HILBERT MEETS RAMANUJAN: SINGULARITY THEORY AND INTEGER PARTITIONS

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ABSTRACT. What can singularities of algebraic varieties say about the decompositions of a positive integer into a sum of positive integers ?

1. INTRODUCTION

In his first letter to Hardy, dated 16 January 1913 ([18], p. 29) Ramanujan stated the formulas

$$1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\ddots}}}=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{1+\sqrt{5}}{2}\right)e^{\frac{2\pi}{5}} \tag{1.1}$$

$$1+\cfrac{e^{-\pi}}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-3\pi}}{\ddots}}}=\left(\sqrt{\frac{5-\sqrt{5}}{2}}-\frac{1-\sqrt{5}}{2}\right)e^{\frac{\pi}{5}} \tag{1.2}$$

about which Hardy writes in the article "The Indian Mathematician Ramanujan" ([34], p. 144):

"[These formulas] defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them."

This article is not exactly about these formulas, but about some identities which are at the heart of their proofs; this allows the author to enjoy writing them and probably the reader (who already knew them or not yet) to enjoy the scene. According to [17], the first proof of these formulas was given by Watson [56]; following [9], let us see how partitions, via the Rogers-Ramaunujan identities, play a fundamental role in the proof. Consider the $q-$difference equation

$$F(x)=F(xq)+xqF(xq^2), \tag{1.3}$$

where $q\in\mathbf{C}^*$ and $F(x)=\sum a_n(q)x^n$ is an analytic function satisfying $F(0)=1$.

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Let $c(x,q) := \frac{F(x)}{F(xq)}$; we have

$$c(x,q) = 1 + \frac{xq}{c(xq,q)} = 1 + \frac{xq}{1 + \frac{xq^2}{c(xq^2,q)}}.$$

Iterating this last identity, we find that the left member of the identity (1.1) is equal to $c(1,e^{-2\pi})$ and that the left member of the identity (1.2) is equal to $c(1,e^{-\pi})$. Now, if we plug $F(x) = \sum a_n(q)x^n$ in the equation (1.3), by comparing the coefficients of x^n on both sides, we get

$$a_n(q) = \frac{q^{n^2}}{(q)_n} = \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

This gives the left equalities in the following two identities:

$$F(1) = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(q)_n} = \prod_{i\equiv\ 1,4 \pmod 5} \frac{1}{1-q^i}. \tag{1.4}$$

$$F(q) = 1 + \sum_{n\geq 1} \frac{q^{n^2+n}}{(q)_n} = \prod_{i\equiv\ 2,3 \pmod 5} \frac{1}{1-q^i}. \tag{1.5}$$

The equalities on the right in (1.4) and (1.5) are two miracles, which are central in this article. They allow us to represent $c(1,q)$ as an infinite product and we may then deduce Ramanujan's continued fraction (1.1),(1.2) by an appeal to the theory of elliptic theta functions.

The "miracles" in (1.4) and (1.5) are called the Rogers-Ramanujan identities; it is magic how they appear "in many different domains": statistical mechanics, combinatorics and number theory, representation theory, probability theory and in Algebraic Geometry and Commutative Algebra; see [12, 16, 20, 22, 30, 33, 29]. Here we will concentrate on the Algebro-Geometric side of the story. But at first, since we have stated the Rogers-Ramanujan identities in terms of $q-$series, let us explain why these are partition identities.

Definition 1.1. A partition of a positive integer n is a decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ such that $\lambda_1 + \cdots + \lambda_r = n$. The λ_i's are called the parts of λ and r is its size.

For instance, 4 has 5 partitions:

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1. \tag{1.6}$$

The combinatorial version of Rogers-Ramanujan identities in terms of integer partitions is due to MacMahon [40] and Schur [53].

Theorem 1.2 (Rogers–Ramanujan identities, combinatorial version)**.** *Let n be a nonnegative integer and set $i \in \{1;2\}$. Denote by $T_{2,i}(n)$ the number of partitions of n such that the difference between two consecutive parts is at least 2 and the part 1 appears at most $i-1$ times. Let $E_{2,i}(n)$ be the number of partitions of n into parts congruent to $\pm 2 + i \mod 5$. Then we have*

$$T_{2,i}(n) = E_{2,i}(n).$$

For example, the partitions of 4 (see 1.6) which are counted by $T_{2,2}(4)$ are 4 and $3+1$; those which are counted by $E_{2,2}(4)$ are 4 and $1+1+1+1$. In particular we have $T_{2,2}(4) = E_{2,2}(4) = 2$, and the theorem says that this is the case for every positive integer n. The relation between the identities (1.4) and (1.5) and theorem 1.2 is that one can prove that the left member of (1.4) (respectively (1.5)) is the generating series of the sequence $T_{2,2}(n)$ (respectively $T_{2,1}(n)$) and it is not a difficult exercise to see that the right member of (1.4) (respectively (1.5)) is the generating series of the sequence $E_{2,2}(n)$ (respectively $E_{2,1}(n)$). Recall here that the generating series of a sequence of integer numbers $(a_n)_{n\in\mathbf{Z}_{\geq 0}}$ is by definition

$$\sum_{n\in\mathbf{Z}_{\geq 0}} a_n q^n.$$

The other important object (with integer partitions) for this article is the arc space, coming from algebraic geometry. Let $X \subset \mathbf{C}^e$ be an algebraic variety: *i.e.*, X is the zero locus in $\mathbf{C}^e$ of a set of polynomials in e variables with coefficients in the field $\mathbf{C}$ of complex numbers. The arc space X_∞ of X is a space which parametrizes the arcs (germs of formal curves) which are traced on X; so a point of X_∞ corresponds to an arc on X. As we will see, this is also an "algebraic variety" (or a scheme) which often is of infinite dimension. Arc spaces (and their finite dimensional approximations) play an important role in singularity theory, for instance via the Nash problem [49], motivic integration [25, 23], birational geometry [47] or equisingularity [45, 46, 37, 38].

This article tells, on the one hand, about a link between arc spaces and partition identities and on the other hand how this link allows one to discover and prove new partition identities. In the second section, we will introduce the arc space and the arc HP-series (the arc Hilbert-Poincaré series) which is an invariant of singularities of algebraic varieties; we will also show how to compute this series in some examples. The third section reveals the relation between the arc HP-series and Rogers-Ramanujan identities: differential algebra and Groebner basis theory play an important role here. The fourth section shows how one can guess and prove new partition identities using the link between arc spaces and integer partitions. The last section is about research directions which are related to the subject of this article but which have not been treated here. The article is meant to be self contained.

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2. The Arc Hilbert-Poincaré series

Let $\mathbf{C}$ be the field of complex numbers (any other field of characteristic zero would be good for this paper). Let $X \subset \mathbf{C}^3$ be an affine algebraic variety; the story is absolutely the same if we replace the 3 in $\mathbf{C}^3$ by an integer number e, modulo more notations; actually later we will consider examples where e (the embedding dimension) is $1, 2$ or 3. For the scope of this paper, we can consider X to be a hypersurface defined by a polynomial $f \in \mathcal{R} = \mathbf{C}[x, y, z]$, i.e.

$$X = \{(a_1, a_2, a_3) \in \mathbf{C}^3 \mid f(a_1, a_2, a_3) = 0\}.$$

Again, not much related to what we will tell changes if we replace the ideal generated by (f) by an ideal generated by a finite number of polynomials. We will also write

$$X = \mathrm{Spec}\frac{\mathbf{C}[x, y, z]}{(f)} = \mathrm{Spec}\frac{\mathcal{R}}{(f)}; \tag{2.1}$$

this latter notation emphasizes, as in modern Algebraic Geometry, on the fact the ring of polynomial functions defined on X with value in $\mathbb{C}$ is given by

$$\mathcal{O}_X = \frac{\mathbf{C}[x, y, z]}{(f)}.$$

For instance, the polynomial function defined by f (or any polynomial in the ideal (f) generated by f) is zero in $\mathcal{O}_X$; this meets the fact that for any $(a_1, a_2, a_3) \in X, f(a_1, a_2, a_3) = 0$. Moreover, the use of the notation Spec allows us to distinguish the variety (or scheme) defined by f from the one defined by f^2 (even though the underlying geometric object is the same); one can think of $\mathrm{Spec}\mathcal{R}/(f^2)$ as a kind of thickening of $\mathrm{Spec}\mathcal{R}/(f)$, since we have more polynomial functions on it, f for instance is not zero in $\mathcal{R}/(f^2)$.

An arc γ on X is defined by a string of power series

$$\gamma(t) = (x(t), y(t), z(t))$$

such that $f(\gamma(t)) = f(x(t), y(t), z(t)) = 0$. This latter equality says that the arc γ which was originally defined as an arc on $\mathbf{C}^3$ is an arc on X. Let us write

$$x(t) = \sum_{i \geq 0} x_i t^i, y(t) = \sum_{i \geq 0} y_i t^i, z(t) = \sum_{i \geq 0} z_i t^i, \tag{2.2}$$

and expand $f(\gamma(t)) =$

$$f(\sum_{i \geq 0} x_i t^i, \sum_{i \geq 0} y_i t^i, \sum_{i \geq 0} z_i t^i) = \sum_{j \geq 0} F_j(x_0, y_0, z_0, \ldots, x_j, y_j, z_j) t^j. \tag{2.3}$$

The data of an arc is then equivalent to the data of the coefficients

$$x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 0},$$

which satisfy the equations $F_j(x_0, y_0, z_0, \ldots, x_j, y_j, z_j) = 0$ for every $j \in \mathbf{Z}_{\geq 0}$. Hence the arc space which is the space of all arcs on X is the algebraic variety X_∞ which is defined in an infinite dimensional affine space (whose coordinates are $x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 0}$) by the polynomials $F_j, j \in \mathbf{Z}_{\geq 0}$. In other terms $X_\infty = \mathrm{Spec}\mathcal{O}_{X_\infty}$ where

$$\mathcal{O}_{X_\infty} = \frac{\mathbb{C}[x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 0}]}{(F_j, j \in \mathbf{Z}_{\geq 0})}.$$

Giving the variables x_i, y_i and z_i the weight i, the polynomials F_j are weighted-homogeneous of degree j : Indeed, if we replace in the equation (2.3) the variables x_i, y_i, z_i by $\lambda^i x_i, \lambda^i y_i, \lambda^i z_i$, it becomes

$$f(\sum_{i \geq 0} \lambda^i x_i t^i, \sum_{i \geq 0} \lambda^i y_i t^i, \sum_{i \geq 0} \lambda^i z_i t^i) = \sum_{j \geq 0} F_j(\lambda^0 x_0, \lambda^0 y_0, \lambda^0 z_0, \ldots, \lambda^j x_j, \lambda^j y_j, \lambda^j z_j) t^j;$$

At the same time, noticing that $\lambda^i t^i = (\lambda t)^i$ we can write the equation as follows

$$f(\sum_{i\geq 0} x_i(\lambda t)^i, \sum_{i\geq 0} y_i(\lambda t)^i, \sum_{i\geq 0} z_i(\lambda t)^i) = \sum_{j\geq 0} F_j(x_0, y_0, z_0, \ldots, x_j, y_j, z_j)(\lambda t)^j;$$

hence, by collecting the coefficients of t^j in both forms of the equation, we have

$$F_j(\lambda^0 x_0, \lambda^0 y_0, \lambda^0 z_0, \ldots, \lambda^j x_j, \lambda^j y_j, \lambda^j z_j) = \lambda^j F_j(x_0, y_0, z_0, \ldots, x_j, y_j, z_j).$$

This gives $\mathcal{O}_{X_\infty}$ a structure of a grading ring, *i.e.*, we have a decomposition

$$\mathcal{O}_{X_\infty} = \bigoplus_{j\geq 0} \mathcal{O}_{X_\infty, j},$$

as a direct sum of subgroups $\mathcal{O}_{X_\infty,j}$ such that the product of an element in $\mathcal{O}_{X_\infty,j}$ with an element in $\mathcal{O}_{X_\infty,j'}$ is an element in $\mathcal{O}_{X_\infty,j+j'}$. The fact that the F_j are weighted-homogeneous is essential, otherwise, we can have two polynomials in $\mathbb{C}[x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 0}]$ which are of different weights but whose images in $\mathcal{O}_{X_\infty}$ are equal. Still, $\mathcal{O}_{X_\infty}$ is not yet our favorite geometric object.

One notices that the data of a morphism of affine algebraic varieties $\phi : X \longrightarrow Y$ (a morphism which is defined by polynomial functions) is equivalent to the data of ring homomorphism $\phi^* : \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ which to a polynomial function h on X, *i.e.* $h \in \mathcal{O}_Y$, associates $\phi^*(h) = \phi \circ h$. Hence the natural ring morphism given by $\mathcal{O}_X \longrightarrow \mathcal{O}_{X_\infty}$ which sends x, y, z respectively to x_0, y_0, z_0, defines a morphism

$$\psi_X : X_\infty \longrightarrow X.$$

We sometimes omit X in the notation ψ_X when X is clear from the context. This is the morphism which to an arc $\gamma(t) = (x(t), y(t), z(t)) \in X_\infty$ associates $\gamma(0) \in X$, the center of γ. Let us assume that the origin $O = (0,0,0) \in X$ (by a change of variable any point $x \in X$ can be considered to be the origin). We are interested in the fiber $\psi^{-1}(O)$ of ψ above O. We have $\psi^{-1}(O) = \mathrm{Spec}\mathcal{A}_\infty$, where

$$\mathcal{A}_\infty = \frac{\mathbb{C}[x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 1}]}{(f_j, j \in \mathbf{Z}_{\geq 1})};$$

the f_j's are obtained from the $F_j's$ by substituting x_0, y_0, z_0 by 0. Hence the f_i's are again weighted-homogeneous when giving x_i, y_i and $z_i, i \in \mathbf{Z}_{>0}$ the weight i and $\mathcal{A}_\infty$ inherits a graded structure $\mathcal{A}_\infty = \bigoplus_{j\geq 0} \mathcal{A}_{\infty,j}$. We are now ready to define our invariant, the arc HP-series.

Definition 2.1. The arc HP-series of X at O is defined by

$$\mathrm{AHP}_{X,O}(q) := \sum_{j\in \mathbf{Z}_{\geq 0}} \dim_{\mathbf{C}} \mathcal{A}_{\infty,j} q^j.$$

Remark 2.2. The reason why we considered the arcs with center at a point (*i.e.* $\psi^{-1}(O)$) and not X_∞ is that the dimension over $\mathbf{C}$ of $\mathcal{O}_{X_\infty,0}$ (the homogeneous component of weight 0) is not finite ($\mathcal{O}_{X_\infty,0}$ is actually isomorphic to $\mathcal{O}_X$). Of course, one could consider the dimension over a generic point of an irreducible component of X, but in that case this series is much less interesting as it will be apparent later.

Example 2.3. The most basic example is the case where $X = \mathrm{Spec}\mathbf{C}[y] = \mathbf{A}^1$ is the affine line and O is the origin. Following the explanation above, we have

$$\mathcal{A}_\infty = \mathbf{C}[y_j, j \in \mathbf{Z}_{>0}],$$

with the graded structure induced from giving y_j the weight j for every $j \in \mathbf{Z}_{>0}$. In particular, $\mathcal{A}_{\infty,j}$ is generated, as a vector space over $\mathbf{C}$ by the monomials

$$y_{j_1}y_{j_2}\cdots y_{j_r}$$

where $j_1 + j_2 + \cdots + j_r = j$ and where we can assume $j_1 \geq j_2 \geq \cdots \geq j_r$. These generators are in bijection with the partitions of j, simply by associating with the monomial $y_{j_1}y_{j_2}\cdots y_{j_r}$ the partition $j = j_1 + j_2 + \cdots + j_r$. Let us use the usual notation $p(j)$ to denote the number of partitions of j, where by convention $p(0) = 1$. We then have

$$\mathrm{AHP}_{\mathbf{A}^1,O}(q) = \sum_{j\in\mathbf{Z}_{\geq 0}} p(j)q^j = \prod_{j\in\mathbf{Z}_{>0}} \frac{1}{1-q^j}$$

The equality to the right in the above equation is a formula which is due to Euler; one can prove it simply by substituting in the product

$$\frac{1}{1-q^j} = (1 + q^j + q^{2j} + \cdots).$$

and then by expanding the product using the usual product of power series. A similar computation gives us

$$\mathrm{AHP}_{\mathbf{A}^d,O}(q) = \prod_{j\in\mathbf{Z}_{>0}} \frac{1}{(1-q^j)^d}$$

Example 2.3 actually allows us to compute the arc HP-series in many examples. To see that, let us use a slightly fancier definition of an arc γ on a variety X : an arc γ on X is a morphism

$$\gamma : \mathrm{Spec}\mathbf{C}[[t]] \longrightarrow X.$$

Here, $\mathbf{C}[[t]]$ is the ring of power series with coefficients in $\mathbf{C}$. One can see it as the completion of the local ring of the affine line at the origin, as follows: the local ring of the affine line $\mathbf{A}^1 = \mathrm{Spec}\mathbf{C}[t]$ is the ring $\mathbf{C}[t]_{(t)}$ which is obtained from $\mathbf{C}[t]$ by inverting all the polynomials $h \in \mathbf{C}[t]$ whose values at the origin O is not 0. This is a local ring with a unique maximal ideal (t); the powers $(t)^n$ of this maximal ideal gives a basis of a topology on $\mathbf{C}[t]_{(t)}$. The completion $\widehat{\mathbf{C}[t]_{(t)}}$ of $\mathbf{C}[t]_{(t)}$ with respect to this topology is $\mathbf{C}[[t]]$. One moral of the story is that $\mathrm{Spec}\mathbf{C}[[t]]$ can be thought as a formal neighborhood of the origin in the affine line $\mathbf{A}^1$, hence the intuition that the image of γ is a germ of a formal curve on X. Now, if we are interested only in the arcs centered at the origin $O \in X$, then such an arc γ corresponds to a morphism $\gamma : \mathcal{O}_{X,0} \longrightarrow \mathbf{C}[[t]]$. Since $\mathbf{C}[[t]]$ is complete the universal property of completeness tells us that γ factors through a morphism $\hat{\gamma} : \widehat{\mathcal{O}_{X,0}} \longrightarrow \mathbf{C}[[t]]$. So, if we assume that the variety X is non-singular at O (for a hypersurface this is equivalent to say that the partial derivatives at O are not all zero) then by Cohen structure theorem ([27], section 7.4), the completion $\widehat{\mathcal{O}_{X,0}}$ is isomorphic to $\mathbb{C}[[y_1, \ldots, y_d]]$, d being the dimension of X at O. It follows that the data of any γ is equivalent to the data of a morphism $\widehat{\gamma}* : \mathbf{C}[[y_1, \ldots, y_d]] \longrightarrow \mathbf{C}[[t]]$ and that $\Psi_X^{-1}(O)$ is isomorphic to $\Psi_{\mathbf{A}^d}^{-1}(O)$. We conclude from example 2.3 the computation of $AHP_{X,O}$; Moreover, one can show that if X is singular at O, $AHP_{X,O} \neq AHP_{\mathbf{A}^d,O}$.

Proposition 2.4. Let X be an algebraic variety and consider a point $O \in X$. We have that X is non-singular at O if and only if

$$\mathrm{AHP}_{X,O}(q) = \prod_{j \in \mathbf{Z}_{>0}} \frac{1}{(1-q^j)^d}$$

Proposition 2.4 tells us that the arc HP-series is an invariant of singularities since it detects singular points from non-singular ones. It also tells us that this series contains more information at singular closed points (the case that we are considering); for instance if X is irreducible, it is non-singular at its generic point and its arc HP-series (where dimensions are considered over the residue field of the generic point) is equal to the series in the proposition; see [44] section 9 for a comparison of the information contained in this invariant with more classical invariants of singularities.

In general, it is quite difficult to compute this series, essentially because the homological complexity of the jet schemes (the finite dimensional approximation of the arc space); for instance even for curves singularities [45], the jet schemes have a lot of irreducible components and they are very far from being equidimensional. We will actually use jet schemes to show how to compute HP-series for some "simple" singularities. For $m \in \mathbf{Z}_{\geq 0}$, an $m-$jet α on X is a morphism $\alpha : \mathrm{Spec}\mathbf{C}[t]/(t^{m+1}) \longrightarrow X$. Following the same reasoning that we made to represent the arc space, we find that for an X like in (2.1) the $m-$th jet scheme of X is

$$X_m = \mathrm{Spec}\mathcal{O}_{X_m} = \mathrm{Spec}\frac{\mathbb{C}[x_i, y_i, z_i, i = 0, \ldots, m]}{(F_j, j = 0, \ldots, m)}.$$

Again, for the same reason as in the arc space case we have a natural morphism $\pi_m : X_m \longrightarrow X$ (again here, when it is clear from the context, we neglect the mentioning of X in the notation π_m) and we have $\pi^{-1}(O) = \mathrm{Spec}\mathcal{A}_m$ where

$$\mathcal{A}_m = \frac{\mathbb{C}[x_i, y_i, z_i, i = 1, \ldots, m]}{(f_1, \ldots, f_m)} \tag{2.4}$$

We are ready to determine the arc HP-series for rational double point surface singularities. These latter are somehow ubiquitous in singularity theory and in algebraic geometry [26]. For instance these are the only locally complete intersection rational surface singularities. Embedded in $\mathbf{C}^3$, They are defined via the equations:

$$A_n, \ n \in \mathbb{N} : xy - z^{n+1} = 0.$$
$$D_n, \ n \in \mathbb{N}, n \geq 4 : z^2 - x(y^2 + x^{n-2}) = 0.$$
$$E_6 : z^2 + y^3 + x^4 = 0$$
$$E_7 : x^2 + y^3 + yz^3 = 0.$$
$$E_8 : z^2 + y^3 + x^5 = 0$$

The following theorem was first proved in [43]; we give here a proof following [22].

Theorem 2.5. *Let X be surface having a rational double point singularity at O. We have*

$$\mathrm{AHP}_{X,O}(q) = \frac{1}{(1-q)^3} \prod_{j \geq 2} \frac{1}{(1-q^j)^2}$$

Proof. We will prove that $\pi_m^{-1}(O) \subset \mathrm{Spec}\mathbf{C}[x_i, y_i, z_i, i = 0, \ldots, m]$ is a complete intersection (*i.e.*, the codimension of all its irreducible components is equal to the number of its defining equations); the result will then follow from [54], knowing that the weight of f_j is j for $j = 1, \ldots, m$, and that by definition the Hilbert-Poincaré series of $\mathcal{A}_m$ is equal to the Hilbert-Poincaré series of $\mathcal{A}_m$ modulo (q^{m+1}). Notice that embedded in $\mathbf{A}_m^3 := \mathrm{Spec}\mathbf{C}[x_i, y_i, z_i, i = 0, \ldots, m]$, $\pi_m^{-1}(O)$ is defined by the ideal $(x_0, y_0, z_0, f_2, \ldots, f_m)$; *i.e.* by the equations given by all the generators of the ideal equal to 0 (f_1 does not appear here because it is equal to 0 modulo (x_0, y_0, z_0)). So the codimension of $\pi_m^{-1}(O)$ in $\mathbf{A}_m^3$ is smaller than or equal to $m+2$, the number of equations. We also know that $\pi_m^{-1}(X_{reg})$ (X_{reg} being the non-singular locus of X) is irreducible of codimension $m + 1$: indeed, one can see that the equations $F_j, j = 0, \ldots, m$ are linear outside $(x, y, z) = (0, 0, 0)$. If the codimension $\pi_m^{-1}(O)$ is smaller than or equal to $m + 1$, then $\pi_m^{-1}(O)$ cannot be included in the Zariski closure $\overline{\pi_m^{-1}(X_{reg})}$ of $\pi_m^{-1}(X_{reg})$ since its dimension is then larger than or equal to the dimension of $\overline{\pi_m^{-1}(X_{reg})}$; the other inclusion is also impossible since $\pi_m^{-1}(O) \subset \{x_0 = 0\}$ while $\overline{\pi_m^{-1}(X_{reg})}$ is not. We deduce that if the codimension $\pi_m^{-1}(O)$ is smaller than or equal to $m + 1$, then X_m has at least two irreducible components; this contradicts the fact that X_m is irreducible since X is locally complete intersection with rational singularities [48]. Hence we deduce that the codimension of any irreducible component of $\pi_m^{-1}(O)$ in $\mathbf{A}_m^3$ is exactly equal to $m + 2$, the number of the defining equations.

□

There are several other instances where the arc HP-series can be determined, see [44].

3. The arc HP-series and the Rogers-Ramanujan identities

The first Rogers-Ramanujan identity comes into the picture when considering one of the most elementary singularities, the one defined by (x^2) in the line. More precisely, from [21], we have:

Theorem 3.1. *Let* $X = \mathrm{Spec}\mathbf{C}[y]/(y^2)$. *We have*

$$\mathrm{AHP}_{X,O}(q) = \prod_{i \equiv 1,4 \mod 5} \frac{1}{1 - q^i}$$

Moreover, let $\mathcal{B}_\infty := \mathcal{A}_\infty/(y_1)$. Again $\mathcal{B}_\infty$ inherits from $\mathcal{A}_\infty$ a graded structure $\mathcal{B}_\infty = \oplus_{j \in \mathbf{Z}_{\geq 0}} \mathcal{B}_{\infty,j}$ and one can consider its Hilbert-Poincaré series

$$HP_{\mathcal{B}_\infty}(q) = \sum_{j \in \mathbf{Z}_{\geq 0}} \dim_{\mathbf{C}} \mathcal{B}_{\infty,j} q^j.$$

Theorem 3.2. *The Hilbert-Poincaré series of* $\mathcal{B}_\infty$ *is*

$$HP_{\mathcal{B}_\infty}(q) = \prod_{i \equiv 2,3 \mod 5} \frac{1}{1 - q^i}$$

We will now give a proof of theorem 3.1. This proof reduces the computations of the arc HP-series via the theory of Groebner basis to the computation of a Hilbert-Poincaré series of a quotient of an infinite dimensional polynomial ring by a monomial ideal. To apply this theory, we use the differential structure of the arc space. Let us say two words about these two concepts, one about each.

Groebner bases. The polynomial ring with one variable, $\mathbf{C}[x]$, is Euclidean, *i.e.* one can apply the Euclidean algorithm which says that given $g, h \in \mathbf{C}[x]$, there exists a unique couple of polynomials (q, r) such that

$$g = hq + r,$$

and $0 \leq \deg(r) < deg(h)$; we have that $r = 0$ if and only if h divides g. This algorithm is very useful to detect whether an element g belongs to an ideal $I \subset \mathbf{C}[x]$: indeed, again thanks to the Euclidean algorithm, $\mathbf{C}[x]$ is principal, $I = (h)$ is generated by one element and $g \in I$ if and only if h divides g, equivalently if $r = 0$. In the polynomial ring $R = \mathbf{C}[x_1, \ldots, x_n]$ with several variables, the ideals are finitely generated (Hilbert Basis theorem) but not principal in general; hence the need of a division algorithm which allows to divide a polynomial by several other polynomials. For that, the degree (which does not define a total ordering of monomials, many monomials may have the same degree) is replaced by a monomial ordering that we denote by $\prec$: this is a total ordering on the monomials of R which satisfies that for monomials m_1, m_2, m_3 if $m_1 \prec m_2$ then $m_1 m_3 \prec m_2 m_3$. We also demand for $\prec$ to be a well ordering, *i.e.*, any set of monomials of R has a smallest element with respect to $\prec$. Unlike the monomial of highest degree, the initial monomial $\text{in}_\prec(h)$ of $h \in R$ with respect to $\prec$ is unique, this is the largest monomial in h with respect to $\prec$. One can then divide a polynomial h by an ordered set of polynomials $(h_1, \ldots, h_s)$, and the result is:

$$h = h_1 q_1 + \ldots h_s q_s + r, \tag{3.1}$$

where $q_1, \ldots, q_s, r \in R$ and there is no monomial appearing in r which is divisible by any of $\text{in}_\prec(h_i), i = 1, \ldots, s$. In general r depends on the order of the $s-$tuple $(h_1, \ldots, h_s)$ and the condition that $r = 0$ is not necessary for f to belong to the ideal generated by $(h_1, \ldots, h_s)$: for instance (see example 5 page 68 of [24]), the division of $x_1 x_2^2 - x_1$ by $(x_1 x_2 - 1, x_2^2 - 1)$ with respect to the lexicographical ordering, where we assume $y \prec x$, is given by:

$$x_1 x_2^2 - x_1 = x_2(x_1 x_2 - 1) + 0.(x_2^2 - 1) + (-x_1 + x_2).$$

The remainder $r = -x_1 + x_2 \neq 0$ but

$$x_1 x_2^2 - x_1 = x_1(x_2^2 - 1) \in (x_1 x_2 - 1, x_2^2 - 1).$$

To fix this problem, one should consider a special (with respect to the chosen monomial order $\prec$) basis $(g_1, \ldots, g_l)$ of the ideal $I = (h_1, \ldots, h_s)$ which satisfies that the initial ideal $\text{in}_\prec(I) := (\text{in}_\prec(h); h \in I)$ is given by

$$\text{in}_\prec(I) := (\text{in}_\prec(g_1), \ldots, \text{in}_\prec(g_l)).$$

Such a basis is called a **Groebner basis** and it ensures when dividing by $(g_1, \ldots, g_l)$ the uniqueness of the remainder r. one notices that in the example above, $I = (h_1, h_2)$ where $h_1 = x_1 x_2 - 1$ and $h_2 = x_2^2 - 1$, the basis (h_1, h_2) is not a Greobner basis (with respect to the lexicographical ordering), indeed:

$$\text{S}(h_1, h_2) := x_2 h_1 - x h_2 = x - y \in I. \tag{3.2}$$

We have $\text{in}_\prec(x-y) = x \notin (\text{in}_\prec(h_1), \text{in}_\prec(h_2)) = (x_1 x_2, x_2^2)$. But the basis $(h_1, h_2, h_3 = x - y)$ is a Groebner basis. The S-polynomial defined in equation (3.2) is made so that one can eliminate the initials of both h_1 and h_2 and search for other elements in the ideal which give new initials that do not belong to the ideal generated by the initials of the generators of the input basis. As one can guess, the S-polynomial is the right tool in general to find a Groebner bases by applying it recursively to all the couple of elements in the basis and by adding them (actually the remainder of their divisions by the basis) to the basis when they are useful. The fact that such an algorithm (the Buchberger algorithm) stops, as for the division algorithm, is related to the property that the monomial order is a well ordering. Now one important thing for us, is that for a graded ring which is the quotient of a polynomial ring R by a (weighted-)homogeneous ideal, the Hilbert-Poincaré series satisfies (see e.g, theorem 5.2.6 in [32])

$$HP_{R/I}(q) = HP_{R/\mathrm{in}_{\prec}(I)}(q). \tag{3.3}$$

Note that the equality (3.3) is somehow natural, since by the discussion above, if we take a Greobner basis $I = (g_1, \ldots, g_l)$, any element in R is congruent by the division algorithm by $(g_1, \ldots, g_l)$ to a unique element r (the remainder) whose terms are not divisible by any $\mathrm{in}_{\prec}(g_i)$, *i.e*, by terms whose image in $R/\mathrm{in}_{\prec}(I)$ is a basis over $\mathbf{C}$. For more about Greobner bases, the reader can consult *e.g* [24, 27, 32].

Differential structure on the arc space. The ring $\mathcal{O}_{X_\infty}$, where X is an affine variety, has a structure of a differential ring. Let us stick to the example of X in section 2 and to the notations there. The ring of global functions on $\mathbf{A}^3_\infty$ is

$$\mathcal{O}_{\mathbf{A}^3_\infty} = \mathbb{C}[x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 0}].$$

We have a derivation D on $\mathcal{O}_{\mathbf{A}^3_\infty}$ defined by $D(x_i) = x_{i+1}, D(y_i) = y_{i+1}, D(z_i) = z_{i+1}$ for $i \in \mathbf{Z}_{\geq 0}$. If we replace in the equation (2.2) the variables x_i by $x_i/i!$ (where $j!$ is the factorial of j), and similarly for y_i and z_i, we find

$$f(\gamma(t)) = \sum_{j\geq 0} \frac{\mathcal{F}_j(x_0, y_0, z_0, \ldots, x_j, y_j, z_j)}{j!} t^j. \tag{3.4}$$

where $\mathcal{F}_0 = f(x_0, y_0, z_0)$ and $\mathcal{F}_j$ is recursively defined by the identity $D(\mathcal{F}_j) = \mathcal{F}_{j+1}$; equation (3.4) follows from the fact that both sides are additive and multiplicative in f and that this equality is obviously true for $f = x, y$ or z. We obtain hence the desired differential structure which is induced by the derivation D on $\mathcal{O}_{X_\infty}$; this is because the rings

$$\frac{\mathbb{C}[x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 0}]}{(F_j, j \in \mathbf{Z}_{\geq 0})} \text{ and } \frac{\mathbb{C}[x_i, y_i, z_i, i \in \mathbf{Z}_{\geq 0}]}{(\mathcal{F}_j, j \in \mathbf{Z}_{\geq 0})}$$

are isomorphic, the isomorphism being given by the change of variables expressed above. Fore more about differential algebras see [36, 52].

Proof. (of theorem 3.1) The ring of $\mathcal{A}_\infty$ of global functions on $\psi^{-1}(O)$ is (modulo an isomorphism) given by

$$\mathcal{A}_\infty = \frac{\mathbf{C}[y_j, j \in \mathbf{Z}_{>0}]}{[y_1^2]}$$

where

$$\begin{aligned}[y_1^2] =&(y_1^2,\\ &2y_1y_2,\\ &2y_2^2+2y_1y_3,\\ &6y_2y_3+2y_1y_4,\\ &6y_3^2+8y_2y_4+2y_1y_5,\ldots)\\ =&(f_2,f_3,\ldots)\end{aligned}$$

is the differential ideal generated by y_1^2 and all its iterated derivatives by the derivation D. For a general singularity $O \in X$, where X is affine, the ring $\mathcal{A}_\infty$ needs not be differential even if $\mathcal{O}_{X_\infty}$ is; in our case, this is true because one can construct an (non-homogeneous) isomorphism between $\mathcal{A}_\infty$ and $\mathcal{O}_{X_\infty}$. Now, when writing the generators f_i of the ideal, we ordered their terms by the weight (in f_i they are all of the same weight i) and by considering as smaller the monomials which make use of larger indices: for instance, y_3^2 is larger than y_2y_4 which is larger than y_1y_5; this order that we denote by $\prec$ sounds to us natural from a geometric point of view since y_3^2 says something about the third neighborhood while y_2y_4 concerns the fourth neighborhood ; so we want to see y_3^2 before y_2y_4. Now if we want to find a Groebner basis, we need to study the S$-$polynomial of the various couples of generators among the $f_i's$. If the the initial monomials of of f_i, f_j are coprime, then their S$-$polynomial will not "give" new initials (see *e.g.* proposition 1, page 106 [24]). So we need to consider the S$-$polynomials for the couples $(f_{2n}, f_{2n+1}), (f_{2n+1}, f_{2n+2})$ and (f_{2n+1}, f_{2n+3}). Let us study the first case, the other being similar. We have

$$\mathrm{S}(f_2,f_3) = 2y_2f_2 - y_1f_3 = 0. \tag{3.5}$$

Now deriving (3.5) iteratively $3n+4$ times, we obtain the equation

$$\sum_{j=1}^{3n-1} c_jy_jf_{3n+1-j} = 0, c_j \in \mathbb{C}. \tag{3.6}$$

Using the Leibniz formula, we find

$$\begin{aligned}c_n &= 2C_{3(n-1)}^{n-2} - C_{3(n-1)}^{n-1}\\ c_{n+1} &= 2C_{3(n-1)}^{n-1} - C_{3(n-1)}^{n}\end{aligned}$$

where $C_n^k := \binom{n}{k}$ denotes the binomial coefficient. Let α_{2n} and α_{2n+1} be respectively the coefficients of y_n^2 in f_{2n} and of y_ny_{n+1} in f_{2n+1}. Since $f_{2n} = D^{2n-2}(f_2)$ and $f_{2n+1} = D^{2n-1}(f_2)$, again using the Leibniz formula we see that the coefficients α_n and α_{n+1} satisfies

$$\begin{aligned}\alpha_{2n} &= C_{2(n-1)}^{n-1}\\ \alpha_{2n+1} &= C_{2n}^{n}\end{aligned}$$

Now, noticing that $\alpha_{2n}c_{n+1} = -\alpha_{2n+1}c_n$ we can rewrite the equation (3.6) as

$$\mathrm{S}(f_{2n}, f_{2n+1}) = \sum_{j=1,\cdots,3n-1; j\neq n,n+1} c_jy_jf_{3n+1-j}.$$

This latter formula says that $\mathrm{S}(f_{2n}, f_{2n+1}), n \geq 2$ does not give new initials (reduces to 0 modulo the basis $(f_2, f_3, \ldots)$, using the terms of [24]). Similarly, we can prove that the S-polynomials of the couples (f_{2n+1}, f_{2n+2}) and $(f_{2n+1}, f_{2n+3}), n \geq 1$ reduce to 0 modulo the basis $(f_2, f_3, \ldots)$ and by theorem 6 page 108 in [24], we deduce that $(f_2, f_3, \ldots)$ is a Groebner basis. Hence, since

$$\mathrm{in}_{\prec}(f_{2n}) = \alpha_{2n} y_n^2 \text{ and } \mathrm{in}_{\prec}(f_{2n+1}) = \alpha_{2n+1} y_n y_{n+1}$$

we have

$$\mathrm{in}_{\prec}([y_1^2]) = (y_n^2, y_n y_{n+1}, n \geq 1).$$

From the equality (3.3), we deduce that the arc HP-series of X at O is equal to the Hilbert-Poincaré series of

$$L := \frac{\mathbf{C}[y_j, j \in \mathbf{Z}_{>0}]}{(y_n^2, y_n y_{n+1}, n \geq 1)},$$

graded by giving the weight j to y_j. The $j-$th (wighted)-homogeneous component L_j of L is generated by the monomials

$$y_{j_1} y_{j_2} \cdots y_{j_s}$$

where $j_1 + j_2 + \cdots + j_s = j$ and where $y_{j_1} y_{j_2} \cdots y_{j_s}$ is not divisible by any monomial of the type y_n^2 or $y_n y_{n+1}$, this is equivalent to say that difference between two consecutive parts of the associated partition $j_1 + j_2 + \cdots + j_s$ of j is at least 2. Using theorem 1.2 and the identity (1.4) we obtain the form of the arc HP-series in the statement of the theorem.

□

Remark 3.3.
(1) The fact that we derived (3.5) $3n+4$ times is not a trick, it is just that we know the weight of $\mathrm{S}(f_{2n}, f_{2n+1})$ and we derived enough times to reach this weight; deriving once make the weight grow of 1.
(2) It worth noticing, that the fact that we considering a non-finitely generated ideal in the above proof is a source of simplification : indeed, if we consider the finitely generated ideals $(f_2, f_3, \ldots, f_m), m \in \mathbf{Z}_{\geq 3})$, then the given basis is no longer a Groebner basis with respect to the considered monomial ordering; it is only when we let m goes to infinity that we have the miracle that the basis is a Groebner basis. This can for instance be seen in the equation 3.6, where some f_i's for $i > 2n+1$ may intervene.

The proof of theorem 3.2 follows the same ideas and computations in the proof of theorem 3.1. The proof above inspires the following approach (see [21]) towards the Rogers-Ramanujan identities. We begin by introducing some notations: Let $I_d = (y_n^2, y_n y_{n+1}, n \geq d)$,

$$L^{(d)} := \frac{\mathbf{C}[y_j, j \in \mathbf{Z}_{\geq d}]}{I_d},$$

graded as above and $h(d) = HP_{L^{(d)}}$. We have an exact sequence

$$0 \longrightarrow \frac{\mathbf{C}[y_j, j \in \mathbf{Z}_{\geq d}]}{(I_d : y_d)}[-d] \longrightarrow \frac{\mathbf{C}[y_j, j \in \mathbf{Z}_{\geq d}]}{I_d} \longrightarrow \frac{\mathbf{C}[y_j, j \in \mathbf{Z}_{\geq d}]}{(I_d, y_d)} \longrightarrow 0 \tag{3.7}$$

where the first non-zero morphism is the multiplication by y_d; the symbol $[-d]$ means that the graded structure is shifted by $-d$, so that the elements of weight 0 after adding the $[-d]$ correspond to those of weight $-d$ if we drop the $[-d]$, and the colon ideal

$$(I_d : y_d) = \{h \in \mathbf{C}[y_j, j \in \mathbf{Z}_{\geq d}] \mid h \cdot y_d \in I_d\}.$$

The shift guarantees that all the morphisms are homogeneous (they send an element of a given weight to an element of the same weight) and hence we have exact sequences at the level of the graded components seen as $\mathbf{C}$-vector spaces. Noticing that

$$\frac{\mathbf{C}[y_j, j \in \mathbf{Z}_{\geq d}]}{(I_d : y_d)} = L^{(d+2)},$$

the rank theorem gives the following

$$h(d) = h(d+1) + q^d \cdot h(d+2),$$

and one deduces (see[21])

Proposition 3.4. The power series $h(1)$ satisfies

$$h(1) = A_d \cdot h(d) + B_{d+1} \cdot h(d+1);$$

for $A_i, B_i \in k[[q]]$ fulfilling the following recursion

$$\begin{array}{rcl} A_d & = & A_{d-1} + B_d \\ B_{d+1} & = & A_{d-1} \cdot q^{d-1} \end{array}$$

with initial conditions $A_1 = A_2 = 1$ and $B_2 = 0$,$B_3 = q$.

Since $ord_q B_d \geq d-2$, both $\lim A_d$ and $\lim B_d$ exist (limits with respect to the $q-$adic topology as sequence of power series), and they satisfy

$$\lim B_d = 0 \text{ and } h(1) = \lim A_d.$$

The recursion from Proposition 3.4 can be simplified to $h(1) = \lim A_d$ where A_d fulfills

$$A_d = A_{d-1} + q^{d-2} \cdot A_{d-2} \tag{3.8}$$

with initial conditions $A_1 = A_2 = 1$.

This last recursion is well-known from [13]. Its limit is the infinite product

$$\prod_{i=1,4 \mathrm{mod}\ 5}^{\infty} \frac{1}{1-q^i},$$

i.e., the generating series of the number of partitions with parts equal to 1 or 4 modulo 5. The construction above gives the generating series G_d defined in [13] an interpretation as a Hilbert-Poincaré series of the quotients $\mathbf{C}[y_j, j \in \mathbf{Z}_{\geq d}]/I_d$. This immediately implies that the series G_d are of the form $G_d = 1 + \sum_{j \geq i} G_{dj} q^j$, the empirical hypothesis of [13].

4. Other Partition identities inspired by this viewpoint

An extension of Rogers-Ramanujan identities. In section 3, we showed that the arc HP-series for one of the simplest singularities is equal to the generating series of the number of partitions appearing in the Rogers-Ramanujan identities. At the heart of the proof, we find a computation of a Groebner basis of the ideal $[y_1^2]$, the defining ideal of the space of arcs centered at $O \in \mathrm{Spec}\mathbf{C}[y]/(y^2)$; this Groebner basis is differentially finite,*i.e.*, it is built from a finite number of elements (here only one) and all their derivatives. The monomial order considered in section 3 is somehow "geometric" (chosen for geometric reasons), but one may also consider another monomial ordering $<$ for which the initial ideal $\mathrm{in}_<([y_1^2])$ of $[y_1^2]$ may vary but the Hilbert-Ponicar series of the quotient

$$HP_{\mathrm{Spec}\mathbf{C}[y_j,j\in\mathbf{Z}_{>0}]/\mathrm{in}_<([y_1^2])} = HP_{\mathrm{Spec}\mathbf{C}[y_j,j\in\mathbf{Z}_{>0}]/([y_1^2])}$$

will not vary, by equality (3.3). In [6], a Groebner basis computation with respect to a weighted lexicographical ordering was considered; but (see theorem 2.2 in [6]) such a basis cannot be differentially finite. This made it very difficult to actually compute a Groebner basis of $[y_1^2]$ with respect to this order; still, from the computation one can guess (without a proof) that the leading ideal should be

$$(y_k y_{i_1}\dots y_{i_k},\ \text{where } k\leq i_1\leq\cdots\leq i_k). \tag{4.1}$$

By playing this game with $[y_i^2]$ for $i\in\mathbf{Z}_{>0}$ and using iteratively exact sequences which are similar to (3.7), on can prove the following (Theorem 1.7 [6]):

Theorem 4.1. *Let $n\geq k$ be a positive integer. The number of partitions of n with parts larger than or equal to k and size less than or equal to (the smallest part minus $k-1$) is equal to the number of partitions of n with parts larger than or equal to k and such that the difference between two consecutive parts is at least 2.*

For $k=1$, theorem 4.1 says that:

For a positive integer $n\geq 1$, the number of partitions of n with size less than or equal to the smallest part is equal to the number of partitions of n such that the difference between two consecutive parts is at least 2; this yields another member of Rogers-Ramanujan identities.

Let us call $G_{2,2}(n)$ the number of partitions of n with size less than or equal to the smallest part. The partitions of 4 (see (1.6) which are counted by $G_{2,2}(4)$ are

$$4 \text{ and } 2+2.$$

In particular we have $T_{2,2}(4)=E_{2,2}(4)=G_{2,2}(4)=2$ (see theorem 1.2 for the notations), and theorem 4.1, for $k=1$, asserts that the equality

$$T_{2,2}(n)=E_{2,2}(n)=G_{2,2}(n)$$

is true for every n.

Remark 4.2. Recently, in [8], using new methods from differential algebra, the authors proved that the ideal appearing in (4.1) is actually the initial ideal of $[y_1^2]$ with respect the weighted lexicographical order. Still, until now we do not have a Groebner basis with respect to this order.

In [2], using similar ideas to those who led to theorem 4.1, the author proved another exciting extension to Rogers-Ramanujan identities, in which the parity (even odd) of the parts of a partition plays an important role.

Gordon's identities and their extensions. In the last section, we kept somewhat hidden the fact that there is a great generalization of theorem 3.1, proved in [22].

Theorem 4.3. *Let $n \in \mathbf{Z}_{\geq 2}$. For $X = Spec\frac{\boldsymbol{K}[y]}{(y^n)}$,*

$$AHP_{X,0}(q) = \prod_{i \not\equiv \ 0,n,n+1 \ mod(2n+1)} \frac{1}{1-q^i}.$$

The proof follows the same strategy of the proof of theorem 3.1 but the differential calculus is much more involved. Another famous family of identities intervenes in the proof, Gordon's identities [31].

Theorem 4.4 (Gordon's identities)**.** *Let r and i be integers such that $r \geq 2$ and $1 \leq i \leq r$. Let $\mathcal{T}_{r,i}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, and at most $i-1$ of the parts λ_j are equal to 1. Let $\mathcal{E}_{r,i}$ be the set of partitions whose parts are not congruent to $0, \pm i \mod (2r+1)$. Let n be a nonnegative integer, and let $T_{r,i}(n)$ (respectively $E_{r,i}(n)$) denote the number of partitions of n which belong to $\mathcal{T}_{r,i}$ (respectively $\mathcal{E}_{r,i}$). Then we have*

$$T_{r,i}(n) = E_{r,i}(n).$$

Using ideas similar to those of section 3, in [3, 1], the author gave an alternative approach to Gordon's identities and conjectured a great generalization of theorem 4.1. This conjecture was proved recently in [5, 4]. Let us give the statement of this theorem: Given an integer $r \geq 2$, for $1 \leq i \leq r$, define the (i, ℓ)-new part of $\lambda = (\lambda_1, \ldots, \lambda_s)$ as follows:

$$p_{i,\ell}(\lambda) := \begin{cases} \lambda_s & \text{if } \ell = 1, \\ \lambda_{s-\sum_{j=1}^{\ell-1} p_{i,j}(\lambda)} & \text{if } 2 \leq \ell \leq i, \\ \lambda_{s+\ell-i-\sum_{j=1}^{\ell-1} p_{i,j}(\lambda)} & \text{if } i < \ell \leq r-1, \end{cases}$$

where $\lambda_j = 0$ for $j \leq 0$, and if $p_{i,\ell}(\lambda) = 0$ then $p_{i,j}(\lambda) = 0$ for $j > \ell$. We denote the number of all non-zero (i, ℓ)-new parts of λ by $N_{r,i}(\lambda)$.

Theorem 4.5. *Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. Let $\mathcal{C}_{r,i}$ be the set of partitions of the form $\lambda = (\lambda_1, \ldots, \lambda_s)$, such that at most $i-1$ of the parts are equal to 1 and either $N_{r,i}(\lambda) < r-1$, or $N_{r,i}(\lambda) = r-1$ and $s \leq \sum_{j=1}^{r-1} p_{i,j}(\lambda) - (r-i)$. Let n be a nonnegative integer, and denote by $C_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{C}_{r,i}$. Then we have*

$$C_{r,i}(n) = T_{r,i}(n) = E_{r,i}(n).$$

The proof uses on the one hand another classification theorem of the partitions in $\mathcal{C}_{r,i}$ in terms of a new type of Durfee dissection (inspired by [11], this is a classification in terms of Ferrers diagrams): the proof of this interpretation uses simple commutative algebra (another purely combinatorial proof of the same result is also given); On the other hand it uses Bailey lattices [15, 55], a very powerful tool for calculus with $q-series$.

Another singularity and its associated family of partition identity. To have a taste of what kind of partition identities can come out of singularities in higher dimensions we give below a family of partition identities which is associated with the singularity at the origin of

$$Y = \text{Spec}\frac{\mathbf{C}[x,y]}{(xy)}.$$

Let us first introduce partitions with 2 colors. Consider that we have two copies of each positive integer m, one is blue and the other is red; we denote these copies by m_b and m_r. We define an order between the colored integers by $m_b > m_r$ (hence $m_b + m_r$ and $m_r + m_b$ are the same); if $m > k$, we set $m_c > k_{c'}$ for $c, c' \in \{b, r\}$.

An integer partition of a positive integer number n is a decreasing sequence (with respect to the order that we have just defined) of positive integers of one color or an other

$$\lambda = (\lambda_{1,c_1} \geq \lambda_{2,c_2} \geq \ldots \geq \lambda_{l,c_l}),$$

where $c_i \in \{b, r\}$ and such that $\lambda_{1,c_1} + \lambda_{2,c_2} + \cdots + \lambda_{l,c_l} = n$. For example, the two colors integer partitions of 2 are:

$$\begin{gathered} 2_b \\ 2_r \\ 1_b + 1_b \\ 1_r + 1_r \\ 1_b + 1_r. \end{gathered}$$

Our singularity Y, sometimes called the node singularity, is somehow related (but still very different in nature) to the singularity $X = \mathrm{Spec}\mathbf{C}[x]/(x^2)$, which led to the Rogers-Ramanujan identities: one can "put them" in a family

$$F : \mathrm{Spec}\frac{\mathbf{C}[x, y, t]}{(x(x - ty))} \longrightarrow \mathrm{Spec}\mathbf{C}[t].$$

The fibers over $t \neq 0$ are isomorphic to Y and the fiber above $t = 0$ is $X \times \mathbf{A}^1$. This can perhaps explain the small similarity of theorem 4.1 with the following theorem from [6]:

Theorem 4.6. *Let j be a positive integer number. The number of partitions of n with 2 colors (say blue and red) of $j, \ldots, 2j - 1$ and only the red color of any other positive integer larger than $2j$ is equal to the number of partitions n whose parts are larger than j and of two colors and such that the number of blue parts is strictly less than its smallest red part (if this latter exists) minus $(j - 1)$.*

5. Omissions

Many other research directions are directly related to the subjects of this article. I can mention the relation between Neighborly partitions, monomial ideals, graphs and hypergraphs [42, 7]; this subject which is a direct continuation of the story told in this article has led recently in [50] to a new proof of Rogers-Ramanujan identities. I can mention the relation with vertex operators and Virasoro Algebras [14, 28, 39]. And the reader possibly sees interactions with other research directions.

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