}, i = 1, \ldots, n,
\]

\[
x_i^{(n+1)} x_j^{(n+1)} - x_i^{(n)} x_j^{(n+1)}, 1 \leq i < j - 1 \leq e - 1.
\]

The ideal in the ring of sections of \( \mathbb{A}^e_{2(n+1)} \) generated by \( (x_i^{(n+1)} x_j^{(n+1)} - x_i^{(n)} x_j^{(n+1)}, 1 \leq i < j - 1 \leq e - 1) \), is isomorphic to the ideal defining \( S \subset \mathbb{A}^e \); hence it is prime and \( \pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1}) \) is irreducible. Since by proposition \( 4.6 \) we have that

\[
\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1} \cap D(x_e^{(n+1)}) \cap D(x_e^{(n+1)}) \neq \emptyset,
\]

then it is dense in \( \pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1}) \), and we deduce that

\[
\pi_{2(n+1),2n+1}^{-1}(C_{i,2n+1}^{n+1,n+1}) = C_{e-1,2(n+1)}^{n+1,n+1},
\]

thus the proposition in this case.

\[\Box\]

Remark 4.11. Note that the argument that we use in the proof of proposition 4.10 for \( e = 4 \) does not work in general. The argument works in the case \( e = 4 \) because the number of equations that define \( S \subset \mathbb{A}^e \) (this number is \( {e-1 \choose 2} \)) is less than or equal to \( e \) if and only if \( e \leq 4 \).

Theorem 4.12. Let \( m \in \mathbb{N}, m \geq 1 \). Modulo the identifications \( C_{i,m}^{s,i} = C_{i+1,m}^{s,m+1} \), the irreducible components of \( S_0^m := \pi_m^{-1}(0) \) are the \( C_{i,m}^{s,l}, i = 2, \ldots, e - 1, s \in \{1, \ldots, \lceil \frac{m}{2} \rceil \} \) and \( l \in \{s, \ldots, m^2 \} \). The irreducible components of \( S_m \) are \( \pi_m^{-1}(S \setminus \{0\}) \) and the irreducible components of \( S_0^m \).
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Proof: By proposition 4.10, \( s^{(0)}_m \) is covered by the \( C^{s, l}_{i,m} \). Consider \( C^{s, l}_{i,m} \) with \( l \neq m^*_i \); since \( l < m^*_i \) this implies that \( m > 2s - 1 \) and \( c_i \neq 2 \). For the generic point \( \gamma \) of \( C^{s, l}_{i,m} \), we have that

\[
E_{i-1,i+1} \circ \gamma = (x_{i-1} \circ \gamma)(x_{i+1} \circ \gamma) - (x_i \circ \gamma)^{c_i} \in (t^{m+1}).
\]

Since \( l < m^*_i \), if \( c_i s \geq m + 1 \), then \( ord_x x_{i-1} \geq m + 1 - l > s \), and if \( c_i s < m + 1 \), then \( ord_x x_{i-1} = c_i s - l > s \). Moreover, for \( 1 \leq j < i - 1 \), we have

\[
E_{ji} \circ \gamma = (x_j \circ \gamma)(x_i \circ \gamma) - (x_{j+1} \circ \gamma)(x_{j+1}^{c_{j+1}} \circ \gamma) \cdots (x_{i-1}^{c_{i-1}} \circ \gamma)(x_{i-1} \circ \gamma) \in (t^{m+1}).
\]

Recalling that \( m + 1 > 2s \) and that \( C^{s, l}_{i,m} \subset \pi_{m,2s-1}(C^{s,s}_{i,2s-1}) \), hence \( ord_x x_{j+1} \geq s \), we get that \( ord_x x_j > s \).

This forbids that \( C^{s, l'}_{i',m} \subset C^{s, l}_{i,m} \) or \( C^{s, l}_{i,m} \subset C^{s, l'}_{i',m} \) for any \( i' \in \{2, \ldots, i-1\} \) because by proposition 4.5, they have the same codimension in \( \mathbb{A}^e_m \), hence they should coincide, so \( ord_x x_{i'} = s \). On the other hand \( C^{s, l}_{i,m} \not\subset C^{s, l'}_{i',m} \) if \( s < s' \), because by proposition 4.8 the \( C^{s, l}_{i,m} \) has non-empty intersection with \( D(x_i^{(s)}) \), but \( C^{s, l'}_{i',m} \subset V(x_i^{(s)}) \). Finally, \( C^{s, l'}_{i',m} \not\subset C^{s, l}_{i,m} \), because by proposition 4.5 the codimension of the first one is less than or equal to the codimension of the second one, and the first statement of the theorem follows. The last statement of the theorem follows from the fact that

\[
\text{codim}(C^{s, l}_{i,m}, \mathbb{A}^e_m) \leq \text{codim}(\pi_{m}^{-1}(S)\backslash 0, \mathbb{A}^e_m).
\]

Indeed: By proposition 4.5, we have \( \text{codim}(C^{s, l}_{i,m}, \mathbb{A}^e_m) = se + (m -(2s-1))(e-2) \). By remark 2.1, we have \( \text{codim}(\pi_{m}^{-1}(S)\backslash 0, \mathbb{A}^e_m) = (m + 1)(e-2) \). Finally, for any \( s \geq 1 \), we have \( se + (m -(2s-1))(e-2) \leq (m + 1)(e-2) \) if and only if \( e \geq 4 \).

Remark 4.13. For \( i = 2, \ldots, e-1 \), \( s \geq 1 \) and \( 2s - 1 \leq 2l - 1 \leq m \leq sc_i - 1 \), we have that

\[
C^{s, m+1-l}_{i,m} = \text{Cont}^s(x_i)_m \cap \text{Cont}^l(x_{i-1})_m.
\]

Proof: Recall that the \( c_i \) are defined by \( q/p = [c_2, \ldots, c_{e-1}] \). We consider the toric surface associated to the continued fraction \([c_{e-1}, \ldots, c_2]\) obtained by reversing the order of the \( c_i \) in the continued fraction of \( q/p \) (by section 2.2, this surface is isomorphic to the original one). From lemma 4.5 and lemma 4.2 applied to the new surface we deduce that \( \text{Cont}^s(x_i)_m \cap \text{Cont}^l(x_{i-1})_m \) is irreducible and is included in \( \pi_{m,2s-1}^{-1}(C^{s, s}_{i,2s-1} \cap D(x_i^{(s)}) \). Moreover it has the same dimension of its irreducible components, therefore it coincides with one of them. But if \( \gamma \) is the generic point of \( \pi_{m,2s-1}^{-1}(C^{s, s}_{i,2s-1} \cap D(x_i^{(s)}) \), we have that \( ord_x x_{i-1} = l \), we deduce the lemma form the fact that \( \gamma \) verifies the equation \( E_{i-1,i+1} \) modulo \( l^{m+1} \) and that the orders of the generic points of the other irreducible components of \( \pi_{m,2s-1}^{-1}(C^{s, s}_{i,2s-1} \cap D(x_i^{(s)}) \) along \( x_{i-1} \) is different from \( l \).
**Definition 4.14.** Let \( m \in \mathbb{N} \), \( m \geq 1 \), and let \( C \) be an irreducible component of \( S^0_m \). By Theorem 4.12, there exists \( s \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor \}, \; l \in \{s, \ldots, m^*_s\} \) and \( i \in \{2, \ldots, e - 1\} \) such that \( C = C_{i,m^*_s} \). Note that \( s = \text{ord}_\gamma(M) := \min_{f \in M} \{ \text{ord}_\gamma(f) \} \) where \( M \) is the maximal ideal of the local ring \( O_{S,0} \) and \( \gamma \) the generic point of \( C \). We say that \( C \) has index of speciality \( s \).

For \( a, b \in \mathbb{N} \), \( b \neq 0 \), we denote by \( \lceil \frac{a}{b} \rceil \) the integral part of \( \frac{a}{b} \). For \( c, m \in \mathbb{N} \), let \( q_c = \lfloor \frac{m}{c} \rfloor \). We set

\[
N^*_c(m) := (sc - (2s - 1)), \text{ for } q_c \neq 0 \text{ and } s = 1, \ldots, q_c;
\]

\[
N^*_s(m) := m - (2s - 2), \text{ for } s = q_c + 1, \ldots, \lfloor \frac{m}{2} \rfloor.
\]

Note that for \( c = c_i \), \( i = 2, \ldots, e - 1 \), we have that \( N^*_c(m) = m^*_i - s + 1 \); in particular if \( c_i = 2 \), then \( N^*_c(m) = 1 \).

For \( m \in \mathbb{N} \), \( m \geq 1 \), we denote by \( N(m) \) the number of irreducible components of \( S^0_m \). Then counting the irreducible components in Theorem 4.12 we find

**Corollary 4.15.** If all the \( c_i \) are equal to 2, then \( N(m) = \lfloor \frac{m}{2} \rfloor \). Otherwise, let \( c_1, \ldots, c_{i_h} \) be the elements in \( \{c_2, \ldots, c_{e-1}\} \) different from 2, then we have

\[
N(m) = \sum_{s=1}^{\lfloor \frac{m}{2} \rfloor} (N^*_{c_1}(m) + (N^*_{c_2}(m) - 1) + \ldots + (N^*_{c_{i_h}}(m) - 1)).
\]

Moreover, for \( s \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor \} \), the number of irreducible components of \( S^0_m \) of index of speciality \( s \) is equal to

\[
N^*_s(c_1)(m) + (N^*_s(c_2)(m) - 1) + \ldots + (N^*_s(c_{i_h})(m) - 1).
\]

The function \( m \rightarrow N(m) \) is increasing.

**Corollary 4.16.** Let \( S \) be a toric surface. The number of irreducible components of \( S^0_m \) for \( m \geq 1 \) and \( \dim S^0_1 \) determine the set \( \{c_i, i = 2, \ldots, e - 1\} \).

**Proof:** We have that \( S^0_1 \) is irreducible and \( \dim(S^0_1) = e \), the embedding dimension of \( S \). If \( e = 3 \), then for \( m \) large enough, we have by theorem 3.1 that \( N(m) = c \) is constant, and we deduce that \( S \) has an \( A_c \) singularity at 0. Suppose that \( e \geq 4 \).

For \( m \geq 1 \), let

\[
\tilde{N}(m) = \sum_{s=1}^{\lfloor \frac{m}{2} \rfloor} ((m + 1 - (2s - 1)) + (e - 3)(m + 1 - (2s - 1) - 1)).
\]

For \( i = 2, \ldots, e - 1 \), we have that \( N^*_c(m) \leq m + 1 - (2s - 1) \), so \( N(m) \leq \tilde{N}(m) \) and \( N(1) = \tilde{N}(1) = 1 \). Let

\[
m_1 = \min\{m : N(m) < \tilde{N}(m)\} \quad \text{and} \quad \alpha_1 = \tilde{N}(m_1) - N(m_1),
\]
then there exists $c_{h_1}, \cdots, c_{h_{a_1}} \in \{c_2, \ldots, c_{e-1}\}$ such that $c_{h_1} = \cdots = c_{h_{a_1}} = m_1$.

If $\alpha_1 = e - 2$, then we have found all the $c_i$. If not, then for $j \geq 2$, we recursively define

$$
\tilde{N}_j(m) = \sum_{s=1}^{\left\lfloor \frac{m}{p} \right\rfloor} (N_{s_{h_1}} (m) + (N_{s_{h_2}} (m) - 1) + \cdots + (N_{s_{h_{a_1}}} (m) - 1) + \cdots +
$$

$$(N_{s_{h_{a_1}+\cdots+a_j-1}} (m) - 1)) + (e - 2 - (\alpha_1 + \cdots + \alpha_{j-1}))(m + 1 - (2s - 1) - 1),$$

$$m_j = \min\{m : N(m) < \tilde{N}_j(m)\} \text{ and } \alpha_j = \tilde{N}_j(m_j) - N(m_j).$$

Therefore there exists $h_{a_1+\cdots+a_{j-1}+1}, \ldots, h_{a_1+\cdots+a_{j-1}+a_j} \in \{c_2, \ldots, c_{e-1}\}$ such that

$$c_{h_{a_1+\cdots+a_{j-1}+1}} = \cdots = c_{h_{a_1+\cdots+a_{j-1}+a_j}} = m_j.$$ 

If $\alpha_1 + \cdots + \alpha_{j-1} + \alpha_j = e - 2$, then we have found all the $c_i$ otherwise we repeat the procedure at most $e - 2$ times.

\[ \square \]

**Remark 4.17.** Corollary 4.16 is to compare with a result of Nicaise (Theorem 2, page 398, [Ni]), where he proved that the motivic Igusa Poincaré series of a toric surface is equivalent to the set $\{c_i, t = 2, \ldots, e - 1\}$, and that the order of the $c_i$ in the continued fraction can not be extracted from this series. It is also clear from the formulas given in proposition 4.5 and corollary 4.15, that the number of irreducible components and their dimensions can not be affected by the order of the $c_i$ in the continued fraction. Note that despite that these informations on the jet schemes are closely related to the informations encoded in the motivic Igusa Poincaré series, they are not equivalent in general.

Below we show how we extract all the $c_i$ and their order or equivalently the analytical type of $S$ from their jet schemes.

As in section 3, we obtain a graph $\Gamma$ by representing every irreducible components of $S^0_{m}, m \geq 1$, by a vertex $v_{i,m}$ and by joining the vertices $v_{i,m+1}$ and $v_{i,m}$ if the morphism $\pi_{m+1,m}$ induces a morphism between the corresponding irreducible components. For every $i = 2, \ldots, i - 1$, and every $s \geq 1$, $\Gamma$ contains a subgraph $\Gamma^s_i$ whose vertices are in $1 - 1$ correspondence with the irreducible components $C_{i,m}, m \geq 2s - 1, l \in \{s, \ldots, m^s_i\}$; the graph $\Gamma^s_i$ coincides with the graph associated to an $A_{e-1}$ singularity in section 3. The identifications $C^s_{i,m} = C^s_{i+1,m}$ induce identifications between infinite lines of $\Gamma^s_i$ and $\Gamma^s_{i+1}$. More precisely, $\Gamma^s_2 = \cdots = \Gamma^s_{e-1}$ if $c_i = 2, i = 2, \ldots, e - 1$; otherwise

$$\bigcup_{i=2, \ldots, e-1} \Gamma^s_i = \Gamma^s_{i_1} \cup \cdots \cup \Gamma^s_{i_h},$$

where $c_{i_1}, \ldots, c_{i_h}$ are those $c_i \neq 2$ as in corollary 4.15 and $\Gamma^s_2 = \cdots = \Gamma^s_{i_1-1} \subset \Gamma^s_{i_1}$, $\Gamma^s_{i_1} = \Gamma^s_{i_1} \cap \Gamma^s_{i_1+1}$ for $i < i < i_{j+1}$ and $\Gamma^s_{i_{j+1}} = \cdots = \Gamma^s_{e-1} \subset \Gamma^s_{i_{j}}$.

As a consequence, we can read off the $c_i \neq 2$ with their order in the continued fraction of $q/p$ from $\Gamma$. 

Example 3. We consider the toric surface singularity defined by the cone generated by 
(0,1) and (4,11). We have that $11/4 = [3,4]$. Below we show the subgraph $\Gamma^1 = \Gamma^1_2 \cup \Gamma^1_3$ of 
The graph Gamma of this singularity. First we show the graphs $\Gamma^1_2$ and $\Gamma^1_3$:

And after the identifications explained above we obtain $\Gamma^1$:
To recover $c_2, \ldots, c_{e-1}$ with their order, we will give weights to the vertices in $\Gamma$ in $1 - 1$ correspondence with the irreducible components of $S_0$ and $S_0^1$. First, $S_0^1$ is isomorphic to $A^e$; we will give weight $e$ to the root of $\Gamma$. Next by theorem 4.12, the irreducible components of $S_0^2$ are $C_{i_1, 2}^{1, 2}, C_{i_1, 2}^{1, 1}, \ldots, C_{i_h, 2}^{1, 1}$.

Back to the equations of $S$, we find that $C_{i_1, 2}^{1, 2} \simeq S^{[i_0, i_1]} \times A^e$, $C_{i_1, 2}^{1, 1} \simeq S^{[i_j, i_j + 1]} \times A^e$ for $j = 1, \ldots, h$ where $i_0 = 1$, $i_{h+1} = e$, $S^{[i_j, i_{j+1}]}$ is the toric surface defined by the 2-minors of the matrix

$$
\begin{pmatrix}
 x_{i_1}^{(1)} & \cdots & x_{i_j - 1}^{(1)} \\
 x_{i_j}^{(1)} & \cdots & x_{i_{j+1}}^{(1)} \\
 x_{i_{j+1}} & \cdots & x_{i_j}^{(1)}
\end{pmatrix}
$$

in Spec $\mathbb{K}[x_{i_1}^{(1)}, \ldots, x_{i_{h+1}}^{(1)}]$ and $A^e = \text{Spec } \mathbb{K}[x_1^{(2)}, \ldots, x_e^{(2)}]$.

Note that for $j = 0, \ldots, h$, the embedding dimension of $S^{[i_j, i_{j+1}]}$ is $i_{j+1} - i_j + 1$, in particular $S^{[i_j, i_{j+1}]}$ is isomorphic to $A^2$ if $i_{j+1} - i_j = 1$.

To the vertex corresponding to an irreducible component of $S_0^1$, we will give weight $1$ its embedding dimension.

**Corollary 4.18.** Let $S$ be a toric surface. The weighted graph that we have associated above to the irreducible components of $S_0^{m, 1} \geq 1$ is equivalent to the analytical type of $S$.

**Remark 4.19.** Note that if we reverse the order of the $c_i$, the analytic type of the corresponding toric surface is unchanged.

Using a theorem of Mustata in [Mus2], we obtain as a byproduct the log canonical threshold $\text{lct}(S, A^e)$ of the pair $S \subset A^e$:

**Corollary 4.20.** Let $S$ be a toric surface of embedding dimension $e$. If $e = 3$ (i.e. $S$ is an $A_n$ singularity) then $\text{lct}(S, A^e) = 1$, otherwise

$$\text{lct}(S, A^e) = \frac{e}{2}$$

**Proof:** By [Mus2] we have that

$$\text{lct}(S, A^e) = \min_{m \in \mathbb{N}} \frac{\text{Codim}(S_m, \mathbb{A}^e_m)}{m + 1}.$$ 

The case $e = 3$ follows from section 3, since in this case we have that $S_m$ is irreducible of codimension $m + 1$ in $\mathbb{A}^e_m$. Let us suppose that $e \geq 4$. If $m$ is odd, $m = 2s - 1$, $s \geq 1$ then the component $C_{i_1, 2s-1}^{s, s}$ is of maximal dimension and we have that

$$\frac{\text{Codim}(C_{i_1, 2s-1}^{s, s}, \mathbb{A}^e_{2s-1})}{2s} = \frac{se}{2s} = \frac{e}{2}.$$ 

If $m$ is even, $m = 2s$, $s \geq 1$ then the components $C_{i, 2s}^{s, i}$, $i = 2, \ldots, e - 1$, $l = s, m_s^e$ are of maximal dimension, and since $e \geq 4$ we have that

$$\frac{\text{Codim}(C_{i, 2s}^{s, i}, \mathbb{A}^e_{2s})}{2s + 1} = \frac{se + e - 2}{2s + 1} \geq \frac{e}{2}.$$
and the lemma follows.

Corollary 4.21. For $m \geq \max \{c_i - 1, \ i = 2, \ldots, e - 1\}$, the number of irreducible components of $S^0_m$, with index of speciality $s = 1$, is equal to the number of exceptional divisors that appear on the minimal resolution of $S$.

Proof. Note that by definition of $N_s^c(m)$, for $m \geq c_i - 1$, we have that $N_s^c(m) = c_i - 1$. By applying corollary 4.15, we find that for $m \geq \max \{c_i - 1, \ i = 2, \ldots, e - 2\}$, the number of irreducible components of $S^0_m$, with index of speciality $s = 1$ is equal to

$$\alpha := (c_{i_1} - 1) + (c_{i_2} - 2) + \cdots + (c_{i_h} - 2),$$

where $c_{i_1}, \ldots, c_{i_h}$ are the elements in $\{c_2, \ldots, c_{e-1}\}$ different from 2. This implies that

$$\alpha = (c_{i_1} - 1) + \sum_{j \neq i_1, j \in \{2, \ldots, e-1\}} (c_j - 2) = c_2 + \cdots + c_{e-1} - 2(e - 2) + 1.$$

The corollary follows from proposition 2.2.

Remark 4.22. Corollary 4.21 is to compare with the bijectivity of the Nash map, due to Ishii and Kollar for this type of Singularities, [IK].

References


REFERENCES


[O] T. Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric
varieties, (3) [Results in Mathematics and Related Areas (3)], 15. Springer-Verlag,

[R] O. Riemenschneider, Zweidimensionale Quotientensingularitäten: Gleichungen und

Amer. Math. Soc. 137 (2009), no. 12, 3953-3967.


[Y] C. Yuen, Jet schemes of determinantal varieties. Algebra, geometry and their interac-

[V] P. Vojta, Jets via Hasse-Schmidt derivations. Diophantine geometry, 335-361, CRM

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