Jet schemes of normal toric surfaces

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Abstract

For $m \in \mathbb{N}, m \geq 1$, we determine the irreducible components of the $m-th$ jet scheme of a normal toric surface S. We give formulas for the number of these components and their dimensions. This permits to determine the log canonical threshold of a toric surface embedded in an affine space. When m varies, these components give rise to projective systems, to which we associate a weighted oriented graph. We prove that, among toric surfaces, the data of this graph is equivalent to the data of the analytical type of S. Besides, we classify these irreducible components by an integer invariant that we call index of speciality. We prove that for m large enough, the set of components with index of speciality 1, is in $1-1$ with the set of exceptional divisors that appear on the minimal resolution of S.

1 Introduction

Nash has introduced the arc space of a variety X in order to investigate the intrinsic data of the various resolutions of singularities of X. The analogy with $p-$adic numbers has led Kontsevich [K], Denef and Loeser [DL1] to invent motivic integration and to introduce several rational series that generalize analogous series in the $p-$adic context [DL2]. The geometric counterpart of the theory of motivic integration has been used by Ein, Mustata and others to obtain formulas controlling discrepancies in terms of invariant of jet schemes -these are finite dimensional approximations of the arc space-[Mus2],[ELM],[EM],[dFEI]. Roughly speaking, while we can extract informations about abstract resolutions of singularities from the arc space and vice versa, we can extract informations about embedded resolutions of singularities from the jet schemes and vice versa. This partly explains why the arc space of a toric variety -which has been intensively studied [KKMS],[L],[B-GS],[I],[IK]- is well understood. Indeed, we know an equivariant abstract resolution of a toric variety, what permits to understand the action of the arc space of the torus on its arc space [I], but an equivariant embedded resolution is less accessible.

The structure of jet schemes of singular algebraic varieties is complicated; despite that they were the subject of numerous article in the last decade, few is known about their geometry for specific class of singularities, except for the following classes: monomial ideals [GS], determinantal varieties [D], plane branches [Mo1], quasi-ordinary singularities [CM].

In this article, we study the jet schemes of a normal toric surface singularity. We determine their irreducible components and we give formulas for their number and dimensions. We give here a brief description of the results. The data of a toric surface singularity S is equivalent to the data of a cone $\sigma \subset N = \mathbb{Z}^2$ generated by $(1,0)$ and (p,q) for two coprime numbers $0 < p < q$. Let $q/p = [c_2, ..., c_{e-1}]$ be the Hirzebruch-Jung continued fraction expansion (see section 2.2); the embedding dimension of S is equal to e; the equations defining the embedding of S in $\mathbb{A}^e = \mathrm{Spec}\mathbb{K}[x_1, \cdots, x_e]$ are described in section 2. Let $m \in \mathbb{N}, m \geq 1$ and let S_m^0 be the space of $m-$jets centered at the singularity of S (see section 2.1 for preliminaries on jet schemes). For $i = 2, \cdots, e-1,\ s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$(i.e. $m \geq 2s-1 \geq 1$) and $l \in \{s, \ldots, L^s_{i,m}\}$, where

$$L^s_{i,m} := min\{(c_i - 1)s, (m+1) - s\},$$

we define

$$D^{s,l}_{i,m} := Cont^s(x_i)_m \cap Cont^l(x_{i+1})_m,$$

where for $p \in \mathbb{N}$, and $f \in \mathbb{K}[x_1, \cdots, x_e]$,

$$Cont^p(f)_m = \{\gamma \in S_m \mid ord_\gamma(f) = p\}.$$

We define $C^{s,l}_{i,m} := \overline{D^{s,l}_{i,m}}$ to be be the Zariski closure of $D^{s,l}_{i,m}$. We find in theorem 4.15 the following.

Theorem. *Let $m \in \mathbb{N},\ m \geq 1$. The irreducible components of S_m^0 are $C^{s,L^s_{i,m}}_{e-1,m}$ and the $C^{s,l}_{i,m},\ i = 2, \cdots, e-1,\ s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$ and $l \in \{s, \ldots, L^s_{i,m} - 1\}\}$.*

The formulas that we obtain for the codimensions of the irreducible components of S_m^0 (see proposition 4.11) enable us, by applying Mustata's formula [Mus2], to determine the log canonical threshold of the pair $S \subset \mathbb{A}^e$ (e is the embedding dimension). For $e = 3$, the log canonical threshold is 1. For $e \geq 4$, we find in corollary 4.27 that

$$lct(S, \mathbb{A}^e) = \frac{e}{2}.$$

Moreover, making use of the truncation morphisms between the jet schemes, we associate with the irreducible components of S_m^0 a graph which is weighted by the codimensions of the irreducible components and the embedding dimension of some of these components. We prove in corollary 4.25 that the data of this graph is equivalent to the analytical type of the surface. Note that motivic invariants of a toric surface singularity do not determine its analytical type [LR],[Ni].

Finally, we classify the irreducible components by a natural invariant that we call index of speciality; this is the order of contact of the generic point of the component with the maximal ideal defining the singular point of S. We prove that for m large enough, the number of irreducible components of S_m^0 is in 1-1 correspondence with the divisors appearing on the minimal abstract resolution of singularities of S. This is to compare with the bijectivness of the Nash map for toric varieties [IK]. This is also related to a jet schemes approach to a conjecture of Teissier on toric resolution of singularities [T]. This approach is explained in [Mo4] (see also [LMR]).

The proof of the main theorem uses heavily the description of the defining equations of the embedding $S \subset \mathbb{A}^e$ ([R],[St]), and some syzygies of these equations that we describe and that are ad hoc to the problem. It also uses known results on the arc space of a toric variety [L],[IK],[I] and it is by induction on m and on the embedding dimension e. In particular it uses a kind of approximation of the toric surface S by toric surfaces with smaller embedding dimensions.

Some of the results of this paper were announced in [Mo3].

The structure of the paper is as follows: in section two we present a reminder on jet schemes and on toric surfaces. In section three we study the jet schemes of the A_n singularities. The last section is devoted to the toric surfaces of embedding dimension bigger or equal to four.

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2 Jet schemes and toric surfaces

2.1 Jet schemes

Let $\mathbb{K}$ be field. Let X be a $\mathbb{K}$-scheme of finite type over $\mathbb{K}$. For $m \in \mathbb{N}$, the functor $F_m : \mathbb{K}-Schemes \longrightarrow Sets$ which to an affine scheme defined by a $\mathbb{K}-$algebra A associates

$$F_m(Spec(A)) = Hom_{\mathbb{K}}(SpecA[t]/(t^{m+1}), X)$$

is representable by a $\mathbb{K}-$scheme X_m [V]. We call X_m the $m-$th jet scheme of X and we have that F_m is isomorphic to its functor of points. In particular the $\mathbb{K}-$points of X_m are in bijection with the $\mathbb{K}[t]/(t^{m+1})-$points of X.
For $m, p \in \mathbb{N}, m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$. These morphisms are affine and for $p < m < q$ they clearly verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$. This yields an inverse system whose limit X_∞ is a scheme called the arc space of X. Note that $X_0 = X$. We denote the canonical projections $X_m \longrightarrow X_0$ by π_m and $X_\infty \longrightarrow X_m$ by Ψ_m. See [EM] for more about jet schemes.

Example 1. *Let* $X = Spec\ \frac{\mathbb{K}[x_1,\cdots,x_n]}{(f_1,\cdots,f_r)}$ *be an affine* $\mathbb{K}-$*scheme. For a* $\mathbb{K}$*-algebra* A*, an*

A-point of X_m is a $\mathbb{K}$-algebra homomorphism

$$\varphi : \frac{\mathbb{K}[x_1, \cdots, x_n]}{(f_1, \cdots, f_r)} \longrightarrow A[t]/(t^{m+1}).$$

This homomorphism is completely determined by the image of $x_i, i = 1, \cdots, n$

$$x_i \longmapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)} t + \cdots + x_i^{(m)} t^m$$

and it should verify that $\varphi(f_l) = f_l(\phi(x_1), \cdots, \phi(x_n)) \in (t^{m+1}),\ l = 1, \cdots, r.$

Therefore if we set

$$f_l(\phi(x_1), \cdots, \phi(x_n)) = \sum_{j=0}^{m} f_l^{(j)}(\underline{x}^{(0)}, \cdots, \underline{x}^{(j)})\ t^j \quad mod \quad (t^{m+1})$$

where $\underline{x}^{(j)} = (x_1^{(j)}, \cdots, x_n^{(j)})$, then we have that

$$X_m = Spec \frac{\mathbb{K}[\underline{x}^{(0)}, \cdots, \underline{x}^{(m)}]}{(f_l^{(j)})_{l=1,\cdots,r}^{j=0,\cdots,m}}$$

Example 2. *From the above example, we see that the m-th jet scheme of the affine space $\mathbb{A}^n$ is isomorphic to $\mathbb{A}^{(m+1)n}$ and that the projection $\pi_{m,m-1} : \mathbb{A}^n_m \longrightarrow \mathbb{A}^n_{m-1}$ is the map that forgets the last n coordinates.*

Remark 2.1. *This a notational remark; in the sequel we will denote the $m-th$ jet scheme of the affine space $\mathbb{A}^n$ by $\mathbb{A}^n_m$.*

Remark 2.2. *Note that in general, if X is a nonsingular variety of dimension n, then all the projections $\pi_{m,m-1} : X_m \longrightarrow X_{m-1}$ are locally trivial fibrations with fiber $\mathbb{A}^n$. In particular X_m is of dimension $n(m+1)$ ([EM]).*

2.2 Toric surfaces

Let S be a singular affine normal toric surface defined over the field $\mathbb{K}$. There exist two coprime integers p and q such that S is defined by the cone $\sigma \subset N = \mathbb{Z}^2$ generated by $(1, 0)$ and (p, q) and $0 < p < q$, i.e. S =Spec$\mathbb{K}[x^u,\ u \in \sigma^\vee \cap M]$ where $\sigma^\vee$ is the dual cone of σ and M is the dual lattice of N ([O]). We have the Hirzebruch-Jung continued fraction expansion in terms of $c_j \geq 2$:

$$\frac{q}{p} = c_2 - \cfrac{1}{c_3 - \cfrac{1}{\cdots - \cfrac{1}{c_{e-1}}}}$$

which we denote by $[c_2, ..., c_{e-1}]$. Let $\theta^\vee$ be the convex hull of $(\sigma^\vee \cap M) \setminus 0$ and let $\partial\theta^\vee$ be its boundary polygon. Let $u_1, u_2, \ldots, u_h$ be the points of M lying in this order on

$\partial\theta^\vee$, with $u_1 = (0, 1)$ and $u_h = (q, -p)$. Then from [O], proposition 1.21 we have that $h = e$ is the embedding dimension of S and the u_i form a minimal system of generators of the semigroup $\sigma^\vee \cap M$. For $i = 1, \ldots, e$, we will denote by x_i the regular function on S defined by x^{u_i}. Riemenschneider has exhibited the generators of the ideal defining S in $\mathbb{A}^e = \mathrm{Spec}\mathbb{K}[x_1, \cdots, x_e]$. They can be given in a quasi-determinantal format [R], [St]:

$$\begin{pmatrix} x_1 & & x_2 & \ldots & x_{e-2} & & x_{e-1} \\ & x_2^{c_2-2} & & \ldots & & x_{e-1}^{c_{e-1}-2} & \\ x_2 & & x_3 & \ldots & x_{e-1} & & x_e \end{pmatrix}$$

where the generalised minors of a quasi-determinant

$$\begin{pmatrix} f_1 & & f_2 & \ldots & f_{k-1} & & f_k \\ & h_{1,2} & & \ldots & & h_{k-1,k} & \\ g_1 & & g_2 & \ldots & g_{k-1} & & g_k \end{pmatrix}$$

are $f_i g_j - g_i(\prod_{n=i}^{j-1} h_{n,n+1}) f_j$.

They can be written as follows:

$$E_{ij} = x_i x_j - x_{i+1} x_{i+1}^{c_{i+1}-2} x_{i+2}^{c_{i+2}-2} \cdots x_{j-2}^{c_{j-2}-2} x_{j-1}^{c_{j-1}-2} x_{j-1},$$

where $\quad 1 \leq i < j - 1 \leq e - 1$.

Let $b_i \in \mathbb{N}, b_i \geq 2$, be such that $q/(q-p) = [b_1, \ldots, b_r]$. Let $l_0 = (1, 0), \ldots, l_{s+1} = (p, q)$ in this order be the elements of N lying on the compact edges of the boundary $\partial\theta$ of the convex hull θ of $(\sigma \cap N) \setminus 0$.

Proposition 2.3. *We have that $r = s$ and is equal to the number of irreducible components of the exceptional curve for the minimal resolution of singularities of S. Moreover we have that*

$$c_2 + \cdots + c_{e-1} - 2(e-2) + 1 = s.$$

See lemma 1.22 and corollary 1.23 in [O] for a proof.

3 Jet schemes of toric surfaces of embedding dimension $e = 3$

Let S be the variety defined in $\mathbb{A}^3$ by the equation $f(x, y, z) = xy - z^{n+1} = 0$. S has an A_n singularity at the origin 0 and is nonsingular elsewhere. Note that an affine toric suface of embedding dimension 3 has this type of singularities (see section 2.1). If we set

$$f(\sum_{i=0}^{m} x^{(i)} t^i, \sum_{i=0}^{m} y^{(i)} t^i, \sum_{i=0}^{m} z^{(i)} t^i) = \sum_{i=0}^{i=m} F^{(i)} t^i \mod t^{m+1}, \qquad (\diamond)$$

then S_m is defined in $\mathbb{A}^{3(m+1)} = \mathbb{A}^3_m$ by the ideal $I_m = (F^{(0)}, F^{(1)}, ..., F^{(m)})$.

By remark 2.2, the morphism $\pi_m^{-1}(S\backslash 0) \longrightarrow S\backslash 0$ is a trivial fibration, therefore we have that $\overline{\pi_m^{-1}(S\backslash 0)}$ is an irreducible component of S_m of codimension $m+1$ in $\mathbb{A}^3_m$. On the other hand, we will prove in the coming lines that the codimension of $S^0_m := \pi_m^{-1}(0)$ in $\mathbb{A}^3_m$ is $m+2$, which means that S_m is irreducible for every $m \in \mathbb{N}$: indeed, since I_m is generated by $m+1$ equations, any irreducible component of S_m could have codimension at most $m+1$. (Note that the irreducibility of S_m follows from [Mus1] because S is locally a complete intersection with a rational singularity, but we give here a direct proof in this simple case.)

We claim that for $m \leq n$, we have $S^0_m = Z^0_m$, where $Z \subset \mathbb{A}^3$ is the hypersurface defined by $xy = 0$. Indeed, a $m-$jet $\gamma_m = (x = \sum_{i=0}^m x^{(i)}t^i, y = \sum_{i=0}^m y^{(i)}t^i, z = \sum_{i=0}^m z^{(i)}t^i) \in (\mathbb{A}^3)_m$ centered at the origin (i.e.$x^{(0)} = y^{(0)} = z^{(0)}$) is in S^0_m if and only if $xy - z^{n+1} \equiv 0 \bmod t^{m+1}$, but since $z_0 = 0$ and $m \leq n$, we have that $ord_t\ z^{n+1} \geq n+1 \geq m+1$, therefore this is equivalent to $ord_t\ xy \geq m+1$ and therefore to $\gamma \in Z^0_m$.

But clearly for $m \leq n$, the irreducible commponents of $Z^0_m = S^0_m$ are the subvarities defined by the ideals

$$I^l_m = (x^{(0)}, ..., x^{(l-1)}, y^{(0)}, ..., y^{(m-l)}, z^{(0)}), l = 1, ..., m.$$

Notice that the codimension of $C^l_m := V(I^l_m)$ in $\mathbb{A}^3_m$ is equal to $m+2$ for $l = 1, ..., m$. We deduce that for $m \leq n, S_m$ is irreducible of codimension $m+1$. On the other hand, for $m \geq n+1$ we have that $C^l_m = \pi^{-1}_{m,n}(V(I^l_n))$ is defined in $(\mathbb{A}^3)_m$ by the ideal $I^l_m = (I^l_n, J^l_{m-(n+1)})$ where $J^l_{m-(n+1)}$ is the ideal obtained from the ideal defining $X_{m-(n+1)}$ in $\mathbb{A}^3_{m-(n+1)}$ by changing variables. Indeed if we set

$$f(\sum_{i=l}^m x^{(i)}t^i, \sum_{i=n-l+1}^m y^{(i)}t^i, \sum_{i=1}^m z^{(i)}t^i) =$$

$$f(t^l(\sum_{i=0}^{m-l} x^{(l+i)}t^i), t^{n-l+1}(\sum_{i=0}^{m-(n-l+1)} y^{(n-l+1+i)}t^i), t(\sum_{i=0}^{m-1} z^{(i+1)}t^i)) =$$

$$t^{n+1}f(\sum_{i=0}^{m-l} x^{(l+i)}t^i, \sum_{i=0}^{m-(n-l+1)} y^{(n-l+1+i)}t^i, \sum_{i=0}^{m-1} z^{(i+1)}t^i)$$

$$= t^{n+1}(\sum_{i=0}^{i=m-(n+1)} G^{(i)}_l t^i) \mod t^{m+1}, \qquad (\diamond\diamond)$$

then $J^l_{m-(n+1)}$ is generated by $G^{(i)}_l, i = 0, \ldots, m-(n+1)$, and by comparing $(\diamond)$ with $(\diamond\diamond)$, we get that

$$G^{(i)}_l = F^{(i)}(x^{(l)}, \ldots, x^{(l+i)}, y^{(n-l+1)}, \ldots, y^{(n-l+1+i)}, z^{(1)}, \ldots, z^{(1+i)}).$$

We deduce that for $l = 1, ..., n$,

$$Codim\ \ (\pi^{-1}_{m,n}(V(I^l_n)), \mathbb{A}^3_m) = n+2+Codim\ \ (S_{m-(n+1)}, \mathbb{A}^3_{m-(n+1)}).$$

This implies by a simple induction that for $l = 1, ..., n$,

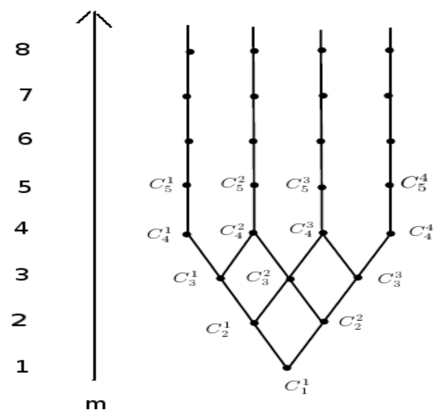
$$Codim \ \ \pi^{-1}_{m,n}(V(I^l_n)) = m + 2.$$

Therefore $Codim \ \ (S^0_m, \mathbb{A}^3_m) = m + 2$, so S_m is irreducible. It follows that $\pi^{-1}_{m,n}(V(I^l_n))$ which is isomorphic to $S_{m-(n+1)} \times \mathbb{A}^{2n+1}$ is irreducible and we conclude:

Theorem 3.1. *Let $m \in \mathbb{N}, n \geq 1$, and let S^0_m be the scheme of $m-$jets centered in the singular locus of an A_n singularity. Then we have the following:*

1. *S^0_m is a locally complete intersection scheme.*
2. *For $m \leq n, S^0_m$ has m irreducible components, $C^l_m, l = 1, \ldots, m$ each of codimension $m + 2$. For $m \geq n + 1$, it has n irreducible components, $C^l_m, l = 1, \ldots, n$, each of codimension $m + 2$.*
3. *The global jet scheme S_m is irreducible.*
4. *For $2 \leq m \leq n$, and $l \in \{1, \ldots, m - 1\}$ we have that $\pi_{m,m-1}(C^l_m) \subset C^l_{m-1}$, $\pi_{m,m-1}(C^l_m) \subset C^{l-1}_{m-1}$ and $\pi_{m,m-1}(C^m_m) \subset C^{m-1}_{m-1}$. For $m \geq n + 1$ we have that $\pi_{m,m-1}(C^l_m) \subset C^l_{m-1}$, for $l \in \{1, \ldots, n\}$. These are all the inclusions induced by $\pi_{m,m-1}$ for $m \geq 2$.*

We obtain a graph Γ by representing every irreducible components of $S^0_m, m \geq 1$, by a vertex $v_{i,m}$ and by joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if the morhphism $\pi_{m+1,m}$ induces a morphism between the corresponding irreducible components. From the theorem 3.1, part 4, we deduce that the graph Γ for the singularity A_4 is the following :



4 Jet schemes of toric surfaces of embedding dimension $e \geq 4$

We keep the notations introduced in section 2 and we begin by introducing some more notations. Let $f \in \mathbb{K}[x_1, \ldots, x_e]$; for $m, p \in \mathbb{N}$ such that $p \leq m$, we set:

$$Cont^p(f)_m(resp.Cont^{>p}(f)_m) := \{\gamma \in S_m \mid ord_\gamma(f) = p(resp. > p)\},$$

$$Cont^p(f) = \{\gamma \in S_\infty \mid ord_\gamma(f) = p\},$$

where $ord_\gamma(f)$ is the $t-$order of $f \circ \gamma$.
For $a, b \in \mathbb{N},\ b \neq 0$, we denote by $\lceil \frac{a}{b} \rceil$ the round-up of $\frac{a}{b}$. For $i = 2, \cdots, e-1,\ s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$(i.e. $m \geq 2s-1 \geq 1$) and $l \in \{s, \ldots, L^s_{i,m}\}$, where

$$L^s_{i,m} := min\{(c_i - 1)s, (m+1) - s\},$$

we set

$$D^{s,l}_{i,m} := Cont^s(x_i)_m \cap Cont^l(x_{i+1})_m,$$

and

$$C^{s,l}_{i,m} := \overline{D^{s,l}_{i,m}}.$$

If R is a ring, $I \subseteq R$ an ideal and $f \in R$, we denote by $V(I)$ the subvariety of $Spec\ R$ defined by I and by $D(f)$ the open set $D(f) := \text{Spec } R_f$.
We will prove that the irreducible components of $S^0_m := \pi_m^{-1}(0)$ are among the closed sets $C^{s,l}_{i,m}$ (see the theorem in the introduction). The irreducibility of the $C^{s,l}_{i,m}$ is proved in proposition 4.7, where we also compute their codimensions. In proposition 4.13 we prove that they cover S^0_m. In lemma 4.12, we prove that there are redundancies between the $C^{s,l}_{i,m}$. The fact that there are no inclusions among them but those of lemma 4.12, is proved in theorem 4.15.

We begin by giving an overview of the strategy of the proof of theorem 4.15.

The first remark is that S^0_1, which is the Zariski tangent space of S at 0, is isomorphic to an affine space (lemma 4.3), more precisely we have:

$$S^0_1 = \text{Spec}\left(\frac{\mathbb{K}[x_1^{(0)}, \ldots, x_e^{(0)}, x_1^{(1)}, \ldots, x_e^{(0)}]}{(x_1^{(0)}, \ldots, x_e^{(0)})}\right). \qquad (\star)$$

A key idea is to stratify it as follows

$$S^0_1 = (S^0_1 \cap D(x_1^{(1)})) \cup \ldots \cup (S^0_1 \cap D(x_1^{(1)})) \cup (S^0_1 \cap (x_1^{(1)}, \ldots, x_e^{(1)}).$$

First we study $\pi^{-1}_{m,1}(S^0_1 \cap D(x_i^{(1)}))$, for $i = 2, \ldots, e-1$ and $m \geq 2$. By using syzygies between the equations defining S (lemma 4.5), we construct in proposition 4.11 a trivial fibration from $\pi^{-1}_{m,1}((S^0_1 \cap D(x_i^{(1)})$, to a constructible subset of the jet schemes of an A_{c_i} singularity. This latter constructible subset is introduced and studied in lemma 4.10, what permits to us to determine the irreducible components of the Zariski closure $\pi^{-1}_{m,1}((S^0_1 \cap D(x_i^{(1)})$, for $i = 2, \ldots, e-1$, namely the $C^{1,l}_{i,m}$. The constructibles $\pi^{-1}_{m,1}((S^0_1 \cap D(x_i^{(1)})$ for $i = 1, e$ are irreducible (proposition 4.11) and included in the Zariski closure of $\pi^{-1}_{m,1}((S^0_1 \cap D(x_i^{(1)}), i = 1, e-1$, (proposition 4.11, part (2)).

It remains to study $\pi^{-1}_{m,1}(S^0_1 \cap (x_1^{(1)}, \ldots, x_e^{(1)})$, for $m \geq 2$. For $m = 2$, we prove that $\pi^{-1}_{2,1}(S^0_1 \cap (x_1^{(1)}, \ldots, x_e^{(1)})$ is included in the Zariski closure of $\pi^{-1}_{2,1}((S^0_1 \cap D(x_i^{(1)})$, for $i = 2, ..., e-1$ (proposition 4.13). The proof of the latter statement in the case where the embedding dimension $e = 4$ is based on dimension arguments, then we use induction on e. For this purpose, we approximate S by toric surfaces which are of less embedding dimensions. For $m = 3, \pi_{3,1}^{-1}(S_1^0 \cap (x_1^{(1)}, \ldots, x_e^{(1)})$(which is equal to $C_{2,3}^{2,2}$ by lemma 4.3) is an irreducible component of S_3^0, and is an affine space that we stratify in a similar way to ($\star$) (see the case $m = 2n+1$ in proposition 4.13). We then as above consider the inverse image by $\pi_{m,3}, m \geq 4$ of each strata. The inverse images by $\pi_{m,3}$ of the open stratas will be understood again by comparison with some subsets of the jet schemes of A_{c_i} singularities and they will give rise to a new generation of irreducible components, namely the $C_{i,m}^{2,l}$. Then we study the inverse image by $\pi_{4,3}$ and $\pi_{5,3}$ of the closed strata. This phenomena is understood by an induction on m, (more precisely on n) which permits us to cover S_m^0 by irreducible subsets. In theorem 4.15 we prove that there are no inclusions between these subsets.

From 4.1 till 4.12, we are preparing the proof of theorem 4.15. Our first aim is to prove the irreducibility of the $C_{i,m}^{s,l}$'s and to compute their codimension in $\mathbb{A}_m^e$, this is the subject of proposition 4.7. We begin by some preparatory lemmas.

Proposition 4.1. *1. For $i = 2, \cdots, e-1$ and $l, s \in \mathbb{N}$ such that $1 \leq s \leq l \leq (c_i - 1)s$, we have that $Cont^s(x_i) \cap Cont^l(x_{i+1}) \neq \emptyset$.*

2. For $s \in \mathbb{N},\ s \geq 1,\ Cont^s(x_1) \cap Cont^s(x_2) \neq \emptyset$.

Proof : (1)-We will prove that there exists an arc h on S, whose generic point lies in the torus, and such that $h \in Cont^s(x_i) \cap Cont^l(x_{i+1})$. Note that with an arc h on S, we can naturally associate a vector $v_h = (a, b) \in \sigma \cap N$ and that for any $v \in \sigma \cap N$ there exists an arc h such that $v = v_h$; moreover, for any $u \in M \cap \sigma^\vee$, we have that $h \in Cont^{v_h.u}(x^u)$, where we denote by $v_h.u$ the scalar product of v_h and u, and by x^u the regular function defined by u on S ([LR], proposition 3.3). Let $u_i, i = 1, \cdots, e$, be the system of minimal generators of $\sigma^\vee \cap M$, defined in 2.2 such that $x^{u_i} = x_i$. Therefore to prove that there exists an arc h as above, it is sufficient to prove that there exists $(a, b) \in \sigma \cap N$ such that $(a, b).u_i = s$ and $(a, b).u_{i+1} = l$. Since u_i and u_{i+1} determine a $\mathbb{Z}-$basis of M, there exists a unique $(a, b) \in N$ such that $(a, b).u_i = s$ and $(a, b).u_{i+1} = l$. Let's prove that (a, b) lies in the interior of σ, i.e. that for $j = 1, \cdots, e$, $(a, b).u_j > 0$. Since $u_{i-1} = c_i u_i - u_{i+1}$, we have that $(a, b).u_{i-1} = c_i s - l$ which is greater than or equal to s because by hypothesis we have $s \leq l \leq s(c_i - 1)$. Similarly we have that $(a, b).u_{i+2} = c_{i+1} l - s$ which is greater than or equal to l. Since $c_i \geq 2$, for $i = 1, \cdots, e$, by descending (respectively ascending) induction we find that $(a, b).u_{j-1} \geq (a, b).u_j$, for $j = 2, \cdots, i$ (respectively $(a, b).u_{j-1} \leq (a, b).u_j$, for $j = i+2, \cdots, e$) and the proposition follows.
(2)-We have that $u_1 = (0, 1), u_2 = (1, 0)$. We need to prove that the unic vector $v = (a, b) \in N$ such that $(a, b).(0, 1) = b = s$ and $(a, b).(1, 0) = a = s$, also belongs to σ; in fact it is is clear that (s, s) belongs to the interior of σ. We also need to prove that for $j = 3, \cdots, e$, we have that $(s, s).u_j \geq s$; since $u_j \in \sigma^\vee$ and $(1, 1)$ lies in the interior of σ, we have that $(1, 1).u_j > 0$, moreover $u_j \in M$ and $(1, 1) \in N$, so $(1, 1).u_j \in \mathbb{Z}$ and $(1, 1).u_j \geq 1$.

□

The following lemma prepares lemma 4.3.

Lemma 4.2. *Let $i = 2, \cdots, e-1,\ m \in \mathbb{N},\ s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$ and $l \in \{s, \ldots, L_{i,m}^s\}$. For $\gamma \in D_{i,m}^{s,l}$, we have*

1. *the inequality* $ord_\gamma x_j \geq s, j = 1, \ldots, e.$
2. *If moreover* $m \neq L^s_{i,m}$, *then for* $j = 1, \ldots, i-1$ *we have* $ord_\gamma x_j > s.$

Proof : Let $\gamma \in D^{s,l}_{i,m}$. This implies that $ord_\gamma E_{i-1,i+1} \geq m+1$. From the expression of $E_{i-1,i+1}$ and the hypothesis $l \in \{s, \ldots, L^s_{i,m}\}$, we get that $ord_\gamma x_{i-1} \geq s$. We also have $ord_\gamma E_{i-2,i} \geq m+1$; using the fact that $ord_\gamma x_{i-1} \geq s$ we get $ord_\gamma x_{i-2} \geq s$. Recursively, by using the conditions, $ord_\gamma E_{j,i} \geq m+1, j = i-3, i-4, \ldots, 1$, we obtain $ord_\gamma x_j \geq s, j = i-3, i-4, \ldots, 1$. Similarly, by using the conditions $ord_\gamma E_{i,j} \geq m+1, j = i+2, \ldots, e$ we obtain $ord_\gamma x_j \geq s, j = i+2, \ldots, s$ and hence the first part of the lemma. The second part follows in the same way using the conditions $ord_\gamma E_{j,i} \geq m+1, j = 1, \ldots, m-1$.

□

Lemma 4.3. *For* $i = 2, \cdots, e-1,\ s \geq 1$, *the ideal defining* $C^{s,s}_{i,2s-1}$ *in* $\mathbb{A}^e_{2s-1}$ *is*

$$I^{s,s}_{i,2s-1} = (x^{(b)}_j, 1 \leq j \leq e, 0 \leq b < s).$$

Note that $C^{s,s}_{i,2s-1}$ *does not depend on* i. *For* $j = 1, e$, *we set*

$$C^{s,s}_{j,2s-1} := C^{s,s}_{i,2s-1},\ i = 2, \cdots, e-1.$$

Proof : Let us prove that $D^{s,s}_{i,2s-1} = V(I^{s,s}_{i,2s-1}) \cap D(x^{(s)}_i x^{(s)}_{i+1})$. Let $\gamma \in \mathbb{A}^e_{2s-1}$ such that $ord_\gamma x_i = ord_\gamma x_{i+1} = s$. Lemma 4.2 gives that $ord_\gamma x_j \geq s,\ j = 1, \ldots e$. We deduce

$$D^{s,s}_{i,2s-1} \subset V(I^{s,s}_{i,2s-1}) \cap D(x^{(s)}_i x^{(s)}_{i+1}).$$

The opposite inclusion comes from the fact that a jet in $V(I^{s,s}_{i,2s-1}) \cap D(x^{(s)}_i x^{(s)}_{i+1}) \subset \mathbb{A}^e_{2s-1}$ satisfies all the equations of S modulo t^{2s}. Since $V(I^{s,s}_{i,2s-1}) \subset \mathbb{A}^e_{2s-1}$ is irreducible, the lemma follows. □

Lemma 4.4. *For* $i = 2, \cdots, e-1,\ m \in \mathbb{N},\ s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$ *and* $l \in \{s, \ldots, L^s_{i,m}\}$, *we have that*

$$C^{s,l}_{i,m} \subset \pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}).$$

Proof : For $\gamma \in D^{s,l}_{i,m}$, it follows from lemma 4.2 (part 1) that $ord_\gamma x_j \geq s, j = 1, \ldots, e$ and hence from lemma 4.3 we deduce that $D^{s,l}_{i,m} \subset \pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1})$. The lemma follows since $\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1})$ is closed.

□

Lemma 4.5. 1. *For* $i = 2, \ldots, e-1,\ m \in \mathbb{N},\ s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$,

$$\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1} \cap D(x^{(s)}_i)) = \{\gamma \in \mathbb{A}^e_m\ ;\ ord_\gamma x_j \geq s,\ j = 1, \cdots, e,\ ord_\gamma x_i = s,$$
$$ord_\gamma E_{i-1,i+1} \geq m+1,\ ord_\gamma E_{j,i} \geq m+1,\ \text{for } 1 \leq j < i-1$$
$$ord_\gamma E_{i,j} \geq m+1,\ \text{for } i < j-1 \leq e-1\}.$$

2. For $i=2,\cdots,e-1,\ m\in\mathbb{N},\ s\in\{1,\ldots,\lceil\frac{m}{2}\rceil\}$ and $l\in\{s,\ldots,L^s_{i,m}\}$, we have

$$D^{s,l}_{i,m}=\{\gamma\in\mathbb{A}^e_m;\ ord_\gamma E_{ij}\geq m+1 \text{ for } i<j-1\leq e-1,$$
$$ord_\gamma E_{ji}\geq m+1 \text{ for } 1\leq j<i-1,$$
$$ord_\gamma E_{i-1,i+1}\geq m+1,\ ord_\gamma x_i=s,\ ord_\gamma x_{i+1}=l\}.$$

Proof : (1) The inclusion " $\subset$ " is an immediate consequence of lemma 4.3. To get the other inclusion, it is enough to check that for every $\gamma\in\mathbb{A}^e_m$ enjoying the conditions listed above, we also have $ord_\gamma E_{jh}\geq m+1$ for $1\leq j<h-1\leq e-1$.
If $i<j$, the syzygie

$$x_iE_{jh}-x_jE_{ih}+x_{j+1}^{c_{j+1}-2}\cdots x_{h-1}^{c_{h-1}-2}x_{h-1}E_{i,j+1}=0 \tag{4.1}$$

implies that $ord_\gamma E_{jh}\geq m+1$, because $ord_\gamma x_j$ and $ord_\gamma x_{h-1}\geq s$ and $ord_\gamma x_i=s$.
Similarly if $h<i$, the syzygie

$$x_iE_{jh}-x_hE_{ji}+x_{j+1}x_{j+1}^{c_{j+1}-2}\cdots x_{h-1}^{c_{h-1}-2}E_{h-1,i}=0 \tag{4.2}$$

implies that $ord_\gamma E_{jh}\geq m+1$, because $ord_\gamma x_h$ and $ord_\gamma x_{j+1}\geq s$ and $ord_\gamma x_i=s$.
Assume now that $1\leq j<i-1$ and $h=i+1$; the syzygie

$$x_{i+1}E_{ji}-x_iE_{j,i+1}+x_{j+1}x_{j+1}^{c_{j+1}-2}\cdots x_{i-1}^{c_{i-1}-2}E_{i-1,i+1}=0 \tag{4.3}$$

implies that $ord_\gamma E_{j,i+1}\geq m+1$.
Similarly if $j=i-1$ and $i+1<h\leq e$, the syzygie

$$x_{i-1}E_{ih}-x_iE_{i-1,h}+x_{i+1}^{c_{i+1}-2}\cdots x_{h-1}^{c_{h-1}-2}x_{h-1}E_{i-1,i+1}=0 \tag{4.4}$$

implies that $ord_\gamma E_{i-1,h}\geq m+1$.
Finally , if $1\leq j<i-1$ and $i+1<h\leq e$, the syzygie

$$x_jE_{ih}-x_iE_{jh}+x_{i+1}^{c_{i+1}-2}\cdots x_{h-1}^{c_{h-1}-2}x_{h-1}E_{j,i+1}=0 \tag{4.5}$$

implies that $ord_\gamma E_{j,h}\geq m+1$, taking into account that we have shown above that $ord_\gamma E_{j,i+1}\geq m+1$.
(2) First, since the ideal defining S in $\mathbb{A}^e$ is generated by $E_{jh},\ 1\leq j<h-1\leq e-1$, we have that

$$D^{s,l}_{i,m}\subset U^{s,l}_{i,m}:=\{\gamma\in\mathbb{A}^e_m;\ ord_\gamma E_{ij}(\text{resp. } ord_\gamma E_{ji})\geq m+1 \text{ for } i<j-1\leq e-1$$
$$(\text{resp. } 1\leq j<i-1),\ ord_\gamma E_{i-1,i+1}\geq m+1,\ ord_\gamma x_i=s,\ ord_\gamma x_{i+1}=l\}.$$

For $\gamma\in U^{s,l}_{i,m}$, we have by the proof of 4.4 that for $j=1,\cdots,e,\ ord_\gamma x_j\geq s$. It follows from the first part of this lemma that $D^{s,l}_{i,m}=U^{s,l}_{i,m}$.

□

Remark 4.6. *Note that the syzygies* (4.1), …, (4.5) *are syzygies in the ring of polynomials and not in the ring of regular functions on S. This is essential for the conclusion in the above lemma.*

Proposition 4.7. *For* $i = 2, \cdots, e-1,\ m \in \mathbb{N},\ s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$ *and* $l \in \{s, \ldots, L^s_{i,m}\}$, $C^{s,l}_{i,m}$ *is irreducible, and its codimension in* $\mathbb{A}^e_m$ *is equal to*

$$se + (m - (2s-1))(e-2).$$

Proof : The irreducibility of $C^{s,l}_{i,m}$ follows from the fact that $D^{s,l}_{i,m}$ is isomorphic to the product of a two dimensional torus by an affine space. Indeed, set $x_j \circ \gamma = \sum_{0 \leq \nu \leq m} x_j^{(\nu)} t^\nu$, $1 \leq j \leq e$. Assume that $ord_\gamma x_i = s$ and $ord_\gamma x_{i+1} = l$; then by lemma 4.5, part (2), $\gamma \in D^{s,l}_{i,m}$ if and only if $ord_\gamma E_{i-1,i+1} \geq m+1, ord_\gamma E_{ij} \geq m+1$ for $i+1 < j \leq e$ and $ord_\gamma E_{ji} \geq m+1$ for $1 \leq j < i-1$. Recall also that we have that $ord_\gamma(x_i) \geq s$ for $i = 1, \ldots, e$.
We begin by examining the condition $ord_\gamma E_{i-1,i+1} \geq m+1$.

If $m+1 \leq c_i s$, we have that $ord_\gamma E_{i-1,i+1} \geq m+1$, if and only if $x_{i-1}^{(\nu)} = 0$ for $0 \leq \nu \leq m-l$; this is due to the fact that we have the $ord_\gamma x_i^{c_i} = c_i s$ and $ord_\gamma x_{i+1} = l$.
If $m+1 > h+1 > c_i s$ then

$$E^{(h)}_{i-1,i+1} = x^{(h-l)}_{i-1} x^{(l)}_{i+1} - H \qquad (\star)$$

where H is a polynomial in $x_i^{(s)}, \ldots, x_i^{(h-c_i s+s)}, x_{i+1}^{(l)}, \ldots, x_{i+1}^{(h-c_i s+l)}$ and $x_{i-1}^{(c_i s-l)}, \ldots, x_{i-1}^{(h-l-1)}$ (where we have put $x_{i-1}^{(\nu)} = 0$ for $0 \leq \nu < c_i s - l$; this follows from the case $m+1 \leq c_i s$). In particular, for $h = c_i s$, we have that $E^{(c_i s)}_{i-1,i+1} = x^{(cis-l)}_{i-1} x^{(l)}_{i+1} - x_i^{(s)^{c}}{}_{i}$. After dividing by $x^{(l)}_{i+1} \neq 0$ we obtain that $E^{(c_i s)}_{i-1,i+1} = 0$ gives that $x^{(cis-l)}_{i-1} = x_i^{(s)^{c}}{}_{i}/x^{(l)}_{i+1}$. Exchanging $x^{(cis-l)}_{i-1}$ by this fraction in $E^{(c_i s+1)}_{i-1,i+1}$ and dividing by $x^{(l)}_{i+1} \neq 0$, we obtain from $(\star)$ that $E^{(c_i s+1)}_{i-1,i+1} = 0$ is equivalent to $x^{(c_i s-l+1)}_{i-1}$ equals a polynomial function in $x_i^{(s)}, x_i^{(s+1)}, 1/x^{(l)}_{i+1}, x^{(l)}_{i+1}, x^{(l+1)}_{i+1}$. Keeping doing this with $E^{(h')}_{i-1,i+1}$ for $c_i s \leq h' < h$ and by replacing in $E^{(h)}_{i-1,i+1}$(see $(\star)$) the variables $x^{(c_i s-l)}_{i-1}, x^{(c_i s-l+1)}_{i-1}, \ldots, x^{(h-l-1)}_{i-1}$ by their expressions as polynomial functions in $x_i^{(s)}, \ldots, x_i^{(h-1-c_i s+s)}, 1/x^{(l)}_{i+1}, x^{(l)}_{i+1}, \ldots, x^{(h-1-c_i s+l)}_{i+1}$, that are obtained form $E^{(h')}_{i-1,i+1}, h' = c_i s, \ldots, h-1$, is an induction on h that permits to express $x^{(h-l)}_{i-1}$ as a polynomial function in the variables $x_i^{(s)}, \ldots, x_i^{(h-c_i s+s)}, 1/x^{(l)}_{i+1}, x^{(l)}_{i+1}, \ldots, x^{(h-c_i s+l)}_{i+1}$. Hence, $ord_\gamma E_{i-1,i+1} \geq m+1$, if and only if $x^{(\nu)}_{i-1} = 0$ for $0 \leq \nu < c_i s - l$ and is a polynomial function of $x_i^{(s)}, \cdots, x_i^{(m-c_i s+s)}, 1/x^{(l)}_{i+1}, x^{(l)}_{(i+1)}, \cdots x^{(m-c_i s+l)}_{(i+1)}$ for $c_i s - l \leq \nu \leq m-l$.

Consider now the conditions $ord_\gamma E_{ij} \geq m+1$ for $i+1 < j \leq e$. For $j = i+2$, notice that $E_{i,i+2}$ has the "same" shape of $E_{i-1,i+1}$. It follows from the study of $ord_\gamma E_{i-1,i+1} \geq m+1$ that $ord_\gamma E_{i,i+2} \geq m+1$ if and only if $x^{(\nu)}_{i+2} = 0$ for $0 \leq \nu < s$ and is a polynomial function of $1/x_i^{(s)}, x_i^{(s)}, \cdots, x_i^{(m-s)}, x^{(l)}_{i+1}, \cdots, x^{(m-l)}_{i+1}$ for $s \leq \nu \leq m-s$. Now by using the expressions of of the $x^{(\nu)}_{i+2}$'s in the equations that defines $ord_\gamma E_{i,i+3} \geq m+1$ (see the shape of the equation $E_{i,i+3}$, for which we can write similar equations as $(\star)$ where H will depend on the $x_i^{(\nu)'}s, x_{i+1}^{(\nu)'}s, x_{i+2}^{(\nu)'}s$ and $x_{i+3}^{(\nu)'}s$), we obtain the expressions of the $x_{i+3}^{(\nu)'}s$ as polynomials in the variables $1/x_i^{(s)}, x_i^{(s)}, \cdots, x_i^{(m-s)}, x_{i+1}^{(l)}, \cdots, x_{i+1}^{(m-l)}$; an induction on $j = i+2, i+3, \ldots, e$ gives that $ord_\gamma E_{ij} \geq m+1$ for $i+1 < j \leq e$ if and only if $x_j^{(\nu)} = 0$ for $0 \leq \nu < s$ and is a polynomial function of $1/x_i^{(s)}, x_i^{(s)}, \cdots, x_i^{(m-s)}, x_{i+1}^{(l)}, \cdots, x_{i+1}^{(m-l)}$ for $s \leq \nu \leq m-s$.

Similarly, $ord_\gamma E_{ji} \geq m+1$, for $1 \leq j < i-1$ if and only if $x_j^{(\nu)} = 0$ for $0 \leq \nu < s$ and is a polynomial function of $1/x_i^{(s)}, x_i^{(s)}, \cdots, x_i^{(m-s)}, x_{i-1}^{(s)}, \cdots, x_{i-1}^{(m-l)}$ for $s \leq \nu \leq m-s$. Taking in considerations that $x_{i-1}^{(m-s)}, \ldots, x_{i-1}^{(m-l)}$ are polynomial functions in the variables $x_i^{(s)}, \cdots, x_i^{(m-c_is+s)}, 1/x_{i+1}^{(l)}, x_{(i+1)}^{(l)}, \cdots x_{(i+1)}^{(m-c_is+l)}$, it follows that a closed point in $D_{i,m}^{s,l}$ determines and is completely determined by the following data:

$$x_i^{(s)}, {x_{i+1}}^{(l)} \in \mathbb{K}^*,$$

$$x_i^{(s+1)}, \ldots, x_i^{(m)} \in \mathbb{K},$$

$$x_{i+1}^{(l+1)}, \ldots, x_{i+1}^{(m)} \in \mathbb{K}$$

$$x_{i-1}^{(m+1-l)}, \ldots, x_{i-1}^{(m)} \in \mathbb{K}$$

$$x_j^{(m+1-s)}, \ldots, x_j^{(m)} \in \mathbb{K}, j = 1, \ldots, i-2, i+2, \ldots, e.$$

As a consequence, the dimension of $D_{i,m}^{s,l}$, hence of its closure $C_{i,m}^{s,l}$ is $d = 2m+s(e-4)+2$. And the formula of the codimension is obtained by considering $(m+1)e-d$.

$\square$

Remark 4.8. *The final presentation of the proof of the proposition 4.7 was suggested by the referee.*

For $i = 2, \ldots, e-2$, let $X^i = \mathrm{Spec}\mathbb{K}[x_{i-1}, x_i, x_{i+1}]/(x_{i-1}x_{i+1} - x_i^{c_i})$. For $s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$, let

$$V_{i,m}^s := \{\gamma \in X_m^i, ord_\gamma(x_j) \geq s,\ j = i-1, i+1,\ ord_\gamma(x_i) = s\},$$

and for $l \in \{s, \ldots, L_{i,m}^s\}$, let

$$\Delta_{i,m}^{s,l} := \{\gamma \in X_m^i, ord_\gamma(x_i) = s,\ ord_\gamma(x_{i+1}) = l\}.$$

The algebraic morphism

$$\frac{\mathbb{K}[x_{i-1}, x_i, x_{i+1}]}{(x_{i-1}x_{i+1} - x_i^{c_i})} \longrightarrow \frac{\mathbb{K}[x_1, \ldots, x_e]}{(E_{ij}, 1 \leq i < j-1 \leq e-1)}$$

induces a natural map $p^i : S \longrightarrow X^i$; the associated map $p^i_m : S_m \longrightarrow X^i_m$ induces morphisms

$$\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1} \cap D(x^{(s)}_i)) \longrightarrow V^s_{i,m} \ \text{ and } \ D^{s,l}_{i,m} \longrightarrow \Delta^{s,l}_{i,m}$$

Now in view of lemma 4.5 (see also the proof of proposition 4.7), we have the following proposition.

Proposition 4.9. *The maps*

$$\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1} \cap D(x^{(s)}_i)) \longrightarrow V^s_{i,m} \ \text{ and } \ D^{s,l}_{i,m} \longrightarrow \Delta^{s,l}_{i,m}$$

are isomorphic to trivial fibrations of rank $s(e-3)$.

Proof : For the second map, this is the geometric translation of lemma 4.5 and proposition 4.7. In particular, the rank of the fibration is determined by the number of free variables

$$x^{(m+1-s)}_j, \ldots, x^{(m)}_j \in \mathbb{K}, j = 1, \ldots, i-2, i+2, \ldots, e,$$

(see the last line of the proof of proposition 4.7): fixing these variables gives a point in the fibre above a fixed point in $\Delta^{s,l}_{i,m}$. The proof concerning the first map is similar. $\square$

The following propositions are preparatory for the proof of proposition 4.13, which states that S^0_m is the union of the $C^{s,l}_{i,m}$.

Lemma 4.10. *For $i = 2, \ldots, e-1$, and $s \in \{1, \ldots, \lceil \frac{m}{2} \rceil\}$, the irreducible components of $\overline{V^s_{i,m}}$ are the $\overline{\Delta^{s,l}_{i,m}}$, $l \in \{s, \ldots, L^s_{i,m}\}$.*

Proof : First, assume that $m+1 \leq c_i s$, so that $L^s_{i,m} = m+1-s$. We have that

$$V^s_{i,m} = \{\gamma \in \mathbb{A}^3_m \ ; \ ord_\gamma x_j \geq s, j = i-1, i+1, ord_\gamma x_i = s$$
$$\text{and } \ ord_\gamma x_{i-1} + ord_\gamma x_{i+1} \geq m+1\}$$

and for $l \in \{s, \ldots, m+1-s\}$,

$$\Delta^{s,l}_{i,m} = \{\gamma \in \mathbb{A}^3_m \ ; \ ord_\gamma x_i = s, ord_\gamma x_{i+1} = l, ord_\gamma x_{i-1} \geq m+1-l\} =$$

$$V(x^{(0)}_{i-1}, \ldots, x^{(m-l)}_{i-1}, x^{(0)}_i, \ldots, x^{(s-1)}_i, x^{(0)}_{i+1}, \ldots, x^{(l-1)}_{i+1}) \cap D(x^{(s)}_i x^{(l)}_{i+1}).$$

Since $s \leq l \leq m+1-s$, we have that $\Delta^{s,l}_{i,m} \subset V^s_{i,m}$, so $\cup_{s \leq l \leq m+1-s} \overline{\Delta^{s,l}_{i,m}} \subset \overline{V^s_{i,m}}$. Now for $\gamma \in V^s_{i,m}$, we have that $ord_\gamma x_i = s, l := ord_\gamma x_{i+1} \geq s$ and $ord_\gamma x_{i-1} \geq m+1-l$. If $l \leq m+1-s$, we thus have that $\gamma \in \Delta^{s,l}_{i,m}$; if $l > m+1-s$, we have that $ord_\gamma x_{i-1} \geq s$, hence $\gamma \in V(x^{(0)}_{i-1}, \ldots, x^{(s-1)}_{i-1}, x^{(0)}_i, \ldots, x^{(s-1)}_i, x^{(0)}_{i+1}, \ldots, x^{(m-s)}_{i+1}) = \overline{\Delta^{s,m+1-s}_{i,m}}$, hence the claim.
Now assume that $c_i s < m+1$, so that $L^s_{i,m} = (c_i-1)s$. For $l \in \{s, \ldots, (c_i-1)s\}$ and $\gamma \in \Delta^{s,l}_{i,m}$, we thus have that $ord_\gamma x_i = s, ord_\gamma x_{i+1} = l \geq s$, and $ord_\gamma x_{i-1} + l = c_i s$, hence $ord_\gamma x_{i-1} = c_i s - l \geq s$, therefore $\Delta^{s,l}_{i,m} \subset V^s_{i,m}$ and $\cup_{s \leq l \leq (c_i-1)s} \overline{\Delta^{s,l}_{i,m}} \subset \overline{V^s_{i,m}}$.

On the other hand $V^s_{i,m} = (\pi^i_{m,c_is-1})^{-1}(V^s_{i,c_is-1})$ where $\pi^i_{m,c_is-1} : X^i_m \longrightarrow X^i_{c_is-1}$ is the natural map. For $s \leq l \leq (c_i-1)s$, we have that $\overline{\Delta^{s,l}_{i,m}} = (\pi^i_{m,c_is-1})^{-1}(\Delta^{s,l}_{i,c_is-1})$.
Now we have just seen that $\overline{V^s_{i,c_is-1}} = \cup_{s\leq l\leq (c_i-1)s}\overline{\Delta^{s,l}_{i,c_is-1}}$ and that
$\overline{\Delta^{s,l}_{i,c_is-1}} = V(x^{(0)}_{i-1},\ldots,x^{(c_is-l-1)}_{i-1},x^{(0)}_i,\ldots,x^{(s-1)}_i,x^{(0)}_{i+1},\ldots,x^{(l-1)}_{i+1})$.

As a consequence $(\pi^i_{m,c_is-1})^{-1}(\overline{\Delta^{s,l}_{i,c_is-1}})$ is isomorphic to the product of an affine space by the space of $(m-c_is)-$jets of the surface Spec$\mathbb{K}[x^{(c_is-l)}_{i-1},x^{(s)}_i,x^{(l)}_{i+1}]/(x_{i-1}{}^{(c_is-l)}x_{i+1}{}^{(l)} - x^{(s)}_i{}^{c_i})$, and this latter is irreducible by section 3, hence coincides with $\overline{\Delta^{s,l}_{i,m}}$. So $\overline{V^s_{i,m}} \subset \cup_{s\leq l\leq (c_i-1)s}\overline{\Delta^{s,l}_{i,m}}$, hence the claim.

□

Proposition 4.11. *Let $m,s \in \mathbb{N}$ such that $s \in \{1,\ldots,\lceil \frac{m}{2}\rceil\}$.*

1. *For $i = 2,\cdots,e-1$, the irreducible components of $\overline{\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x^{(s)}_i))}$ are the $C^{s,l}_{i,m}, l \in \{s,\cdots,L^s_{i,m}\}$.*

2. *For $i = 1,e$, we have that $\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x^{(s)}_i))$ is irreducible of codimension*
$$se + (m-(2s-1))(e-2)$$
in $\mathbb{A}^e_m$. Moreover we have that
$$\overline{\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x^{(s)}_1))} = C^{s,L^s_{i,m}}_{2,m}$$
and
$$\overline{\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x^{(s)}_e))} = C^{s,s}_{e-1,m}.$$

Proof : (1) By the lemmas 4.4 and 4.5, we have that $D^{s,l}_{i,m} \subset \pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x^{(s)}_i)) =$
$$\{\gamma \in \mathbb{A}^e_m \ ; \ ord_\gamma x_j \geq s, \ j=1,\cdots,e, \ ord_\gamma x_i = s, \ ord_\gamma E_{i-1,i+1} \geq m+1,$$
$$ord_\gamma E_{j,i}(resp. ord_\gamma E_{i,j}) \geq m+1, \text{ for } 1 \leq j < i-1 (resp. \ i < j-1 \leq e-1)\}.$$
Now in view of proposition 4.9, the maps
$$\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x^{(s)}_i)) \longrightarrow V^s_{i,m} \text{ and } D^{s,l}_{i,m} \longrightarrow \Delta^{s,l}_{i,m}$$
are isomorphic to a trivial fibration of rank $s(e-3)$. By lemma 4.10, the irreducible components of $\overline{V^s_{i,m}}$ are the $\overline{\Delta^{s,l}_{i,m}}$, $l \in \{s,\ldots,L^s_{i,m}\}$. Since $V^s_{i,m} = \overline{V^s_{i,m}} \cap D(x^{(s)}_i)$, we thus have $V^s_{i,m} = \cup_l(\overline{\Delta^{s,l}_{i,m}} \cap D(x^{(s)}_i))$; so $\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x^{(s)}_i)) \simeq \cup_l \Omega^{s,l}_{i,m}$ where $\Omega^{s,l}_{i,m} = (\overline{\Delta^{s,l}_{i,m}} \cap D(x^{(s)}_i)) \times \mathbb{A}^{s(e-3)}$. As a consequence $\Omega^{s,l}_{i,m}$ is irreducible and we have that $D^{s,l}_{i,m} \subset \Omega^{s,l}_{i,m}$. Moreover
$$Codim(\Omega^{s,l}_{i,m},\mathbb{A}^e_m) = (e-3)(m+1) + (m+s+1) - s(e-3) =$$

$$(m+1)(e-2)-s(e-4)=Codim(C^{s,l}_{i,m},\mathbb{A}^e_m),$$

hence $C^{s,l}_{i,m}=\overline{\Omega^{s,l}_{i,m}}$ and the claim follows since $C^{s,l}_{i,m}\neq C^{s,l'}_{i,m}$ for $l\neq l'$.

(2) Assume $i=1$, the case $i=e$ follows in the same way. We first check that

$$\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x_1^{(s)}))=$$
$$\{\gamma\in\mathbb{A}^e_m, ord_\gamma(x_j)\geq s,\ j=1,\ldots,e,\ ord_\gamma(x_1)=s,$$
$$ord_\gamma E_{1j}\geq m+1\ \text{ for } 3\leq j\leq e\}.$$

The inclusion " $\subset$ " is clear. To get the opposite inclusion we have to prove that the conditions just listed imply that $ord_\gamma E_{jh}\geq m+1$ for $2\leq j<h-1\leq e-1$. This is an immediate consequence of the syzygie

$$x_1E_{jh}-x_jE_{1h}+x_{j+1}^{c_{j+1}-2}\cdots x_{h-1}^{c_{h-1}-2}x_{h-1}E_{1,j+1}=0.$$

Therefore, $\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x_1^{(s)}))$ is isomorphic to the product of $\mathbb{K}^*$ by an affine space of dimension $(m-s)+(m-s+1)+s(e-2)$ and its Zariski closure is irreducible of codimension $(m+1)(e-2)-s(e-4)$ in $\mathbb{A}^e_m$.
Now the equality

$$\overline{\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x_1^{(s)}))}=C^{s,L^s_{i,m}}_{2,m}$$

follows from the fact that by proposition 4.1 we have that $Cont^s(x_1)\cap Cont^s(x_2)\neq\emptyset$, hence $\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x_2^{(s)}))\cap\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x_1^{(s)}))\neq\emptyset$; since this latter is irreducible, its generic point γ coincides with the generic point of one of the irreducible components of $\pi^{-1}_{m,2s-1}(C^{s,s}_{i,2s-1}\cap D(x_2^{(s)}))$. The condition $ord_\gamma E_{1,2}\geq m+1$ shows that this irreducible component is $C^{s,L^s_{i,m}}_{2,m}$. The other equality in the statement has a similar proof.

$\square$

Lemma 4.12. *For $i=2,\ldots,e-2$, we have that*

$$C^{s,s}_{i,m}=C^{s,L^s_{i+1,m}}_{i+1,m}.$$

Proof : If $m+1\leq c_{i+1}s$, by definition $m^s_{i+1}=m+1-s$, and in view of lemma 4.3 and lemma 4.4, we have that $D^{s,s}_{i,m}\subset\overline{\pi^{-1}_{m,2s-1}(C^{s,s}_{i+1,2s-1}\cap D(x_{i+1}^{(s)}))}$. Now by proposition 4.11, the irreducible components of $\pi^{-1}_{m,2s-1}(C^{s,s}_{i+1,2s-1}\cap D(x_{i+1}^{(s)}))$ are the $C^{s,l}_{i+1,m}$ for $l\in\{s,\ldots,L^s_{i+1,m}\}$. Since $C^{s,s}_{i,m}=\overline{D^{s,s}_{i,m}}$ is irreducible, and its codimension in $\mathbb{A}^e_m$ coincides with the codimension of any of the $C^{s,l}_{i+1,m}$, there exists l such that $C^{s,s}_{i,m}=C^{s,l}_{i+1,m}$ with $s\leq l\leq m+1-s$. So $D^{s,s}_{i,m}$ and $D^{s,l}_{i+1,m}$ are dense open subsets of $C^{s,s}_{i,m}$ and there exists $\gamma\in D^{s,s}_{i,m}\cap D^{s,l}_{i+1,m}$. We thus have $ord_\gamma x_i=ord_\gamma x_{i+1}=s$, and $ord_\gamma x_{i+2}=l$. But $E_{i,i+2}=x_ix_{i+2}-x_{i+1}^{c_{i+1}}$ and $ord_\gamma E_{i,i+2}\geq m+1$. Since $m+1\leq c_{i+1}s$, this implies

$ord_\gamma x_{i+2} = l \geq m+1-s$, so $l = m+1-s$, i.e. $C^{s,s}_{i,m} = C^{s,L^s_{i+1,m}}_{i+1,m}$.
Assume now that $m+1 > c_{i+1}s$; for any $\gamma \in D^{s,s}_{i,m}$, we have that $ord_\gamma x_i = ord_\gamma x_{i+1} = s$ and $ord_\gamma E_{i,i+2} \geq m+1$, hence $ord_\gamma x_{i+2} = (c_{i+1}-1)s = L^s_{i+1,m}$ which implies that $D^{s,s}_{i,m} \subset D^{s,L^s_{i+1,m}}_{i+1,m}$. Since both are irreducible and have the same dimension, we deduce by passing to the closure that $C^{s,s}_{i,m} = C^{s,L^s_{i+1,m}}_{i+1,m}$. $\square$

Let $S^0_m := \pi_m^{-1}(O)$, where O is the singular point of S. Note that $\overline{\pi_m^{-1}(S-\{0\})}$ is an irreducible component of S_m of codimension $(m+1)(e-2)$ in $\mathbb{A}^e_m$; we will see that the irreducible components of S^0_m have codimension less than or equal to $(m+1)(e-2)$, therefore they are irreducible components of S_m.

Proposition 4.13.

$$S^0_m = \bigcup_{i\in\{2,\ldots,e-1\},s\in\{1,\ldots,\lceil\frac{m}{2}\rceil\},l\in\{s,\ldots,L^s_{i,m}\}} C^{s,l}_{i,m}.$$

Proof : We first look at **the case m=2n+1**, $n \geq 0$. We claim that

$$S^0_{2n+1} = \bigcup_{i\in\{1,\ldots,e\},s\in\{1,\ldots,n\}} \pi^{-1}_{2n+1,2s-1}(C^{s,s}_{i,2s-1} \cap D(x_i^{(s)})) \cup C^{n+1,n+1}_{i,2n+1}. \qquad (\diamond)$$

The proof of the claim is by induction on n. By lemma 4.3, we have that $S^0_1 = C^{1,1}_{i,1}$ for any $i = 1,...,e$, hence the case $n=0$. Using the inductive hypothesis for $n-1$, and the fact that for $s \in \{1,\ldots,n-1\}$ we have that $\pi_{2n-1,2s-1} \circ \pi_{2n+1,2n-1} = \pi_{2n+1,2s-1}$, we obtain:

$$S^0_{2n+1} = \pi^{-1}_{2n+1,2n-1}(S^0_{2n-1}) =$$

$$\bigcup_{i\in\{1,\ldots,e\},s\in\{1,\ldots,n-1\}} \pi^{-1}_{2n+1,2s-1}(C^{s,s}_{i,2s-1} \cap D(x_i^{(s)})) \cup \pi^{-1}_{2n+1,2n-1}(C^{n,n}_{i,2n-1}).$$

The claim follows from the stratification
$C^{n,n}_{i,2n-1} = \bigcup_{j=1,\cdots,e}(C^{n,n}_{i,2n-1} \cap D(x_j^{(n)})) \cup (C^{n,n}_{i,2n-1} \cap V(x_1^{(n)},\cdots,x_e^{(n)}))$,
and from the fact that by lemma 4.3, $\pi^{-1}_{2n+1,2n-1}(C^{n,n}_{i,2n-1} \cap V(x_1^{(n)},\cdots,x_e^{(n)})) = C^{n+1,n+1}_{i,2n+1}$.
We conclude the proof of the proposition for $m = 2n+1$ from proposition 4.11 (1) and (2).

The case m =2(n+1), $n \geq 0$: by $(\diamond)$ we just need to prove that for $n \geq 0$, and $i = 1,\ldots,e$ we have that

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}) = \cup_{\{i=2,\cdots,e-1\ ;\ l=n+1,\cdots,L^{n+1}_{i,2(n+1)}\}} C^{n+1,l}_{i,2(n+1)}.$$

First note that by lemma 4.3 and 4.4, we have the inclusion

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}) \supset \cup_{\{i=2,\cdots,e-1\ ;\ l=n+1,\cdots,L^{n+1}_{i,2(n+1)}\}} C^{n+1,l}_{i,2(n+1)}. \quad (\diamond)$$

The proof of the opposite inclusion is by induction on the embedding dimension e of S. First assume that $e = 4$; the equations defining S in $\mathbb{A}^4$ are E_{13}, E_{14}, E_{24}. So the ideal defining $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1})$ in $\mathbb{A}^4_{2(n+1)}$ is generated by

$$(x_j^{(0)}, \ldots, x_j^{(n)}, E_{13}^{(2n+2)}, E_{14}^{(2n+2)}, E_{24}^{(2n+2)};\ j = 1, \ldots, 4),$$

hence every irreducible component of $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1})$ has codimension in $\mathbb{A}^4_{2(n+1)}$ less than or equal to $4(n + 1) + 3 = 4n + 7$.
Now we have that

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}) = \bigcup_{j=1,\ldots,4} \pi^{-1}_{2(n+1),2n+1}((C^{n+1,n+1}_{i,2n+1} \cap D(x_j^{(n+1)})))$$

$$\cup\ \pi^{-1}_{2(n+1),2n+1}((C^{n+1,n+1}_{i,2n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)})))$$

$$= \bigcup_{j=1,\ldots,4} \overline{\pi^{-1}_{2(n+1),2n+1}((C^{n+1,n+1}_{i,2n+1} \cap D(x_j^{(n+1)})))}$$

$$\cup\ \pi^{-1}_{2(n+1),2n+1}((C^{n+1,n+1}_{i,2n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)}))).$$

Moreover by proposition 4.11 part (2), indices 1 and 4 are superfluous. In addition by lemma 4.3 and proposition 4.11. 1), we have that for $j = 2, 3$,

$$\overline{\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap D(x_j^{(n+1)}))} = \bigcup_{l=n+1,\ldots,(2(n+1))_j^{n+1}} C^{n+1,l}_{j,2(n+1)}.$$

Hence $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}) =$

$$\bigcup_{l=n+1,\ldots,(2(n+1))_j^{n+1};\ j=2,3} C^{n+1,l}_{j,2n+1}\ \cup\ \pi^{-1}_{2(n+1),2n+1}((C^{n+1,n+1}_{i,2n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)}))).$$

Finally we have that $\pi^{-1}_{2(n+1),2n+1}((C^{n+1,n+1}_{i,2n+1} \cap V(x_1^{(n+1)}, \ldots, x_4^{(n+1)}))) =$

$$\{\gamma \in S_{2(n+1)},\ ord_\gamma x_j \geq n+2,\ j = 1, \ldots, 4\} = \{\gamma \in \mathbb{A}^4_{2(n+1)},\ ord_\gamma x_j \geq n+2,\ j = 1, \ldots, 4\}$$

$$= V(x_j^{(0)}, \ldots, x_j^{(n+1)};\ j = 1, \ldots, 4)$$

is irreducible of codimension $4(n + 2)$ in $\mathbb{A}^4_{2(n+1)}$. Since $4(n + 2) > 4n + 7$, it is not an irreducible component of $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1})$, hence the claim.
We now assume the lemma to be true for toric surfaces $\tilde{S}$ of embedding dimension $\tilde{e}$ with $4 \leq \tilde{e} \leq e - 1$. We have that $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}) =$

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap D(x_e^{(n+1)}))\ \cup\ \pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap V(x_e^{(n+1)}).$$

By proposition 4.11, part (2), $\overline{\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap D(x^{(n+1)}_e))} \subset C^{s,s}_{e-1,2(n+1)}$, so it remains to determine $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap V(x^{(n+1)}_e))$. The discussion splits into two cases:
i) There exists $h \in \{3,\dots,e\}$ such that $c_{h-1} > 2$ and $c_h = \cdots = c_{e-1} = 2$.
By lemma 4.3, we have that $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap V(x^{(n+1)}_e) =$

$$\{\gamma \in S_{(2n+1)};\ ord_\gamma x_j \geq n+1,\ 1 \leq j \leq e-1,\ ord_\gamma x_e \geq n+2\} =$$

$$\{\gamma \in \mathbb{A}^e_{(2n+1)};\ ord_\gamma x_j \geq n+1,\ 1 \leq j \leq e-1,\ ord_\gamma x_e \geq n+2,$$
$$ord_\gamma E_{jk} \geq 2n+3,\ 1 \leq j < k-1 \leq e-1\}.$$

Now recall that $E_{e-2,e} = x_{e-2}x_e - x^{c_{e-1}}_{e-1}$. If $h < e$, we have that $c_{e-1} = 2$, so for $\gamma \in \mathbb{A}^e_{2(n+1)}$ such that $ord_\gamma x_{e-2} \geq n+1, ord_\gamma x_e \geq n+2$ and $ord_\gamma E_{e-2,e} \geq 2n+3$, we thus have that $2ord_\gamma x_{e-1} \geq 2n+3$ hence $ord_\gamma x_{e-1} \geq n+2$. Similarly, if $i \geq h$, for $\gamma \in \mathbb{A}^e_{2(n+1)}$ such that $ord_\gamma x_{i-1} \geq n+1, ord_\gamma x_{i+1} \geq n+2$ and $ord_\gamma E_{i-1,i+1} \geq 2n+3$, we get that $ord_\gamma x_i \geq n+2$. By descending induction on i, this shows that

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap V(x^{(n+1)}_e) \subset V(x^{(n+1)}_h, \dots, x^{(n+1)}_e).$$

Note that this inclusion is verified by definition when $h = e$. Moreover, for $\gamma \in \mathbb{A}^e_{2(n+1)}$ such that $ord_\gamma x_j \geq n+1$(resp. $n+2$) for $1 \leq j < h$(resp. $h \leq j \leq e$), we have that $ord_\gamma E_{jk} \geq 2n+3$ if $h \leq k \leq e$, indeed we have that

$$ord_\gamma x_j x_k \geq n+1+n+2 = 2n+3, \quad \text{and}$$

$$ord_\gamma x_{j+1}x^{c_{j+1}-2}_{j+1} \dots x^{c_{k-1}-2}_{k-1} x_{k-1} \geq 3(n+1) \text{ (resp. } n+1+n+2)$$

for $k = h$(resp. $k > h$). Therefore we have that $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap V(x^{(n+1)}_e) =$

$$\{\gamma \in \mathbb{A}^e_{2(n+1)};\ ord_\gamma x_j \geq n+1, 1 \leq j \leq h-1, ord_\gamma x_j \geq n+2, h \leq j \leq e,$$
$$ord_\gamma E_{jk} \geq 2n+3, 1 \leq j < k-1 \leq h-2\}. \quad (\diamond\diamond)$$

If $h \geq 5$, this can be interpreted geometrically as follows: Let $\tilde{S}$ be the toric surface in $\mathbb{A}^{h-1} = \mathrm{Spec}[x_1,\dots,x_{h-1}]$ defined by the ideal generated by $(E_{jk}, 1 \leq j < k-1 \leq h-2)$ and for $i = 2,\dots,h-2$, $m \in \mathbb{N}, s \in \{1,\dots,\lceil \frac{m}{2} \rceil\}, l \in \{s,\dots,L^s_{i,m}\}$ let

$$\tilde{D}^{s,l}_{i,m} = \{\gamma \in \tilde{S}_m;\ ord_\gamma x_i = s, ord_\gamma x_{i+1} = l\}$$

and $\tilde{C}^{s,l}_{i,m} = \overline{\tilde{D}^{s,l}_{i,m}}$; finally for $m > p$, let $\tilde{\pi}_{m,p} : \tilde{S}_m \longrightarrow \tilde{S}_p$ be the canonical projection. By lemma 4.3 again, we have that

$$\tilde{\pi}^{-1}_{2(n+1),2n+1}(\tilde{C}^{n+1,n+1}_{i,2n+1}) = \{\gamma \in \mathbb{A}^{h-1}_{2(n+1)};\ ord_\gamma x_j \geq n+1, 1 \leq j \leq h-1,$$
$$ord_\gamma E_{jk} \geq 2n+3, 1 \leq j < k-1 \leq h-2\}.$$

Therefore we deduce that $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}\cap V(x_e^{(n+1)})=$

$$\tilde{\pi}^{-1}_{2(n+1),2n+1}(\tilde{C}^{n+1,n+1}_{i,2n+1})\times \mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=h,\ldots,e],$$

which by the inductive hypothesis is equal to

$$\bigcup_{i=2,\ldots,h-2;\ l=n+1,\ldots,L^{n+1}_{i,2(n+1)}}\tilde{C}^{n+1,l}_{i,2(n+1)}\times \mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=h,\ldots,e].$$

Newt we claim that

$$\bigcup_{i=2,\ldots,h-2;\ l=n+1,\ldots,L^{n+1}_{i,2(n+1)}}C^{n+1,l}_{i,2(n+1)}\subset V(x_h^{(n+1)},\ldots,x_e^{(n+1)}).$$

Indeed, let $\gamma\in D^{n+1,l}_{i,2(n+1)}$ for some i and l in the above union. We have that $\gamma\in\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1})$, i.e. $ord_\gamma x_j\geq n+1$ for $1\leq j\leq e$, $ord_\gamma x_i=n+1$ and $ord_\gamma E_{ie}\geq 2n+3$. Since $i\leq h-2$ and $c_{h-1}>2$, this implies that

$$ord_\gamma x_{i+1}x_{i+1}^{c_{i+1}-2}\ldots x_{e-1}^{c_{e-1}-2}x_{e-1}\geq 2n+3,$$

therefore $ord_\gamma x_ix_e\geq 2n+3$, thus $ord_\gamma x_e\geq n+2$, and since we have proved that

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}\cap V(x_e^{(n+1)})\subset V(x_h^{(n+1)},\ldots,x_e^{(n+1)}),$$

we deduce that $C^{n+1,l}_{i,2(n+1)}=\overline{D^{n+1,l}_{i,2(n+1)}}\subset V(x_h^{(n+1)},\ldots,x_e^{(n+1)})$.
Finally by proposition 4.7, $C^{n+1,l}_{i,2(n+1)}$(resp. $\tilde{C}^{n+1,l}_{i,2(n+1)}$) is irreducible of codimension $(n+1)e+e-2$(resp. $(n+1)(h-1)+h-3$) in $\mathbb{A}^e_{2(n+1)}$(resp. $\mathbb{A}^{h-1}_{2(n+1)}$), therefore

$$\dim\ C^{n+1,l}_{i,2(n+1)}=\dim\ \tilde{C}^{n+1,l'}_{i',2(n+1)}\times \mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=h,\ldots,e]$$

for any $i'\in\{2,\ldots h-2\}, l'\in\{n+1,\ldots,L^{n+1}_{i',2(n+1)}\}$, and we deduce from the first inclusion $(\diamond)$ that $C^{n+1,l}_{i,2(n+1)}$ coincides with $\tilde{C}^{n+1,l'}_{i',2(n+1)}\times \mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=h,\ldots,e]$ for some $i'\in\{2,\ldots h-2\}$, and $l'\in\{n+1,\ldots,L^{n+1}_{i',2(n+1)}\}$.
But we have that $ord_\gamma x_i=n+1$, $ord_\gamma(x_{i+1})=l$ for γ the generic point of $C^{n+1,l}_{i,2(n+1)}$, therefore since $i+1\leq h-1$, we have that $ord_{\tilde{\gamma}}x_i=n+1$ and $ord_{\tilde{\gamma}}x_{i+1}=l$ for $\tilde{\gamma}$ the generic point of $\tilde{C}^{n+1,l'}_{i',2(n+1)}$. Therefore $\tilde{\gamma}\in\tilde{C}^{n+1,l}_{i,2(n+1)}$ and we deduce that $\tilde{C}^{n+1,l'}_{i',2(n+1)}\subset\tilde{C}^{n+1,l}_{i,2(n+1)}$. But since they are irreducible of the same codimension they are equal, so we have that

$$C^{n+1,l}_{i,2(n+1)}=\tilde{C}^{n+1,l}_{i,2(n+1)}\times \mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=h,\ldots,e].$$

We thus have that

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}\cap V(x_e^{(n+1)}))=\bigcup_{i=2,\ldots,h-2;l=n+1,\ldots,L^{n+1}_{i,2(n+1)}}C^{n+1,l}_{i,2(n+1)},$$

and the claim follows.(Note that we get that

$$\bigcup_{i=2,\ldots,h-2,e-1,l=n+1;\ \ldots,L^{n+1}_{i,2(n+1)}} C^{n+1,l}_{i,2(n+1)} = \bigcup_{i=2,\ldots,e-1,l=n+1;\ \ldots,L^{n+1}_{i,2(n+1))}} C^{n+1,l}_{i,2(n+1)}$$

as an immediate consequence of lemma 4.3 and lemma 4.12.)

If $h=4$, let $\tilde{S}$ be the toric surface in $\mathbb{A}^3=\mathrm{Spec}\mathbb{K}[x_1,x_2,x_3]$ defined by the ideal $(E_{1,3})$ and let $\tilde{C}^{n+1}_{2(n+1)}=\{\gamma\in\tilde{S}_{2(n+1)};\ ord_\gamma x_j\geq n+1,\ j=1,2,3\}$. The equality $(\diamond\diamond)$ reduces to

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}\cap V(x_e^{(n+1)}))=\tilde{C}^{n+1}_{2(n+1)}\times\mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=4,\ldots,e].$$

Since $E_{13}=x_1x_3-x_2^{c_2}$, if $c_2>2$, $\tilde{C}^{n+1}_{2(n+1)}\subset\mathrm{Spec}\mathbb{K}[x_j^{(n+1)},\ldots,x_j^{(2(n+1))},j=1,\ldots,3]$ is defined by the ideal $(x_1^{(n+1)}x_3^{(n+1)})$, so $\tilde{C}^{n+1}_{2(n+1)}=V(x_1^{(n+1)})\cup V(x_3^{(n+1)})$ while it is irreducible if $c_2=2$.
We check as above that

$$\bigcup_{l=n+1,\ldots,L^{n+1}_{2,2(n+1)}} C^{n+1,l}_{2,2(n+1)}\subset V(x_4^{(n+1)},\ldots,x_e^{(n+1)})$$

and that $\dim C^{n+1,l}_{2,2(n+1)}$ coincides with the dimension of any irreducible components of $\tilde{C}^{n+1}_{2(n+1)}\times\mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=4,\ldots,e]$. Again in view of $(\diamond)$, each $C^{n+1,l}_{2,2(n+1)}$ is an irreducible component of $\tilde{C}^{n+1}_{2(n+1)}\times\mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=4,\ldots,e]$.
If $c_2=2$, then $L^{n+1}_{2,2(n+1)}=n+1$ and we thus have

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}\cap V(x_e^{(n+1)}))=C^{n+1,n+1}_{2,2(n+1)}.$$

If $c_2>2$, we have that $L^{n+1}_{2,2(n+1)}=n+2$, and the same argument as above shows that

$$C^{n+1,n+1}_{2,2(n+1)}=V(x_1^{(n+1)})\times\mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=4,\ldots,e]$$

$$C^{n+1,n+2}_{2,2(n+1)}=V(x_3^{(n+1)}\times\mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=4,\ldots,e].$$

We thus have

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}\cap V(x_e^{(n+1)}))=\bigcup_{l=n+1;\ \ldots,L^{n+1}_{i,2(n+1))}} C^{n+1,l}_{2,2(n+1)}$$

hence the claim.
Finally if $h=3$, by $(\diamond\diamond)$ we have that $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}\cap V(x_e^{(n+1)}))=$

$$\mathrm{Spec}\mathbb{K}[x_j^{(n+1)},\ldots,x_j^{(2(n+1))},j=1,2]\times\mathrm{Spec}\mathbb{K}[x_j^{(n+2)},\ldots,x_j^{(2(n+1))},j=3,\ldots,e].$$

Now we have that $C^{n+1,n+1}_{2,2(n+1)} \subset V(x_3^{(n+1)},\ldots,x_e^{(n+1)})$. Indeed, for $\gamma \in D^{n+1,n+2}_{i,2n+1}$, we have that $ord_\gamma x_2 = n+1, ord_\gamma x_3 = n+2$, $ord_\gamma x_j \geq n+1, j = 4,\ldots,e$ and $ord_\gamma E_{2j} \geq 2n+3$ for $j = 4,\ldots,e$. Since $c_3 = \ldots = c_{e-1} = 2$, this implies that $ord_\gamma x_j \geq n+2$ for $j = 4,\ldots,e$, so $\gamma \in V(x_3^{(n+1)},\ldots,x_e^{(n+1)})$. We conclude that $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1} \cap V(x_e^{(n+1)})) = C^{n+1,n+2}_{2,2(n+1)}$ because both sets are irreducible and have the same dimension, and the claim follows in this case.

ii) If $c_2 = \cdots = c_{e-1} = 2$ then

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}) = V(x_i^{(0)},\ldots,x_i^{(n)},\ i = 1,\ldots,n,$$
$$x_i^{(n+1)}x_j^{(n+1)} - x_{i-1}^{(n+1)}x_{j-1}^{(n+1)}, 1 \leq i < j-1 \leq e-1).$$

The ideal generated by $(x_i^{(n+1)}x_j^{(n+1)} - x_{i-1}^{(n+1)}x_{j-1}^{(n+1)},\ 1 \leq i < j-1 \leq e-1)$, is isomorphic to the ideal defining S in $\mathbb{A}^e$, hence it is prime and $\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1})$ is irreducible. It follows from proposition 4.11, part (2) that

$$\pi^{-1}_{2(n+1),2n+1}(C^{n+1,n+1}_{i,2n+1}) = C^{n+1,n+1}_{e-1,2(n+1)},$$

thus the proposition in this case.

□

Remark 4.14. *Note that the argument that we use in the proposition 4.13 for $e = 4$ does not work in general. The argument works in the case $e = 4$ because the number of equations that define $S \subset \mathbb{A}^e$ (this number is $\binom{2}{e-1}$) is less or equal to e if and only if $e \leq 4$.*

Theorem 4.15. *Let $m \in \mathbb{N},\ m \geq 1$. Modulo the identifications $C^{s,s}_{i,m} = C^{s,L^s_{i+1,m}}_{i+1,m}$, the irreducible components of $S^0_m := \pi_m^{-1}(0)$ are the $C^{s,l}_{i,m},\ i = 2,\cdots,e-1,\ s \in \{1,\ldots,\lceil \frac{m}{2} \rceil\}$ and $l \in \{s,\ldots,L^s_{i,m}\}\}$. The irreducible components of S_m are $\overline{\pi_m^{-1}(S\backslash 0)}$ and the irreducible components of S^0_m.*

Proof : By proposition 4.13, S_m is covered by the $C^{s,l}_{i,m}$. Consider $C^{s,l}_{i,m}$ with $l \neq L^s_{i,m}$; since $l < L^s_{i,m}$ this implies that $m > 2s-1$ and $c_i \neq 2$. For the generic point γ we know from lemma 4.2 (part 2) that for $1 \leq j \leq i-1, ord_\gamma x_j > s$.

This forbids that $C^{s,l'}_{i',m} \subset C^{s,l}_{i,m}$ or $C^{s,l}_{i,m} \subset C^{s,l'}_{i',m}$ for $i' \in \{2,\ldots,i-1\}$ because by proposition 4.7, they have the same codimension in $\mathbb{A}^e_m$, hence an inclusion as above implies that they shoud coincide, so $ord_\gamma x_{i'} = s$. On the other hand, $C^{s,l}_{i,m} \not\subset C^{s',l'}_{i',m}$, if $s < s'$, because by proposition 4.11, $C^{s,l}_{i,m}$ has non-empty intersection with $D(x_i^{(s)})$, but $C^{s',l'}_{i',m} \subset V(x_i^{(s)})$. Finally, $C^{s',l'}_{i',m} \not\subset C^{s',l'}_{i',m}$ because by proposition 4.7 the codimension of the first one, is less then or equal to the codimension of the second one, and the first statement of the theorem follows. The last statement of the theorem follows from the fact that

$$codim(C^{s,l}_{i,m},\mathbb{A}^e_m) \leq codim(\overline{\pi_m^{-1}(S\backslash 0)},\mathbb{A}^e_m).$$

Indeed : By proposition 4.7, $codim(C^{s,l}_{i,m},\mathbb{A}^e_m) = se+(m-(2s-1))(e-2)$. By remark 2.2, we have that $codim(\overline{\pi_m^{-1}(S\backslash 0)},\mathbb{A}^e_m) = (m+1)(e-2)$. and we have that for $s\geq 1, se+(m-(2s-1))(e-2)\leq (m+1)(e-2)$ if and only if $e\geq 4$. $\square$

Definition 4.16. *Let $m\in\mathbb{N},\ m\geq 1$, and let C be an irreducible component of S^0_m and γ be its generic point. By Theorem 4.15, there exist $s\in\{1,\ldots,\lceil\frac{m}{2}\rceil\}$, $l\in\{s,\ldots,L^s_{i,m}\}$ and $i\in\{2,\cdots,e-1\}$ such that $C=C^{s,l}_{i,m}$. We say that C has index of speciality s.*

Note that $s=ord_\gamma(M):=\min_{f\in M}\{ord_\gamma(f)\}$ where M is the maximal ideal of the local ring $O_{S,0}$ and γ the generic point of C.

For $i=2,\ldots,e-1$, and $m\in\mathbb{N}$, we set

$$N^s_{c_i}(m):=L^s_{i,m}-s+1.$$

For $m\in\mathbb{N},\ m\geq 1$, we call $N(m)$ the number of irreducible component of S^0_m. Then counting the irreducible components in the Theorem 4.15 we find

Corollary 4.17. *If all the c_i are equal to 2, then $N(m)=\lceil\frac{m}{2}\rceil$. Otherwise let $c_{i_1},...,c_{i_h}$ be the elements in $\{c_2,\ldots,c_{e-2}\}$ different from 2, then we have*

$$N(m)=\sum_{s=1}^{\lceil\frac{m}{2}\rceil}(N^s_{c_{i_1}}(m)+(N^s_{c_{i_2}}(m)-1)+\ldots+(N^s_{c_{i_h}}(m)-1)).$$

Moreover, for $s\in\{1,\ldots,\lceil\frac{m}{2}\rceil\}$, the number of irreducible components of S^0_m of index of speciality s is equal to

$$N^s_{c_{i_1}}(m)+(N^s_{c_{i_2}}(m)-1)+\ldots+(N^s_{c_{i_h}}(m)-1.$$

Corollary 4.18. *Let S be a toric surface. The number of irreducible components of S^0_m and their dimensions determine the embedding dimension e of S and the set $\{c_t,t=2,\ldots,e-2\}$.*

Proof : We have that $\dim(S^0_1)=e$, the embedding dimension of S. If $e=3$, then for m big enough, we have by theorem 3.1 that $N(m)=c$ is constant, and we deduce that S is an A_c singularity. Suppose the $e\geq 4$.
For $m\geq 1$, let

$$\tilde{N}_1(m)=\sum_{s=1}^{\lceil\frac{m}{2}\rceil}((m+1-(2s-1))+(e-3)(m+1-(2s-1)-1).$$

We have that $N(m)\leq\tilde{N}_1(m)$ and $N(1)=\tilde{N}_1(1)=1$. Let

$$m_1=min\{m\ ;\ N(m)<\tilde{N}_1(m)\}\ \text{ and }\ \alpha_1=\tilde{N}_1(m_1)-N(m_1),$$

then there exists $i_1,\cdots,i_{\alpha_1}\in\{c_2,\ldots,c_{e-1}\}$ such that $c_{i_1}=\cdots=c_{i_{\alpha_1}}=m_1$.
If $\alpha_1=e-2$, then we have found all the c_i. If not, then for $j\geq 2$, we recursively define

$$\tilde{N}_j(m)=\sum_{s=1}^{\lceil\frac{m}{2}\rceil}(N^s_{c_{i_1}}(m)+(N^s_{c_{i_2}}(m)-1)+\cdots+(N^s_{c_{i_{\alpha_1}}}(m)-1)+\cdots+$$

$$(N^s_{c_{i_{\alpha_1+\cdots+\alpha_{j-1}}}}(m)-1))+(e-2-(\alpha_1+\cdots+\alpha_{j-1}))(m+1-(2s-1)-1),$$

$$m_j=min\{m\ ;\ N(m)<\tilde{N}_j(m)\}\ \text{ and }\ \alpha_j=\tilde{N}_j(m_j)-N(m_j).$$

Therefore there exists $i_{\alpha_1+\cdots+\alpha_{j-1}+1},\cdots,i_{\alpha_1+\cdots+\alpha_{j-1}+\alpha_j}\in\{c_2,\ldots,c_{e-1}\}$ such that

$$c_{i_{\alpha_1+\cdots+\alpha_{j-1}+1}}=\cdots=c_{i_{\alpha_1+\cdots+\alpha_{j-1}+\alpha_j}}=m_j.$$

If $\alpha_1+\cdots+\alpha_{j-1}+\alpha_j=e-2$, then we have found all the c_t, otherwise we repeat the procedure at most $e-2$ times. □

Remark 4.19. *Corollary* 4.18 *is to compare with the result of Nicaise in [Ni], where he proved that the motivic Igusa Poincaré series of a toric surface is equivalent to the set* $\{c_t,t=2,\ldots,e-2\}$, *and that the order of the* c_i *in the continued fraction can not be extracted from this series. It is clear also from the formulas given in proposition 4.7 and corollary 4.17, that the number of irreducible components and their dimensions is not affected by the order of the* c_i *in the continued fraction. Note that despite that these informations on the jet schemes are closely related to the informations encoded in the motivic Igusa Poincaré series, they are not equivalent in general.*

Below we show how we extract all the c_i and their order or equivalently the analytical type of S from their jet schemes. We first explain in the next proposition how the components $C^{1,l}_{i,m}$ behave under the truncation morphisms $\pi_{m,m-1}$. The proof follows from section 3 and propositions 4.10,4.13.

Proposition 4.20. *Let* $m\in\mathbb{N},m\geq 1$. *Let* $i\in 2,\ldots,e-1$, *and* $l\in\{1,\ldots,L^1_{i,m}\}$. *For* $2\leq m\leq c_i-1$, *we have the following inclusions*

$$\pi_{m,m-1}(C^{1,l}_{i,m})\subset C^{1,l-1}_{i,m-1},$$

whenever $l-1\in\{1,\ldots,L^1_{i,m-1}\}$.

$$\pi_{m,m-1}(C^{1,l}_{i,m})\subset C^{1,l}_{i,m-1},$$

whenever $l\in\{1,\ldots,L^1_{i,m-1}\}$. *For* $m\geq c_i$, *we have*

$$\pi_{m,m-1}(C^{1,l}_{i,m})\subset C^{1,l}_{i,m-1},$$

for $l\in\{1,\ldots,c_i-1\}$. *And these are all the inclusions between components of index of speciality* 1 *induced by* $\pi_{m,m-1},m\geq 1$.

$$\pi_{3,2}(C^{2,2}_{i,3})\subset C^{1,l'}_{i',2},\ \textit{for}\ \ i'\in\{2,\ldots,e-1\},l'\in\{1,L^1_{i',2}\},$$

This means that it is included in all the irreducible components of the level 2 *jet scheme.*

As in section 3, we now attach to the structure of the jet schemes of S a **weighted graph** that detects the invariants of the singularity S.

Definition 4.21. *1. The weighted graph of the jet schemes of S is the levelled weighted graph Γ obtained by*

- *representing every irreducible components of $S_m^0, m \geq 1$, by a vertex $v_{i,m}$, where the sub-index m is the level of the vertex;*
- *joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if the morphism $\pi_{m+1,m}$ induces a morphism between the corresponding irreducible components;*
- *weighting each vertex by the dimension of the corresponding irreducible component.*

2. The index 1 weighted graph of the jet schemes of S is the subgraph Γ^1 of Γ whose vertices are those associated with the components of index of speciality equal to 1. It is obtained from Γ by deleting the other vertices (those corresponding to irreducible components of index of speciality different from 1) and edges with at least one of the extremities not corresponding to an irreducible component of index of speciality 1.

We first will describe the subgraph Γ^1. The last inclusion in the proposition 4.20 implies that we can detect the vertex associated with the component $C_{i,3}^{2,2}$. We then can extract the graph Γ^1 from Γ by deleting all the vertices and edges which are connected to the vertex associated with $C_{i,3}^{2,2}$, and whose index of speciality is not 1. Then, applying proposition 4.20, we find that Γ^1 can be constructed from the c_i's as follows: for every $i = 2, \ldots, i-1$, let Γ_i^1 be the graph whose vertices are in 1 - 1 correspondence with the irreducible components $C_{i,m}^{1,l}, m \geq 1$, and $l \in \{1, \ldots, L_{i,m}^1\}$; the graph Γ_i^1 coincides with the graph associated with an $A_{c_{i-1}}$ singularity in section 3. The identifications $C_{i,m}^{1,1} = C_{i+1,m}^{1,L_{i+1,m}^1}$, induce identifications between infinite lines of Γ_i^1 and Γ_{i+1}^1 (See the next example). Then Γ^1 is the union of Γ_i^1 modulo the identifications. We then obtain :

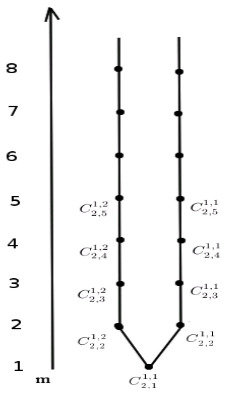
Corollary 4.22. *Let S be a toric surface.*

1. *The weighted graph Γ determines the embedding dimension e of S and the set $\{c_t, t = 2, \ldots, e-2\}$.*
2. *The order of the c_i's which are different from 2.*

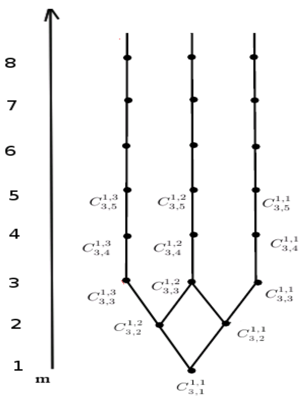
Proof. The first part follows from corollary 4.18 knowing only the weight of the vertex corresponding to S_1^0, i.e. its dimension. It follows from the discussion above (mainly from the last inclusion in the proposition 4.20) that, given Γ, we can extract Γ^1 from it. The order of the c_i's which are different from 2 is then extracted from Γ^1 thanks to the identifications described above (see the figure in the next example for an illustration). □

Remark 4.23. *Notice that if $c_i = 2$, then Γ_i^1 looks like a line and is not possible to detect where it sits on Γ^1 after the identifications.*

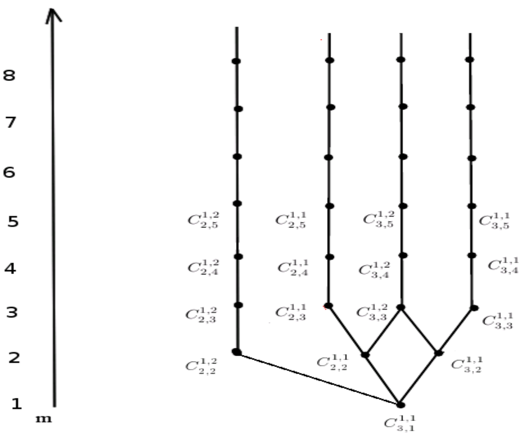
Example 3. *We consider the toric surface singularity defined by the cone generated by the vectors* $(1,0)$ *and* $(4,11)$*. We have that* $11/4 = [3,4]$*. Below we show the subgraph* $\Gamma^1 = \Gamma^1_2 \cup \Gamma^1_3$ *of the graph* Γ *of this singularity. First we show the graphs* Γ^1_2 *and* Γ^1_3*. To keep the figure the simplest possible, we do not weight the graph here with the codimensions; but he weights are essential to detect the invariants of* S*. And after the identifications we obtain* Γ^1 *:*



Γ^1_2



Γ^1_3



Γ^1

To recover $c_2, \ldots, c_{e-1}$ with their order (mainly the order where the c_i's which are equal to 2 appear), we will need to put more weights on the vertices of Γ associated with the irreducible components of S^0_2; these are $C^{1,2}_{i_1,2}, C^{1,1}_{i_2,2}, C^{1,1}_{i_2,2}, \ldots, C^{1,1}_{i_h,2}$ where $i_1, \ldots, i_h$ are like in corollary 4.17. Back to the equations of S, we find that $C^{1,2}_{i_1,2} \simeq S^{[i_0,i_1]} \times \mathbb{A}^e, C^{1,1}_{i_j,2} \simeq S^{[i_j,i_{j+1}]} \times \mathbb{A}^e$, for $j = 1, \ldots, h$ where $i_0 = 1, i_{h+1} = e, S^{[i_j,i_{j+1}]}$ is the toric surface defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x^{(1)}_{i_j} & \dots & x^{(1)}_{i_{j+1}-1} \\ x^{(1)}_{i_j+1} & \dots & x^{(1)}_{i_{j+1}} \end{pmatrix}$$

in $\mathbb{K}[x^{(1)}_{i_j},\dots,x^{(1)}_{i_{j+1}}]$ and $\mathbb{A}^e=\mathbb{K}[x^{(2)}_1,\dots,x^{(2)}_e]$. Note that for $j=0,\dots,h$, the embedding dimension of $S^{[i_j,i_{j+1}]}$ is $i_{j+1}-i_j+1$, in particular $S^{[i_j,i_{j+1}]}$ is isomorphic to $\mathbb{A}^2$ if and only if $i_{j+1}-i_j=1$. Hence, after weighting the vertices corresponding to irreducible components of S^0_2 by their embedding dimensions, we see how the $c_i's$ which are equal to 2 are distributed between the other $c_i's$. Hence we define an other weighted graph as follows:

Definition 4.24. *We denote by $E\Gamma$ the weighted graph which is obtained from Γ by weighting the vertices of Γ associated with the irreducible components of S^0_2 by their embedding dimensions (note that by the definition of Γ, these vertices are also weighted by their dimensions).*

Hence we obtain:

Corollary 4.25. *Let S be a toric suface. The data of the weighted graph $E\Gamma$ of S^0_m is equivalent to the data of all the c_i and of their order in the continued fraction, or equivalently to the analytical type of S.*

Remark 4.26. *Note that if we reverse the order of the c_t, the obtained toric surface will be isomorphic to the original one.*

Using a theorem of Mustata in [Mus2], we obtain as a by-product the log canonical threshold $lct(S,\mathbb{A}^e)$ of the pair $S\subset\mathbb{A}^e$:

Corollary 4.27. *Let S be a toric surface of embedding dimension e. If $e=3$ (i.e. S is an A_n singularity) then $lct(S,\mathbb{A}^e)=1$, otherwise*

$$lct(S,\mathbb{A}^e)=\frac{e}{2}$$

Proof : By [Mus2] we have that

$$lct(S,\mathbb{A}^e)=\min_{m\in\mathbb{N}}\frac{Codim(S_m,\mathbb{A}^e_m)}{m+1}.$$

The case $e=3$ follows from section 3, since in this case we have that S_m is irreducible of codimension $m+1$. Let us suppose that $e\geq 4$. If m is odd, $m=2s-1,\ s\geq 1$ then the component $C^{s,s}_{i,2s-1}$ is of maximal dimension and we have that

$$\frac{Codim(C^{s,s}_{i,2s-1},\mathbb{A}^e_{2s-1})}{2s}=\frac{se}{2s}=\frac{e}{2}.$$

If m is even, $m=2n,\ n\geq 0$ then the components $C^{n,l}_{i,2n},\ i=2,\dots,e-1,\ l=n, L^n_{i,m}$ are of maximal dimension, and since $e\geq 4$ we have that

$$\frac{Codim(C^{n,l}_{i,2n},\mathbb{A}^e_{2n})}{2n+1}=\frac{ne+e-2}{2n+1}\geq\frac{e}{2},$$

and the lemma follows.

□

Corollary 4.28. *For $m \geq max\{c_i,\ i = 2, \cdots, e-1\}$, the number of irreducible components of S^0_m, with index of speciality $s = 1$, is equal to the number of exceptional divisors that appear on the minimal resolution of S.*

Proof. This comes from the comparison of corollary 4.17 with proposition 2.3. □

Remark 4.29. *The corollary 4.28 is to compare with the bijectivity of the Nash map, due to Ishii and Kollar for this type of Singularities, [IK]. Actually, the projective limits of the systems $(C^{1,l}_{i,m})_m$ gives rise to the irreducible components of the space of arcs centred at the singular point of S.*

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