JET SCHEMES OF QUASI-ORDINARY SURFACE SINGULARITIES

HELENA COBO AND HUSSEIN MOURTADA

ABSTRACT. We describe the irreducible components of the jet schemes with origin in the singular locus of a two-dimensional quasi-ordinary hypersurface singularity. A weighted graph is associated with these components and with their embedding dimensions and their codimensions in the jet schemes of the ambient space. We prove that the data of this weighted graph is equivalent to the data of the topological type of the singularity. We also determine a component of the jet schemes (equivalent to a divisorial valuation on $\mathbb{A}^3$), that computes the log canonical threshold of the singularity embedded in $\mathbb{A}^3$. This provides us with pairs $X \subset \mathbb{A}^3$ whose log canonical thresholds are not computed by monomial divisorial valuations. Note that for a pair $C \subset \mathbb{A}^2$, where C is a plane curve, the log canonical threshold is always computed by a monomial divisorial valuation (in suitable coordinates of $\mathbb{A}^2$).

1. INTRODUCTION

By definition, a complex analytic quasi-ordinary singularity $(X, 0)$ of dimension d comes with a finite projection $p : X \longrightarrow \mathbb{A}^d$, whose discriminant is a normal crossing divisor. These singularities appear in the Jungian approach to resolution of singularities (see [31]). We are interested in irreducible quasi-ordinary surfaces X, defined by $f \in \mathbb{C}\{x_1, x_2\}[z]$. Thanks to the Abhyankar-Jung theorem, we know that a hypersurface of this type is parametrized in the form $x_i = x_i$ for $i = 1, 2$ and $z = \zeta(x_1, x_2)$, where ζ is an element in $\mathbb{C}\{x_1^{1/n}, x_2^{1/n}\}, n$ being the degree of f as a polynomial in z. Moreover, some special exponents (called the characteristic exponents) which belong to the support of the series ζ, are complete invariants of the topological type of the singularity (see [15]). In particular, they determine invariants which come from resolution of singularities, like the log canonical threshold or the Motivic zeta functions ([3], [9], [8], [18]). They also give insights about the construction of a resolution of singularities ([6], [7], [34], [17]).

Our aim is to construct some comparable complete invariants for all types of singularities. Since in general, we cannot have a parametrization, we search for such invariants in the jet schemes. For $m \in \mathbb{N}$, the m-th jet scheme, denoted by X_m, is a scheme that parametrizes morphisms Spec $\mathbb{C}[t]/(t^{m+1}) \longrightarrow X$. Intuitively

2010 *Mathematics Subject Classification.* 14E18,14J17.
Key words and phrases. Singularities, Jet schemes, Quasi-ordinary singularities, log canonical threshold.

we can think of it as a scheme parametrizing arcs in an ambient space, which have contact at least $m+1$ with X. We know already that some invariants which come from resolution of singularities are encoded in the jet schemes ([29], [13]).

We want to extract from the jet schemes information about the singularity, which can be expressed in terms of invariants of resolutions of singularities. For specific types of singularities, the knowledge of the irreducible components of the jet schemes X_m of a singular variety X, together with some of their invariants, such as dimension or embedding dimension, allows us to determine deep invariants of the singularity of X: the topological type in the case of curves (see [24]), and the analytical type in the case of normal toric surfaces (see [25]). Moreover, in the case of irreducible plane curves, the minimal embedded resolution can be constructed from the jet schemes ([21]), and the same holds for rational double point singularities ([28]).

Understanding the structure of jet schemes for particular singularities is an interesting problem. It has been studied in [35] and [12] for determinantal varieties, in [24] for plane curve singularities, in [25] for normal toric surfaces, in [26] for rational double point surface singularities, and in [33] for commuting matrix pairs schemes.

In this paper, we study jet schemes of a two-dimensional, irreducible quasi-ordinary hypersurface singularity $X=\{f=0\}$, with $f\in\mathbb{C}\{x_1,x_2\}[z]$. We give a combinatorial description of the irreducible components of the set of m-jets with center in the singular locus of X, in terms of invariants of the singularity extracted from the characteristic exponents of X. We define the candidates to be the irreducible components C_m^ν, but there are many inclusions among these candidates. We study these inclusions by defining on $\mathbb{Z}^2_{\geq 0}$ a subtle relation depending on m and expressed in terms of the invariants cited above. It reflects the evolution of the singular loci of quasi-ordinary surfaces approximating our surface X.

Then, with the minimal elements with respect to this relation we define a set $F_m\subset\mathbb{Z}^2$, and for any $\nu\in F_m$, we have a component $C_m^\nu\subset X_m$. We prove that these are the irreducible components of m-jets through the singular locus.

Theorem 1.1. *Let X be a quasi-ordinary hypersurface of dimension two. For any $m\in\mathbb{Z}_{>0}$, the scheme of $m-$th jet of X with center in its singular locus has the following decomposition into irreducible components*

$$\left(\pi_m^{-1}(X_{Sing})\right)_{red}=\bigcup_{\nu\in F_m} C_m^\nu,$$

where $\pi_m : X_m \longrightarrow X$ is induced by projection.

Note that if we choose an affine variety $Y\subset\mathbb{C}^3$ which has a quasi-ordinary singularity at a point x, then after shrinking Y into a small enough neighbourhood of x, this gives us the decomposition of Y_m into irreducible components, modulo adding the component obtained as the Zariski closure of the set of jets whose center is in the regular locus of Y.

In general, for any algebraic variety V, the irreducible components of the jet schemes V_m fit in natural projective systems, to which we associate a weighted graph. Graphs are a powerful tool for studying surface singularities (see [32] for a nice and historical introduction on this topic). The vertices of our graph correspond to irreducible components, and to every vertex we attach the corresponding embedding dimension and codimension in the jet scheme of the ambient space. We will prove the following result.

Theorem 1.2. *Let X be a quasi-ordinary hypersurface of dimension two. The weighted graph associated with the irreducible components of jets through the singular locus determines and it is determined by the topological type of the singularity.*

This theorem achieves one of our goals for this type of singularities: constructing a complete invariant of the topological type of the singularity from its jet schemes; while the graph of the jet schemes is defined in general, the characteristic exponents, which are also a complete invariant of quasi-ordinary singularities, does not have a meaning for more general singularities for two reasons: 1) for a general singularity we only have parametrizations of parts of the singularity (wedges), 2) the shape of these parametrizations is more complicated than the shape of parametrizations of quasi-ordinary singularities.

It is also important to stress that other invariants involving arcs and jets, like motivic zeta functions, do not determine the topological type in the case of quasi ordinary singularities, see [9] and [18].

We devote Section 4 to study in detail the case of quasi-ordinary surfaces with only one characteristic exponent, and in next section we deal with the general case.

In another direction, using Mustaţa's formula ([29]), we determine an irreducible component of an m-th jet scheme, or equivalently a divisorial valuation on the ambient space $\mathbb{A}^3$, which computes the log canonical threshold of the pair $X \subset \mathbb{A}^3$ (the log canonical threshold for such a pair has been computed in [8], looking at the poles of the motivic zeta function). This provides us with pairs $X \subseteq \mathbb{A}^3$ whose log canonical threshold is not computed by a monomial divisorial valuation. The quasi-ordinary surface in $\mathbb{A}^3$ defined by $f = (z^2 - x_1x_2)^2 - x_1^3x_2z$ is such a pair. Note that for a pair $C \subseteq \mathbb{A}^2$, where C is a plane curve, the log canonical threshold is always computed by a monomial valuation. See [4] and [2] for the computation of the log canonical threshold for plane curves.

Using same ideas of [27], it seems possible to construct an embedded resolution of singularities of X from the data of the graph constructed in this paper. We think that such a resolution would shed light on the resolution of singularities obtained by González Pérez in [17], and would make more precise his answer to the question of Lipman (see [23]) on the construction of a canonical resolution of singularities of a quasi-ordinary hypersurface from its characteristic exponents.

Acknowledgments. We are grateful to P.D. González Pérez and a number of referees, for comments and suggestions which improved enormously the content and presentation of the paper. HC was supported by ERC Grant Agreement 246903 NMNAG, and HM was supported by ANR-17-CE40-0023.

2. Jet schemes

In this section we define jet schemes of an affine scheme X, see [13] and [19] for details. Let $X = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/I$ be an affine scheme of finite type. For $m \in \mathbb{Z}_{>0}$ the functor F_m : $\mathbb{C}$-Schemes $\longrightarrow$ Sets which, with an affine scheme defined by a $\mathbb{C}$-algebra A, associates

$$F_m\big(\operatorname{Spec}(A)\big) = \operatorname{Hom}_{\mathbb{C}}\big(\operatorname{Spec}(A[t]/(t^{m+1})), X\big),$$

is representable by a $\mathbb{C}$-scheme, denoted by X_m. This is the scheme of m-jets. Its closed points are morphisms of the form

$$\gamma : \operatorname{Spec}(\mathbb{C}[t]/(t^{m+1})) \longrightarrow X.$$

Such a morphism γ is equivalent to a $\mathbb{C}$-algebra homomorphism

$$\gamma^* : \mathbb{C}[x_1, \ldots, x_n]/I \longrightarrow \mathbb{C}[t]/(t^{m+1}).$$

If we fix a set of generators $f_1, \ldots, f_r$ for the ideal I, the map γ^* is determined by the image of the x_i

$$x_i \mapsto x_i^{(0)} + x_i^{(1)} t + \cdots + x_i^{(m)} t^m,\ 1 \leq i \leq n,$$

where the relations

$$f_i\big(x_1^{(0)} + \cdots + x_1^{(m)} t^m, \ldots, x_n^{(0)} + \cdots + x_n^{(m)} t^m\big) \equiv 0 \bmod t^{m+1} \tag{1}$$

must hold for each f_i, with $1 \leq i \leq r$. If we write

$$\begin{aligned} &f_i\big(x_1^{(0)} + x_1^{(1)} t + \cdots + x_1^{(m)} t^m, \ldots, x_n^{(0)} + x_n^{(1)} t + \cdots + x_n^{(m)} t^m\big) = \\ &= \textstyle\sum_{j=0}^m F_i^{(j)}(x_1^{(0)}, \ldots, x_1^{(j)}, \ldots, x_n^{(0)}, \ldots, x_n^{(j)})\ t^j \bmod t^{m+1}, \end{aligned} \tag{2}$$

we have that giving a closed point of X_m is equivalent to giving a point in

$$V(F_l^{(j)})_{0 \leq j \leq m, 1 \leq l \leq r} \subset \mathbb{A}_m^n,$$

where $\mathbb{A}_m^n = \operatorname{Spec}(\mathbb{C}[x_i^{(0)}, \ldots, x_i^{(m)}]_{i=1,\ldots,n})$. Hence we can make the following identification

$$X_m = \operatorname{Spec} \left(\frac{\mathbb{C}[x_i^{(0)}, \ldots, x_i^{(m)}]_{i=1,\ldots,n}}{\big(F_l^{(j)}\big)_{0 \leq j \leq m,\ 1 \leq l \leq r}} \right). \tag{3}$$

We can give a useful relation among the $F_l^{(j)}$ in terms of derivations. Let δ be the $\mathbb{C}$-derivation on $\mathbb{C}[x_i^{(0)}, \ldots, x_i^{(m)}]_{i=1,\ldots,n}$ defined by

$$\delta(x_i^{(m)}) = 0 \text{ and } \delta(x_i^{(j)}) = x_i^{(j+1)} \text{ for } 0 \leq j < m.$$

For $f \in \mathbb{C}[x_1, \dots, x_n]$ let $f^{(0)} = f(x_1^{(0)}, \dots, x_n^{(0)})$, $f^{(1)} := \delta(f)$ and recursively $f^{(m)} = \delta(f^{(m-1)})$. By using the change of variables

$$\begin{array}{cccc} \phi: & \mathbb{C}[x_i^{(0)}, \dots, x_i^{(m)}]_{1 \le i \le n} & \longrightarrow & \mathbb{C}[x_i^{(0)}, \dots, x_i^{(m)}]_{1 \le i \le n} \\ & x_i^{(r)} & \longmapsto & r! x_i^{(r)} \end{array}$$

we can prove that

$$\phi(f^{(r)}) = r! F^{(r)}.$$

Hence we have the following description of the jet schemes, equivalent to (3), coming from differential algebra.

Proposition 2.1. *(See Proposition 2.3 in* [24]*) Let* $X = Spec\left(\frac{\mathbb{C}[x_1,\dots,x_n]}{(f_1,\dots,f_r)}\right)$ *and* $m \in \mathbb{Z}_{>0}$*, then*

$$X_m = Spec\left(\frac{\mathbb{C}[x_i^{(0)}, \dots, x_i^{(m)}]_{i=1,\dots,n}}{\left(f_i^{(j)}\right)_{1 \le i \le r,\ 0 \le j \le m}}\right).$$

Corollary 2.2. *Every polynomial* $F^{(l)}$ *is non-zero and quasi-homogeneous of degree* l *in* $x_k^{(0)}, \dots, x_k^{(l)}$*, for* $1 \le k \le n$*. In* $F^{(0)}, \dots, F^{(l)}$ *the variables* $x_k^{(l)}$ *for* $1 \le k \le n$ *appear only in* $F^{(l)}$*, and with exponent one.*

Example 2.3. *Let* X *be the quasi-ordinary surface defined by the polynomial* $f = z^3 - x_1^3 x_2^2$*. The equations defining the* 3*-jets are (in both descriptions):*

$$\begin{aligned}
F^{(0)} &= {z^{(0)}}^3 - {x_1^{(0)}}^3 {x_2^{(0)}}^2 = f^{(0)} \\
F^{(1)} &= 3{z^{(0)}}^2 z^{(1)} - 3{x_1^{(0)}}^2 x_1^{(1)} {x_2^{(0)}}^2 - 2{x_1^{(0)}}^3 x_2^{(0)} x_2^{(1)} = f^{(1)} \\
F^{(2)} &= 3{z^{(0)}}^2 z^{(2)} + 3z^{(0)} {z^{(1)}}^2 - 6{x_1^{(0)}}^2 x_1^{(1)} x_2^{(0)} x_2^{(1)} - 2{x_1^{(0)}}^3 x_2^{(0)} x_2^{(2)} - 3{x_1^{(0)}}^2 x_1^{(2)} {x_2^{(0)}}^2 \\
&\quad - {x_1^{(0)}}^3 {x_2^{(1)}}^2 - 3x_1^{(0)} {x_1^{(1)}}^2 {x_2^{(0)}}^2 = \tfrac{1}{2}\phi(f^{(2)}) = \tfrac{1}{2}\phi(\delta(f^{(1)})) \\
F^{(3)} &= {z^{(1)}}^3 + 6z^{(0)} z^{(1)} z^{(2)} + 3{z^{(0)}}^2 z^{(3)} - 2{x_1^{(0)}}^3 x_2^{(0)} x_2^{(3)} - 2{x_1^{(0)}}^3 x_2^{(1)} x_2^{(2)} \\
&\quad - 6{x_1^{(0)}}^2 x_1^{(1)} x_2^{(0)} x_2^{(2)} - 3{x_1^{(0)}}^2 x_1^{(1)} {x_2^{(1)}}^2 - 6{x_1^{(0)}}^2 x_1^{(2)} x_2^{(0)} x_2^{(1)} \\
&\quad - 6x_1^{(0)} {x_1^{(1)}}^2 x_2^{(0)} x_2^{(1)} - 3{x_1^{(0)}}^2 x_1^{(3)} {x_2^{(0)}}^2 - 6x_1^{(0)} x_1^{(1)} x_1^{(2)} {x_2^{(0)}}^2 - {x_1^{(1)}}^3 {x_2^{(0)}}^2 \\
&= \tfrac{1}{3!}\phi(f^{(3)}) = \tfrac{1}{3!}\phi(\delta^2(f^{(1)}))
\end{aligned}$$

For $m > n \ge 0$, we have a canonical projection $\pi_{m,n} : X_m \longrightarrow X_n$ induced by the projection $\mathbb{C}[t]/t^{m+1} \longrightarrow \mathbb{C}[t]/t^{n+1}$, and we denote $\pi_{m,0}$ simply by $\pi_m : X_m \longrightarrow X$.

Proposition 2.4. *(see* [11] *and* [13]*) If* X *is a non-singular variety of dimension* d *then for any* $m \ge 0$ *the projections* $\pi_{m+1,m} : X_{m+1} \longrightarrow X_m$ *are locally trivial with fiber* $\mathbb{A}^d$*. In particular* X_m *is a non-singular variety of dimension* $(m+1)d$*.*

The above construction of jet schemes in the algebraic case can be done analogously in the analytic case. Indeed, as we will see in the next section, we will deal with $f \in \mathbb{C}\{x_1, x_2\}[z]$. Then, for $l \in \mathbb{Z}_{\geq 0}$, denoting by

$$R^{(l)} := \mathbb{C}\{x_1^{(0)}, x_2^{(0)}\}[x_k^{(1)}, \dots, x_k^{(l)}, z^{(0)}, \dots, z^{(l)}]_{k=1,2},$$

we have that $F^{(l)} \in R^{(l)}$, and

$$X_m = \operatorname{Spec}\left(\frac{R^{(m)}}{(F^{(0)}, \dots, F^{(m)})}\right).$$

We will anyway speak of the *polynomials* $F^{(l)}$ defining the space of m-jets.

Remark 2.5. *To describe the components of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$, since the level m is clear from the context, we will use the notation $V(I)$ instead of the more accurate one Spec $\left(\frac{R^{(m)}}{I}\right)$.*

3. Quasi-ordinary surface singularities

In this section we collect some well known facts about quasi-ordinary hypersurface singularities of dimension two. We state everything for the case of surfaces, though the definitions and results hold in any dimension.

An equidimensional germ $(X, 0)$, of dimension two, is *quasi-ordinary* (q.o. for short) if there exists a finite projection $p : (X, 0) \to (\mathbb{C}^2, 0)$ which is a local isomorphism outside a normal crossing divisor. If $(X, 0)$ is a hypersurface there is an embedding $(X, 0) \subset (\mathbb{C}^3, 0)$, where X is defined by an equation $f = 0$, and $f \in \mathbb{C}\{x_1, x_2\}[z]$ is a *quasi-ordinary polynomial*; that is, a Weierstrass polynomial with discriminant $\Delta_z f$ of the form $\Delta_z f = x_1^{\delta_1} \cdot x_2^{\delta_2} \epsilon$ for a unit ϵ in the ring $\mathbb{C}\{x_1, x_2\}$ of convergent power series and $(\delta_1, \delta_2) \in \mathbb{Z}^2_{\geqslant 0}$. In these coordinates the projection p is the restriction of the projection

$$\mathbb{C}^3 \to \mathbb{C}^2, \quad (x_1, x_2, z) \mapsto (x_1, x_2).$$

From now on we assume $(X, 0)$ to be analytically irreducible, that is, $f \in \mathbb{C}\{x_1, x_2\}[z]$ is irreducible (see [5] and [14] for criteria of irreducibility of q.o. polynomials). The Jung-Abhyankar theorem guarantees that the roots of a q.o. polynomial f, called *q.o. branches*, are fractional power series in $\mathbb{C}\{x_1^{1/n}, x_2^{1/n}\}$, for $n = \deg f$ (see [1]). The difference $\zeta^{(i)} - \zeta^{(j)}$ of two different roots of f divides the discriminant of f in the ring $\mathbb{C}\{x_1^{1/n}, x_2^{1/n}\}$. Therefore $\zeta^{(i)} - \zeta^{(j)} = x_1^{\lambda_{ij}^{(1)}} x_2^{\lambda_{ij}^{(2)}} u_{ij}$ where u_{ij} is a unit in $\mathbb{C}\{x_1^{1/n}, x_2^{1/n}\}$. The exponents $\lambda_{ij} = \left(\lambda_{ij}^{(1)}, \lambda_{ij}^{(2)}\right)$ are characterized in the following Lemma.

Lemma 3.1. (see [15], Prop. 1.3.) *Let $f \in \mathbb{C}\{x_1, x_2\}[z]$ be an irreducible q.o. polynomial. Let ζ be a root of f with expansion:*

$$\zeta = \sum \beta_\lambda \mathbf{x}^\lambda. \tag{4}$$

There exists $0 \neq \lambda_1, \dots, \lambda_g \in \mathbb{Q}^2_{\geqslant 0}$ *such that* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_g$*, and if* $M_0 := \mathbb{Z}^2$ *and* $M_j := M_{j-1} + \mathbb{Z}\lambda_j$ *for* $j = 1, \dots, g$*, then:*

(i) $\beta_{\lambda_i} \neq 0$ *and if* $\beta_\lambda \neq 0$ *then* $\lambda \in M_j$ *where* j *is the unique integer such that* $\lambda_j \leqslant \lambda$ *and* $\lambda_{j+1} \nleq \lambda$ *(where* $\leqslant$ *means coordinate-wise and we convey that* $\lambda_{g+1} = \infty$*).*

(ii) *For* $j = 1, \dots, g$*, we have* $\lambda_j \notin M_{j-1}$*, hence the index* $n_j = [M_{j-1} : M_j]$ *is* > 1*.*

Moreover if $\zeta \in \mathbb{C}\{x_1^{1/n}, x_2^{1/n}\}$ *is a fractional power series satisfying the conditions above, then* ζ *is a quasi-ordinary branch.*

Definition 3.2. *The exponents* $\lambda_1, \dots, \lambda_g$ *in Lemma 3.1 are called* characteristic exponents *of the q.o. branch* ζ*. We denote by* M *the lattice* M_g *and we call it the lattice associated to the q.o. branch* ζ*. We denote by* N *(resp.* N_i*) the dual lattice of* M *(resp.* M_i *for* $i = 1, \dots, g$*). For convenience we set* $\lambda_0 := (0, 0)$ *and* $n_0 := 1$*. Moreover we set* $\lambda_{g+1} = \infty$*.*

In [15] Gau proved that the characteristic exponents determine and are determined by the embedded topological type of $(X, 0)$.

As a consequence of Lemma 3.1 we have the following result:

Lemma 3.3. *If* ζ *is a quasi-ordinary branch of the form (4) then the series* $\zeta_{j-1} := \sum_{\lambda \ngeq \lambda_j} \beta_\lambda \mathbf{x}^\lambda$ *is a quasi-ordinary branch with characteristic exponents* $\lambda_1, \dots, \lambda_{j-1}$*, for* $j = 1, \dots, g$*.*

Definition 3.4. *For* $0 \leq j \leq g-1$ *we have the germ of quasi-ordinary hypersurface* $(X^{(j)}, 0)$*, where* $X^{(j)}$ *is parametrized by the branch* ζ_j*. For convenience we also denote* ζ *by* ζ_g *and* X *by* $X^{(g)}$*.*

Without loss of generality we relabel the variables x_1, x_2 in such a way that if $\lambda_j = \big(\lambda_j^{(1)}, \lambda_j^{(2)}\big) \in \mathbb{Q}^2$ for $j = 1, \dots, g$, then we have:

$$\big(\lambda_1^{(1)}, \dots, \lambda_g^{(1)}\big) \geqslant_{\text{lex}} \big(\lambda_1^{(2)}, \dots, \lambda_g^{(2)}\big), \tag{5}$$

where $\geqslant_{\text{lex}}$ is lexicographic order. The q.o. branch ζ is said to be normalized if λ_1 is not of the form $\big(\lambda_1^{(1)}, 0\big)$ with $\lambda_1^{(1)} < 1$. Lipman proved that the germ $(X, 0)$ can be parametrized by a normalized q.o. branch (see [15], Appendix). We assume from now on that the q.o. branch ζ is normalized.

The semigroup $\mathbb{Z}^2_{\geqslant 0}$ has a minimal set of generators v_1, v_2, which is a basis of the lattice M_0. The dual basis, $\{w_1, w_2\}$, is a basis of the dual lattice N_0, and it spans a regular cone σ in $N_{0,\mathbb{R}} = N_0 \otimes_{\mathbb{Z}} \mathbb{R}$. It follows that $\mathbb{Z}^2_{\geqslant 0} = \sigma^\vee \cap M_0$, where $\sigma^\vee = \mathbb{R}^2_{\geqslant 0}$ is the dual cone of σ. The $\mathbb{C}$-algebra $\mathbb{C}\{x_1, x_2\}$ is isomorphic to the $\mathbb{C}$–algebra

$$\mathbb{C}\{\sigma^\vee \cap M_0\} = \Big\{ \sum c_\lambda \mathbf{x}^\lambda \mid c_\lambda \in \mathbb{C},\ \lambda \in \sigma^\vee \cap M_0 \Big\}.$$

The local algebra $\mathcal{O}_X = \mathbb{C}\{x_1, x_2\}[z]/(f)$ of the singularity $(X, 0)$ is isomorphic to $\mathbb{C}\{\sigma^\vee \cap M_0\}[\zeta]$. By Lemma 3.1 the series ζ can be viewed as an element $\sum \beta_\lambda \mathbf{x}^\lambda$ of the algebra $\mathbb{C}\{\sigma^\vee \cap M\}$.

The elements of M defined by:

$$\gamma_1 = \lambda_1 \text{ and } \gamma_{j+1} - n_j\gamma_j = \lambda_{j+1} - \lambda_j \text{ for } j = 1, \dots, g-1, \tag{6}$$

span the semigroup $\Gamma := \mathbb{Z}^2_{\geqslant 0} + \gamma_1\mathbb{Z}_{\geqslant 0} + \cdots + \gamma_g\mathbb{Z}_{\geqslant 0} \subset \sigma^\vee \cap M$. Analogously to λ_0 and λ_{g+1}, we set $\gamma_0 = (0,0)$ and $\gamma_{g+1} = \infty$, for convenience.

The semigroup Γ defines an analytic invariant of the germ $(X, 0)$ (see [16],[30],[20]).

Definition 3.5. *The monomial variety associated to* $(X, 0)$ *is the toric variety*

$$X^\Gamma := \mathit{Spec}\ \mathbb{C}[\Gamma].$$

Moreover we associate with the characteristic exponents the following sequence of semigroups:

$$\Gamma_j = \sigma^\vee \cap M + \gamma_1\mathbb{Z}_{\geq 0} + \cdots + \gamma_j\mathbb{Z}_{\geq 0}, \text{ for } j = 0, \dots, g.$$

And we have the corresponding monomial varieties X^{Γ_j} associated to Γ_j. We denote by $e_{i-1} := n_i \cdots n_g$ for $1 < i \leq g$ and set $e_g := 1$. Notice that, by (5) and the definition of $\gamma_1, \dots, \gamma_g$, we deduce that

$$\big(\gamma_1^{(1)}, \dots, \gamma_g^{(1)}\big) \geqslant_{\text{lex}} \big(\gamma_1^{(2)}, \dots, \gamma_g^{(2)}\big). \tag{7}$$

The following Lemma gathers some important facts about the generators γ_j and the semigroups Γ_j.

Lemma 3.6. *(see Lemma 3.3 in* [16]*)*

(i) *We have that* $\gamma_j > n_{j-1}\gamma_{j-1}$ *for* $j = 2, \dots, g$, *where* $<$ *means* $\neq$ *and* $\leq$ *coordinate-wise.*

(ii) *If a vector* $u_j \in \sigma^\vee \cap M_j$, *then we have* $u_j + n_j\gamma_j \in \Gamma_j$.

(iii) *The vector* $n_j\gamma_j$ *belongs to the semigroup* Γ_{j-1} *for* $j = 1, \dots, g$. *Moreover, we have a unique relation*

$$n_j\gamma_j = \alpha^{(j)} + r_1^{(j)}\gamma_1 + \cdots + r_{j-1}^{(j)}\gamma_{j-1} \tag{8}$$

such that $0 \leq r_i^{(j)} \leq n_i - 1$ *and* $\alpha^{(j)} \in M_0$ *for* $j = 1, \dots, g$.

Definition 3.7. *Given two irreducible q.o. polynomials* f *and* g *in* $\mathbb{C}\{x_1, x_2\}[z]$ *such that* fg *is a q.o. polynomial, we say that* f *and* g *have order of coincidence* $\alpha \in \mathbb{Q}^2$ *if* α *is the largest exponent on the set*

$$\left\{\lambda_{ij} \mid f(\zeta^{(i)}) = g(\zeta^{(j)}) = 0\right\},$$

where $\zeta^{(i)}$ *and* $\zeta^{(j)}$ *are roots of* fg.

Definition 3.8. *We associate to f a set of semi-roots*

$$z = f_0, f_1, \ldots, f_g = f \in \mathbb{C}\{x_1, x_2\}[z].$$

Every f_j is an irreducible q.o. polynomial of degree $n_0 \cdots n_j$ with order of coincidence with f equal to λ_{j+1} for $j = 0, \ldots, g$.

They are parametrized by truncations of a root $\zeta(x_1^{1/n}, x_2^{1/n})$ of f in the following sense:

Proposition 3.9. *(see [16]) Let $q \in \mathbb{C}\{x_1, x_2\}[z]$ be a monic polynomial of degree $n_0 \cdots n_j$. Then q is a j-th semi-root of f if and only if $q(\zeta) = \mathbf{x}^{\gamma_{j+1}} \epsilon_j$ for a unit ϵ_j in $\mathbb{C}\{x_1, x_2\}[z]$.*

Corollary 3.10. *The quasi-ordinary polynomials $f_j \in \mathbb{C}\{x_1, x_2\}[z]$ defining $X^{(j)}$ (see Definition 3.4) for $j = 0, \ldots, g-1$ form a system of semiroots of f. More precisely f_j is a j-th semiroot of f.*

Semi-roots play an important role in the understanding of quasi-ordinary singularities. In what follows we state some results about quasi-ordinary polynomials and semi-roots.

Lemma 3.11. *(See Lemma 35 in [17]) The expansion of semi-roots is of the following form:*

$$f_j = f_{j-1}^{n_j} - c_j x_1^{\alpha_1^{(j)}} x_2^{\alpha_2^{(j)}} f_0^{r_1^{(j)}} \cdots f_{j-2}^{r_{j-1}^{(j)}} + \sum c_{\underline{\alpha}, \underline{r}} x_1^{\alpha_1} x_2^{\alpha_2} f_0^{r_1} \cdots f_{j-1}^{r_j}, \tag{9}$$

where $c_j \in \mathbb{C}^*$, $0 \le r_i^{(j)}, r_i < n_i$ for $i = 1, \ldots, j$, and

$$n_j \gamma_j = (\alpha_1^{(j)}, \alpha_2^{(j)}) + r_1^{(j)} \gamma_1 + \cdots + r_{j-1}^{(j)} \gamma_{j-1} < (\alpha_1, \alpha_2) + r_1 \gamma_1 + \cdots + r_j \gamma_j.$$

As a consequence we have the following description of f.

Lemma 3.12. *For $0 \le l \le g-1$ we have*

$$f = f_l^{e_l} - d_l x_1^{\beta_1^{(l)}} x_2^{\beta_2^{(l)}} f_0^{s_1^{(l)}} \cdots f_{l-1}^{s_l^{(l)}} + \sum d_{\underline{\beta}, \underline{s}} x_1^{\beta_1} x_2^{\beta_2} f_0^{s_1} \cdots f_l^{s_{l+1}}$$

where $d_l \in \mathbb{C}^*$, $0 \le s_i^{(l)}, s_i < e_i$, and

$$\begin{aligned} n_{l+1} e_{l+1} \gamma_{l+1} &= (\beta_1^{(l)}, \beta_2^{(l)}) + s_1^{(l)} \gamma_1 + \cdots + s_l^{(l)} \gamma_l \\ &\le (\beta_1, \beta_2) + s_1 \gamma_1 + \cdots + s_{l+1} \gamma_{l+1} \end{aligned}$$

Sometimes we will write

$$f = f_l^{e_l} + \sum d_{\underline{\beta}, \underline{s}} x_1^{\beta_1} x_2^{\beta_2} f_0^{s_1} \cdots f_l^{s_{l+1}},$$

with $n_{l+1} e_{l+1} \gamma_{l+1} \le (\beta_1, \beta_2) + s_1 \gamma_1 + \cdots + s_{l+1} \gamma_{l+1}$, taking into account that for $\underline{\beta} = \left(\beta_1^{(l)}, \beta_2^{(l)}\right)$ and $\underline{s} = \left(s_1^{(l)}, \ldots, s_l^{(l)}, 0\right)$ we have $d_{\underline{\beta}, \underline{s}} \neq 0$.

Definition 3.13. *We define*

$$Z_i = X \cap \{x_i = 0\}, \qquad \textit{for } i = 1,2$$

$$Z_{12} = X \cap \{x_1 = x_2 = 0\}.$$

Moreover, the smallest number $c \in \{1,2\}$ with the property that

$$\lambda_i^{(j)} = 0, \ \textit{for all } 1 \leq i \leq g \textit{ and } c+1 \leq j \leq 2$$

is called the equisingular dimension of the quasi-ordinary projection p.

By condition (5) we have that c gives the number of variables appearing in the monomials $\mathbf{x}^{\lambda_1}, \dots, \mathbf{x}^{\lambda_g}$. In [22] Lipman proved that the spaces Z_1, Z_2 and Z_{12} are irreducible and described the singular locus of a q.o. singularity in terms of them. We state his result here for the particular case of surfaces.

Theorem 3.14. *(See Theorem 7.3 in [22]) Let X be a quasi-ordinary surface singularity with characteristic exponents $\lambda_1, \dots, \lambda_g$. Then we have:*

(i) *$X_{Sing} = Z_{12}$ if and only if $g = 1$ and $\lambda_1 = \left(\frac{1}{n}, \frac{1}{n}\right)$.*
(ii) *If $c = 1$ then $X_{Sing} = Z_1$.*
(iii) *Otherwise $c = 2$, and since $\lambda_1^{(1)} \neq 0$, $Z_1 \subset X$ is a component of X_{Sing}. Moreover, if we do not have simultaneously $\lambda_k^{(2)} = 0$ for all $1 \leq k \leq g-1$ and $\lambda_g^{(2)} = \frac{1}{n_g}$, the singular locus is reducible of the form $X_{Sing} = Z_1 \cup Z_2$.*

Definition 3.15. *Let X be a quasi-ordinary surface singularity with $g \geq 1$ characteristic exponents. We define the integers $g_1 \geq 0$ and $g_2 \in \{g_1, g_1+1\}$ as follows:*

$$\textit{if } \ c = 1 \textit{ we set } g_1 = g_2 = g+1,$$

otherwise (recall that we set $\gamma_0 = (0,0)$),

$$\gamma_{g_1}^{(2)} = 0 \textit{ and } \gamma_{g_1+1}^{(2)} \neq 0,$$

$$g_2 = \begin{cases} g_1 + 1 & \textit{if } \ \gamma_{g_1+1}^{(2)} = \frac{1}{n_{g_1+1}} \\ \\ g_1 & \textit{otherwise} \end{cases}$$

Remark 3.16. *The integers g_1 and g_2 describe completely the singular locus of $X^{(j)}$ for $1 \leq j \leq g$. Indeed, first notice that*

$$Z_1 = \{x_1 = z = 0\}$$

$$Z_2 = \{x_2 = f_{g_1} = 0\}$$

$$Z_{12} = \{(0,0,0)\}$$

and hence the singular locus of a quasi-ordinary surface singularity X is either a point, or a line, or two lines, or a line and a singular curve. Moreover, for $1 \leq j \leq g$, we have

$$X^{(j)}_{Sing} = \begin{cases} Z_{12} & \textit{if } \ j = 1 \textit{ and } \lambda_1 = \left(\frac{1}{n_1}, \frac{1}{n_1}\right) \\ Z_1 & \textit{if } \ j \leq g_2 \\ Z_1 \cup Z_2 & \textit{if } \ g_2 < j \leq g \end{cases} \tag{10}$$

Then, geometrically, the meaning of the integer g_2 is to measure the irreducibility of the singular locus of the semi-roots, since $X^{(j)}_{Sing}$ is irreducible if and only if $1 \leq j \leq min\{g_2, g\}$.

Now we define a sequence of semi-open cones keeping track of the singular locus of the quasi-ordinary hypersurfaces $X^{(j)}$ for $j = 1, \ldots, g$.

Definition 3.17. *Recall that $\sigma = \mathbb{R}^2_{\geq 0}$. Let $\rho_1 = (1,0)\mathbb{R}_{\geq 0}$ and $\rho_2 = (0,1)\mathbb{R}_{\geq 0}$ be its one-dimensional closed faces. For $1 \leq j \leq g$*

$$\sigma_{Sing,j} = \begin{cases} \overset{\circ}{\sigma} & \textit{if} \quad X^{(j)}_{Sing} = Z_{12} \\ \sigma \setminus \rho_2 & \textit{if} \quad X^{(j)}_{Sing} = Z_1 \\ \sigma \setminus \{(0,0)\} & \textit{if} \quad X^{(j)}_{Sing} = Z_1 \cup Z_2 \end{cases}, \qquad \textit{and } \sigma_{Reg,j} = \sigma \setminus \sigma_{Sing,j}.$$

For convenience we define $\sigma_{Reg,j} = \rho_1 \cup \rho_2$ for $j = -1, 0$. Moreover we denote $\sigma_{Reg,g}$ and $\sigma_{Sing,g}$ simply by σ_{Reg} and σ_{Sing}.

The sequence $\{\sigma_{Reg,-1}, \ldots, \sigma_{Reg,g}\}$ is not very complicated, in the sense that most of the elements are the same. Since by definition $\gamma^{(2)}_{g_1+1} = \lambda^{(2)}_{g_1+1}$, it is clear by definition and by (10) that

$$\text{for } -1 \leq j \leq g_2 \qquad \sigma_{Reg,j} = \begin{cases} \rho_1 \cup \rho_2 & \text{if } j < 1 \text{ or if } j = 1 \text{ and } \gamma_1 = \left(\frac{1}{n_1}, \frac{1}{n_1}\right) \\ \rho_2 & \text{otherwise} \end{cases}$$

$$\text{for } g_2 + 1 \leq j \leq g \quad \sigma_{Reg,j} = \{(0,0)\}$$

Moreover notice that, by definition, we have $\sigma_{Sing,j} \subseteq \sigma_{Sing,j+1}$.

Definition 3.18. *Given $\nu \in \sigma \cap N_0$, we define the following integer*

$$i(\nu) = \begin{cases} g+1 & \textit{if } \nu \in N_g \\ min\ \{1 \leq i \leq g \mid \nu \notin N_i\} & \textit{otherwise} \end{cases}$$

We finish the section with another definition.

Definition 3.19. *For $\nu \in \sigma \cap N_0$ we define the ring*

$$R_\nu = \begin{cases} \mathbb{C}\{x_1^{(0)}, x_2^{(0)}\} & \text{if } \nu = (0,0) \\ \mathbb{C}\{x_1^{(0)}\}[x_2^{(\nu_2)}] & \text{if } \nu \in \rho_2 \\ \mathbb{C}\{x_2^{(0)}\}[x_1^{(\nu_1)}] & \text{if } \nu \in \rho_1 \\ \mathbb{C}[x_1^{(\nu_1)}, x_2^{(\nu_2)}] & \text{otherwise} \end{cases}$$

4. Jet schemes of q.o. surface singularities: the case of one characteristic exponent

We describe the irreducible components of the m–jet schemes through the singular locus of a q.o. surface with one characteristic exponent. First we define certain algebraic varieties C_m^ν and prove its irreducibility. Since they cover the whole $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$, they are candidates to be the irreducible components of m-jets through the singular locus, and we have to study the inclusions among them.

Finally we construct a graph Γ representing the decomposition of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ for every m, with a suitable decoration. We prove that this graph is equivalent to the topological type of the singularity, i.e., to the characteristic exponent λ. All these results will be generalized in Section 5. Since in that section we will work with the generators of the semigroup Γ rather than with the characteristic exponents, we will use now the notation γ instead of λ (recall that by definition $\gamma = \lambda$).

In this section X is a q.o. surface defined by the polynomial

$$f = z^n - x_1^a x_2^b + \sum_{(i,j)+k\gamma > n\gamma} c_{ijk} x_1^i x_2^j z^k,$$

where $\gamma = \left(\frac{a}{n}, \frac{b}{n}\right)$ with $a \geq b \geq 0$ is the characteristic exponent. We have that $\gcd(a,b,n) = 1$ because we assume the q.o. surface to be irreducible. Moreover, if $b = 0$ then we have that $a > n$, since the branch is normalized.

Remark 4.1. *Note that $f = z^n - x_1^a x_2^b$ defines a toric surface, non-normal in general (it is normal if and only if $a = b = 1$). Therefore, in particular, in this section we describe the m–jets through the singular locus of a family of non-normal toric surfaces.*

Let us look at some examples.

Example 4.2. *Let X be the surface defined by the q.o. polynomial*

$$f = z^3 - x_1^4 + x_1^4 x_2 + x_1^3 z + x_1^2 x_2 z^2$$

with characteristic exponent $\gamma = \left(\frac{4}{3}, 0\right)$. We have that $X_{Sing} = \{z = x_1 = 0\}$, and then

$$\pi_m^{-1}(X_{Sing}) = V\left(x_1^{(0)}, z^{(0)}, F^{(1)}, \ldots, F^{(m)}\right),$$

since $F^{(0)} \equiv 0$ *mod* $\left(x_1^{(0)}, z^{(0)}\right)$. *Moreover*

$$\begin{aligned} F^{(1)} = \; & 3{z^{(0)}}^2 z^{(1)} - 4{x_1^{(0)}}^3 x_1^{(1)} + 4{x_1^{(0)}}^3 x_1^{(1)} x_2^{(0)} + {x_1^{(0)}}^4 x_2^{(1)} + 3{x_1^{(0)}}^2 x_1^{(1)} z^{(0)} + {x_1^{(0)}}^3 z^{(1)} \\ & + 2x_1^{(0)} x_1^{(1)} x_2^{(0)} {z^{(0)}}^2 + {x_1^{(0)}}^2 x_2^{(1)} {z^{(0)}}^2 + 2{x_1^{(0)}}^2 x_2^{(0)} z^{(0)} z^{(1)} \\ \equiv \; & 0 \; mod \left(x_1^{(0)}, z^{(0)}\right). \end{aligned}$$

Analogously $F^{(2)} \equiv 0$ *mod* $\left(x_1^{(0)}, z^{(0)}\right)$, *but* $F^{(3)} \equiv {z^{(1)}}^3$ *mod* $\left(x_1^{(0)}, z^{(0)}\right)$. *Hence we deduce that*

$$\left(\pi_\ell^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, z^{(0)}\right), \quad \textit{for } \ell = 1, 2$$

$$\left(\pi_3^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, z^{(0)}, z^{(1)}\right)$$

Note that, though they are defined by the same ideal $\left(x_1^{(0)}, z^{(0)}\right)$, *we have that*

$$\left(\pi_1^{-1}(X_{Sing})\right)_{red} \neq \left(\pi_2^{-1}(X_{Sing})\right)_{red}$$

since $\left(\pi_1^{-1}(X_{Sing})\right)_{red} \subset \mathbb{A}_1^3$ *while* $\left(\pi_2^{-1}(X_{Sing})\right)_{red} \subset \mathbb{A}_2^3$ *(see Remark 2.5). For* $m = 4$ *we have*

$$F^{(4)} \equiv -{x_1^{(1)}}^4 + {x_1^{(1)}}^4 x_2^{(0)} \; mod \left(x_1^{(0)}, z^{(0)}, z^{(1)}\right).$$

Any jet through the singular locus has origin $(0, x_2^{(0)}, 0) \in X$, *and since we are dealing with a germ of q.o. surface* X, *we deduce that* $x_2^{(0)}$ *is small enough so that* $-1 + x_2^{(0)} \neq 0$, *or in other words,* $-1 + x_2^{(0)}$ *is a unit in the ring* $\mathbb{C}\{x_2^{(0)}\}$. *Therefore, from the equation* $-{x_1^{(1)}}^4 + {x_1^{(1)}}^4 x_2^{(0)} = 0$, *we deduce that* $x_1^{(1)}$ *must vanish, and hence*

$$\left(\pi_4^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\right).$$

Moreover, with analogous arguments we have

$$F^{(5)} \equiv 0 \; mod \left(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\right)$$

$$F^{(6)} \equiv {z^{(2)}}^3 \; mod \left(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\right)$$

$$F^{(7)} \equiv 0 \; mod \left(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}, z^{(2)}\right)$$

$$F^{(8)} \equiv (-1 + x_2^{(0)}){x_1^{(2)}}^4 \; mod \left(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}, z^{(2)}\right)$$

$$F^{(9)} \equiv {z^{(3)}}^3 \; mod \left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}\right)$$

$$F^{(\ell)} \equiv 0 \; mod \left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\right) \quad \textit{for} \quad \ell = 10, 11$$

$$F^{(12)} \equiv {z^{(4)}}^3 - {x_1^{(3)}}^4 + {x_1^{(3)}}^4 x_2^{(0)} \; mod \left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\right)$$

Hence we deduce the following decomposition in irreducible components:

$$\left(\pi_5^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\right)$$

$$\left(\pi_\ell^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}, z^{(2)}\right) \quad \textit{for} \quad \ell = 6, 7$$

$$\left(\pi_8^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}\right)$$

$$\left(\pi_\ell^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\right) \quad \textit{for} \quad \ell = 9, 10, 11$$

$$\left(\pi_{12}^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}, {z^{(4)}}^3 - {x_1^{(3)}}^4 + {x_1^{(3)}}^4 x_2^{(0)}\right)$$

For $m \leq 12$ *we have seen that* $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ *is irreducible (note that* $z^3 - x_1^4 + x_1^4 x_2$ *is an irreducible q.o. polynomial and therefore* $\left(\pi_{12}^{-1}(X_{Sing})\right)_{red}$ *is irreducible), but for* $m > 12$ *this is no longer true. Indeed,*

$$F^{(13)} \equiv 3{z^{(4)}}^2 z^{(5)} - 4{x_1^{(3)}}^4 x_1^{(4)} + {x_1^{(3)}}^4 x_2^{(1)} + 4{x_1^{(3)}}^3 x_1^{(4)} x_2^{(0)} + {x_1^{(3)}}^3 z^{(4)} \ \textit{mod}\ I,$$

where we set $I = \left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\right)$*, and* $\left(\pi_{13}^{-1}(X_{Sing})\right)_{red}$ *has two irreducible components:*

$$\overline{V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}, F^{(12)}, F^{(13)}\right) \cap \{x_1^{(3)} \neq 0\}}$$

$$V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}\right)$$

The irreducibility of the first component follows by Proposition 2.4, since its generic part

$$V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}, F^{(12)}, F^{(13)}\right) \cap \{x_1^{(3)} \neq 0\}$$

projects by $\pi_{13,12}$ *into the non-singular locus*

$$\mathrm{Reg}\left(V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}, F^{(12)}\right)\right).$$

Note that

$$F^{(12)} \equiv {z^{(4)}}^3 - {x_1^{(3)}}^4 + {x_1^{(3)}}^4 x_2^{(0)} \textit{mod}\ \left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\right),$$

and hence the regular part is contained in $\{x_1^{(3)} \neq 0\}$.

The same kind of argument implies that

$$\overline{V\left(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}, F^{(12)}, \ldots, F^{(m)}\right) \cap \{x_1^{(3)} \neq 0\}},$$

is irreducible, and we will prove that it is indeed an irreducible component of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ *for any* $m \geq 13$.

In the example above we have components defined by the annihilation of hyperplane coordinates in $\mathbb{A}_m^3 = \mathrm{Spec}\ \mathbb{C}[x_1^{(i)}, x_2^{(i)}, z^{(i)}]_{i=0,\ldots,m}$. They have the property of staying irreducible when lifted from level m to $m+1$. We see next an example where this is not always the case. This difference will turn out to be important later, when studying the graph in Lemma 4.19.

Example 4.3. *Let X be the surface defined by the q.o. polynomial*

$$f = z^4 - x_1^6 x_2 + x_1^5 x_2 z + x_1^3 x_2 z^2 + x_1^2 x_2 z^3,$$

with characteristic exponent $\gamma = (\frac{6}{4}, \frac{1}{4})$. We have that $X_{Sing} = \{z = x_1 = 0\}$, and then

$$\pi_m^{-1}(X_{Sing}) = V\big(x_1^{(0)}, z^{(0)}, F^{(1)}, \dots, F^{(m)}\big),$$

since $F^{(0)} \equiv 0$ mod $\big(x_1^{(0)}, z^{(0)}\big)$. We have that

$$\begin{aligned} F^{(1)} = \quad & 4z^{(0)}z^{(1)} - 6{x_1^{(0)}}^5 x_1^{(1)} x_1^{(1)} x_2^{(0)} - {x_1^{(0)}}^6 x_2^{(1)} + 5{x_1^{(0)}}^4 x_1^{(1)} x_2^{(0)} z^{(0)} \\ & + {x_1^{(0)}}^5 x_2^{(1)} z^{(0)} + {x_1^{(0)}}^5 x_2^{(0)} z^{(1)} + 3{x_1^{(0)}}^2 x_1^{(1)} x_2^{(0)} {z^{(0)}}^2 + {x_1^{(0)}}^3 x_2^{(1)} {z^{(0)}}^2 \\ & + 2{x_1^{(0)}}^3 x_2^{(0)} z^{(0)} z^{(1)} + 2x_1^{(0)} x_1^{(1)} x_2^{(0)} {z^{(0)}}^3 + {x_1^{(0)}}^2 x_2^{(1)} {z^{(0)}}^3 \\ & + 3{x_1^{(0)}}^2 x_2^{(0)} {z^{(0)}}^2 z^{(1)} \\ \\ \equiv \quad & 0 \ mod \ \big(x_1^{(0)}, z^{(0)}\big) \end{aligned}$$

Analogously we have that $F^{(2)} \equiv 0$ mod $\big(x_1^{(0)}, z^{(0)}\big)$ and $F^{(3)} \equiv 0$ mod $\big(x_1^{(0)}, z^{(0)}\big)$, but $F^{(4)} \equiv {z^{(1)}}^4$ mod $\big(x_1^{(0)}, z^{(0)}\big)$. Moreover $F^{(5)} \equiv 0$ mod $\big(x_1^{(0)}, z^{(0)}, z^{(1)}\big)$ and $F^{(6)} \equiv -{x_1^{(1)}}^6 x_2^{(0)}$ mod $\big(x_1^{(0)}, z^{(0)}, z^{(1)}\big)$, which implies that

$$\big(\pi_6^{-1}(X_{Sing})\big)_{red} = V\big(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\big) \cup V\big(x_1^{(0)}, x_2^{(0)}, z^{(0)}, z^{(1)}\big).$$

Note how $V\big(x_1^{(0)}, z^{(0)}, z^{(1)}\big)$ is a component at level $m = 5$, and it is defined by hyperplane coordinates, but $\pi_{6,5}^{-1}\big(V(x_1^{(0)}, z^{(0)}, z^{(1)})\big)$ is no longer irreducible. Now, to lift these two components to level 7 we study the polynomial $F^{(7)}$. We have that

$$F^{(7)} \equiv \begin{cases} \qquad 0 \qquad mod \ \big(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\big) \\ \\ -{x_1^{(1)}}^6 x_2^{(1)} \ mod \ \big(x_1^{(0)}, x_2^{(0)}, z^{(0)}, z^{(1)}\big) \end{cases}$$

Then

$$\begin{aligned} \pi_{7,6}^{-1}\big(V\big(x_1^{(0)}, x_2^{(0)}, z^{(0)}, z^{(1)}\big)\big) = V\big(x_1^{(0)}, x_1^{(1)}, x_2^{(0)}, z^{(0)}, z^{(1)}\big) \bigcup \\ \bigcup V\big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}\big) \end{aligned}$$

and since $V\big(x_1^{(0)}, x_1^{(1)}, x_2^{(0)}, z^{(0)}, z^{(1)}\big) \subseteq V\big(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\big)$ we conclude that

$$\big(\pi_7^{-1}(X_{Sing})\big)_{red} = V\big(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\big) \cup V\big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}\big)$$

At level $m = 8$ we have

$$F^{(8)} \equiv \begin{cases} \qquad {z^{(2)}}^4 \qquad mod \ \big(x_1^{(0)}, x_1^{(1)}, z^{(0)}, z^{(1)}\big) \\ \\ {z^{(2)}}^4 - {x_1^{(1)}}^6 x_2^{(2)} \ mod \ \big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}\big) \end{cases}$$

and

$$\pi_{9,8}^{-1}\big(V\big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}, F^{(8)}\big)\big)_{red} =$$

$$= \pi_{9,8}^{-1}\big(Sing(V\big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}, F^{(8)}\big))\big) \bigcup$$
$$\bigcup \overline{\pi_{9,8}^{-1}\big(Reg(V\big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}, F^{(8)}\big))\big)} =$$
$$= V\big(x_1^{(0)}, x_1^{(1)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}, z^{(2)}\big) \bigcup$$
$$\bigcup \overline{V\big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}, z^{(1)}, F^{(8)}, F^{(9)}\big) \cap \{x_1^{(1)} \neq 0\}}.$$

We will describe the irreducible decomposition of m–jets through the singular locus as

$$\big(\pi_m^{-1}(X_{Sing})\big)_{red} = \bigcup_{\nu \in F_m} C_m^{\nu}$$

for a certain finite set $F_m \subseteq \mathbb{Z}^2$ and certain irreducible sets C_m^{ν} that we proceed to define. First we recall, in the case of only one characteristic exponent, some objects described in Section 3 in general.

The notion of equisingular dimension c was given in Definition 3.13. By Theorem 3.14 we have the following description of the singular locus of X (recall that the characteristic exponent is $\gamma = \big(\frac{a}{n}, \frac{b}{n}\big)$ with $a \geq b \geq 0$),

$$X_{Sing} = \begin{cases} \{z = x_1 = 0\} & \text{if } c = 1 \ \big(\text{i.e. } b = 0\big) \\ \{(0,0,0)\} & \text{if } c = 2 \text{ and } a = b = 1 \\ \{z = x_1 = 0\} & \text{if } c = 2 \text{ and } a > 1, b = 1 \\ \{z = x_1 = 0\} \cup \{z = x_2 = 0\} & \text{if } c = 2 \text{ and } a, b > 1 \end{cases} \tag{11}$$

From Definition 3.17 we have that

$$\sigma_{Sing} = \begin{cases} \overset{\circ}{\sigma} & \text{if } \gamma = \big(\frac{1}{n}, \frac{1}{n}\big) \\ \sigma \setminus \rho_2 & \text{if } \gamma = \big(\frac{a}{n}, \frac{1}{n}\big) \text{ or } \gamma = \big(\frac{a}{n}, 0\big) \\ \sigma \setminus \{(0,0)\} & \text{otherwise} \end{cases}$$

where recall that $\sigma = \mathbb{R}^2_{\geq 0}$ and $\rho_1 = (1,0)\mathbb{R}_{\geq 0}$ and $\rho_2 = (0,1)\mathbb{R}_{\geq 0}$ are its one dimensional faces.

Given $\gamma(t) \in X_m$ with $x_i \circ \gamma(t) \neq 0$ for $i = 1, 2$, we have that $\mathrm{ord}_t(x_i \circ \gamma(t)) \geq 0$. Hence

$$\nu := \big(\mathrm{ord}_t(x_1 \circ \gamma(t)), \mathrm{ord}_t(x_2 \circ \gamma(t))\big) \in \sigma \cap N_0.$$

If we add the condition $\pi_m(\gamma(t)) \in X_{Sing}$ then $\nu \in \sigma_{Sing} \cap N_0$. Moreover it is clear that $0 \leq \nu_i \leq m$ for $i = 1, 2$.

Definition 4.4. *Given a positive integer m and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$, we define an algebraic variety $C_m^\nu \subseteq \mathbb{A}_m^3$ as follows (recall Remark 2.5).*

- *If $m < \langle \nu, n\gamma \rangle$,*

$$C_m^\nu := V\left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{([m/n])}\right)$$

Note that C_m^ν is a non-singular algebraic variety of $\mathbb{A}_m^3$.

- *If $m = \langle \nu, n\gamma \rangle$ and $\nu \in N$,*

$$C_{\langle \nu, n\gamma \rangle}^\nu := V\left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{(\langle \nu,\gamma \rangle -1)}, F^{(\langle \nu, n\gamma \rangle)}\right)$$

Note that C_m^ν is not well defined if $\nu \in N_0 \setminus N$ since $\langle \nu, \gamma \rangle$ is not an integer.

The polynomial $F^{(\langle \nu, n\gamma \rangle)}$ modulo the ideal

$$\left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{(\langle \nu,\gamma \rangle -1)}\right)$$

is studied in Lemma 4.5, and it turns out that $C_{\langle \nu, n\gamma \rangle}^\nu$ is a singular algebraic variety of $\mathbb{A}_m^3$.

- *If $m > \langle \nu, n\gamma \rangle$ and $\nu \in N$,*

$$C_m^\nu := \overline{\pi_{m,\langle \nu, n\gamma \rangle}^{-1}\left(Reg\ (C_{\langle \nu, n\gamma \rangle}^\nu)\right)}$$

where the overline denotes the Zariski closure and Reg stands for regular locus.

It turns out to be crucial to understand the variety $C_{\langle \nu, n\gamma \rangle}^\nu$.

Lemma 4.5. *For $\nu \in \sigma_{Sing} \cap N$ we define*

$$F_\nu^{(\langle \nu, n\gamma \rangle)} = {z^{(\langle \nu, \gamma \rangle)}}^n - {x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b + \sum c_{ijk}\ {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j {z^{(\langle \nu, \gamma \rangle)}}^k$$

where the sum runs over i, j, k subject to the conditions: the monomial $c_{ijk} x_1^i x_2^j z^k$ appears in the q.o. polynomial f and $\langle \nu, (i,j) + k\gamma \rangle = \langle \nu, n\gamma \rangle$. Note that, if $\nu \in \mathring{\sigma}$, the sum $\sum c_{ijk} {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j {z^{(\langle \nu, \gamma \rangle)}}^k$ in $F_\nu^{(\langle \nu, n\gamma \rangle)}$ is zero.

We have that

$$F^{(\langle \nu, n\gamma \rangle)} \equiv F_\nu^{(\langle \nu, n\gamma \rangle)}\ mod\ \left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{(\langle \nu,\gamma \rangle -1)}\right),$$

and hence

$$C_{\langle \nu, n\gamma \rangle}^\nu = V\left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{(\langle \nu,\gamma \rangle -1)}, F_\nu^{(\langle \nu, n\gamma \rangle)}\right).$$

In particular observe that $F_\nu^{(\langle \nu, n\gamma \rangle)}$ is a q.o. polynomial in $R_\nu[z^{(\langle \nu, \gamma \rangle)}]$ (see Definition 3.19).

If $\nu \in \sigma_{Sing} \cap \left(N_0 \setminus N\right)$ we define

$$F_\nu^{(\langle \nu, n\gamma \rangle)} = -{x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b + \sum c_{ij0} {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j$$

where $c_{ij0}x_1^i x_2^j$ *is a monomial in* f *and* $\langle \nu, (i,j)\rangle = \langle \nu, n\gamma\rangle$*. Then*

$$F^{(\langle \nu, n\gamma\rangle)} \equiv F_\nu^{(\langle \nu, n\gamma\rangle)} \ mod \ \left(x_1^{(0)}, \dots, x_1^{(\nu_1-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2-1)}, z^{(0)}, \dots, z^{([\langle \nu, n\gamma\rangle])}\right)$$

The sum $\sum c_{ij0} {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j$ *is non-zero if and only if* $\nu \in \rho_1 \cup \rho_2$*. Moreover, if* $\nu \in \rho_1 \cup \rho_2$

$$F_\nu^{(\langle \nu, n\gamma\rangle)} = -{x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b \cdot U$$

where U *is a unit in* R_ν*.*

Proof. If $\nu \in N$, we have by definition that the polynomial $F^{(\langle \nu, n\gamma\rangle)}$ modulo the ideal $\left(x_1^{(0)}, \dots, x_1^{(\nu_1-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2-1)}, z^{(0)}, \dots, z^{(\langle \nu, \gamma\rangle-1)}\right)$ is of the form

$$ {z^{(\langle \nu, \gamma\rangle)}}^n - {x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b + \sum c_{ijk}\, {x_1^{(\alpha)}}^i {x_2^{(\beta)}}^j {z^{(\delta)}}^k$$

where the sum runs subject to the conditions

$$(i,j) + k\gamma > n\gamma$$

$$i\alpha + j\beta + k\delta = \langle \nu, n\gamma\rangle$$

$$\alpha \geq \nu_1,\ \beta \geq \nu_2,\ \delta \geq \langle \nu, \gamma\rangle$$

If there exists at least one $c_{ijk} \neq 0$ under these conditions

$$\langle \nu, n\gamma\rangle = i\alpha + j\beta + k\delta \geq i\nu_1 + j\nu_2 + k\langle \nu, \gamma\rangle = \langle \nu, (i,j) + k\gamma\rangle \geq \langle \nu, n\gamma\rangle,$$

then all inequalities must be equalities and we deduce that

$$\alpha = \nu_1,\ \beta = \nu_2 \text{ and } \delta = \langle \nu, \gamma\rangle.$$

Then, the condition $i\alpha + j\beta + k\delta = \langle \nu, n\gamma\rangle$ is $\langle \nu, (i,j) + k\gamma\rangle = \langle \nu, n\gamma\rangle$, and this only holds if $\nu \in \rho_1 \cup \rho_2$.

If $\nu \in N_0 \setminus N$, first note that

$$[\langle \nu, \gamma\rangle] < \langle \nu, \gamma\rangle < [\langle \nu, \gamma\rangle] + 1.$$

By definition $F^{(\langle \nu, n\gamma\rangle)}$ mod $\left(x_1^{(0)}, \dots, x_1^{(\nu_1-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2-1)}, z^{(0)}, \dots, z^{([\langle \nu, \gamma\rangle])}\right)$ is of the form

$$-{x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b + \sum c_{ijk} {x_1^{(\alpha)}}^i {x_2^{(\beta)}}^j {z^{(\delta)}}^k$$

where the sum runs subject to the conditions

$$(i,j) + k\gamma > n\gamma$$

$$i\alpha + j\beta + k\delta = \langle \nu, n\gamma\rangle$$

$$\alpha \geq \nu_1,\ \beta \geq \nu_2,\ \delta \geq [\langle \nu, \gamma\rangle] + 1$$

If there exists at least one $c_{ijk} \neq 0$ with i, j, k under these conditions and moreover with $k > 0$, then

$$\langle \nu, n\gamma\rangle = i\alpha + j\beta + k\delta \geq i\nu_1 + j\nu_2 + k([\langle \nu, \gamma\rangle] + 1) > \langle \nu, (i,j) + k\gamma\rangle \geq \langle \nu, n\gamma\rangle$$

Therefore we must have $k = 0$, and then

$$\langle \nu, n\gamma \rangle = i\alpha + j\beta \geq \langle \nu, (i,j) \rangle \geq \langle \nu, n\gamma \rangle$$

Hence $\alpha = \nu_1$ and $\beta = \nu_2$. As in the case $\nu \in N$, the sum $\sum c_{ij0} {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j$ is non-zero if and only if $\nu \in \rho_1 \cup \rho_2$ since the conditions $(i,j) > n\gamma$ and $\langle \nu, (i,j) \rangle = \langle \nu, n\gamma \rangle$ are compatible if and only if $\nu \in \sigma_{Sing} \cap (\rho_1 \cup \rho_2)$.

If $\nu \in \rho_1$ then $F_\nu^{(\langle \nu, n\gamma \rangle)} = -{x_1^{(\nu_1)}}^a {x_2^{(0)}}^b + \sum c_{ij0} {x_1^{(\nu_1)}}^i {x_2^{(0)}}^j$ with the conditions $(i,j) > (a,b)$ and $\langle \nu, n\gamma \rangle = \langle \nu, (i,j) \rangle$. Then $i = a$, and therefore $j > b$. Hence

$$F_\nu^{(\langle \nu, n\gamma \rangle)} = -{x_1^{(\nu_1)}}^a {x_2^{(0)}}^b \Big(1 + \sum c_{ij0} {x_2^{(0)}}^{j-b}\Big)$$

and $1 + \sum c_{ij0} {x_2^{(0)}}^{j-b}$ is a unit in $\mathbb{C}\{x_2^{(0)}\}$.

If $\nu \in \rho_2$ the proof is analogous, simply noticing (by the definition of g_2) that, since $\nu \in \sigma_{Sing}$, we must be in the case $b > 1$. $\square$

Example 4.6. *Consider the q.o. surface defined by $f = z^2 - x_1^4 x_2^3 + x_1^2 x_2^2 z$, with characteristic exponent $\lambda = \left(\frac{4}{2}, \frac{3}{2}\right)$. For $\nu = (2,0)$ we have*

$$F_\nu^{(8)} = {z^{(4)}}^2 - {x_1^{(2)}}^4 {x_2^{(0)}}^3 + {x_1^{(2)}}^2 {x_2^{(0)}}^2 z^{(4)}.$$

Obviously we can not write $F_\nu^{(8)}$ as ${z^{(4)}}^2 - {x_1^{(2)}}^4 {x_2^{(0)}}^3 \cdot U$ with U a unit, as we have proved in Lemma 4.5 that it is the case when $\nu \notin N$. Notice however that

$$V\big(F_\nu^{(8)}\big) \cap \{x_1^{(2)} \neq 0\} \cap \{x_2^{(0)} \neq 0\} \subseteq \{z^{(4)} \neq 0\},$$

this will turn out to be crucial (see Corollary 5.16 for a complete statement in the general case).

To understand completely the sets C_m^ν for $m > \langle \nu, n\gamma \rangle$, we need to study the regular part $\mathrm{Reg}\left(C^\nu_{\langle \nu, n\gamma \rangle}\right)$. It is closely related to the regular locus of X, described in (11).

Lemma 4.7. *For $\nu \in \sigma_{Sing} \cap N$, let us denote*

$$J^\nu = \big(x_1^{(0)}, \ldots, x_1^{(\nu_1 - 1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2 - 1)}, z^{(0)}, \ldots, z^{(\langle \nu, \gamma \rangle - 1)}\big)$$

We have:

(i) *if $\gamma = \left(\frac{1}{n}, \frac{1}{n}\right)$,*

$$\overline{Reg\left(C^\nu_{\langle \nu, n\gamma \rangle}\right)} = C^\nu_{\langle \nu, n\gamma \rangle}$$

and, as a consequence, for $m > \langle \nu, n\gamma \rangle$,

$$C_m^\nu = V\left(J^\nu, F^{(\langle \nu, n\gamma \rangle)}, \ldots, F^{(m)}\right)$$

(ii) *otherwise $\gamma = \left(\frac{a}{n}, \frac{b}{n}\right)$ with $a > 1$ and*

$$Reg\left(C^\nu_{\langle \nu, n\gamma \rangle}\right) = \begin{cases} C^\nu_{\langle \nu, n\gamma \rangle} \cap \{x_1^{(\nu_1)} \neq 0\} & \textit{if } b \in \{0,1\} \\[2ex] C^\nu_{\langle \nu, n\gamma \rangle} \cap \{x_1^{(\nu_1)} \neq 0\} \cap \{x_2^{(\nu_2)} \neq 0\} & \textit{otherwise} \end{cases}$$

As a consequence, for $m > \langle \nu, n\gamma\rangle$ and $b = 0$ or 1, we have that

$$C_m^\nu = \overline{V\left(J^\nu, F^{(\langle \nu, n\gamma\rangle)}, \ldots, F^{(m)}\right) \cap \{x_1^{(\nu_1)} \neq 0\}}$$

while for $m > \langle \nu, n\gamma\rangle$ and $b \neq 0, 1$,

$$C_m^\nu = \overline{V\left(J^\nu, F^{(\langle \nu, n\gamma\rangle)}, \ldots, F^{(m)}\right) \cap \{x_1^{(\nu_1)} \neq 0\} \cap \{x_2^{(\nu_2)} \neq 0\}}$$

Proof. We distinguish cases according to the description in (11) of the singular locus.

(i) If $\gamma = \left(\frac{1}{n}, \frac{1}{n}\right)$ the claim of the Lemma is clear, since the singular locus of such a q.o. surface is the origin.

(ii) If $\gamma = \left(\frac{a}{n}, 0\right)$ we have to prove that, in $C^\nu_{\langle \nu, n\gamma\rangle}$, the conditions $x_1^{(\nu_1)} \neq 0$ and $z^{(\langle \nu,\gamma\rangle)} \neq 0$ are equivalent. By Lemma 4.5,

$$F_\nu^{(\langle \nu, n\gamma\rangle)} = {z^{(\langle \nu,\gamma\rangle)}}^n - {x_1^{(\nu_1)}}^a + \sum c_{ijk} {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j {z^{(\langle \nu,\gamma\rangle)}}^k$$

is a defining equation of $C^\nu_{\langle \nu, n\gamma\rangle}$. If the sum in $F_\nu^{(\langle \nu, n\gamma\rangle)}$ is empty the claim is obvious. Otherwise $\nu \in \rho_1 \cup \rho_2$ and, since $\nu \in \sigma_{Sing}$, we deduce that $\nu \in \rho_1$. Then

$$F_\nu^{(\langle \nu, n\gamma\rangle)} = {z^{(\langle \nu,\gamma\rangle)}}^n - {x_1^{(\nu_1)}}^a + \sum c_{ijk} {x_1^{(\nu_1)}}^i {x_2^{(0)}}^j {z^{(\langle \nu,\gamma\rangle)}}^k$$

with $\langle \nu, (i,j) + k\gamma\rangle = \langle \nu, n\gamma\rangle$, or equivalently

$$\nu_1 \left(i + k\frac{a}{n}\right) = \nu_1 n \frac{a}{n}$$

This implies that $i + k\frac{a}{n} = a$, and since $0 \leq k < n$ and $\gcd(a, n) = 1$, we deduce that $i = a$ and $k = 0$. Then $j > 0$ for any $c_{aj0} \neq 0$ and we can write $F_\nu^{(\langle \nu, n\gamma\rangle)}$ as

$$\begin{aligned} F_\nu^{(\langle \nu, n\gamma\rangle)} &= {z^{(\langle \nu,\gamma\rangle)}}^n - {x_1^{(\nu_1)}}^a + \textstyle\sum_j c_{aj0} {x_1^{(\nu_1)}}^a {x_2^{(0)}}^j \\ &= {z^{(\langle \nu,\gamma\rangle)}}^n - {x_1^{(\nu_1)}}^a \left(1 + \textstyle\sum_j c_{aj0} {x_2^{(0)}}^j\right) \end{aligned}$$

Since $1 + \sum c_{aj0} {x_2^{(0)}}^j$ is a unit in $R_\nu = \mathbb{C}\{x_2^{(0)}\}[x_1^{(\nu_1)}]$, it does not vanish, and the claim follows.

(iii) If $\gamma = \left(\frac{a}{n}, \frac{1}{n}\right)$, by Lemma 4.5 we have that

$$F_\nu^{(\langle \nu, n\gamma\rangle)} = {z^{(\langle \nu,\gamma\rangle)}}^n - {x_1^{(\nu_1)}}^a x_2^{(\nu_2)} + \sum c_{ijk} {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j {z^{(\langle \nu,\gamma\rangle)}}^k$$

where the sum runs over (i, j, k) such that $(i, j) + k\gamma > (a, 1)$. Therefore we deduce that for any such (i, j, k) we have $i > 0$ because $0 \leq k < n$. It follows that if $x_1^{(\nu_1)} = 0$ and $F_\nu^{(\langle\nu,n\gamma\rangle)} = 0$ then we have that $z^{(\langle\nu,\gamma\rangle)} = 0$. Hence

$$\text{Reg}\left(C^\nu_{\langle\nu,n\gamma\rangle}\right) = C^\nu_{\langle\nu,n\gamma\rangle} \cap \{z^{(\langle\nu,\gamma\rangle)} \neq 0\} \bigcup C^\nu_{\langle\nu,n\gamma\rangle} \cap \{x_1^{(\nu_1)} \neq 0\} = C^\nu_{\langle\nu,n\gamma\rangle} \cap \{x_1^{(\nu_1)} \neq 0\}$$

(iv) If $\gamma = \left(\frac{a}{n}, \frac{b}{n}\right)$ with $b > 1$, we have that

$$\text{Reg}\left(C^\nu_{\langle\nu,n\gamma\rangle}\right) = C^\nu_{\langle\nu,n\gamma\rangle} \cap \{x_1^{(\nu_1)} \neq 0\} \cap \{x_2^{(\nu_2)} \neq 0\} \bigcup C^\nu_{\langle\nu,n\gamma\rangle} \cap \{z^{(\langle\nu,\gamma\rangle)} \neq 0\}$$

and we claim that $C^\nu_{\langle\nu,n\gamma\rangle} \cap \{z^{(\langle\nu,\gamma\rangle)} \neq 0\} = C^\nu_{\langle\nu,n\gamma\rangle} \cap \{x_1^{(\nu_1)} \neq 0\} \cap \{x_2^{(\nu_2)} \neq 0\}$. Indeed, by Lemma 4.5 it follows that

$$F_\nu^{(\langle\nu,n\gamma\rangle)} = {z^{(\langle\nu,\gamma\rangle)}}^n - {x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b + \sum c_{ijk}\, {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j {z^{(\langle\nu,\gamma\rangle)}}^k$$

is a defining equation of $C^\nu_{\langle\nu,n\gamma\rangle}$. If the sum is zero, the claim is obvious. Otherwise $\nu \in \rho_1 \cup \rho_2$. Let us suppose that $\nu \in \rho_1$ (the case $\nu \in \rho_2$ is completely analogous). The monomials on the sum are of the form $c_{ijk} {x_1^{(\nu_1)}}^i {x_2^{(0)}}^j {z^{(\langle\nu,\gamma\rangle)}}^k$ with $i + k\frac{a}{n} = a$ and $j + k\frac{b}{n} > b$. In particular we deduce that $i, j > 0$. Therefore if $F_\nu^{(\langle\nu,n\gamma\rangle)} = 0$ and either $x_1^{(\nu_1)} = 0$ or $x_2^{(0)} = 0$ it follows that $z^{(\langle\nu,\gamma\rangle)} = 0$. And conversely, if $z^{(\langle\nu,\gamma\rangle)} = 0$, then

$$F_\nu^{(\langle\nu,n\gamma\rangle)}\left(z^{(\langle\nu,\gamma\rangle)} = 0\right) = -{x_1^{(\nu_1)}}^a {x_2^{(0)}}^b \cdot U$$

with U a unit in $\mathbb{C}\{x_2^{(0)}\}$ (because if we impose $k = 0$ on the conditions $i + k\frac{a}{n} = a$ and $j + k\frac{b}{n} > b$ we obviously obtain $i = a$ and $j > b$). Then the result follows.

□

Definition 4.8. *We define the set* $A_m \cup B_m \subseteq \sigma_{Sing} \cap [0,m]^2 \cap N_0$ *as*

$$A_m = \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0 \mid \langle\nu, n\gamma\rangle > m\}$$

$$B_m = \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N \mid \langle\nu, n\gamma\rangle \leq m\}$$

Moreover, we decompose the set B_m *as* $B_m = B_m^= \cup B_m^<$, *where*

$$B_m^= = \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N \mid \langle\nu, n\gamma\rangle = m\}$$

$$B_m^< = \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N \mid \langle\nu, n\gamma\rangle < m\}$$

Remark 4.9. *Notice that, for* $m \in \mathbb{Z}_{>0}$, *the set* B_m *is*

$$\begin{aligned} B_m &= \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N \mid a\nu_1 + b\nu_2 \leq m\} \\ &= \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0 \mid \tfrac{a\nu_1 + b\nu_2}{n} \in \mathbb{Z} \text{ and } a\nu_1 + b\nu_2 \leq m\} \end{aligned}$$

For any $m > 0$ and any $\nu \in A_m \cup B_m$, we have defined an algebraic variety C_m^ν in Definition 4.4. These are the candidates to be the irreducible components of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$. To prove this assertion we need to make sure that these sets are irreducible, and that they cover $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$.

Lemma 4.10. *For $m \in \mathbb{Z}_{>0}$ and $\nu \in A_m \cup B_m$ we have that C_m^ν is irreducible. Moreover its codimension is:*

$$Codim(C_m^\nu) = \begin{cases} \nu_1 + \nu_2 + [\frac{m}{n}] & \text{if } \nu \in A_m \\ \\ \nu_1 + \nu_2 + \langle \nu, \gamma \rangle + m - \langle \nu, n\gamma \rangle + 1 & \text{if } \nu \in B_m \end{cases}$$

Proof. For $\nu \in A_m \cup B_m^=$ the claim is clear by definition and by Lemma 4.5. If $\nu \in B_m^<$ it follows by Proposition 2.4. □

Lemma 4.11. *For $m \in \mathbb{Z}_{>0}$,*

$$\left(\pi_m^{-1}(X_{Sing})\right)_{red} = \bigcup_{\nu \in A_m \cup B_m} C_m^\nu$$

Proof. By definition we have that $\cup_{\nu \in A_m \cup B_m} C_m^\nu \subseteq \left(\pi_m^{-1}(X_{Sing})\right)_{red}$. We have to prove that any m-jet $\gamma(t) \in \left(\pi_m^{-1}(X_{Sing})\right)_{red}$ belongs to certain C_m^ν with $\nu \in A_m \cup B_m$.

• Suppose first that $x_i \circ \gamma(t) \neq 0$ for $i = 1, 2$. Then we set $\nu := \big(\mathrm{ord}_t(x_1 \circ \gamma(t)), \mathrm{ord}_t(x_2 \circ \gamma(t))\big)$. We have that $\nu \in \sigma_{Sing} \cap [0, m]^2 \cap N_0$, and we only need to prove that if $\langle \nu, n\gamma \rangle \leq m$ then $\nu \in N$. Indeed, let us suppose the contrary, that $m \geq \langle \nu, n\gamma \rangle$ and $\nu \in N_0 \setminus N$. We define the ideal

$$J = \big(x_1^{(0)}, \ldots, x_1^{(\nu_1 - 1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2 - 1)}\big).$$

Note that $\gamma(t) \in V(J) \cap \{x_1^{(\nu_1)} \neq 0\} \cap \{x_2^{(\nu_2)} \neq 0\}$. Using that $f \circ \gamma(t) \equiv 0 \mod t^{m+1}$ we deduce that

$$\gamma(t) \in V\big(J + (z^{(0)}, \ldots, z^{([\langle \nu, \gamma \rangle])})\big) \cap \{x_1^{(\nu_1)} \neq 0\} \cap \{x_2^{(\nu_2)} \neq 0\}$$

and that

$$F^{(\langle \nu, n\gamma \rangle)} \equiv -{x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b + \sum c_{ijk} {x_1^{(\nu_1)}}^i {x_2^{(\nu_2)}}^j {z^{([\langle \nu, \gamma \rangle] + 1)}}^k \mod J + \big(z^{(0)}, \ldots, z^{([\langle \nu, \gamma \rangle])}\big)$$

where the sum runs under the conditions $(i, j) + k\gamma > n\gamma$ and $\langle \nu, (i, j) \rangle + k([\langle \nu, \gamma \rangle] + 1) = \langle \nu, n\gamma \rangle$. But, since $\nu \notin N$ we have $[\langle \nu, \gamma \rangle] + 1 > \langle \nu, \gamma \rangle$, and then

$$\langle \nu, (i, j) \rangle + k([\langle \nu, \gamma \rangle] + 1) > \langle \nu, (i, j) + k\gamma \rangle \geq \langle \nu, n\gamma \rangle$$

and we deduce that $F^{(\langle \nu, n\gamma \rangle)} \equiv -{x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b \mod J + \big(z^{(0)}, \ldots, z^{([\langle \nu, \gamma \rangle])}\big)$. Since we have that ${x_1^{(\nu_1)}}^a {x_2^{(\nu_2)}}^b$ is non-zero, this contradicts the fact that $\gamma(t) \in X_m$, because $\langle \nu, n\gamma \rangle \leq m$.

• Suppose that $x_1\circ\gamma(t)=0$ and $x_2\circ\gamma(t)\neq 0$. We set $\nu_2:=\mathrm{ord}_t\big(x_2\circ\gamma(t)\big)\leq m$. Note that $\nu:=(m,\nu_2)\in A_m$ and since $\gamma(t)\in C_m^{\nu}$ we are done.

• If $x_1\circ\gamma(t)\neq 0$ and $x_2\circ\gamma(t)=0$ then we set $\nu:=(\mathrm{ord}_t(x_1\circ\gamma(t)),m)$. We have that if $b\neq 0$ then $\langle\nu,n\gamma\rangle>m$ and therefore $\nu\in A_m$. If $b=0$ and $a\nu_1\leq m$ we can prove, arguing as in the case $x_i\circ\gamma(t)\neq 0$ for $i=1,2$, that $\nu\in N$. Then $\nu\in B_m$. In both cases ($b=0$ and $b\neq 0$) we have that $\gamma(t)\in C_m^{\nu}$.

• If $x_i\circ\gamma(t)\neq 0$ for $i=1,2$, we set $\nu:=(m,m)$. We have that $\nu\in A_m$ and $\gamma(t)\in C_m^{\nu}$. □

The description given in Lemma 4.11 is not the decomposition in irreducible components, we still have to study the inclusions among the sets C_m^{ν}.

Let us denote by $\leq$ the coordinate-wise order:

$$\begin{aligned}\nu\leq\nu' &\iff \nu'\in\nu+\sigma\\ &\iff \nu_i\leq\nu_i' \text{ for } i=1,2\end{aligned}\tag{12}$$

Then, given $\nu,\nu'\in A_m\cup B_m$ such that $\nu\not\leq\nu'$ it is clear that $C_m^{\nu'}\nsubseteq C_m^{\nu}$, since for any ν, by definition, we have

$$C_m^{\nu}\subseteq V\big(x_1^{(0)},\ldots,x_1^{(\nu_1-1)},x_2^{(0)},\ldots,x_2^{(\nu_2-1)}\big).$$

Therefore we have to consider $\nu,\nu'\in A_m\cup B_m$ with $\nu\leq\nu'$ and study wether we have the inclusion $C_m^{\nu'}\subseteq C_m^{\nu}$ or not.

Definition 4.12. *We define, for $m\in\mathbb{Z}_{>0}$, the relation $\leq_m$ on $A_m\cup B_m$ as follows,*

$$\nu\leq_m\nu' \textit{ if and only if } \begin{cases}\nu'-\nu\in\sigma_{Reg,0} & \textit{if } \nu,\nu'\in A_m\cup B_m^{=}\\ \nu'-\nu\in\sigma_{Reg,1} & \textit{otherwise}\end{cases}$$

Remark 4.13. *Note that if $\nu\leq_m\nu'$ then $\nu\leq\nu'$.*

We have defined, for every $m\in\mathbb{Z}_{>0}$, a partial order $\leq_m$ on $\mathbb{Z}^2_{\geq 0}$. Hence, given any subset $R\subseteq\mathbb{Z}^2_{\geq 0}$, we may consider the set

$$\min_{\leq_m}R=\{v\in R\mid\nexists\, w\in R \text{ such that } w\leq_m v\}$$

Theorem 4.14. *The decomposition of $\big(\pi_m^{-1}(X_{Sing})\big)_{red}$ in irreducible components is given by*

$$\big(\pi_m^{-1}(X_{Sing})\big)_{red}=\bigcup_{\nu\in F_m}C_m^{\nu}$$

where $F_m=\min_{\leq_m}\{A_m\cup B_m\}$.

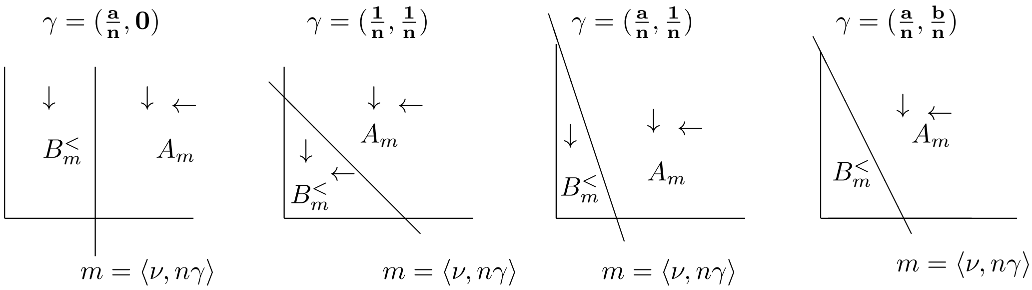

FIGURE 1. A sketch of the different orderings $\leq_m$ in the case of one characteristic exponent.

Proof. Notice first that, for $\nu, \nu' \in A_m$, we have $\nu \leq_m \nu'$ if and only if $\nu \leq \nu'$, simply because $A_m \subseteq N_0$ and $\sigma_{Reg,0} = \rho_1 \cup \rho_2$. See Figure 1 for a sketch of how the relation $\leq_m$ acts on $A_m \cup B_m$ for the different cases depending on γ. Moreover, for any $\nu \in A_m$, by definition we have

$$C_m^\nu = V\big(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{([m/n])}\big).$$

Then it is clear that, given $\nu, \nu' \in A_m$,

$$\nu' \in \nu + \sigma \iff C_m^{\nu'} \subseteq C_m^\nu,$$

and we deduce that

$$\bigcup_{\nu \in A_m} C_m^\nu = \bigcup_{\nu \in \min_{\leq} A_m} C_m^\nu$$

where recall that by definition $\nu \leq \nu'$ if and only if $\nu' \in \nu + \sigma$.

To prove the statement we distinguish cases depending on γ.

(i) If $\gamma = (\frac{1}{n}, \frac{1}{n})$, the relation is

$$\nu \leq_m \nu' \text{ if and only if } \nu' - \nu \in \rho_1 \cup \rho_2$$

for any $\nu, \nu' \in A_m \cup B_m$. Moreover $\sigma_{Sing} = \overset{\circ}{\sigma}$ and

$$A_m = \{\nu \in \overset{\circ}{\sigma} \cap [0, m]^2 \cap N_0 \mid \nu_1 + \nu_2 > m\}$$

$$B_m = \{\nu \in \overset{\circ}{\sigma} \cap [0, m]^2 \cap N \mid \nu_1 + \nu_2 \leq m\}$$

We distinguish two cases, $m < n$ and $m \geq n$.

If $m < n$ then $B_m = \emptyset$ (since $\langle \nu, \gamma \rangle = \frac{\nu_1+\nu_2}{n} \in \mathbb{N}$ and $\nu_1 + \nu_2 \leq m < n$ are incompatible conditions) and therefore

$$\begin{aligned} F_m &= \min_{\leq_m} A_m = \min_{\leq} A_m \\ &= \{\nu \in \overset{\circ}{\sigma} \cap N_0 \mid \nu_1 + \nu_2 = m+1\} \\ &= \{(1,m),(2,m-1),\dots,(m,1)\} \end{aligned}$$

If $m \geq n$, we have that

$$B^0 := \{(1,n-1),\dots,(n-1,1)\} \subseteq B_m$$

and

$$A_m \cup B_m \subseteq \bigcup_{\nu \in B^0} (\nu + \sigma)$$

Let $\nu' \in A_m \cup B_m \setminus B^0$, we will prove that there exists $\nu \in B^0$ such that $C_m^{\nu'} \subseteq C_m^{\nu}$. There are two cases:

- If $\nu' \in A_m$, then

$$C_m^{\nu'} = V\big(x_1^{(0)},\dots,x_1^{(\nu_1'-1)},x_2^{(0)},\dots,x_2^{(\nu_2'-1)},z^{(0)},\dots,z^{([m/n])}\big)$$

Let ν be any point in B^0 such that $\nu' \in \nu + \sigma$. Then

$$C_m^{\nu} = V\big(x_1^{(0)},\dots,x_1^{(\nu_1-1)},x_2^{(0)},\dots,x_2^{(\nu_2-1)},z^{(0)},F^{(n)},\dots,F^{(m)}\big)$$

We claim that $C_m^{\nu'} \subseteq C_m^{\nu}$. Indeed, first it is clear that for $i = 1,2$

$$x_i^{(0)},\dots,x_i^{(\nu_i-1)},z^{(0)} \in \big(x_1^{(0)},\dots,x_1^{(\nu_1'-1)},x_2^{(0)},\dots,x_2^{(\nu_2'-1)},z^{(0)},\dots,z^{([m/n])}\big)$$

We have to prove that, for $n \leq l \leq m$

$$F^{(l)} \in \big(x_1^{(0)},\dots,x_1^{(\nu_1'-1)},x_2^{(0)},\dots,x_2^{(\nu_2'-1)},z^{(0)},\dots,z^{([m/n])}\big)$$

How does $F^{(l)}$ look like? It consists of monomials of the form:

$z^{(a_1)}\cdots z^{(a_n)}$	with $a_i \leq l$ and $a_1 + \cdots + a_n = l$
$x_1^{(b_1)}x_2^{(b_2)}$	with $b_1, b_2 \leq l$ and $b_1 + b_2 = l$
$x_1^{(r_1)}\cdots x_1^{(r_{\alpha_1})}x_2^{(s_1)}\cdots x_2^{(s_{\alpha_2})}z^{(t_1)}\cdots z^{(t_k)}$	with $r_i, s_i, t_i \leq l$, and $\sum r_i + \sum s_i + \sum t_1 = l$

with the condition $(\alpha_1, \alpha_2) + k\gamma > n\gamma$ (we are just deriving the equation $F^{(0)}$ l times and forgetting about the coefficient of each monomial). Let us impose now the conditions

$$a_i, t_j \geq [\tfrac{m}{n}] + 1$$

$$b_1, r_i \geq \nu'_1$$

$$b_2, s_i \geq \nu'_2$$

which correspond to the fact that we are interested in the equation $F^{(l)}$ modulo the ideal

$$\big(x_1^{(0)}, \ldots, x_1^{(\nu'_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu'_2-1)}, z^{(0)}, \ldots, z^{([m/n])}\big).$$

Then we have that

$$l = a_1 + \cdots + a_n \geq n\big([\frac{m}{n}] + 1\big) > m$$

which is impossible. Moreover

$$l = b_1 + b_2 \geq \nu'_1 + \nu'_2 > m$$

since $\nu' \in A_m$, and this is a contradiction. Finally

$$\begin{aligned} l &= r_1 + \cdots + r_{\alpha_1} + s_1 + \cdots + s_{\alpha_2} + t_1 + \cdots + t_k \\ &\geq \langle \nu', (\alpha_1, \alpha_2)\rangle + k([\tfrac{m}{n}] + 1) \\ &\geq \langle \nu', (1,1) - k\gamma\rangle + k([\tfrac{m}{n}] + 1) \\ &= (\nu'_1 + \nu'_2)(1 - \tfrac{k}{n}) + k([\tfrac{m}{n}] + 1) \\ &> m + k([\tfrac{m}{n}] + 1 - \tfrac{m}{n}) > m \end{aligned}$$

Then we have proved that for $l \leq m$

$$F^{(l)} \equiv 0 \bmod \big(x_1^{(0)}, \ldots, x_1^{(\nu'_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu'_2-1)}, z^{(0)}, \ldots, z^{([m/n])}\big).$$

• If $\nu' \in B_m$, the strategy is the same, and we can prove that $C_m^{\nu'} \subseteq C_m^{\nu}$ for any $\nu \in B^0$ such that $\nu' \in \nu + \sigma$. Indeed,

$$C_m^{\nu'} = V\big(x_1^{(0)}, \ldots, x_1^{(\nu'_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu'_2-1)}, z^{(0)}, \ldots, z^{(\langle \nu', \gamma\rangle - 1)}, F^{(\langle \nu', n\gamma\rangle)}, \ldots, F^{(m)}\big)$$

and

$$C_m^{\nu} = V\big(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, F^{(n)}, \ldots, F^{(m)}\big)$$

We only have to prove that for $n \leq l < \langle \nu', n\gamma\rangle$

$$F^{(l)} \in \big(x_1^{(0)}, \ldots, x_1^{(\nu'_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu'_2-1)}, z^{(0)}, \ldots, z^{(\langle \nu', \gamma\rangle - 1)}\big)$$

In this case note that $\nu'_1 + \nu'_2 \leq m$ and $\nu \in N$. The monomials of $F^{(l)}$ are described in the previous case, but now the conditions we impose are

$$a_i, t_i \geq \langle \nu', \gamma\rangle$$

$$b_1, r_i \geq \nu'_1$$

$$b_2, s_i \geq \nu'_2$$

Then

$$l = a_1 + \cdots + a_n \geq n\langle \nu', \gamma \rangle$$

which is impossible. Moreover

$$l = b_1 + b_2 \geq \nu_1' + \nu_2' = n\langle \nu', \gamma \rangle$$

another contradiction. And finally,

$$\begin{array}{ll} l & = r_1 + \cdots + r_{\alpha_1} + s_1 + \cdots + s_{\alpha_2} + t_1 + \cdots + t_k \\ & \geq \alpha_1 \nu_1' + \alpha_2 \nu_2' + k\langle \nu' \gamma \rangle \\ & = \langle \nu', (\alpha_1, \alpha_2) + k\gamma \rangle \geq n\langle \nu', \gamma \rangle \end{array}$$

Then we have proved that

$$F^{(l)} \equiv 0 \bmod \big(x_1^{(0)}, \ldots, x_1^{(\nu_1'-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2'-1)}, z^{(0)}, \ldots, z^{(\langle \nu', \gamma \rangle - 1)}\big),$$

and the claim follows.

(ii) If $\gamma = (\frac{a}{n}, 0)$, we have that $\gcd(a, n) = 1$ and $a > n$ (recall that the q.o. surface is irreducible and the branch is normalized). We have that

$$A_m = \{\nu \in \sigma_{Sing} \cap [0, m]^2 \cap N_0 \mid a\nu_1 > m\}$$

$$B_m = \{\nu \in \sigma_{Sing} \cap [0, m]^2 \cap N \mid a\nu_1 \leq m\}$$

Then, in this case, $\min_{\leq_m} A_m$ consists of a single element, and

$$\bigcup_{\nu \in A_m} C_m^\nu = C_m^{\nu^*}$$

where $\nu^* = ([\frac{m}{a}] + 1, 0)$ is the smallest element (with respect to $\leq$) in A_m.

For $\nu, \nu' \in B_m^<$ with $\nu' - \nu \in \rho_2$ we have that $C_m^{\nu'} \subseteq C_m^\nu$. Indeed, we have that $\nu' = \nu + (0, r)$ with $r \in \mathbb{Z}_{>0}$, and then $\langle \nu, \gamma \rangle = \langle \nu', \gamma \rangle$. Then (recall the notation $J^\nu = \big(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{(\langle \nu, \gamma \rangle - 1)}\big)$)

$$C_m^\nu = \overline{V\big(J^\nu, F^{(\langle \nu, n\gamma \rangle)}, \ldots, F^{(m)}\big) \cap \{x_1^{(\nu_1)} \neq 0\}}$$

$$C_m^{\nu'} = \overline{V\big(J^\nu, F^{(\langle \nu, n\gamma \rangle)}, \ldots, F^{(m)}\big) \cap \{x_1^{(\nu_1)} \neq 0\}}$$

and the claim follows. Therefore,

$$\bigcup_{\nu \in B_m^<} C_m^\nu = \bigcup_{\nu \in \min_{\leq_m} B_m^<} C_m^\nu$$

and using that in this case

$$\nu \in N \text{ if and only if } \nu + r(0, 1) \in N, \text{ with } r \in \mathbb{Z}$$

we deduce that

$$\min_{\leq_m} B_m^< \subseteq \mathbb{N} \times \{0\}$$

So far we have that

$$\left(\pi_m^{-1}(X_{Sing})\right)_{red}=\bigcup_{\nu\in\{\nu^*\}\cup B_m^{=}\cup\ \min_{\leq_m} B_m^{<}} C_m^{\nu}$$

Given $\nu,\nu'\in\min_{\leq_m}B_m^{<}$ with $\nu\leq\nu'$, then $\nu=(\nu_1,0)$ and $\nu'=(\nu'_1,0)$ with $\nu_1<\nu'_1$ and $\nu_1\frac{a}{n},\nu'_1\frac{a}{n}\in\mathbb{Z}$, and by Lemma 4.10 we have

$$\operatorname{Codim}(C_m^{\nu})-\operatorname{Codim}(C_m^{\nu'})=(\nu'_1-\nu_1)\big(a-1-\frac{a}{n}\big)$$

and since $a>n>2$ we have that $an>a+n$ and we deduce that $\operatorname{Codim}(C_m^{\nu})>\operatorname{Codim}(C_m^{\nu'})$ and therefore $C_m^{\nu'}\nsubseteq C_m^{\nu}$.

- If $B_m^{=}=\emptyset$, it is because $m\not\equiv 0 \bmod a$, and then

$$F_m=\{\nu^*\}\cup\min_{\leq_m}B_m^{<}$$

since ν^* is not comparable by $\leq_m$ with any element in $B_m^{<}$. We have to prove that for any $\nu\in\min_{\leq_m}B_m^{<}$ we have $C_m^{\nu^*}\nsubseteq C_m^{\nu}$. By Lemma 4.10 we have

$$\begin{aligned}\operatorname{Codim}(C_m^{\nu})-\operatorname{Codim}(C_m^{\nu^*}) &=\nu_1+\nu_1\tfrac{a}{n}+m-\nu_1 a-[\tfrac{m}{a}]-[\tfrac{m}{n}]\\ &=m-\nu_1\tfrac{an-a-n}{n}-[\tfrac{m}{a}]-[\tfrac{m}{n}]\\ &=\tfrac{m}{a}+\tfrac{m}{n}+m(1-\tfrac{1}{a}-\tfrac{1}{n})-\nu_1\tfrac{an-a-n}{n}-[\tfrac{m}{a}]-[\tfrac{m}{n}]\\ &=\tfrac{m}{a}-[\tfrac{m}{a}]+\tfrac{m}{n}-[\tfrac{m}{n}]+(\tfrac{m}{a}-\nu_1)\tfrac{an-a-n}{n}\end{aligned}$$

which is positive since $m>a\nu_1$ and $\frac{an-a-n}{n}>0$. Hence $\dim(C_m^{\nu^*})>\dim(C_m^{\nu})$ and therefore $C_m^{\nu^*}\nsubseteq C_m^{\nu}$.

- Suppose now that $B_m^{=}\neq\emptyset$ (i.e. $m\equiv 0 \bmod a$) and let us denote $\nu^\circ=(m/a,0)$ its smallest element. Then $\nu^*=(m/a+1,0)$ and $\nu^\circ\leq\nu^*$. We claim that $C_m^{\nu^*}\subseteq C_m^{\nu^\circ}$. Indeed, we have

$$C_m^{\nu^*}=V\big(x_1^{(0)},\ldots,x_1^{(m/a)},z^{(0)},\ldots,z^{(m/n)}\big)$$

$$C_m^{\nu^\circ}=V\big(x_1^{(0)},\ldots,x_1^{(m/a-1)},z^{(0)},\ldots,z^{(m/n-1)},F^{(m)}\big)$$

and, by Lemma 4.5,

$$F^{(m)}\equiv {z^{(m/n)}}^n-{x_1^{(m/a)}}^a+\sum c_{ijk}{x_1^{(m/a)}}^i{x_2^{(0)}}^j{z^{(m/n)}}^k \bmod J^{\nu^\circ}$$

Since there are no monomials of the form $c_{0j0}{x_2^{(0)}}^j$, $F^{(m)}\equiv 0 \bmod J^{\nu^*}$ and the inclusion $C_m^{\nu^*}\subseteq C_m^{\nu^\circ}$ follows. Then

$$F_m=\{\nu^\circ\}\cup\min_{\leq_m}B_m^{<}$$

This is the description in irreducible components, or in other words, there are no more inclusions among the sets C_m^{ν}. We only need to prove that for any $\nu \in B_m^<$ we have $C_m^{\nu^\circ} \nsubseteq C_m^\nu$. And this follows since

$$\mathrm{Codim}(C_m^\nu) - \mathrm{Codim}(C_m^{\nu^\circ}) = \big(\frac{m}{a} - \nu_1\big)\frac{an-a-n}{n} > 0$$

(iii) If $\gamma = (\frac{a}{n}, \frac{1}{n})$, we have

$$A_m = \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0 \mid a\nu_1 + \nu_2 > m\}$$

$$B_m = \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N \mid a\nu_1 + \nu_2 \leq m\}$$

If $\nu, \nu' \in B_m^<$ with $\nu \leq_m \nu'$, then $\nu' = \nu + (0, rn)$. Let us prove that $C_m^{\nu'} \subseteq C_m^\nu$. We have

$$C_m^\nu = \overline{V\big(J^\nu, F^{(\langle\nu,n\gamma\rangle)}, \ldots, F^{(m)}\big) \cap \{x_1^{(\nu_1)} \neq 0\}}$$

$$C_m^{\nu'} = \overline{V\big(J^{\nu'}, F^{(\langle\nu',n\gamma\rangle)}, \ldots, F^{(m)}\big) \cap \{x_1^{(\nu'_1)} \neq 0\}}$$

and since $\nu'_1 = \nu_1$, $\nu'_2 > \nu_2$ and $\langle\nu',\gamma\rangle > \langle\nu,\gamma\rangle$, it is enough to prove that for $\langle\nu, n\gamma\rangle \leq l < \langle\nu', n\gamma\rangle$,

$$F^{(l)} \in \big(x_1^{(0)}, \ldots, x_1^{(\nu'_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu'_2-1)}, z^{(0)}, \ldots, z^{(\langle\nu',\gamma\rangle-1)}\big)$$

The monomials in $F^{(l)}$ are of the form

$z^{(c_1)} \cdots z^{(c_n)}$	with $c_i \leq l$ and $c_1 + \cdots + c_n = l$
$x_1^{(b_1)} \cdots x_1^{(b_a)} x_2^{(b_{a+1})}$	with $b_i \leq l$ and $b_1 + \cdots + b_{a+1} = l$
$x_1^{(r_1)} \cdots x_1^{(r_{\alpha_1})} x_2^{(s_1)} \cdots x_2^{(s_{\alpha_2})} z^{(t_1)} \cdots z^{(t_k)}$	with $r_i, s_i, t_i \leq l$ and $\sum r_i + \sum s_i + \sum t_i = l$

with the condition $(\alpha_1, \alpha_2) + k\gamma > n\gamma$. Imposing the conditions

$$c_i, t_j \geq \langle\nu',\gamma\rangle$$

$$b_i, r_j \geq \nu'_1$$

$$b_{a+1}, s_j \geq \nu'_2$$

on the monomials of $F^{(l)}$, we have

$$l = c_1 + \cdots + c_n \geq \langle\nu', n\gamma\rangle$$

$$l = b_1 + \cdots + b_a + b_{a+1} \geq a\nu'_1 + \nu'_2 = \langle\nu', n\gamma\rangle$$

$$\begin{aligned} l &= r_1 + \cdots + r_{\alpha_1} + s_1 + \cdots + s_{\alpha_2} + t_1 + \cdots + t_k \\ &\geq \alpha_1\nu'_1 + \alpha_2\nu'_2 + k\langle\nu',\gamma\rangle \\ &= \langle\nu', (\alpha_1,\alpha_2) + k\gamma\rangle \geq \langle\nu', n\gamma\rangle \end{aligned}$$

and hence we have proved that for $\langle \nu, n\gamma\rangle \leq l < \langle \nu', n\gamma\rangle$

$$F^{(l)} \equiv 0 \bmod \big(x_1^{(0)}, \dots, x_1^{(\nu'_1-1)}, x_2^{(0)}, \dots, x_2^{(\nu'_2-1)}, z^{(0)}, \dots, z^{(\langle \nu', \gamma\rangle-1)}\big)$$

If there exists $\nu^\circ \in B_m^{=}$ and $\nu^* \in \min_{\leq_m} A_m$ such that $\nu^\circ \leq \nu^*$, we claim that $C_m^{\nu^*} \subseteq C_m^{\nu^\circ}$. Indeed,

$$C_m^{\nu^\circ} = V\big(x_1^{(0)}, \dots, x_1^{(\nu_1^\circ-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2^\circ-1)}, z^{(0)}, \dots, z^{(\langle \nu^\circ, \gamma\rangle-1)}, F^{(m)}\big)$$
$$C_m^{\nu^*} = V\big(x_1^{(0)}, \dots, x_1^{(\nu_1^*-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2^*-1)}, z^{(0)}, \dots, z^{[m/n]}\big)$$

and since $m = \langle \nu^\circ, n\gamma\rangle$, and

$$\nu_i^\circ \leq \nu_i^*, \quad \text{for } i = 1, 2$$

$$\langle \nu^\circ, \gamma\rangle - 1 < [\tfrac{m}{n}] = \langle \nu^\circ, \gamma\rangle$$

by Lemma 4.5 it follows that

$$F^{(m)} \equiv 0 \bmod \big(x_1^{(0)}, \dots, x_1^{(\nu_1^*-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2^*-1)}, z^{(0)}, \dots, z^{(\langle \nu^*, \gamma\rangle)}\big).$$

To finish we have to prove that given $\nu, \nu' \in F_m$ with $\nu \leq \nu'$ we have that $C_m^{\nu'} \nsubseteq C_m^{\nu}$. Notice that the only choice is that $\nu \in B_m^{<}$, while $\nu' \in A_m \cup B_m$.

First consider $\nu \in B_m^{<}$ and $\nu' \in B_m$. By the definition of $\leq_m$ we have that $\nu_1 \neq \nu'_1$ and $\nu_2, \nu'_2 < n$ (since $\nu \in N$ if and only if $\nu - (0, n) \in N$). By Lemma 4.10,

$$\begin{aligned} \mathrm{Codim}(C_m^{\nu}) - \mathrm{Codim}(C_m^{\nu'}) &= \langle \nu' - \nu, n\gamma - \gamma - (1,1)\rangle \\ &= (\nu'_1 - \nu_1)\tfrac{an-a-n}{n} - (\nu'_2 - \nu_2)\tfrac{1}{n} \end{aligned}$$

and we have $\mathrm{Codim}(C_m^{\nu}) - \mathrm{Codim}(C_m^{\nu'}) \geq 0$ since $\frac{\nu'_2-\nu_2}{n} < 1$, $\nu'_1 - \nu_1 > 0$ and $\frac{an-a-n}{n} \geq 0$, and $\mathrm{Codim}(C_m^{\nu}) - \mathrm{Codim}(C_m^{\nu'})$ is an integer.

Suppose now that $\nu' \in A_m$. By the inequality above it is enough to prove that $C_m^{\nu'} \nsubseteq C_m^{\nu}$ for $\nu \in B_m^{<}$ with ν_2 maximal. Then $\nu'_2 - \nu_2 \leq n$. If $\nu' = (\nu_1, 0)$ then $\nu = (\nu_1, 0)$ and the proof goes as in case (ii). Otherwise $\nu' - (1, 0) \notin A_m \cup B_m$, and then $\langle \nu' - (1,0), n\gamma\rangle \leq m$, therefore

$$\langle \nu', n\gamma\rangle = m + 1$$

since $\nu' \in A_m$. We have

$$\begin{aligned} \mathrm{Codim}(C_m^{\nu}) - \mathrm{Codim}(C_m^{\nu'}) &= \langle \nu' - \nu, n\gamma - \gamma - (1,1)\rangle + 1 \\ &= (\nu'_1 - \nu_1)\tfrac{an-a-n}{n} - (\nu'_2 - \nu_2)\tfrac{1}{n} + 1 \end{aligned}$$

(iv) If $\gamma = (\frac{a}{n}, \frac{b}{n})$ with $b > 1$, we have

$$A_m = \{\nu \in \sigma_{Sing} \cap [0, m]^2 \cap N_0 \mid a\nu_1 + b\nu_2 > m\}$$

$$B_m = \{\nu \in \sigma_{Sing} \cap [0, m]^2 \cap N \mid a\nu_1 + b\nu_2 \leq m\}$$

If $\nu, \nu' \in A_m \cup B_m^{=}$ we have that $C_m^{\nu'} \subseteq C_m^{\nu}$ if and only if $\nu \leq \nu'$, as in the other cases. Moreover, since $\sigma_{Reg,1} = \{0\}$, we have

$$\min_{\leq_m} B_m^{<} = B_m^{<}.$$

For any $\nu, \nu' \in B_m^{<}$ with $\nu \leq \nu'$, we have

$$\text{(13)} \quad \operatorname{Codim}(C_m^{\nu}) - \operatorname{Codim}(C_m^{\nu'}) = (\nu_1' - \nu_1)\frac{an-a-n}{n} + (\nu_2' - \nu_2)\frac{bn-b-n}{n} \geq 0,$$

since $an \geq a + n$ and $bn \geq b + n$. Hence $C_m^{\nu'} \nsubseteq C_m^{\nu}$.

We still have to prove that for $\nu \in B_m^{<}$ and $\nu' \in \min_{\leq_m} A_m \cup B_m^{=}$ with $\nu \leq \nu'$,

$$\operatorname{Codim}(C_m^{\nu}) - \operatorname{Codim}(C_m^{\nu'}) \geq 0.$$

Note that, by equation (13), it is enough to prove it for $\nu \in B_m^{<}$ maximal with respect to $\leq_m$. We set $m_0 := \langle \nu, n\gamma \rangle < m$. We have that $\nu \in B_{m_0}^{=}$ and

$$\left(\pi_{m_0+1,m_0}^{-1}(C_{m_0}^{\nu})\right)_{red} = C_{m_0+1}^{\nu} \cup C_{m_0+1}^{\nu+(1,0)} \cup C_{m_0+1}^{\nu+(0,1)}$$

where

$$\operatorname{Codim}\left(C_{m_0+1}^{\nu}\right) = \operatorname{Codim}\left(C_{m_0+1}^{\nu+(1,0)}\right) = \operatorname{Codim}\left(C_{m_0+1}^{\nu+(0,1)}\right)$$

Since we have

$$C_m^{\nu} = \overline{V\left(J^{\nu}, F^{(m_0)}, \ldots, F^{(m)}\right) \cap \{x_1^{(\nu_1)} \neq 0\} \cap \{x_2^{(\nu_2)} \neq 0\}}$$

then

$$\operatorname{Codim}\left(C_m^{\nu}\right) = \nu_1 + \nu_2 + \langle \nu, \gamma \rangle + m - m_0 + 1 = \operatorname{Codim}\left(C_{m_0}^{\nu}\right) + m - m_0.$$

The component associated to ν' (i.e. $C_m^{\nu'}$) must come from either $\nu + (1,0)$ or $\nu + (0,1)$ (or even both). More precisely, when lifting the component, say $C_{m_0}^{\nu+(1,0)}$, to higher levels we will pass from $\nu + (1,0)$ to ν' as follows, if we set $m_1 := m_0 + 1$ and $\nu^{(1)} := \nu + (1,0)$,

$$C_{m_1}^{\nu^{(1)}} \longrightarrow C_{m_2}^{\nu^{(2)}} \longrightarrow \cdots \longrightarrow C_{m_r}^{\nu^{(r)}}$$

with $m_r = m$ and $\nu^{(r)} = \nu'$,

$$m_0 + 1 = m_1 < m_2 < \cdots < m_r = m$$

and

$$\nu^{(i)} \in A_{m_i}.$$

Moreover we have

$$\operatorname{Codim}\left(C_{m_i}^{\nu_i}\right) = \operatorname{Codim}\left(C_{m_i-1}^{\nu^{(i-1)}}\right) + 1$$

Here we use that ν is maximal in $B_m^{<}$ and ν' minimal in $A_m \cup B_m^{=}$, and therefore there may not exists $\widetilde{\nu} \in B_m$ such that

$$\nu \leq \widetilde{\nu} \leq \nu'$$

Hence

$$\begin{aligned}\operatorname{Codim}\big(C_m^{\nu'}\big) &\leq \operatorname{Codim}\big(C_{m_0+1}^{\nu+(1,0)}\big) + m - m_0 - 1 \\ &= \operatorname{Codim}\big(C_{m_0+1}^{\nu}\big) + m - m_0 - 1 \\ &= \operatorname{Codim}\big(C_{m_0}^{\nu}\big) + m - m_0 \\ &= \operatorname{Codim}\big(C_{m}^{\nu}\big)\end{aligned}$$

□

Remark 4.15. *This result is to be compared with the case of plane curves with one characteristic pair studied in* [24] *(Corollary 4.4), and with the case of A_n-singularities studied in* [26].

Remark 4.16. *If $\gamma = \left(\frac{1}{n}, \frac{1}{n}\right)$ we have just proved that*

$$\left(\pi_m^{-1}(X_{Sing})\right)_{red} = \begin{cases} \bigcup_{\nu\in\mathring{\sigma},\nu_1+\nu_2=m+1} C_m^{\nu} & \text{if } m < n \\ \bigcup_{\nu\in\mathring{\sigma},\nu_1+\nu_2=n} C_m^{\nu} & \text{if } m \geq n \end{cases}$$

that is, the number of irreducible components of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ is m if $m < n$ and $n-1$ otherwise. In particular, observe that this number stabilizes. If $\gamma \neq \left(\frac{1}{n}, \frac{1}{n}\right)$, the cardinal of F_m does not stabilize.

4.1. **The graph.** As we pointed out in Remark 4.15, the result in Theorem 4.14 has to be compared with some particular cases in [24] and in [26]. In those papers it was proved that the structure of the jet schemes determines the topological type of the singularity. We devote this section to prove the same result, for any q.o. surface with only one characteristic exponent.

Definition 4.17. *We construct a graph Γ by representing each irreducible component of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ by a vertex $V_{i,m}$, and joining two vertices $V_{i,m}$ and $V_{j,m+1}$ if $\pi_{m+1,m}$ induces a map between the corresponding irreducible components (see Definition 5.34 for the general definition). We weight the graph by giving the embedding dimension (e) and the codimension (c) of any component. Then a vertex at level m is denoted by $V_m(e,c)$.*

We say that there is a splitting in the graph at level m whenever there is more that one vertex at level m projecting to the same vertex $V_{m-1}(e,c)$ at level $m-1$. If $e+c = 3m$ then we say that the splitting is of first type, and otherwise the splitting is of second type.

Remark 4.18. *Notice that if C_m^ν is a component of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ such that $\left(\pi_{m+1,m}^{-1}(C_m^\nu)\right)_{red}$ is reducible, there is a splitting only if the components of the lifting $\left(\pi_{m+1,m}^{-1}(C_m^\nu)\right)_{red}$ are also components of $\left(\pi_{m+1}^{-1}(X_{Sing})\right)_{red}$.*

For instance, let $f = z^5 - x_1^3 x_2$ be the q.o. polynomial with exponent $\gamma = \left(\frac{3}{5}, \frac{1}{5}\right)$. At level $m = 3$ we have

$$\left(\pi_3^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, z^{(0)}\right) \cup V\left(x_1^{(0)}, x_2^{(0)}, z^{(0)}\right) = C_3^{(2,0)} \cup C_3^{(1,1)},$$

and $\left(\pi_{4,3}^{-1}(C_3^{(1,1)})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, x_2^{(0)}, z^{(0)}\right) \cup V\left(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, z^{(0)}\right) = C_4^{(2,1)} \cup C_4^{(1,2)}$. But this does not correspond to a splitting in Γ, since $\left(\pi_{4,3}^{-1}(C_3^{(2,0)})\right)_{red} = V\left(x_1^{(0)}, x_1^{(1)}, z^{(0)}\right) = C_4^{(2,0)}$ and $C_4^{(2,1)} \subseteq C_4^{(2,0)}$. Therefore

$$\left(\pi_4^{-1}(X_{Sing})\right)_{red} = C_4^{(2,0)} \cup C_4^{(1,2)}.$$

We prove next how these splittings permit to extract information about the q.o. singularity, more concretely, about the characteristic exponent.

Lemma 4.19. *Let Γ be the graph describing the jet schemes through the singular locus of a q.o. surface with one normalized characteristic exponent $\left(\frac{a}{n}, \frac{b}{n}\right)$.*

- (i) *If there are splittings where three vertices at level $m+1$ project into a vertex at level m, then $b > 1$.*
- (ii) *Otherwise $b \in \{0, 1\}$ and we have the following possibilities.*
  - (ii.a) *If every splitting is of first type, then $a = b = 1$.*
  - (ii.b) *If every splitting is of second type then either we have $b = 0$ or we have $b = 1$ and n divides a.*
  - (ii.c) *If there are both types of splittings then $b = 1$ and n does not divide a.*

Proof. First note that if Γ is the graph describing the jets through the singular locus of a q.o. surface with one normalized exponent γ, then in Γ there must be splittings. Indeed, it follows by Remark 4.16. Suppose first that $\gamma = \left(\frac{1}{n}, \frac{1}{n}\right)$, then at level $m = 1$ there is only one irreducible component $V\left(x_1^{(0)}, x_2^{(0)}, z^{(0)}\right)$, and for m big enough there are $n \geq 2$ irreducible components. When $\gamma \neq \left(\frac{1}{n}, \frac{1}{n}\right)$ we have one or two irreducible components at level $m = 1$ and the number of components is not bounded as m grows.

Note also that with the data of the weights we can deduce that the vertex $V_m(e, c)$ corresponds either to a component C_m^ν with $\nu \in A_m$ (if $e + c = 3(m+1)$) or $\nu \in B_m$ (otherwise). Hence we can also define the types of splittings as follows, if there is a splitting at level m projecting to a vertex $V_{m-1}(e, c)$ corresponding to the component C_{m-1}^ν, we have:

- splitting of the first type if and only if $\nu \in A_{m-1}$,
- splitting of the second type if and only if $\nu \in B_{m-1}$.

First we prove that if $a \geq b > 1$ there are always splittings (necessarily of second type) where three vertices project into one vertex of the graph. Later we will see that this only occurs in this case. Since the relation $\leq_m$ on $B_m^<$ in this case is $\nu \leq_m \nu'$ if and only if $\nu = \nu'$, we deduce that for m big enough, there exists $\nu \in B_m^<$ (and therefore $\nu \in F_m$) of the form $\nu = (\nu_1, 0)$. Set $m_0 = \langle \nu, n\gamma \rangle$, then $\nu \in B_{m_0}^=$ and

$$\pi_{m_0+1,m_0}^{-1}\big(C_{m_0}^\nu\big) = C_{m_0+1}^\nu \cup C_{m_0+1}^{\nu+(1,0)} \cup C_{m_0+1}^{\nu+(0,1)}$$

We clearly have $\nu \in F_{m_0+1}$, and the question is whether $\nu + (1,0)$ and $\nu + (0,1)$ belong to F_{m_0+1} or not, to know whether we have a true splitting or not (see Remark 4.18). We have that $\nu + (1,0), \nu + (0,1) \in A_{m_0+1}$, since

$$\langle \nu + (1,0), n\gamma \rangle = m_0 + a > m_0 + 1$$

$$\langle \nu + (0,1), n\gamma \rangle = m_0 + b > m_0 + 1$$

Moreover we have that $\nu + (0,1) \in F_{m_0+1}$ since there is no $\nu' \in A_{m_0+1}$ with $\nu' \leq \nu$ because $\nu = (\nu_1, 0)$, and there is no $\nu' = (\nu_1', 0) \in B_{m_0+1}^=$ with $\nu_1' < \nu_1$, since this would contradict that $\nu \in B_{m_0}^=$. To finish we have to prove that $\nu + (0,1) \in F_{m_0+1}$. Suppose there exits $r > 0$ such that $\nu'' = \nu + (0,1) - (r,0) \in A_{m_0+1} \cup B_{m_0+1}^=$ (this would imply that $C_{m_0+1}^{\nu+(0,1)} \subseteq C_{m_0+1}^{\nu''}$), then

$$m_0 + b - ra \geq m_0 + 1$$

or equivalently $b \geq ra + 1$, which is impossible if $r > 0$.

To study the splittings at level $m+1$, we have to study the irreducibility of $\pi_{m+1,m}^{-1}(C_m^\nu)$ with $\nu \in F_m \subseteq A_m \cup B_m$. We distinguish cases:

(i) If $\nu \in F_m \cap A_m$, we have the following possibilities.

(a) If $m + 1 < \langle \nu, n\gamma \rangle$, then $\nu \in A_{m+1}$, and

$$\pi_{m+1,m}^{-1}(C_m^\nu) = C_{m+1}^\nu \quad \text{irreducible}$$

Indeed, by definition $C_m^\nu = V\big(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{([m/n])}\big)$. Then

$$\pi_{m+1,m}^{-1}(C_m^\nu) = V\big(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{([m/n])}, F^{(m+1)}\big)$$

where

$$F^{(m+1)} \bmod \big(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{([m/n])}\big) \equiv$$

$$\equiv \begin{cases} z^{([m/n]+1)^n} & \text{if } m+1 \equiv 0 \bmod n \\ \\ 0 & \text{otherwise} \end{cases}$$

since $m + 1 < \langle \nu, n\gamma \rangle$. Notice that if $m + 1 \equiv 0 \bmod n$ then $[\frac{m}{n}] + 1 = \frac{m+1}{n}$.

(b) If $m+1 = \langle \nu, n\gamma \rangle$ and $\nu \in N$, then $\nu \in B^{=}_{m+1}$ and

$$\pi^{-1}_{m+1,m}(C^{\nu}_{m}) = C^{\nu}_{m+1} \quad \text{irreducible}$$

since $\pi^{-1}_{m+1,m}(C^{\nu}_{m}) = V\big(J^{\nu}, F^{(m+1)}\big)$ and by Lemma 4.5 we have

$$F^{(m+1)} \equiv F^{(m+1)}_{\nu} \mod \big(x_1^{(0)}, \dots, x_1^{(\nu_1-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2-1)}, z^{(0)}, \dots, z^{(\langle \nu, \gamma \rangle - 1)}\big)$$

where $F^{(m+1)}_{\nu}$ is an irreducible polynomial.

(c) If $m+1 = \langle \nu, n\gamma \rangle$ and $\nu \notin N$, then $\nu \notin A_{m+1} \cup B_{m+1}$ and

$$\big(\pi^{-1}_{m+1,m}(C^{\nu}_{m})\big)_{red} = \begin{cases} \text{is reducible (i.e. splitting)} & \text{if } b \neq 0 \\ \text{is irreducible} & \text{otherwise} \end{cases}$$

Indeed, first note that

$$\left[\frac{m+1}{n}\right] = \left[\frac{m}{n}\right] = [\langle \nu, \gamma \rangle]$$

We have that

$$\pi^{-1}_{m+1,m}(C^{\nu}_{m}) = V\big(x_1^{(0)}, \dots, x_1^{(\nu_1-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2-1)}, z^{(0)}, \dots, z^{([m/n])}, F^{(m+1)}\big),$$

and, by Lemma 4.5

$$F^{(m+1)} \equiv F^{(m+1)}_{\nu} \mod \big(x_1^{(0)}, \dots, x_1^{(\nu_1-1)}, x_2^{(0)}, \dots, x_2^{(\nu_2-1)}, z^{(0)}, \dots, z^{([m/n])}\big)$$

where

$$F^{(m+1)}_{\nu} = -{x_1^{(\nu_1)}}^{a} {x_2^{(\nu_2)}}^{b} \cdot U$$

where U is a unit in R_{ν}. Therefore, whenever $b \neq 0$, $\big(\pi^{-1}_{m+1,m}(C^{\nu}_{m})\big)_{red} = C^{\nu+(1,0)}_{m+1} \cup C^{\nu+(0,1)}_{m+1}$, and to have a splitting of first type we need to argue that $\nu + (1,0)$, $\nu + (0,1) \in F_{m+1}$, and the splitting is of the form two vertices projecting to one vertex.

(ii) If $\nu \in F_m \cap B^{=}_{m}$, then

$$\big(\pi^{-1}_{m+1,m}(C^{\nu}_{m})\big)_{red} = \begin{cases} \text{irreducible} & \text{if } \gamma = (\frac{1}{n}, \frac{1}{n}) \\ \text{reducible} & \text{otherwise} \end{cases}$$

Indeed, if $\gamma = \big(\frac{1}{n}, \frac{1}{n}\big)$, then

$$\pi^{-1}_{m+1,m}(C^{\nu}_{m}) = V\big(x_1^{(0)}, \dots, x_1^{(\nu_1)}, x_2^{(0)}, \dots, x_2^{(\nu_2-1)}, z^{(0)}, \dots, z^{(\langle \nu, \gamma \rangle - 1)}, F^{(m)}, F^{(m+1)}\big)$$

and by Lemma 4.7 we have that $\pi^{-1}_{m+1,m}(C^{\nu}_{m}) = C^{\nu}_{m+1}$, with $\nu \in B^{<}_{m+1}$. By Lemma 4.10 $\pi^{-1}_{m+1,m}(C^{\nu}_{m})$ is irreducible.

Suppose then that $\gamma \neq \left(\frac{1}{n}, \frac{1}{n}\right)$, i.e. $a > 1$. Then, from Lemma 4.7 we deduce

$$\left(\pi_{m+1,m}^{-1}(C_m^\nu)\right)_{red} = \begin{cases} C_{m+1}^\nu \cup C_{m+1}^{\nu+(1,0)} & \text{if } b = 0, 1 \\ C_{m+1}^\nu \cup C_{m+1}^{\nu+(1,0)} \cup C_{m+1}^{\nu+(0,1)} & \text{if } b > 1 \end{cases}$$

Hence if $b = 0, 1$ we have two vertices projecting to one vertex, while if $b > 1$ we have three vertices projecting to one vertex.

(iii) If $\nu \in F_m \cap B_m^<$, then $\pi_{m+1,m}^{-1}(C_m^\nu) = C_{m+1}^\nu$ irreducible.

To finish, notice that if $\gamma = \left(\frac{a}{n}, \frac{1}{n}\right)$ and n divides a then $F_m \subseteq \mathbb{Z} \times \{0\}$ and for every $\nu \in F_m \cap A_m$ we have $\nu \in N$, therefore we never have the situation described in (c).

□

Remark 4.20. *The splittings of second type at level $m+1$ correspond to the following situation, there is a component C_m^ν with $\nu \in B_m^=$, such that $\pi_{m+1,m}^{-1}(C_m^\nu)$ is reducible. Then, by definition, C_m^ν is a singular algebraic variety, and the decomposition of $\pi_{m+1,m}^{-1}(C_m^\nu)$ in irreducible components has one component (precisely C_{m+1}^ν) projecting to the regular locus of C_m^ν, while the rest of the components (one or two, depending on the singular locus of C_m^ν) project to the singular locus of C_m^ν. In this situation we say that C_m^ν splits at level $m+1$ through the singular locus.*

Theorem 4.21. *The graph Γ describing the structure of jet schemes through the singular locus of a q.o. surface singularity with one normalized characteristic exponent λ, determines and it is determined by λ.*

Proof. Recall that $\lambda = \gamma$. By Lemma 4.19, looking at the splittings in Γ, we are able to distinguish the four cases:

(i) $\gamma = \left(\frac{a}{n}, 0\right)$ or $\gamma = \left(\frac{a}{n}, \frac{1}{n}\right)$ with $a \equiv 0 \bmod n$

(ii) $\gamma = \left(\frac{1}{n}, \frac{1}{n}\right)$

(iii) $\gamma = \left(\frac{a}{n}, \frac{1}{n}\right)$ with $a \not\equiv 0 \bmod n$

(iv) $\gamma = \left(\frac{a}{n}, \frac{b}{n}\right)$ with $a \geq b > 1$

Now we recover γ on each case. We will see how, roughly speaking, the splittings of the first type give information about a and b, while the splittings of the second type give information about n. Recall that with the data of the codimension and the embedding dimension we can deduce if a vertex $V_m(c, e)$ corresponds to a component C_m^ν with $\nu \in A_m$ or with $\nu \in B_m$.

(i) Case $\gamma = \big(\frac{a}{n}, 0\big)$ or $\gamma = \big(\frac{a}{n}, \frac{1}{n}\big)$ with $a \equiv 0 \bmod n$. At level $m = 1$ we have only one vertex. Looking at its codimension as m grows, we know that at level $m = n$ the codimension grows for the first time. If this vertex corresponds to a component C_m^ν with $\nu \in B_m$ then it must be $a = n$ and then

$$\gamma = \Big(1, \frac{1}{n}\Big)$$

Otherwise, we have recovered the multiplicity n and we know that $a > n$.

If $\gamma = \big(\frac{a}{n}, 0\big)$ with $\gcd(a, n) = 1$, for $\nu = (n, 0) \in N$ and level $\langle \nu, n\gamma\rangle = an$ we have $\nu \in F_{an}$, with

$$C_{an}^\nu = V\big(x_1^{(0)}, \ldots, x_1^{(\nu_1 - 1)}, z^{(0)}, \ldots, z^{(a-1)}, F^{(an)}\big)$$

Then at level $an + 1$ we have the first splitting, and it splits as

$$\begin{aligned}\big(\pi^{-1}_{an+1,an}(C_{an}^{(n,0)})\big)_{red} &= \big(V\big(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, z^{(0)}, \ldots, z^{(a-1)}, F^{(an)}, F^{(an+1)}\big)\big)_{red} \\ &= C_{an+1}^{(n,0)} \cup C_{an+1}^{(n+1,0)}\end{aligned}$$

We can read the number an from the graph, and since we know n, we recover a too.

If $\gamma = \big(\frac{a}{n}, \frac{1}{n}\big)$ with $a \equiv 0 \bmod n$, we have, for $\nu = (1, 0) \in N$ and level $\langle \nu, n\gamma\rangle = a$, that $\nu \in F_a$, with

$$C_a^\nu = V\big(x_1^{(0)}, z^{(0)}, \ldots, z^{(\langle \nu,\gamma\rangle - 1)}, F^{(a)}\big)$$

At level $a + 1$ we have the first splitting, as follows

$$\big(\pi^{-1}_{a+1,a}(C_a^\nu)\big)_{red} = \big(V\big(x_1^{(0)}, z^{(0)}, \ldots, z^{(\langle \nu,\gamma\rangle-1)}, F^{(a)}, F^{(a+1)}\big)\big)_{red} = C_{a+1}^{(1,0)} \cup C_{a+1}^{(2,0)}$$

and we can read the number a.

Note that in both cases the first splitting is of second type. How do we distinguish these two cases? We have at level $m = 1$ only one component and of codimension 2. In the second case we have at level $m = a - 1$ only one component, of codimension $\langle \nu, \gamma\rangle + 1 = \frac{a}{n} + 1 > 2$. Hence we must have jumps in codimension, but all are level $m \equiv 0 \bmod n$, more precisely, at level $m = n$ when passing from the component $V\big(x_1^{(0)}, z^{(0)}\big)$ to $V\big(x_1^{(0)}, z^{(0)}, z^{(1)}\big)$, at level $m = 2n$ when passing to the component $V(x_1^{(0)}, z^{(0)}, z^{(1)}, z^{(2)}\big)$, and so on. However, in the case $\gamma = \big(\frac{a}{n}, 0\big)$ we have at level $an - 1$ the component $C_{an-1}^{(n,0)}$ of codimension $n + a > 2$, and there must jumps in codimension at certain levels $m \not\equiv 0 \bmod n$, more precisely, when passing from $C_m^{(1,0)}$ to $C_{m+1}^{(2,0)}$ and so on. Here it is crucial that $\gcd(a, n) = 1$.

(ii) Case $\gamma = \big(\frac{1}{n}, \frac{1}{n}\big)$. See Remark 4.16, the number of irreducible components stabilizes at value $n - 1$ at level $m = n - 1$. Then we read easily n from the graph.

(iii) Case $\gamma = \left(\frac{a}{n}, \frac{1}{n}\right)$ with $a \not\equiv 0 \bmod n$. Let us look at the first part of the graph, before there is a splitting. Let m_0 be the level at which we find the first splitting. By Lemma 4.19 the splitting can be of first or second type. We recover a as follows. We claim that the splitting is of first type and at level $m_0 = a$. Indeed, notice that $(1,0) \notin N$, since $a \not\equiv 0 \bmod n$. Then

$$C_{a-1}^{(1,0)} = V\left(x_1^{(0)}, z^{(0)}, \ldots, z^{([\frac{a-1}{n}])}\right)$$

and

$$\pi_{a,a-1}^{-1}(C_{a-1}^{(1,0)}) = V\left(x_1^{(0)}, z^{(0)}, \ldots, z^{([\frac{a-1}{n}])}, F^{(a)}\right)$$

where, by Lemma 4.5,

$$F^{(a)} \equiv -{x_1^{(1)}}^a x_2^{(0)}\left(1 - \sum c_{aj0}{x_2^{(0)}}^{j-1}\right) \bmod \left(x_1^{(0)}, z^{(0)}, \ldots, z^{([\frac{a-1}{n}])}\right)$$

And since $1 - \sum c_{aj0}{x_2^{(0)}}^{j-1}$ is a unit in $\mathbb{C}\{x_2^{(0)}\}$, we deduce that at level a there is a splitting

$$\pi_{a,a-1}^{-1}(C_{a-1}^{(1,0)}) = C_a^{(2,0)} \cup C_a^{(1,1)}$$

of first type.

We still have to find the value of n from the graph. Notice that, since $X_{Sing} = \{x_1 = z = 0\}$ irreducible, and since $a > n$, we have that $\left(\pi_m^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, z^{(0)}\right)$ for $1 \leq m < n$, and $\left(\pi_n^{-1}(X_{Sing})\right)_{red} = V\left(x_1^{(0)}, z^{(0)}, z^{(1)}\right)$. Therefore the number n is the first time in the graph where the codimension grows.

(iv) Case $\gamma = \left(\frac{a}{n}, \frac{b}{n}\right)$ with $a \geq b > 1$.

It is clear that $a = b$ if and only if the graph is symmetric. Suppose we have $a = b$, then $\gcd(a, n) = 1$, and, as in the previous cases, the first time we have a jump in codimension without splittings, is necessarily at level n. While the first splitting is at level $a = b$.

Suppose now that $a > b$. First we recover the multiplicity n. At level $m = n$ it is the first time in the graph that we have a jump in codimension in all components at this level (there might be more than two components if $a < n$). Indeed, at level $m = n$ there must be a jump in every component since ${z^{(1)}}^n$ appears in $F^{(m)}$. Of course, there might be jumps in codimension in previous components, but since $b \neq a$, there may not be in every component.

Now we will distinguish in which component the graph projects to $\{x = z = 0\}$ and which to $\{y = z = 0\}$ (recall that in this case the singular locus of X is reducible and has two components, therefore the graph has two components, one describing the lifting of $V\left(x_1^{(0)}, z^{(0)}\right)$ and the other describing the lifting of $V\big(x_2^{(0)}, z^{(0)}\big)\big)$. Again, it is crucial that $b < a$. At level $m = 1$ we have in both components one vertex and with codimension 2. The first time that this situation changes (meaning, at least one component either splitts or its codimension jumps), must occur in the branch projecting to $\{z^{(0)} = x_2^{(0)} = 0\}$ and at level $m = b$ (if the splitting is of first type, or if there is a jump in codimension) or at level $m + 1 = b$ (if the splitting is of second type).

Looking at the other component of the branch, we recover analogously, the number a.

□

We end this section with a couple of examples illustrating the previous result. We will draw an arrow in the graph at level m_0, when a component associated with certain ν, gives rise to a component for every $m \geq m_0$, i.e., $\nu \in F_m$ for every $m \geq m_0$.

Example 4.22. *Consider the graph Γ drawn in Figure 2, representing the structure of m-jet schemes through the singular locus of a q.o. singularity. Recall that the vertices are weighted with e the embedded dimension and c the codimension.*

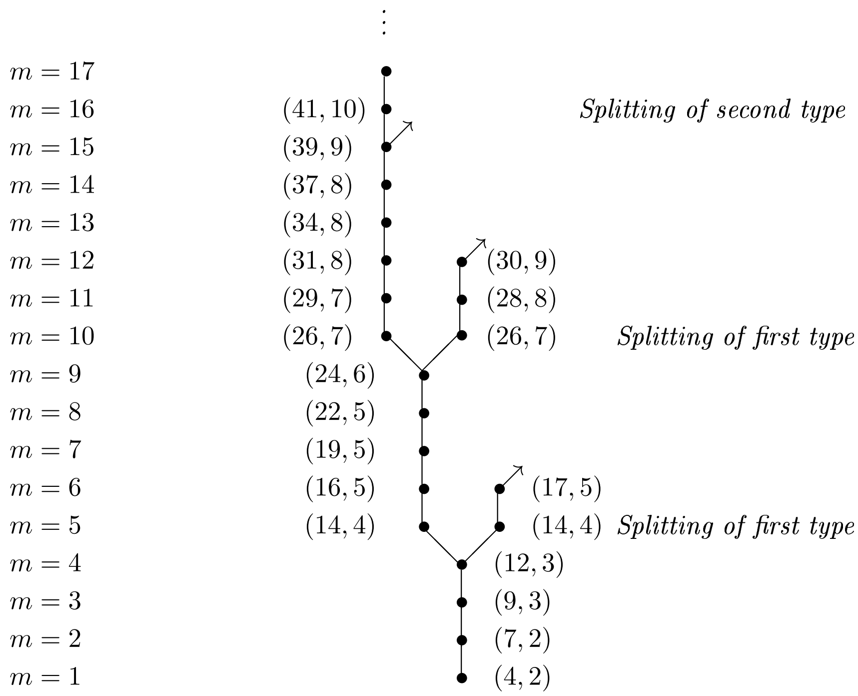

FIGURE 2. The graph of the irreducible components of jets through the singular locus of a q.o. surface singularity.

Since there are splittings of both types, but it never happens that one component splits into three components, we deduce that $\gamma = \left(\frac{a}{n}, \frac{1}{n}\right)$ *with* $a > n$*. The first splitting is of first type at level* $m = 5$*, hence* $a = 5$*. To compute* n *it is enough to find the first time we have a jump in the codimension. Therefore we have*

$$\gamma = \left(\frac{5}{3}, \frac{1}{3}\right).$$

Example 4.23. *In Figure 3 the graph associated with the jet schemes of a q.o. singularity is drawn. Since the graph is more complicated than the one in the previous example, we will only decorate it with the codimension, but instead we will say the type of splitting whenever there is one (recall that for this we use the embedding dimension). Let us recover the data of the characteristic exponent. There are splittings of second type where three vertices at level* m *project into one*

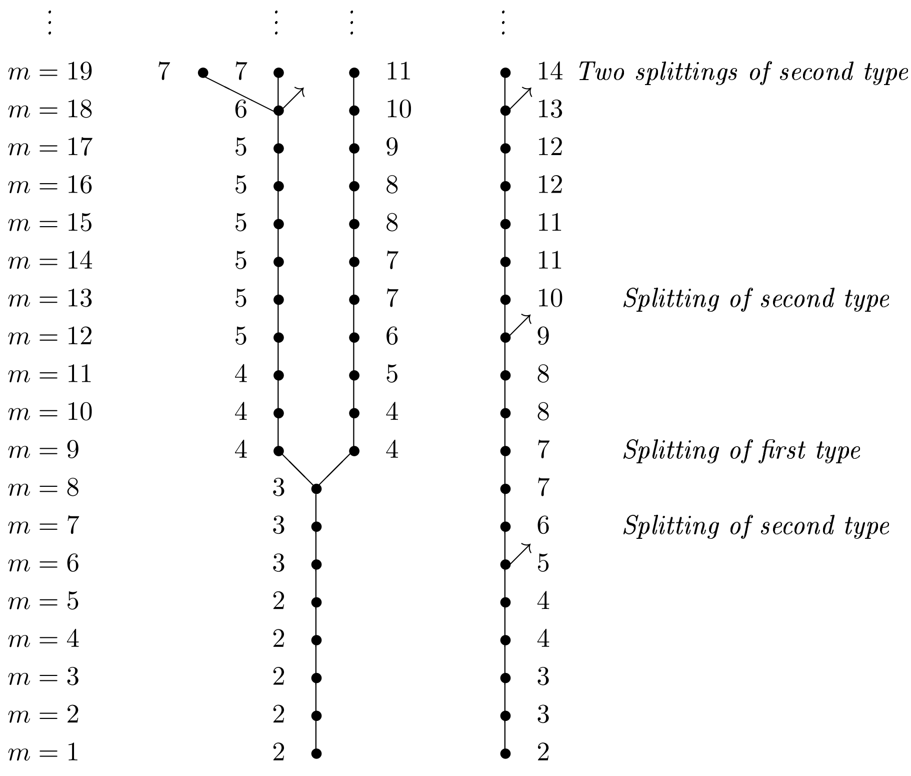

FIGURE 3. The graph of the irreducible components of jets through the singular locus of a q.o. surface singularity, decorated only with the codimension.

vertex at level $m-1$ (we can see one at level 19 in Figure 3). Therefore the characteristic exponent is of the form $\gamma = \left(\frac{a}{n}, \frac{b}{n}\right)$ with $a \geq b > 1$. Since the graph is obviously asymmetric, we deduce that $a > b$.

The multiplicity is $n = 6$ because at level 6 we can see the first jump in codimension in all components. Since the first jump in codimension is at level $m = 2$ and only in one of the components of the graph, we deduce that $b = 2$, because $b < a$. Now, we recover a looking at the first splitting in the other component of the graph, it occurs at level $m = 9$ and it is a splitting of first type, therefore $a = 9$, and the graph represented in Figure 3 describes the structure of irreducible components through the singular locus of a q.o. surface with characteristic exponent

$$\gamma = \left(\frac{9}{6}, \frac{2}{6}\right).$$

5. Jet schemes of quasi-ordinary surface singularities: the general case

We generalize the results of the previous section to the case of any number of characteristic exponents. Let X be a q.o. surface defined by a polynomial f with g characteristic exponents. We describe the decomposition of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ in irreducible components as

$$\left(\pi_m^{-1}(X_{Sing})\right)_{red} = \cup_{\nu \in F_m} C_m^{\nu}$$

analogously as for the case of one characteristic exponent. First we will define the candidates C_m^{ν}, we prove its irreducibility and finally study the inclusions among them, to define the set F_m.

Let us look first at some examples.

Example 5.1. *Consider the q.o. surface X defined by $f = (z^2 - x_1^3)^3 - x_1^{10}x_2^4$. The generators of the semigroup are $\gamma_1 = (\frac{3}{2}, 0)$ and $\gamma_2 = (\frac{10}{3}, \frac{4}{3})$, and the singular locus is $X_{Sing} = \{x_1 = z = 0\} \cup \{x_2 = z^2 - x_1^3 = 0\}$. Let us look at the component $Z_2 = \{x_2 = z^2 - x_1^3 = 0\}$ of the singular locus. If we lift Z_2 to level m we have*

$$\pi_m^{-1}(Z_2) = V\left(x_2^{(0)}, {z^{(0)}}^2 - {x_1^{(0)}}^3, F^{(1)}, \ldots, F^{(m)}\right)$$

since $F^{(0)} = \left({z^{(0)}}^2 - {x_1^{(0)}}^3\right)^3 - {x_1^{(0)}}^{10}{x_2^{(0)}}^4 \equiv 0 \ mod \ \left(x_2^{(0)}, {z^{(0)}}^2 - {x_1^{(0)}}^3\right)$. This last congruence is easier to handle if we use the first approximated root $f_1 = z^2 - x_1^3$. It is clear that we can write

$$F^{(0)} = {F_1^{(0)}}^3 - {x_1^{(0)}}^{10}{x_2^{(0)}}^4 \equiv 0 \ mod \ \left(x_2^{(0)}, F_1^{(0)}\right).$$

What it is not that clear is that

$$F^{(1)} = 3{F_1^{(0)}}^2 F_1^{(1)} - 10{x_1^{(0)}}^9 x_1^{(1)}{x_2^{(0)}}^4 - 4{x_1^{(0)}}^{10}{x_2^{(0)}}^3 x_2^{(1)}.$$

In the example above we are, roughly speaking, considering f_1 as a variable in the expansion of f:

$$f = {f_1}^3 - x_1^{10} x_2^4$$

Let us formalize this idea. Consider the following embedding of $\mathbb{A}^3$ in $\mathbb{A}^{3+g}$ with coordinates $(\mathbf{x}, u_0, \ldots, u_g)$. The embedding is defined in terms of the semi-roots as follows. Let us denote, for $0 \leq j \leq g-1$ (see Lemma 3.11),

$$h_j = -u_{j+1} + {u_j}^{n_{j+1}} - c_{j+1} \mathbf{x}^{\alpha^{(j+1)}} u_0^{r_1^{(j+1)}} \cdots u_{j-1}^{r_j^{(j+1)}} + \sum c_{\underline{\alpha}, \underline{r}}^{(j+1)} \mathbf{x}^{\alpha} u_0^{r_1} \cdots u_j^{r_{j+1}}$$

We can embed $\mathbb{A}^3$ in $\mathbb{A}^{3+g}$ as $V(h_0, \ldots, h_{g-1})$, and, if we set $h_g = u_g$, then (see [17]) the embedding of X in $\mathbb{A}^{3+g}$ is defined by

$$V(h_0, \ldots, h_g)$$

We abuse of notation and denote by X the embedding of our q.o. surface in $\mathbb{A}^{3+g}$. Note that we are not dealing with a hypersurface anymore. The jet scheme X_m is now defined by

$$X_m = \text{ Spec } \left(\frac{\mathbb{C}\{x_1^{(0)}, x_2^{(0)}\}[x_i^{(1)}, \ldots, x_i^{(m)}, u_0^{(0)}, \ldots, u_0^{(m)}, \ldots, u_g^{(0)}, \ldots, u_g^{(m)}]_{i=1,2}}{\left(H_0^{(0)}, \ldots, H_0^{(m)}, H_1^{(0)}, \ldots, H_1^{(m)}, \ldots, H_g^{(0)}, \ldots, H_g^{(m)}\right)} \right)$$

We denote, for $0 \leq j \leq g-1$, $q_{j+1} \in \mathbb{C}\{x_1, x_2\}[u_0, \ldots, u_j]$ such that

$$h_j = -u_{j+1} + q_{j+1}(x_1, x_2, u_0, \ldots, u_j)$$

holds. Then (recall notations in Section 2, where we used capital letters for polynomials, but not for variables) we have that, for $0 \leq j < g$ and $l \geq 0$

$$H_j^{(l)} = -u_{j+1}^{(l)} + Q_{j+1}^{(l)}$$

Consider the ring

$$R_j^{(l)} = \mathbb{C}\{x_1^{(0)}, x_2^{(0)}\}[x_1^{(1)}, \ldots, x_1^{(l)}, x_2^{(1)}, \ldots, x_2^{(l)}, u_0^{(0)}, \ldots, u_0^{(l)}, \ldots, u_j^{(0)}, \ldots, u_j^{(l)}]$$

for $0 \leq j \leq g$ and $l \geq 0$. We can identify $R_0^{(l)}$ with $R^{(l)}$ (see Section 2). Since the elements $Q_j^{(l)}$ belong to the ring $R_j^{(l)}$, it makes sense to define the following evaluation map defined by giving suitable values to the variables:

$$\begin{array}{rccll} ev_j: & \mathbb{C}\{x_1, x_2\}[u_0, \ldots, u_j] & \longrightarrow & \mathbb{C}\{x_1, x_2\}[z] & \\ & x_i & \mapsto & x_i, & \text{for } i = 1, 2 \\ & u_i & \mapsto & f_i, & \text{for } i = 0, \ldots, j \end{array}$$

(recall that $f_0 = z$), and at the level of jets:

$$\begin{array}{rccll} ev_j^{(m)}: & R_j^{(m)} & \longrightarrow & R^{(m)} & \\ & x_i^{(l)} & \mapsto & x_i^{(l)}, & \text{for } i = 1, 2 \text{ and } 0 \leq l \leq m \\ & u_i^{(l)} & \mapsto & F_i^{(l)}, & \text{for } i = 0, \ldots, j \text{ and } 0 \leq l \leq m \end{array}$$

We have then the following result.

Lemma 5.2. *For $0 \leq j \leq g$ and $0 \leq l \leq m$*
$$F_j^{(l)} = ev_j^{(m)}\big(Q_j^{(l)}\big).$$

This permits to describe the equations of the jets using derivations and considering the approximated roots as variables, as illustrated in Example 5.1.

Remark 5.3. *As Corollary 2.2 shows the linearity of equations*
$$F_i^{(l)}\big(x_1^{(0)}, \dots, x_1^{(l)}, x_2^{(0)}, \dots, x_2^{(l)}, z^{(0)}, \dots, z^{(l)}\big)$$
in $x_1^{(l)}, x_2^{(l)}$ and $z^{(l)}$, by Lemma 5.2 we deduce the linearity of
$$F_i^{(l)}\big(x_1^{(0)}, \dots, x_1^{(l)}, x_2^{(0)}, \dots, x_2^{(l)}, z^{(0)}, \dots, z^{(l)}, F_1^{(0)}, \dots, F_1^{(l)}, \dots, F_{i-1}^{(0)}, \dots, F_{i-1}^{(l)}\big)$$
in $x_1^{(l)}, x_2^{(l)}, z^{(l)}, F_1^{(l)}, \dots, F_{i-1}^{(l)}$, meaning that they appear in $F_i^{(l)}$ with exponent one.

Example 5.4. *We continue with Example 5.1. If we lift the component of the singular locus*
$$Z_2 = \{x_2 = f_1 = 0\}$$
at level 3, we have that
$$\pi_3^{-1}(Z_2) = \pi_3^{-1}\big(V(x_2^{(0)}, F_1^{(0)})\big) = V\big(x_2^{(0)}, F_1^{(0)}, F^{(1)}, F^{(2)}, F^{(3)}\big),$$
where $F_1^{(0)} = {z^{(0)}}^2 - {x_1^{(0)}}^3$. We can easily check that
$$\begin{aligned} F^{(1)} &= 3{F_1^{(0)}}^2 F_1^{(1)} - 10{x_1^{(0)}}^9 x_1^{(1)} {x_2^{(0)}}^4 - 4{x_1^{(0)}}^{10} {x_2^{(0)}}^3 x_2^{(1)} \\ &\equiv 0 \ mod \ \big(x_2^{(0)}, F_1^{(0)}\big) \\ \\ F^{(2)} &= 2{F_1^{(0)}}^2 F_1^{(2)} + F_1^{(0)} {F_1^{(1)}}^2 + \cdots \\ &\equiv 0 \ mod \ \big(x_2^{(0)}, F_1^{(0)}\big) \\ \\ F^{(3)} &\equiv {F_1^{(1)}}^3 \ mod \ \big(x_2^{(0)}, F_1^{(0)}\big), \end{aligned}$$
and then $\big(\pi_3^{-1}(Z_2)\big)_{red} = V\big(x_2^{(0)}, F_1^{(0)}, F_1^{(1)}\big)$. Notice that it is not irreducible. Indeed, it decomposes as
$$\begin{aligned} \big(\pi_3^{-1}(Z_2)\big)_{red} &= \pi_{3,2}^{-1}\big(Sing(V\big(x_2^{(0)}, F_1^{(0)}\big))\big) \bigcup \overline{\pi_{3,2}^{-1}\big(Reg(V\big(x_2^{(0)}, F_1^{(0)}\big))\big)} \\ &= V\big(x_1^{(0)}, x_2^{(0)}, z^{(0)}\big) \bigcup \overline{V\big(x_2^{(0)}, F_1^{(0)}, F_1^{(1)}\big) \cap \{x_1^{(0)} \neq 0\}} \end{aligned}$$
But, since $V\big(x_1^{(0)}, z^{(0)}\big)$ is an irreducible component projecting to $Z_1 = \{z = x_1 = 0\}$ and we have that $V\big(x_1^{(0)}, x_2^{(0)}, z^{(0)}\big) \subseteq V\big(x_1^{(0)}, z^{(0)}\big)$, we deduce that is not an irreducible component of $\big(\pi_3^{-1}(X_{Sing})\big)_{red}$.

Let us denote by C_3 *the component* $\overline{V\big(x_2^{(0)}, F_1^{(0)}, F_1^{(1)}\big) \cap \{x_1^{(0)} \neq 0\}}$*, and set* $C_4 := \big(\pi_{4,3}^{-1}(C_3)\big)_{red}$*. We have that* C_4 *is irreducible, because*

$$C_4 = \overline{V\big(x_2^{(0)}, F_1^{(0)}, F_1^{(1)}, F^{(4)}\big) \cap \{x_1^{(0)} \neq 0\}},$$

with

$$F^{(4)} \equiv -{x_1^{(1)}}^{10} {x_2^{(1)}}^{4} \ mod \ \big(x_2^{(0)}, F_1^{(0)}, F_1^{(1)}\big)$$

and hence $C_4 = \overline{V\big(x_2^{(0)}, x_2^{(1)}, F_1^{(0)}, F_1^{(1)}\big) \cap \{x_1^{(0)} \neq 0\}}$*. With the same arguments it is not difficult to see that if we lift to level* 12*, we have*

$$C_{12} := \big(\pi_{12,3}^{-1}(C_3)\big)_{red} = \overline{V\big(x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, F_1^{(0)}, F_1^{(1)}, F_1^{(2)}, F_1^{(3)}, F^{(12)}\big) \cap \{x_1^{(0)} \neq 0\}}$$

where

$$F^{(12)} \equiv {F_1^{(4)}}^3 - {x_1^{(0)}}^{10} {x_2^{(3)}}^4 \ mod \ \big(x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, F_1^{(0)}, F_1^{(1)}, F_1^{(2)}, F_1^{(3)}\big)$$

Then C_{12} *is irreducible, since* ${F_1^{(4)}}^3 - {x_1^{(0)}}^{10} {x_2^{(3)}}^4$ *is irreducible, but if we lift to next level, we have that*

$$\big(\pi_{13,12}^{-1}(C_{12})\big)_{red} = \overline{V\big(x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, F_1^{(0)}, F_1^{(1)}, F_1^{(2)}, F_1^{(3)}, F^{(12)}, F^{(13)}\big) \cap \{x_1^{(0)} \neq 0\}},$$

which is not irreducible, since it splits through the singular locus of the variety $V\big(x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, F_1^{(0)}, F_1^{(1)}, F_1^{(2)}, F_1^{(3)}, F^{(12)}\big)$*. Then* $\big(\pi_{13,12}^{-1}(C_{12})\big)_{red} = C_{13} \cup C'_{13}$*, where*

$$\begin{aligned} C_{13} &= \overline{V\big(x_2^{(0)}, \ldots, x_2^{(3)}, F_1^{(0)}, \ldots, F_1^{(4)}\big) \cap \{x_1^{(0)} \neq 0\}} \\ C'_{13} &= \overline{V\big(x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, F_1^{(0)}, \ldots, F_1^{(3)}, F^{(12)}, F^{(13)}\big) \cap \{x_1^{(0)} \neq 0\} \cap \{x_2^{(3)} \neq 0\}} \end{aligned}$$

To formalize all the ideas illustrated in the examples we need to introduce some notation.

Definition 5.5. *For* $\nu \in \sigma \cap N_0$ *and* $m \in \mathbb{Z}_{\geq 0}$ *we define the ideal*

$$J_m^\nu = Rad\left(x_1^{(0)}, \ldots, x_1^{(\nu_1 - 1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2 - 1)}, F^{(0)}, \ldots, F^{(m)}\right).$$

For convenience we set

$$J_{-1}^\nu = \big(x_1^{(0)}, \ldots, x_1^{(\nu_1 - 1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2 - 1)}\big)$$

Moreover we define the integer $j(m,\nu) \in \{0, \ldots, g\}$*, defined by the inequalities*

$$\langle \nu, e_{j-1}\gamma_j \rangle \leq m < \langle \nu, e_j \gamma_{j+1} \rangle,$$

and the integer $j'(m,\nu) \in \{-1, 0, \ldots, j(m,\nu)\}$ *defined by*

$$\langle \nu, e_{j-1}\gamma_j \rangle + e_j \leq m < \langle \nu, e_j \gamma_{j+1} \rangle + e_{j+1},$$

where we have to set

$$\gamma_{-1} := (0,0), \ e_{-1} := 0, \ e_{-2} := 0$$

Recall that we convey $\gamma_{g+1} = \infty$*.*

We denote by $D(h)$ the open set

$$D(h) = Spec\ R_h$$

where R is the ring $R = \mathbb{C}\{x_1^{(0)}, x_2^{(0)}\}[x_1^{(j)}, x_2^{(j)}, z^{(0)}, z^{(j)}]_{j>0}$. Recall that, for $l > 0$ we denote by $R^{(l)}$ the subring $R^{(l)} = \mathbb{C}\{x_1^{(0)}, x_2^{(0)}\}[x_1^{(j)}, x_2^{(j)}, z^{(0)}, z^{(j)}]_{0<j\leq l}$.

We need to introduce the artificial notation of γ_{-1}, e_{-1} and e_{-2} to be able to define $j'(m,\nu) = -1$, which will cover the range $0 \leq m < n$ for any ν. Now, for any ν and m, the integers $j(m,\nu)$ and $j'(m,\nu)$ are defined.

With the definition of the integer $j'(m,\nu)$ we can write in a compact form, the relation $\leq_m$ given in Definition 4.12, for the case of one characteristic exponent, as

$$\nu \leq_m \nu' \text{ if and only if } \nu' - \nu \in \sigma_{Reg,j'(m,\nu)},$$

because if $g = 1$, we have that:

- $\nu \in A_m \cup B_m^{=}$ is equivalent to $j'(m,\nu) \leq 0$
- $\nu \in B_m^{<}$ is equivalent to $j'(m,\nu) = 1$

We are going to prove that this is the relation that controls the inclusions among the candidates C_m^ν to be irreducible components also in the general case, but the proof is much more involved. First we have to define the candidates to be the irreducible components.

Definition 5.6. *For $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$ we set (recall Remark 2.5)*

$$D_m^\nu = \begin{cases} V(J_m^\nu) & \text{if } \sigma_{Reg,j'(m,\nu)} = \rho_1 \cup \rho_2 \\ V(J_m^\nu) \cap D(x_1^{(\nu_1)}) & \text{if } \sigma_{Reg,j'(m,\nu)} = \rho_2 \\ V(J_m^\nu) \cap D(x_1^{(\nu_1)}) \cap D(x_2^{(\nu_2)}) & \text{if } \sigma_{Reg,j'(m,\nu)} = \{(0,0)\} \end{cases}$$

where $j' = j'(m,\nu)$. Moreover we define $C_m^\nu = \overline{D_m^\nu}$ its Zariski closure.

Note that D_m^ν is reduced since the ideals J_l^ν are radical.

With these sets C_m^ν we can cover $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$. Indeed, given a jet $\gamma \in X_m$, if $x_i \circ \gamma \neq 0$ for $i = 1,2$, the vector $\nu = \left(\mathrm{ord}_t(x_1 \circ \gamma), \mathrm{ord}_t(x_2 \circ \gamma)\right)$ belongs to $\sigma \cap N_0$ and $0 \leq \nu_i \leq m$. Moreover it is clear that $\gamma \in D_m^\nu \subseteq C_m^\nu$, and we deduce

$$X_m = \bigcup_{\nu \in \sigma \cap [0,m]^2 \cap N_0} C_m^\nu,$$

where $[0,m]$ denotes the closed interval, and $[0,m]^2$ the square $[0,m] \times [0,m]$. We are interested in m-jets with origin at the singular locus, and this introduces some constraints in the possible values of ν.

Lemma 5.7. *For $m \in \mathbb{Z}_{>0}$ we have that $\left(\pi_m^{-1}(X_{Sing})\right)_{red} = \bigcup_{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0} C_m^{\nu}$.*

Proof. Given $\gamma(t) \in \left(\pi_m^{-1}(X_{Sing})\right)_{red}$, suppose first that $x_i \circ \gamma(t) \neq 0$ for $i = 1, 2$. Then we define $\nu := \left(\mathrm{ord}_t(x_1 \circ \gamma(t)), \mathrm{ord}_t(x_2 \circ \gamma(t))\right) \in [0,m]^2 \cap N_0$ and obviously $\gamma(t) \in D_m^{\nu} \subseteq C_m^{\nu}$. We have to prove that $\nu \in \sigma_{Sing} \cap N_0$, and this follows easily from Definition 3.17, by distinguishing cases.

Now we deal with the other cases. If $x_i \circ \gamma(t) = 0$ for $i = 1, 2$, then $\gamma(t) \in C_m^{\nu}$ for any $\nu \in \sigma_{Sing} \cap N_0$ with $0 \leq \nu_i \leq m$ for $i = 1, 2$.

If $x_1 \circ \gamma(t) = 0$ and $x_2 \circ \gamma(t) \neq 0$, then we denote $\alpha := \mathrm{ord}_t(x_2 \circ \gamma(t))$. We have $0 \leq \alpha \leq m$, and $\gamma(t) \in C_m^{\nu}$ for any $\nu \in \sigma_{Sing} \cap N_0$, with $0 \leq \nu_i \leq m$ for $i = 1, 2$, and $\nu_2 \leq \alpha$.

The left case $x_1 \circ \gamma(t) \neq 0$ and $x_2 \circ \gamma(t) = 0$ is analogous to the last one.

We prove the other inclusion. If $\gamma(t) \in X_m \setminus \pi_m^{-1}(X_{Sing})$, then $\gamma(0) \notin X_{Sing}$. Again distinguishing cases depending on the singular locus, we can prove that $\nu = \left(\mathrm{ord}_t(x_1 \circ \gamma(t)), \mathrm{ord}_t(x_2 \circ \gamma(t))\right) \notin \sigma_{Sing}$. □

The examples at the beginning of this section together with the discussion in Section 4 for the case of one characteristic exponent, illustrate that the main point is to study carefully the equations defining the m-jets. More concretely, we have to study

$$F^{(l)} \bmod J_{l-1}^{\nu}$$

for $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$ and $l \geq 0$.

We have seen in the examples how the semi-roots f_i appear in the sequence $F^{(0)}, \ldots, F^{(m)}$ modulo the ideal J_{m-1}^{ν}. By definition

$$F_i^{(l)} \in R^{(l)} = \mathbb{C}\{x_1^{(0)}, x_2^{(0)}\}[x_k^{(1)}, \ldots, x_k^{(l)}, z^{(0)}, \ldots, z^{(l)}]_{k=1,2}.$$

However, by Lemma 3.11 and Lemma 5.2, we can see $F_i^{(l)}$ as an element in

$$\mathbb{C}\{x_1^{(0)}, x_2^{(0)}\}[x_k^{(1)}, \ldots, x_k^{(l)}]_{k=1,2}[F_r^{(0)}, \ldots, F_r^{(l)}]_{0 \leq r < i}.$$

Definition 5.8. *For $\nu \in \sigma_{Sing} \cap N_0$ and $l \in \mathbb{Z}_{\geq 0}$, we define $F_{0,\nu}^{(l)} = F_0^{(l)}$, and, for $1 \leq i \leq g$ we define, by recurrence, $F_{i,\nu}^{(l)}$ as the polynomial $F_i^{(l)}$ once we set*

$$x_k^{(0)} = \cdots = x_k^{(\nu_k - 1)} = 0, \ k = 1, 2,$$

$$F_{j,\nu}^{(r_j)} = 0 \ \text{ for } 0 \leq j < i \text{ and } 0 \leq r_j < \langle \nu, \gamma_{j+1} \rangle.$$

By definition we have

$$F_i^{(l)} \equiv F_{i,\nu}^{(l)} \bmod \left(J_{-1}^{\nu}, F_{j,\nu}^{(r_j)}\right)_{0 \leq j < i, \ 0 \leq r_j < \langle \nu, \gamma_{j+1} \rangle} \tag{14}$$

Let us study carefully the polynomials $F_{i,\nu}^{(l)}$, since they are the interesting equations in J_m^ν, the defining equations of the sets C_m^ν. The next result is the generalization of Lemma 4.5 to the case of $g \geq 1$ characteristic exponents.

Lemma 5.9. *For any $\nu \in \sigma_{Sing} \cap N_0$ and $1 \leq i \leq g$, we have that*

$$F_{i,\nu}^{(l)} = 0 \text{ for } 0 \leq l < \langle \nu, n_i\gamma_i \rangle.$$

For $l \geq \langle \nu, n_i\gamma_i \rangle$, the polynomial $F_{i,\nu}^{(l)}$ is non-zero and quasi-homogeneous of degree l. More precisely, for $l = \langle \nu, n_i\gamma_i \rangle$ we have the following description of $F_{i,\nu}^{(\langle \nu, n_i\gamma_i \rangle)}$.

(i) *If $\nu \in \overset{\circ}{\sigma}$, then, for $1 \leq i \leq i(\nu)$, the polynomial $F_{i,\nu}^{(\langle \nu, n_i\gamma_i \rangle)}$ is*

$$\begin{cases} {F_{i-1,\nu}^{(\langle \nu,\gamma_i \rangle)}}^{n_i} - c_i {x_1^{(\nu_1)}}^{\alpha_1^{(i)}} {x_2^{(\nu_2)}}^{\alpha_2^{(i)}} {F_{0,\nu}^{(\langle \nu,\gamma_1 \rangle)}}^{r_1^{(i)}} \cdots {F_{i-2,\nu}^{(\langle \nu,\gamma_{i-1} \rangle)}}^{r_{i-1}^{(i)}}, & \text{if } i < i(\nu) \\ -c_{i(\nu)} {x_1^{(\nu_1)}}^{\alpha_1^{(i(\nu))}} {x_2^{(\nu_2)}}^{\alpha_2^{(i(\nu))}} {F_{0,\nu}^{(\langle \nu,\gamma_1 \rangle)}}^{r_1^{(i(\nu))}} \cdots {F_{i(\nu)-2,\nu}^{(\langle \nu,\gamma_{i(\nu)-1} \rangle)}}^{r_{i(\nu)-1}^{(i(\nu))}} & \text{if } i = i(\nu) \end{cases}$$

(ii) *If $\nu \in \rho_1 \cup \rho_2$, the description of $F_{i,\nu}^{(\langle \nu, n_i\gamma_i \rangle)}$ is more complicated. We have, for $1 \leq i \leq i(\nu)$,*

$$\begin{cases} {F_{i-1,\nu}^{(\langle \nu,\gamma_i \rangle)}}^{n_i} - c_i {x_1^{(\nu_1)}}^{\alpha_1^{(i)}} {x_2^{(\nu_2)}}^{\alpha_2^{(i)}} {F_{0,\nu}^{(\langle \nu,\gamma_1 \rangle)}}^{r_1^{(i)}} \cdots {F_{i-2,\nu}^{(\langle \nu,\gamma_{i-1} \rangle)}}^{r_{i-1}^{(i)}} + G_{i,\nu}^{(\langle \nu,n_i\gamma_i \rangle))}, \\ -c_{i(\nu)} {x_1^{(\nu_1)}}^{\alpha_1^{(i(\nu))}} {x_2^{(\nu_2)}}^{\alpha_2^{(i(\nu))}} {F_{0,\nu}^{(\langle \nu,\gamma_1 \rangle)}}^{r_1^{(i(\nu))}} \cdots {F_{i(\nu)-2,\nu}^{(\langle \nu,\gamma_{i(\nu)-1} \rangle)}}^{r_{i(\nu)-1}^{(i(\nu))}} + G_{i(\nu),\nu}^{(\langle \nu,n_{i(\nu)}\gamma_{i(\nu)} \rangle)} \end{cases}$$

where, for $1 \leq i < i(\nu)$,

$$G_{i,\nu}^{(\langle \nu,n_i\gamma_i \rangle)} = \sum c_{\underline{\alpha},\underline{r}} {x_1^{(\nu_1)}}^{\alpha_1} {x_2^{(\nu_2)}}^{\alpha_2} {F_{0,\nu}^{(\langle \nu,\gamma_1 \rangle)}}^{r_1} \cdots {F_{i-1,\nu}^{(\langle \nu,\gamma_i \rangle)}}^{r_i},$$

$c_{\underline{\alpha},\underline{r}}$ are the coefficients appearing in the expansion of f_i given in Lemma 3.11, such that

$$\langle \nu, (\alpha_1, \alpha_2) + r_1\gamma_1 + \cdots + r_i\gamma_i \rangle = \langle \nu, n_i\gamma_i \rangle,$$

while when $i = i(\nu)$,

$$G_{i(\nu),\nu}^{(\langle \nu,n_{i(\nu)}\gamma_{i(\nu)} \rangle)} = \sum_{\underline{r}=(r_1,\ldots,r_{i(\nu)-1},0)} c_{\underline{\alpha},\underline{r}} {x_1^{(\nu_1)}}^{\alpha_1} {x_2^{(\nu_2)}}^{\alpha_2} {F_{0,\nu}^{(\langle \nu,\gamma_1 \rangle)}}^{r_1} \cdots {F_{i(\nu)-2,\nu}^{(\langle \nu,\gamma_{i(\nu)-1} \rangle)}}^{r_{i(\nu)-1}},$$

subject to the same conditions as before. Moreover, in this case the polynomial $F_{i(\nu),\nu}^{(\langle \nu,n_{i(\nu)}\gamma_{i(\nu)} \rangle)}$ can be written as

$$F_{i(\nu),\nu}^{(\langle \nu,n_{i(\nu)}\gamma_{i(\nu)} \rangle)} = -c_{i(\nu)} {x_1^{(\nu_1)}}^{\alpha_1^{(i(\nu))}} {x_2^{(\nu_2)}}^{\alpha_2^{(i(\nu))}} {F_{0,\nu}^{(\langle \nu,\gamma_1 \rangle)}}^{r_1^{(i(\nu))}} \cdots {F_{i(\nu)-2,\nu}^{(\langle \nu,\gamma_{i(\nu)-1} \rangle)}}^{r_{i(\nu)-1}^{(i(\nu))}} \cdot U$$

where U is a unit in R_ν.

For $i > i(\nu)$ we can sometimes describe some polynomials $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}$. There exists an integer $r(\nu) \geq 0$ (which is always 0 when $\nu \in \overset{\circ}{\sigma}$) such that

$$\langle \nu, n_{i(\nu)} \cdots n_{i(\nu)+r(\nu)}\gamma_{i(\nu)}\rangle = \langle \nu, n_{i(\nu)+r(\nu)}\gamma_{i(\nu)+r(\nu)}\rangle < \langle \nu, \gamma_{i(\nu)+r(\nu)+1}\rangle$$

and then, for $i(\nu) < i \leq i(\nu) + r(\nu)$ the polynomial $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}$ has the form described above for $i < i(\nu)$.

Before proving the Lemma we illustrate the content in the next example.

Example 5.10. *Let us consider the q.o. polynomial $f = \left((z^2 - x_1^2x_2)^3 - x_1^7x_2^3\right)^2 - x_1^{15}x_2^{17}$, with characteristic exponents*

$$\gamma_1 = \left(1, \frac{1}{2}\right),\ \gamma_2 = \left(\frac{7}{3}, 1\right),\ \gamma_3 = \left(\frac{15}{2}, \frac{7}{2}\right)$$

Then $n_1 = 2$, $n_2 = 3$ and $n_3 = 2$. For $\nu = (0, 1)$, we have that $i(\nu) = 1$, since $\nu \notin N_1$. Moreover $r(\nu) = 1$, because

$$3 = \langle \nu, n_1n_2\gamma_1\rangle = \langle \nu, n_2\gamma_2\rangle < \langle \nu, \gamma_3\rangle = \frac{7}{2}$$

We have that

$$F_{1,\nu}^{(\langle \nu, n_1\gamma_1\rangle)} = F_{1,\nu}^{(1)} = -{x_1^{(0)}}^2 x_2^{(1)}$$

$$F_{2,\nu}^{(\langle \nu, n_2\gamma_2\rangle)} = F_{2,\nu}^{(3)} = {F_{1,\nu}^{(1)}}^3 - {x_1^{(0)}}^7 {x_2^{(1)}}^3$$

and the polynomial $F_{2,\nu}^{(3)}$ can be written as

$$F_{2,\nu}^{(3)} = -{x_1^{(0)}}^6 {x_2^{(1)}}^3 \left(1 + x_1^{(0)}\right)$$

with $1 + x_1^{(0)}$ a unit in $\mathbb{C}\{x_1^{(0)}\}$, as the Lemma above claims.

Notice that, despite the fact that $\nu \in \rho_1 \cup \rho_2$, we have $G_{1,\nu}^{(1)} = G_{2,\nu}^{(3)} = 0$. This is due to the fact that the q.o. polynomial f is very simple, it is enough to consider the following polynomial with the same characteristic exponents

$$h = \left((z^2 - x_1^2x_2 + 5x_1^3x_2)^3 - x_1^7x_2^3 + 2x_1^6x_2^3z\right)^2 - x_1^{15}x_2^{17}.$$

to have non-zero polynomials $G_{i,\nu}^{(l)}$.

Proof of Lemma 5.9. For $1 \leq i \leq i(\nu)$ we will use the expansion of the semiroot f_i given in Lemma 3.11. Notice that by definition, $F_i^{(l)}$ consists of *monomials* of the form

$$F_{i-1}^{(a_1)} \cdots F_{i-1}^{(a_{n_i})}$$

$$x_1^{(b_1)} \cdots x_1^{(b_{\alpha_1})} x_2^{(c_1)} \cdots x_2^{(c_{\alpha_2})} F_0^{(s_1^{(0)})} \cdots F_0^{(s_{r_1}^{(0)})} \cdots F_{i-1}^{(s_1^{(i-1)})} \cdots F_{i-1}^{(s_{r_i}^{(i-1)})}$$

with $0 \leq a_1 \leq \cdots \leq a_{n_i} \leq l$ and $a_1 + \cdots + a_{n_i} = l$, with $0 \leq b_1 \leq \cdots \leq b_{\alpha_1} \leq l$, $0 \leq s_1^{(j)} \leq \cdots \leq s_{r_{j+1}}^{(j)} \leq l$ and with $b_1 + \cdots + b_{\alpha_1} + c_1 + \cdots + c_{\alpha_2} + s_1^{(0)} + \cdots +$

$s_{r_1}^{(0)}+\cdots+s_1^{(i-1)}+\cdots+s_{r_i}^{(i-1)}=l$. Setting $x_k^{(l)}=0$ and $F_{j,\nu}^{(r_j)}=0$, as in Definition 5.8, amounts to impose the conditions

$$a_j \geq \langle \nu, \gamma_i \rangle$$

$$b_j \geq \nu_1$$

$$c_j \geq \nu_2$$

$$s_j^{(k)} \geq \langle \nu, \gamma_{k+1} \rangle$$

Then, the first type of *monomials* have order

$$l = a_1 + \cdots + a_{n_i} \geq n_i \langle \nu, \gamma_i \rangle$$

while the second type of *monomials* have order

$$\begin{aligned} l &= b_1 + \cdots + b_{\alpha_1} + \cdots + \cdots + s_1^{(i-1)} + \cdots + s_{r_i}^{(i-1)} \\ &\geq \alpha_1 \nu_1 + \alpha_2 \nu_2 + r_1 \langle \nu, \gamma \rangle + \cdots + r_i \langle \nu, \gamma_i \rangle \\ &\geq \langle \nu, n_i \gamma_i \rangle \end{aligned}$$

Hence we are left with the *monomials* of order $\geq \langle \nu, n_i\gamma_i \rangle$. Therefore $F_{i,\nu}^{(l)}=0$ for $0 \leq l < \langle \nu, n_i\gamma_i\rangle$ as claimed.

The expression of $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}$ for $i<i(\nu)$ follows since these are the *monomials* of order exactly $\langle \nu, n_i\gamma_i\rangle$. For $i=i(\nu)$ we have to notice that $\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle \in \mathbb{Z}$ but $\langle \nu, \gamma_{i(\nu)}\rangle \notin \mathbb{Z}$, and since in $F_{i(\nu),\nu}^{(\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle)}$ we have to set $F_{i(\nu)-1,\nu}^{(r)}=0$ for $0 \leq r < \langle \nu, \gamma_{i(\nu)}\rangle$, the *term* $f_{i-1}^{n_i}$ does not contribute at level $\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle$, because $n_{i(\nu)}\lceil \langle \nu, \gamma_{i(\nu)}\rangle \rceil > \langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle$ (where $\lceil x \rceil$ denotes the smallest integer bigger or equal than x).

The special form of the polynomial $F_{i(\nu),\nu}^{(\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle)}$ as a *monomial* times a unit, is proved with the same kind of arguments. Notice that the formula for $G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}$ still holds for $i=i(\nu)$, and it is straightforward to prove that for any term appearing in $G_{i(\nu),\nu}^{(\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle)}$ we must have $r_{i(\nu)}=0$. Suppose now that $\nu \in \rho_1$ (the case $\nu \in \rho_2$ is completely analogous), then again with the same kind of arguments as before, and using the condition

$$\langle \nu, (\alpha_1, \alpha_2) + r_1\gamma_1 + \cdots + r_{i(\nu)-1}\gamma_{i(\nu)-1} \rangle = \langle \nu, n_{i(\nu)}\gamma_{i(\nu)} \rangle$$

we can prove that

$$\alpha_1 > \alpha_1^{(i(\nu))}$$

$$r_j = r_j^{(i(\nu))}, \quad \text{for } 1 \leq j \leq i(\nu)-1$$

and the result follows.

Finally, we have to prove the claim for $F_{i(\nu)+l,\nu}^{(\langle \nu, n_{i(\nu)+l}\gamma_{i(\nu)+l}\rangle)}$ for $i(\nu) < l < i(\nu) + r(\nu)$ when $\nu \in \rho_1 \cup \rho_2$ and $r(\nu) > 0$. Notice that the condition defining $r(\nu)$ is equivalent to the set of $r(\nu)+1$ conditions

$$\begin{aligned}
&\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle = \langle \nu, \gamma_{i(\nu)+1}\rangle \\
&\langle \nu, n_{i(\nu)+1}\gamma_{i(\nu)+1}\rangle = \langle \nu, \gamma_{i(\nu)+2}\rangle \\
&\qquad\qquad \vdots \\
&\langle \nu, n_{i(\nu)+r(\nu)-1}\gamma_{i(\nu)+r(\nu)-1}\rangle = \langle \nu, \gamma_{i(\nu)+r(\nu)}\rangle \\
&\langle \nu, n_{i(\nu)+r(\nu)}\gamma_{i(\nu)+r(\nu)}\rangle < \langle \nu, \gamma_{i(\nu)+r(\nu)+1}\rangle
\end{aligned}$$

and therefore $\langle \nu, \gamma_{i(\nu)+l}\rangle \in \mathbb{Z}_{>0}$ for $1 \leq l \leq r(\nu)$, even though $\nu \notin N_{i(\nu)+l}$. And the proof goes as in the case $i < i(\nu)$. $\square$

Corollary 5.11. *In the same spirit of Remark 5.3 we have that, for $r > 0$ and $1 \leq i \leq g$, $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r)}$ is linear in*

$$x_1^{(\nu_1+r)}, x_2^{(\nu_2+r)}, z^{(\langle \nu,\gamma_1\rangle + r)}, F_1^{(\langle \nu,\gamma_2\rangle + r)}, \ldots, F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle + r)}$$

Proposition 5.12. *Given $\nu \in \sigma_{Sing} \cap N_0$, for $0 \leq l \leq \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle$, we have that,*

$$F^{(l)} \equiv \begin{cases} F_{j(l,\nu),\nu}^{(\frac{l}{e_{j(l,\nu)}})^{e_{j(l,\nu)}}} & \textit{mod } J_{l-1}^{\nu} \quad \textit{if } \; l \equiv 0 \textit{ mod } e_{j(l,\nu)} \\ \\ 0 & \textit{mod } J_{l-1}^{\nu} \quad \textit{otherwise} \end{cases}$$

Before proving this result we deduce an interesting consequence, where we give a smaller set of generators of the ideal J_m^{ν}.

Corollary 5.13. *Given $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$ such that $m \leq \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle$, we have that*

$$J_m^{\nu} = \left(J_{-1}^{\nu}, F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r_i)}\right)_{0 \leq i \leq j(m,\nu)},$$

for $0 \leq r_i < \langle \nu, \gamma_{i+1} - n_i\gamma_i\rangle$ if $0 \leq i < j(m,\nu)$, and for $i = j(m,\nu)$ and $0 \leq r_{j(m,\nu)} \leq [\frac{m - \langle \nu, e_{j(m,\nu)-1}\gamma_{j(m,\nu)}\rangle}{e_{j(m,\nu)}}]$.

Proof. By Proposition 5.12 we have that $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r_i)} \in J_{e_i(\langle \nu, n_i\gamma_i\rangle + r_i)}^{\nu}$. Since, by definition, $J_{l-1}^{\nu} \subseteq J_l^{\nu}$, it is enough to notice that for $0 \leq i < j(m,\nu)$ and $0 \leq r_i < \langle \nu, \gamma_{i+1} - n_i\gamma_i\rangle$, and for $i = j(m,\nu)$ and $0 \leq r_{j(m,\nu)} \leq [\frac{m - \langle \nu, e_{j(m,\nu)-1}\gamma_{j(m,\nu)}\rangle}{e_{j(m,\nu)}}]$ we have that $e_i(\langle \nu, n_i\gamma_i\rangle + r_i) \leq m$. $\square$

If we consider the analogous definition of J_m^ν and D_m^ν for each of the approximated roots f_j (which are q.o. themselves) and the corresponding surfaces $X^{(j)}$ (see Definition 3.4), then we can define the sets $D_{j,m}^\nu$, and we have the following result, which is a consequence of Proposition 5.12 and can be seen as its geometric counterpart.

Proposition 5.14. *For $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$ such that $m \leq \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)} \rangle$, we have that*

$$D_m^\nu = \left(\pi^a_{m,[\frac{m}{e_j}]}\right)^{-1}\left(D^\nu_{j,[\frac{m}{e_j}]}\right)$$

where $j = j(m,\nu)$, for $q > p$, $\pi^a_{q,p} : \mathbb{A}^3_q \longrightarrow \mathbb{A}^3_p$ is the projection on the jet schemes of the affine ambient space.

Proof. It follows by Proposition 5.12 and the fact that if $j(m,\nu) = j$ then $\langle \nu, n_j\gamma_j \rangle \leq \frac{m}{e_j} < \langle \nu, \gamma_{j+1} \rangle$. $\square$

Hence, for $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$ with $m \leq \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)} \rangle$, if $j(m,\nu) = j$, the geometry of C_m^ν is determined by the geometry of the j-th semi-root.

Proof of Proposition 5.12. Note that we have $j(l,\nu) \leq i(\nu)$, and then $\langle \nu, n_i\gamma_i \rangle \in \mathbb{Z}$ for $1 \leq i \leq j(l,\nu)$.

• We start by dealing with the case $\nu \notin \rho_1 \cup \rho_2$, which is the easiest. In this case we have for any $0 \leq i < g$

$$\langle \nu, n_i\gamma_i \rangle < \langle \nu, \gamma_{i+1} \rangle.$$

We proceed by induction on l. For $l = 0$ we have

$$F^{(0)} = {F_0^{(0)}}^n + \sum_{(i,j)+k\gamma_1 \geq n\gamma_1} d_{ijk} {x_1^{(0)}}^i {x_2^{(0)}}^j {F_0^{(0)}}^k$$

since $f = z^n + \sum_{(i,j)+k\gamma_1 \geq n\gamma_1} d_{ijk} x_1^i x_2^j z^k$. We have $\nu_1, \nu_2 > 0$, and in the previous expansion of f we have that $(i,j) \neq (0,0)$, since $k < n$. Therefore we deduce that

$$F^{(0)} \equiv {F_{0,\nu}^{(0)}}^n \mod J_{-1}^\nu.$$

Recall that $F_{0,\nu}^{(l)} = F_0^{(l)}$ and that $J_{-1}^\nu = \left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}\right)$. As a consequence $F_0^{(0)} \in J_0^\nu \subseteq J_i^\nu$ for any $i \geq 0$, and therefore for any $\gamma(t) \in D_i^\nu$ with $i \geq 0$, $\mathrm{ord}_t\big(f_0 \circ \gamma(t)\big) > 0$.

Suppose that the claim is true for $F^{(0)}, \ldots, F^{(l)}$. Then, by induction hypothesis, for any $i \geq l$ we have

$$F_{s,\nu}^{(\langle \nu, n_s\gamma_s \rangle)}, \ldots, F_{s,\nu}^{(\langle \nu, \gamma_{s+1} \rangle - 1)} \in J_i^\nu, \text{ for } 0 \leq s < j$$

$$F_{j,\nu}^{(\langle \nu, n_j\gamma_j \rangle)}, \ldots, F_{j,\nu}^{(\langle \nu, n_j\gamma_j \rangle + r)} \in J_i^\nu$$

with $r=[\frac{l-\langle\nu,e_{j-1}\gamma_j\rangle}{e_j}]$ and $j=j(l,\nu)$. By (14) the same holds for $F_s^{(l)}$, and we deduce that for any $\gamma(t)\in D_i^{\nu}$ with $i\geq l$,

$$\mathrm{ord}_t\big(f_s\circ\gamma(t)\big)\geq\langle\nu,\gamma_{s+1}\rangle,\ \text{for } 0\leq s<j,$$

$$\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)>\langle\nu,n_j\gamma_j\rangle+[\tfrac{l-\langle\nu,e_{j-1}\gamma_j\rangle}{e_j}]=[\tfrac{l}{e_j}]$$

where $j=j(l,\nu)$. The last equality implies that $\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\geq\frac{l+1}{e_j}$. There are two cases:

(i) If $j(l+1,\nu)=j(l,\nu)=j$, i.e. $\langle\nu,e_{j-1}\gamma_j\rangle\leq l<l+1<\langle\nu,e_j\gamma_{j+1}\rangle$. Then $l+1=\langle\nu,e_{j-1}\gamma_j\rangle+\alpha$ with $\alpha>0$. We have two possibilities:

(a) If $l+1\equiv 0 \bmod e_j$, then we can write

$$l+1=\langle\nu,e_{j-1}\gamma_j\rangle+re_j$$

with $r>0$. By Lemma 3.12

$$f=f_j^{e_j}-d_jx_1^{\beta_1^{(j)}}x_2^{\beta_2^{(j)}}f_0^{s_1^{(j)}}\cdots f_{j-1}^{s_j^{(j)}}+\sum d_{\underline{\beta},\underline{s}}x_1^{\beta_1}x_2^{\beta_2}f_0^{s_1}\cdots f_j^{s_{j+1}},$$

and then, for any $\gamma(t)\in D_i^{\nu}$ with $i\geq l+1$ we have

$$\begin{aligned}\mathrm{ord}_t\big(f_j^{e_j}\circ\gamma(t)\big)&=e_j\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\geq l+1\\ \mathrm{ord}_t\big(d_jx_1^{\beta_1^{(j)}}x_2^{\beta_2^{(j)}}f_0^{s_1^{(j)}}\cdots f_{j-1}^{s_j^{(j)}}\circ\gamma(t)\big)&\geq\langle\nu,(\beta_1^{(j)},\beta_2^{(j)})+s_1^{(j)}\gamma_1+\cdots+s_j^{(j)}\gamma_j\rangle\\ &=\langle\nu,e_j\gamma_{j+1}\rangle>l+1\end{aligned}$$

Suppose that there exists certain coefficient $d_{\underline{\beta},\underline{s}}\neq 0$ such that

$$\mathrm{ord}_t\big(d_{\underline{\beta},\underline{s}}x_1^{\beta_1}x_2^{\beta_2}f_0^{s_1}\cdots f_j^{s_{j+1}}\circ\gamma(t)\big)\leq l+1.$$

Then

$$\begin{aligned}l+1&\geq\langle\nu,(\beta_1,\beta_2)\rangle+s_1\mathrm{ord}_t\big(f_0\circ\gamma(t)\big)+\cdots+s_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\\ &\geq\langle\nu,(\beta_1,\beta_2)+s_1\gamma_1+\cdots+s_j\gamma_j\rangle+s_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\\ &\geq\langle\nu,e_j\gamma_{j+1}\rangle-s_{j+1}\langle\nu,\gamma_{j+1}\rangle+s_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\\ &\geq\langle\nu,e_j\gamma_{j+1}\rangle-s_{j+1}\langle\nu,\gamma_{j+1}\rangle+s_{j+1}\tfrac{l+1}{e_j}\end{aligned}$$

Then $\langle\nu,(e_j-s_{j+1})\gamma_{j+1}\rangle\leq(l+1)(1-\frac{s_{j+1}}{e_j})$, and, since $s_{j+1}<e_{j+1}<e_j$, we deduce

$$\langle\nu,e_j\gamma_{j+1}\rangle\leq l+1$$

which is a contradiction. Then we have proved that

$$F^{(l+1)}\equiv F_{j,\nu}^{(\langle\nu,n_j\gamma_j\rangle+r)^{e_j}} \bmod J_l^{\nu}.$$

(b) If $l+1\not\equiv 0 \bmod e_j$, then we have

$$\mathrm{ord}_t\big(f_j^{e_j}\circ\gamma(t)\big)>l+1$$

and arguing as before we deduce that

$$F^{(l+1)} \equiv 0 \bmod J_l^\nu$$

(ii) If $j(l+1,\nu) = j(l,\nu)+1 = j+1$, i.e. $\langle \nu, e_{j-1}\gamma_j\rangle \le l < \langle \nu, e_j\gamma_{j+1}\rangle \le l+1 < \langle \nu, e_{j+1}\gamma_{j+2}\rangle$, then $l+1 = \langle \nu, e_j\gamma_{j+1}\rangle$, and $j<g$. Hence $\mathrm{ord}_t\big(f_j \circ \gamma(t)\big) \ge \langle \nu, \gamma_{j+1}\rangle$ and $l+1 \equiv 0 \bmod e_{j+1}$. By Lemma 3.11, $f_{j+1} = f_j^{n_{j+1}} - c_j x_1^{\alpha_1^{(j+1)}} x_2^{\alpha_2^{(j+1)}} f_0^{r_1^{(j+1)}} \cdots f_{j-1}^{r_j^{(j+1)}} + \sum s_{\underline{\alpha},\underline{r}} x_1^{\alpha_1} x_2^{\alpha_2} f_0^{r_1} \cdots f_j^{r_{j+1}}$. By induction hypothesis

$$\mathrm{ord}_t\big(f_j^{n_{j+1}} \circ \gamma(t)\big) \ge n_{j+1}\tfrac{l+1}{e_j} = \langle \nu, n_{j+1}\gamma_{j+1}\rangle$$

$$\mathrm{ord}_t\big(c_j x_1^{\alpha_1^{(j+1)}} x_2^{\alpha_2^{(j+1)}} f_0^{r_1^{(j+1)}} \cdots f_{j-1}^{r_j^{(j+1)}} \circ \gamma(t)\big) \ge \langle \nu, n_{j+1}\gamma_{j+1}\rangle$$

If there were $c_{\underline{\alpha},\underline{r}} \neq 0$ such that $\mathrm{ord}_t\big(x_1^{\alpha_1} x_2^{\alpha_2} f_0^{r_1} \cdots f_j^{r_{j+1}} \circ \gamma(t)\big) < \langle \nu, n_{j+1}\gamma_{j+1}\rangle$, then

$$\begin{aligned}\langle \nu, n_{j+1}\gamma_{j+1}\rangle &> \langle \nu, (\alpha_1,\alpha_2)\rangle + r_1\mathrm{ord}_t\big(f_0\circ\gamma(t)\big) + \cdots + r_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\\ &\ge \langle \nu, (\alpha_1,\alpha_2) + r_1\gamma_1 + \cdots + r_j\gamma_j\rangle + r_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\\ &> \langle \nu, n_{j+1}\gamma_{j+1}\rangle - r_{j+1}\langle \nu, \gamma_{j+1}\rangle + r_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big)\\ &\ge \langle \nu, n_{j+1}\gamma_{j+1}\rangle - r_{j+1}\langle \nu, \gamma_{j+1}\rangle + r_{j+1}\langle \nu, \gamma_{j+1}\rangle = \langle \nu, n_{j+1}\gamma_{j+1}\rangle\end{aligned}$$

which is a contradiction. Hence $\mathrm{ord}_t\big(f_{j+1}\circ\gamma(t)\big) \ge \langle \nu, n_{j+1}\gamma_{j+1}\rangle$. Now consider the expansion

$$f = f_{j+1}^{e_{j+1}} - d_{j+1} x_1^{\beta_1^{(j+1)}} x_2^{\beta_2^{(j+1)}} f_0^{s_1^{(j+1)}} \cdots f_j^{s_{j+1}^{(j+1)}} + \sum d_{\underline{\beta},\underline{s}} x_1^{\beta_1} x_2^{\beta_2} f_0^{s_1} \cdots f_{j+1}^{s_{j+2}}$$

given in Lemma 3.12. With the same argument as in the previous case we prove that $F^{(l+1)} \equiv F_{j+1,\nu}^{(\langle \nu, n_{j+1}\gamma_{j+1}\rangle)^{e_{j+1}}} \bmod J_l^\nu$.

- Now we consider the case $\nu \in \rho_1 \cup \rho_2$.

(i) If $\nu = (\nu_1, 0)$. Then we have

$$F^{(0)} \equiv {F_0^{(0)}}^n + \sum_{(0,j)+k\gamma_1 \ge n\gamma_1} d_{0jk} {x_2^{(0)}}^j {F_0^{(0)}}^k \bmod J_{-1}^\nu$$

but since the condition $(0,j)+k\gamma_1 \ge n\gamma_1$ with $k<n$ is impossible, we deduce

$$F^{(0)} \equiv {F_0^{(0)}}^n \bmod J_{-1}^\nu$$

and the proof goes as in the case $\nu \notin \rho_1\cup\rho_2$, with the difference that it might be that $j(l,\nu) = i$ while $j(l+1,\nu) > i+1$. This is because even though $\gamma_{i+1} > n_i\gamma_i$, if $\nu = (\nu_1, 0)$, we may have the equality $\langle \nu, n_i\gamma_i\rangle = \langle \nu, \gamma_{i+1}\rangle$.

(ii) If $\nu = (0, \nu_2)$, then by definition $\langle \nu, e_{g_1-1}\gamma_{g_1}\rangle = 0 < \langle \nu, e_{g_1}\gamma_{g_1+1}\rangle$, and

$$F^{(0)} \equiv {F_{g_1,\nu}^{(0)}}^{e_{g_1}} + \sum d_{\underline{\beta},\underline{s}} {x_1^{(0)}}^{\beta_1} {F_{0,\nu}^{(0)}}^{s_1} \cdots {F_{g_1,\nu}^{(0)}}^{s_{g_1+1}} \bmod J_{-1}^\nu$$

with $\langle \nu, e_{g_1}\gamma_{g_1+1}\rangle \leq \langle \nu, (\beta_1, 0) + s_1\gamma_1 + \cdots + s_{g_1+1}\gamma_{g_1+1}\rangle = s_{g_1+1}\langle \nu, \gamma_{g_1+1}\rangle$, and no matter whether s_{g_1+1} is zero or not, since $\langle \nu, \gamma_{g_1+1}\rangle \neq 0$, we deduce $e_{g_1} \leq s_{g_1+1}$, which is impossible, since $s_{g_1+1} < e_{g_1+1}$. Hence $F^{(0)} \equiv {F^{(0)}_{g_1,\nu}}^{e_{g_1}} \mod J^{\nu}_{-1}$, and the first step of induction is proved. The rest of the proof goes as the case $\nu \notin \rho_1 \cup \rho_2$ with the differences explained in the case $\nu \in \rho_1$.

□

By the congruence in (14) we deduce that for $m \in \mathbb{Z}_{>0}$, $\nu \in \sigma_{Sing} \cap [0, m]^2 \cap N_0$ and $0 \leq i \leq j(m, \nu)$,

$$\operatorname{ord}_t\big(f_i \circ \gamma(t)\big) \geq \langle \nu, n_i\gamma_i\rangle \tag{15}$$

for any $\gamma(t) \in D^{\nu}_m$. But we can be more precise, as the following result claims.

Lemma 5.15. *Given $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0, m]^2 \cap N_0$, for any m-jet $\gamma(t) \in D^{\nu}_m$,*

$$\mathit{ord}_t\big(f_i \circ \gamma(t)\big) = \langle \nu, \gamma_{i+1}\rangle, \quad \textit{for } 0 \leq i < j(m, \nu)$$

$$\mathit{ord}_t\big(f_i \circ \gamma(t)\big) > \tfrac{m}{e_i}, \quad \textit{for } j(m, \nu) \leq i \leq g$$

Proof. By (14) and Corollary 5.13 we have that for $l \in \mathbb{Z}_{>0}$

$$F^{(l)}_{i,\nu} \equiv F^{(l)}_i \mod J^{\nu}_m.$$

Hence we will use the following equivalence, for any jet $\gamma(t) \in D^{\nu}_m$ we have $\operatorname{ord}_t\big(f_i \circ \gamma(t)\big) \leq l$ if and only if $F^{(l)}_i \notin J^{\nu}_m$, or equivalently $F^{(l)}_{i,\nu} \notin J^{\nu}_m$. Note that,

$$\operatorname{ord}_t\big(f_i \circ \gamma(t)\big) \geq \langle \nu, \gamma_{i+1}\rangle, \quad \text{for } 0 \leq i < j(m, \nu).$$

Indeed, it follows by Corollary 5.13 if $\langle \nu, \gamma_{i+1} - n_i\gamma_i\rangle > 0$, and by (15) otherwise. Now we prove by induction on $i < j(m, \nu)$ that $F^{(\langle \nu, \gamma_{i+1}\rangle)}_{i,\nu} \notin J^{\nu}_m$ and hence the equality $\operatorname{ord}_t\big(f_i \circ \gamma(t)\big) = \langle \nu, \gamma_{i+1}\rangle$ follows. We can divide the part $G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}$ of $F^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}$ as $G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu} = G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}(1) + G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}(2)$, where

$$G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}(1) = \sum c_{\underline{\alpha},\underline{r}} {x_1^{(\nu_1)}}^{\alpha_1} {x_2^{(\nu_2)}}^{\alpha_2} {F^{(\langle \nu, \gamma_1\rangle)}_{0,\nu}}^{r_1} \cdots {F^{(\langle \nu, \gamma_{i-1}\rangle)}_{i-2,\nu}}^{r_{i-1}}$$

with $\langle \nu, (\alpha_1, \alpha_2) + r_1\gamma_1 + \cdots + r_{i-1}\gamma_{i-1}\rangle = \langle \nu, n_i\gamma_i\rangle$, and

$$G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}(2) = \sum c_{\underline{\alpha},\underline{r}} {x_1^{(\nu_1)}}^{\alpha_1} {x_2^{(\nu_2)}}^{\alpha_2} {F^{(\langle \nu, \gamma_1\rangle)}_{0,\nu}}^{r_1} \cdots {F^{(\langle \nu, \gamma_i\rangle)}_{i-1,\nu}}^{r_i}$$

with $\langle \nu, (\alpha_1, \alpha_2) + r_1\gamma_1 + \cdots + r_i\gamma_i\rangle = \langle \nu, n_i\gamma_i\rangle$ and $r_i \neq 0$. Then we can write

$$\begin{aligned} F^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu} &= \Big({F^{(\langle \nu, \gamma_i\rangle)}_{i-1,\nu}}^{n_i} + G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}(2)\Big) + \\ &+ \Big(-c_i {x_1^{(\nu_1)}}^{\alpha_1^{(i)}} {x_2^{(\nu_2)}}^{\alpha_2^{(i)}} {F^{(\langle \nu, \gamma_1\rangle)}_{0,\nu}}^{r_1^{(i)}} \cdots {F^{(\langle \nu, \gamma_{i-1}\rangle)}_{i-2,\nu}}^{r_{i-1}^{(i)}} + G^{(\langle \nu, n_i\gamma_i\rangle)}_{i,\nu}(1)\Big) \end{aligned}$$

where in the second part $F^{(\langle \nu, \gamma_i\rangle)}_{i-1,\nu}$ does not appear.

First step of induction. We distinguish two cases.

• If $\min\,\{1 \leq i \leq g \mid \langle \nu, n_i\gamma_i\rangle < \langle \nu, \gamma_{i+1}\rangle\} = 1$, then, by Corollary 5.13, $F_{1,\nu}^{(\langle \nu, n_1\gamma_1\rangle)} \in J_m^\nu$, i.e., the first non-monomial equation among the generators of J_m^ν is $F_{1,\nu}^{(\langle \nu, n_1\gamma_1\rangle)}$. Suppose that $F_{0,\nu}^{(\langle \nu, \gamma_1\rangle)} \in J_m^\nu$, then

$$-c_1 {x_1^{(\nu_1)}}^{\alpha_1^{(1)}} {x_2^{(\nu_2)}}^{\alpha_2^{(1)}} + G_{1,\nu}^{(\langle \nu, n_i\gamma_1\rangle)}(1) \in J_m^\nu$$

If $G_{1,\nu}^{(\langle \nu, n_1\gamma_1\rangle)}(1) = 0$ then $-c_1 {x_1^{(\nu_1)}}^{\alpha_1^{(1)}} {x_2^{(\nu_2)}}^{\alpha_2^{(1)}} \in J_m^\nu$, which is a contradiction, since it can not be a defining equation of D_m^ν. Otherwise, $\nu \in \rho_1 \cup \rho_2$ and we have the equation

$$-c_1 {x_1^{(\nu_1)}}^{\alpha_1^{(1)}} {x_2^{(\nu_2)}}^{\alpha_2^{(1)}} + \sum c_{\underline{\alpha},0} {x_1^{(\nu_1)}}^{\alpha_1} {x_2^{(\nu_2)}}^{\alpha_2} = 0$$

where $(\alpha_1, \alpha_2) > n_1\gamma_1$ and $\langle \nu, (\alpha_1, \alpha_2)\rangle = \langle \nu, n_1\gamma_1\rangle$.

(i) If $\nu \in \rho_1$, then the condition $\langle \nu, (\alpha_1, \alpha_2)\rangle = \langle \nu, n_1\gamma_1\rangle$ gives $\alpha_1 = \alpha_1^{(1)}$, and hence the equation is

$$ {x_2^{(0)}}^{\alpha_2^{(1)}} + \sum c_{\underline{\alpha},0} {x_2^{(0)}}^{\alpha_2} = 0$$

since $x_1^{(\nu_1)} \neq 0$. But this equation is invertible in $\mathbb{C}\{x_2^{(0)}\}$ and it can not be zero.

(ii) If $\nu \in \rho_2$ the same argument holds.

We have proved that $-c_1 {x_1^{(\nu_1)}}^{\alpha_1^{(1)}} {x_2^{(\nu_2)}}^{\alpha_2^{(1)}} + G_{1,\nu}^{(\langle \nu, n_1\gamma_1\rangle)}(1) \notin J_m^\nu$ and hence $F_{0,\nu}^{(\langle \nu, \gamma_1\rangle)} \notin J_m^\nu$. Therefore we have that $\mathrm{ord}_t\big(f_0 \circ \gamma(t)\big) = \langle \nu, \gamma_1\rangle$.

• If $\min\,\{1 \leq i \leq g \mid \langle \nu, n_i\gamma_i\rangle < \langle \nu, \gamma_{i+1}\rangle\} > 1$, then we have that $\nu \in \rho_1 \cup \rho_2$. Denoting $m_0(\nu) = \min\,\{1 \leq i \leq g \mid \langle \nu, n_i\gamma_i\rangle < \langle \nu, \gamma_{i+1}\rangle\}$ we have that $F_{m_0(\nu),\nu}^{(\langle \nu, n_{m_0(\nu)}\gamma_{m_0(\nu)}\rangle)}$ is the first non-monomial equation among the generators of J_m^ν. Moreover for $1 \leq l < m_0(\nu)$

$$F_{l,\nu}^{(\langle \nu, \gamma_{l+1}\rangle)} = F_{l,\nu}^{(\langle \nu, n_l\gamma_l\rangle)}$$

and hence we can write $F_{m_0(\nu),\nu}^{(\langle \nu, n_{m_0(\nu)}\gamma_{m_0(\nu)}\rangle)}$ as a function on $x_1^{(\nu_1)}, x_2^{(\nu_2)}$ and $F_{0,\nu}^{(\langle \nu, \gamma_1\rangle)}$. Suppose that $F_{0,\nu}^{(\langle \nu, \gamma_1\rangle)} \in J_m^\nu$, then we have an equation

$$G(x_1^{(\nu_1)}, x_2^{(\nu_2)}) = 0$$

where the monomials of G are of the form ${x_1^{(\nu_1)}}^{\alpha_1} {x_2^{(\nu_2)}}^{\alpha_2}$ with $\langle \nu, (\alpha_1, \alpha_2)\rangle = \langle \nu, n_{m_0(\nu)}\gamma_{n(\nu)}\rangle = n_{m_0(\nu)} \cdots n_l \langle \nu, \gamma_l\rangle$ for $1 \leq l \leq m_0(\nu)$.

If $\nu = (\nu_1, 0) \in \rho_1$, then $\nu_1\alpha_1 = \nu_1 n_{m_0(\nu)} \cdots n_l \gamma_l^{(1)}$ and hence α_1 is fixed in all monomials of G. Then, since $x_1^{(\nu_1)} \neq 0$, we can write the equation as an equation in $x_2^{(0)}$, which is invertible in $\mathbb{C}\{x_2^{(0)}\}$ and hence not zero.

If $\nu = (0, \nu_2) \in \rho_2$ the proof is analogous.

Suppose that $\mathrm{ord}_t\big(f_i \circ \gamma(t)\big) = \langle \nu, \gamma_{i+1}\rangle$ for $0 \leq i < j$ and we will prove it for $j < j(m,\nu)$. We distinguish two cases.

• If $\langle \nu, \gamma_{j+2}\rangle > \langle \nu, n_{j+1}\gamma_{j+1}\rangle$, then by Corollary 5.13 we have $F_{j+1,\nu}^{(\langle \nu, n_{j+1}\gamma_{j+1}\rangle)} \in J_m^\nu$. Suppose that $F_{j,\nu}^{(\langle \nu,\gamma_{j+1}\rangle)} \in J_m^\nu$, then

$$-c_{j+1} x_1^{(\nu_1)^{\alpha_1^{(j+1)}}} x_2^{(\nu_2)^{\alpha_2^{(j+1)}}} F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)^{r_1^{(j+1)}}} \cdots F_{j-1,\nu}^{(\langle \nu,\gamma_j\rangle)^{r_j^{(j+1)}}} + G_{j+1,\nu}^{(\langle \nu, n_{j+1}\gamma_{j+1}\rangle)}(1) \in J_m^\nu.$$

This is a contradiction if $G_{j+1,\nu}^{(\langle \nu, n_{j+1}\gamma_{j+1}\rangle)}(1) = 0$. Otherwise we have that $\nu \in \rho_1 \cup \rho_2$.

If $\nu = (\nu_1, 0) \in \rho_1$, then

$$G_{j+1,\nu}^{(\langle \nu, n_{j+1}\gamma_{j+1}\rangle)}(1) = \sum c_{\underline{\alpha},\underline{r}} x_1^{(\nu_1)^{\alpha_1}} x_2^{(0)^{\alpha_2}} F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)^{r_1}} \cdots F_{j-1,\nu}^{(\langle \nu,\gamma_j\rangle)^{r_j}}$$

with

$$\begin{aligned} &\alpha_1 + r_1\gamma_1^{(1)} + \cdots + r_j\gamma_j^{(1)} = n_{j+1}\gamma_{j+1}^{(1)} \\ &\alpha_2 + r_1\gamma_1^{(2)} + \cdots + r_j\gamma_j^{(2)} > n_{j+1}\gamma_{j+1}^{(2)} \end{aligned} \tag{16}$$

Recall that $n_{j+1}\gamma_{j+1} = (\alpha_1^{(j+1)}, \alpha_2^{(j+1)}) + r_1^{(j+1)}\gamma_1 + \cdots + r_j^{(j+1)}\gamma_j$, where the integers $(\alpha_1^{(j+1)}, \alpha_2^{(j+1)}), r_1^{(j+1)}, \ldots, r_j^{(j+1)}$ are unique by Lemma 3.6. Then we deduce from (16) that

$$\alpha_1 = \alpha_1^{(j+1)}$$

$$r_l = r_l^{(j+1)} \text{ for } 1 \leq l \leq j$$

$$\alpha_2 > \alpha_2^{(j+1)}$$

and we are done, since $-c_{j+1} x_1^{(\nu_1)^{\alpha_1^{(j+1)}}} x_2^{(0)^{\alpha_2^{(j+1)}}} F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)^{r_1^{(j+1)}}} \cdots F_{j-1,\nu}^{(\langle \nu,\gamma_j\rangle)^{r_j^{(j+1)}}} + G_{j+1,\nu}^{(\langle \nu, n_{j+1}\gamma_{j+1}\rangle)}(1)$ can be written as

$$x_1^{(\nu_1)^{\alpha_1^{(j+1)}}} x_2^{(0)^{\alpha_2^{(j+1)}}} F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)^{r_1^{(j+1)}}} \cdots F_{j-1,\nu}^{(\langle \nu,\gamma_j\rangle)^{r_j^{(j+1)}}} P(x_2^{(0)})$$

which is never zero, by induction hypothesis, and by the fact that $P(0) = -c_{j+1}$ and since we consider germs of quasi-ordinary singularities $P(x_2^{(0)})$ is invertible in $\mathbb{C}\{x_2^{(0)}\}$.

If $\nu = (0, \nu_2) \in \rho_2$ the proof is completely analogous.

• If $\langle \nu, \gamma_{j+2}\rangle = \langle \nu, n_{j+1}\gamma_{j+1}\rangle$, we are in the case $\nu \in \rho_1 \cup \rho_2$, and analogously as we did with $m_0(\nu)$ in the first step of induction, we define the integer $m_{j+2}(\nu) = \min\{j+2 < i \leq g \mid \langle \nu, n_i\gamma_i\rangle = \langle \nu, \gamma_{i+1}\rangle\}$. Then, by Corollary 5.13 we have $F_{m_{j+2}(\nu),\nu}^{(\langle \nu, n_{m_{j+2}(\nu)}\gamma_{m_{j+2}(\nu)}\rangle)} \in J_m^\nu$, and we can write it as a polynomial in $x_1^{(\nu_1)}, x_2^{(\nu_2)}$, $F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)}, \ldots, F_{j,\nu}^{(\langle \nu,\gamma_{j+1}\rangle)}$. Suppose that $F_{j,\nu}^{(\langle \nu,\gamma_{j+1}\rangle)} \in J_m^\nu$. Then the monomials in

$F_{j,\nu}^{(\langle\nu,\gamma_{j+1}\rangle)}$ are of the form

$$x_1^{(\nu_1)^{\alpha_1}} x_2^{(\nu_2)^{\alpha_2}} F_{0,\nu}^{(\langle\nu,\gamma_1\rangle)^{r_1}} \cdots F_{j-1,\nu}^{(\langle\nu,\gamma_j\rangle)^{r_j}}$$

with the conditions

$$\begin{aligned} \langle\nu,(\alpha_1\alpha_2)+r_1\gamma_1+\cdots+r_j\gamma_j\rangle &= \langle\nu, n_{m_{j+2}(\nu)}\gamma_{m_{j+2}(\nu)}\rangle \\ &= \langle\nu, n_{m_{j+2}(\nu)-1} n_{m_{j+2}(\nu)}\gamma_{m_{j+2}(\nu)-1}\rangle \\ &\vdots \\ &= \langle\nu, n_{j+1}\cdots n_{m_{j+2}(\nu)}\gamma_{j+1}\rangle. \end{aligned}$$

If $\nu=(\nu_1,0)\in\rho_1$, we deduce that $\alpha_1, r_1,\ldots,r_j$ are fixed and α_2 varies. Hence we have that

$$x_1^{(\nu_1)^{\alpha_1}} x_2^{(0)^{\alpha_2}} F_{0,\nu}^{(\langle\nu,\gamma_1\rangle)^{r_1}} \cdots F_{j-1,\nu}^{(\langle\nu,\gamma_j\rangle)^{r_j}} P(x_2^{(0)})=0$$

is one defining equation of D_m^{ν}, and this is a contradiction.

The case $\nu\in\rho_2$ is analogous.

Now we prove the second part of the statement of the Lemma.

By Corollary 5.13, for any $\gamma(t)\in D_m^{\nu}$, $\mathrm{ord}_t\big(f_{j(m,\nu)}\circ\gamma(t)\big)>\frac{m}{e_{j(m,\nu)}}$. To prove the claim for $j(m,\nu)+1$ consider the expansion, denoting $j(m,\nu)$ by j to simplify notation,

$$f_{j+1}=f_j^{n_{j+1}}-c_{j+1}x_1^{\alpha_1^{(j+1)}}x_2^{\alpha_2^{(j+1)}}f_0^{r_1^{(j+1)}}\cdots f_{j-1}^{r_j^{(j+1)}}+\sum c_{\underline{\alpha},\underline{r}}x_1^{\alpha_1}x_2^{\alpha_2}f_0^{r_1}\cdots f_j^{r_{j+1}}$$

with $n_{j+1}\gamma_{j+1}=(\alpha_1^{(j+1)},\alpha_2^{(j+1)})+r_1^{(j+1)}\gamma_1+\cdots+r_j^{(j+1)}\gamma_j<(\alpha_1,\alpha_2)+r_1\gamma_1+\cdots+r_{j+1}\gamma_{j+1}$. Then,

$$\mathrm{ord}_t\big(f_j^{n_{j+1}}\circ\gamma(t)\big) > \tfrac{m}{e_{j+1}}$$

$$\begin{aligned} \mathrm{ord}_t\big(x_1^{\alpha_1^{(j+1)}}x_2^{\alpha_2^{(j+1)}}f_0^{r_1^{(j+1)}}\cdots f_{j-1}^{r_j^{(j+1)}}\circ\gamma(t)\big) &= \langle\nu,(\alpha_1^{(j+1)},\alpha_2^{(j+1)})+\textstyle\sum_{i=1}^{j} r_i^{(j+1)}\gamma_i\rangle \\ &= \langle\nu, n_{j+1}\gamma_{j+1}\rangle \\ &> \tfrac{m}{e_{j+1}} \end{aligned}$$

Suppose that there exists $c_{\underline{\alpha},\underline{r}}\neq 0$ such that $\mathrm{ord}_t\big(c_{\underline{\alpha},\underline{r}}x_1^{\alpha_1}x_2^{\alpha_2}f_0^{r_1}\cdots f_j^{r_{j+1}}\circ\gamma(t)\big)\leq\frac{m}{e_{j+1}}$, then

$$\begin{aligned} \tfrac{m}{e_{j+1}} &\geq \langle\nu,(\alpha_1,\alpha_2)+r_1\gamma_1+\cdots+r_j\gamma_j\rangle+r_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big) \\ &> \langle\nu,(n_{j+1}-r_{j+1})\gamma_{j+1}\rangle+r_{j+1}\mathrm{ord}_t\big(f_j\circ\gamma(t)\big) \\ &> \langle\nu,(n_{j+1}-r_{j+1})\gamma_{j+1}\rangle+r_{j+1}\tfrac{m}{e_j} \end{aligned}$$

Therefore $\langle\nu,(n_{j+1}-r_{j+1})\gamma_{j+1}\rangle<(1-\frac{r_{j+1}}{n_{j+1}})\frac{m}{e_{j+1}}$, and since $r_{j+1}<n_{j+1}$, we deduce that $\langle\nu,\gamma_{j+1}\rangle<\frac{m}{e_j}$, which contradicts the definition of $j(m,\nu)$. Then we have proved that

$$\mathrm{ord}_t\big(f_{j+1}\circ\gamma(t)\big)>\frac{m}{e_{j+1}}$$

Recursively we prove the rest of the inequalities for $j(m,\nu)+1<k\leq g$. $\square$

Corollary 5.16. *For $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$, such that $m < \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle$, we have the following. If $1 \leq j(m,\nu) \leq g_1$, then we have*

$$V(F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)})_{1\leq i\leq j(m,\nu),\langle \nu,\gamma_{i+1}-n_i\gamma_i\rangle>0} \cap D(x_1^{(\nu_1)}) \subset \bigcap_{i=0}^{j(m,\nu)-1} D\big(F_{i,\nu}^{(\langle \nu,\gamma_{i+1}\rangle)}\big),$$

while if $g_1 < j(m,\nu) \leq g$ we have

$$V(F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)})_{1\leq i\leq j(m,\nu),\langle \nu,\gamma_{i+1}-n_i\gamma_i\rangle>0} \cap D(x_1^{(\nu_1)}) \cap D(x_2^{(\nu_2)}) \subset \bigcap_{i=0}^{j(m,\nu)-1} D\big(F_{i,\nu}^{(\langle \nu,\gamma_{i+1}\rangle)}\big).$$

Example 5.17. *Let X be a quasi-ordinary surface defined by $f = ((z^2 - x_1^3x_2^2)^2 - x_1^6x_2^4z)^3 - x_1^{23}x_2^{14}z$. The generators of the semigroup Γ are $\gamma_1 = (\frac{3}{2},1)$, $\gamma_2 = (\frac{15}{4},\frac{5}{2})$ and $\gamma_3 = (\frac{49}{6},5)$. Notice that $\nu = (0,3) \notin N_2$, and $\langle \nu, e_1\gamma_2\rangle = \langle \nu, e_2\gamma_3\rangle = 45$. At level $m = 45$ we have the set*

$$D_{45}^{(0,3)} = V(x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, F_{1,\nu}^{(6)}, F_{1,\nu}^{(7)}, F_{3,\nu}^{(45)}) \cap D(x_1^{(0)}) \cap D(x_2^{(3)}),$$

where

$$\begin{aligned} F_{1,\nu}^{(6)} &= {z^{(3)}}^2 - {x_1^{(0)}}^3 {x_2^{(3)}}^2 \\ F_{3,\nu}^{(45)} &= {F_{2,\nu}^{(15)}}^3 - {x_1^{(0)}}^{23} {x_2^{(3)}}^{14} z^{(3)} \\ &= ({x_1^{(0)}}^6 {x_2^{(3)}}^4 z^{(3)})^3 - {x_1^{(0)}}^{23} {x_2^{(3)}}^{14} z^{(3)} \end{aligned}$$

Since $D_{45}^{(0,3)} \subset D(x_1^{(0)}) \cap D(x_2^{(3)}) \cap D(z^{(3)})$, we have that $F_{3,\nu}^{(45)} = 0$ if and only if ${z^{(3)}}^2 - {x_1^{(0)}}^5 {x_2^{(3)}}^2 = 0$. This equation, together with $F_{1,\nu}^{(6)} = 0$, implies ${x_1^{(0)}}^3 {x_2^{(3)}}^2 ({x_1^{(0)}}^2 - 1) = 0$, and since ${x_1^{(0)}}^2 - 1$ is a unit in $\mathbb{C}\{x_1^{(0)}\}$, we deduce $D_{45}^{(0,3)} = \emptyset$.

This example illustrates the fact that we are looking at jet schemes of a germ of quasi-ordinary singularity, instead of jet schemes of the whole affine surface. If we looked at the whole surface there would be other irreducible components that we do not consider here. This is expectable because the components we consider are determined by the invariants of the topological type at the origin, so they describe only what happens in a small neighbourhood of the origin. Actually the other components that may appear when looking at the whole affine surface, will project on closed points, different from the origin, of the singular locus.

Lemma 5.18. *For $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$, we have $D_m^\nu = \emptyset$ if and only if $m \geq \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle$ and $\sigma_{Reg,j'(m,\nu)} \neq \rho_1 \cup \rho_2$.*

Proof. Notice that, by definition, $D_m^\nu \neq \emptyset$ if $\sigma_{Reg,j'(m,\nu)} = \rho_1 \cup \rho_2$. Moreover, if $m < \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle$, then we will prove in Proposition 5.25 that D_m^ν is non-empty.

For the other implication, suppose that $m \geq \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle$ and $\sigma_{Reg,j'(m,\nu)} \neq \rho_1 \cup \rho_2$. We have by Proposition 5.12 that $F_{i(\nu),\nu}^{(\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle)} \in J_m^\nu$, and by Lemma 5.9 that

$$\begin{aligned} F_{i(\nu),\nu}^{(\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle)} &= -c_{i(\nu)} {x_1^{(\nu_1)}}^{\alpha_1^{(i(\nu))}} {x_2^{(\nu_2)}}^{\alpha_2^{(i(\nu))}} {F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)}}^{r_1^{(i(\nu))}} \cdots {F_{i(\nu)-2}^{(\langle \nu,\gamma_{i(\nu)-1}\rangle)}}^{r_{i(\nu)-1}^{(i(\nu))}} + \\ &\quad + G_{i(\nu),\nu}^{(\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle)} \\ &= {x_1^{(\nu_1)}}^{\alpha_1^{(i(\nu))}} {x_2^{(\nu_2)}}^{\alpha_2^{(i(\nu))}} {F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)}}^{r_1^{(i(\nu))}} \cdots {F_{i(\nu)-2}^{(\langle \nu,\gamma_{i(\nu)-1}\rangle)}}^{r_{i(\nu)-1}^{(i(\nu))}} \cdot U \end{aligned}$$

where U is a unit in R_ν. Now, applying Corollary 5.16 to ν and $m = \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle - 1$ we deduce that

$$D_m^\nu \subset V\big(F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}\big)_{1\leq i \leq j(m,\nu), \langle \nu, \gamma_{i+1} - n_i\gamma_i\rangle > 0} \cap D(M) \subset \bigcap_{i=0}^{j(m,\nu)-1} D\big(F_{i,\nu}^{(\langle \nu,\gamma_{i+1}\rangle)}\big)$$

where $j(m,\nu) = i(\nu) - 1$ and $M = x_1^{(\nu_1)}$ if $j(m,\nu) \leq g_1$ and $M = x_1^{(\nu_1)}x_2^{(\nu_2)}$ otherwise. Then $F_{i(\nu),\nu}^{(\langle \nu, n_{i(\nu)}\gamma_{i(\nu)}\rangle)}$ can not be zero (note that $\alpha_2^{(i(\nu))} = 0$ if $i(\nu) \leq g_1$) and therefore $D_m^\nu = \emptyset$. $\square$

Remark 5.19. *For $m \in \mathbb{Z}_{>0}$ and $\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0$, if $D_m^\nu \neq \emptyset$ then $\nu \in N_{j(m,\nu)}$.*

Definition 5.20. *Given $m \in \mathbb{Z}_{>0}$ we define the set:*

$$L_m = \{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0 \mid m < \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)}\rangle\}$$

and for $0 \leq j \leq g$

$$L_m^{(j)} = \{\nu \in L_m \mid j(m,\nu) = j\}$$

Remark 5.21. *By Corollary 5.16, if $\nu \in L_m^{(0)}$, the ideal J_m^ν is monomial, more precisely*

$$J_m^\nu = \left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu_2-1)}, z^{(0)}, \ldots, z^{([m/n])}\right).$$

Lemma 5.22. *For $m \in \mathbb{Z}_{>0}$, we have that $L_m \neq \emptyset$, and*

$$\pi_m^{-1}(X_{Sing}) = \bigcup_{\nu \in L_m} C_m^\nu.$$

Proof. It follows by Lemma 5.7 and Lemma 5.18, since we have that

$$\bigcup_{\nu \in \sigma_{Sing} \cap [0,m]^2 \cap N_0} D_m^\nu = \bigcup_{\nu \in L_m} D_m^\nu$$

where the unions are finite, and therefore it is enough to take the Zariski closure.
□

Notation 5.23. *For $0 \leq i < g$, we denote $k_i(\nu) = \langle \nu, \gamma_{i+1} - n_i\gamma_i \rangle$, or simply k_i if ν is clear in the context.*

Remark 5.24. *For $m \in \mathbb{Z}_{>0}$ and $\nu \in L_m$, we have that $k_{j(m,\nu)}(\nu) > 0$ and $k_{j'(m,\nu)}(\nu) > 0$.*

Now we can prove the irreducibility of the sets C_m^ν.

Proposition 5.25. *For any $m \in \mathbb{Z}_{>0}$ and $\nu \in L_m$, the set C_m^ν is irreducible and*

$$Codim(C_m^\nu) = \nu_1 + \nu_2 + \sum_{k=0}^{j(m,\nu)-1} \langle \nu, \gamma_{k+1} - n_k\gamma_k \rangle + \left[\frac{m}{e_{j(m,\nu)}}\right] - \langle \nu, n_{j(m,\nu)}\gamma_{j(m,\nu)} \rangle + 1$$

Proof. We will denote along this proof $j(m,\nu)$ just by j.

The irreducibility follows by Proposition 2.4 and the definition of C_m^ν. Let us prove the formula of the codimension.

• If $\nu \in L_m^{(0)}$ it follows from Remark 5.21 that $C_m^\nu = V\left(J_{-1}^\nu, z^{(0)}, \ldots, z^{([m/n])}\right)$. The claim about the codimension follows trivially.

• If $\nu \in L_m^{(j)}$ with $j > 0$, then

$$J_m^\nu = \left(x_1^{(0)}, \ldots, x_1^{(\nu_1-1)}, x_2^{(0)}, \ldots, x_2^{(\nu-1)}, F_{i,\nu}^{(\langle \nu, n_i\gamma_i \rangle + r_i)}\right)_{0 \leq i \leq j}$$

for $0 \leq r_i < k_i(\nu)$ if $i < j$ and $0 \leq r_j \leq [m/e_j] - \langle \nu, n_j\gamma_j \rangle$. It is not a monomial ideal. We divide the set of non-monomial generators in two sets:

$$\mathcal{C}_1 = \left\{F_{i,\nu}^{(\langle \nu, n_i\gamma_i \rangle)}\right\}_{1 \leq i \leq j,\ k_i > 0}$$

$$\mathcal{C}_2 = \left\{F_{i,\nu}^{(\langle \nu, n_i\gamma_i \rangle + r)}\right\}_{(i,r) \in A_2}$$

where $A_2 = \{(i,r) \mid 1 \leq i < j,\ 0 < r < k_i\} \cup \{(j,r) \mid 0 < r \leq [\frac{m}{e_j}] - \langle \nu, n_j\gamma_j \rangle\}$.

We claim that $V(\langle \mathcal{C}_1 \rangle) \simeq Z^{\Gamma_m^\nu}$, the toric variety defined by the semigroup Γ_m^ν generated by

$$\{\gamma_i\}_{1 \leq i \leq j(m,\nu),\ k_i > 0}$$

If $\nu \notin \rho_1 \cup \rho_2$ then any $k_i > 0$ and hence $\Gamma_m^\nu = \Gamma_{j(m,\nu)}$. Then $V(\langle \mathcal{C}_1 \rangle)$ is isomorphic to the monomial variety associated to $X^{(j(m,\nu))}$ (see Definition 3.5).

To deal with $\mathcal{C}_2$ we need to study the elements $F_{i,\nu}^{(\langle \nu, n_i\gamma_i \rangle + r_i)}$ with $r_i > 0$ (note that we do not describe those in Lemma 5.9). For i such that $k_i(\nu) > 0$ we know that $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)} \in J_m^\nu$, and we can write, as we did in the proof of Lemma 5.15,

$$\begin{aligned} F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)} = & \left(F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle)^{n_i}} + G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}(2)\right) + \\ & + \left(-c_i x_1^{(\nu_1)^{\alpha_1^{(i)}}} x_2^{(\nu_2)^{\alpha_2^{(i)}}} F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)^{r_1^{(i)}}} \cdots F_{i-2,\nu}^{(\langle \nu,\gamma_{i-1}\rangle)^{r_{i-1}^{(i)}}} + G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}(1)\right) \end{aligned}$$

where

$$G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}(1) = \sum c_{\underline{\alpha},\underline{r}} x_1^{(\nu_1)^{\alpha_1}} x_2^{(\nu_2)^{\alpha_2}} F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)^{r_1}} \cdots F_{i-2,\nu}^{(\langle \nu,\gamma_{i-1}\rangle)^{r_{i-1}}}$$

with $\langle \nu, (\alpha_1,\alpha_2) + r_1\gamma_1 + \cdots + r_{i-1}\gamma_{i-1}\rangle = \langle \nu, n_i\gamma_i\rangle$, and

$$G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}(2) = \sum c_{\underline{\alpha},\underline{r}} x_1^{(\nu_1)^{\alpha_1}} x_2^{(\nu_2)^{\alpha_2}} F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)^{r_1}} \cdots F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle)^{r_i}}$$

with $\langle \nu, (\alpha_1,\alpha_2) + r_1\gamma_1 + \cdots + r_i\gamma_i\rangle = \langle \nu, n_i\gamma_i\rangle$ and $r_i \neq 0$. Then in the second part of $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}$ the $F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle)}$ does not appear, and, by the definition of the derivation δ, that we can write

$$F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r)} = P \cdot F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle + r)} + Q$$

where $F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle + r)}$ does not appear neither in P nor in Q. Moreover we have that $P \neq 0$, or in other words, $P \notin J_m^\nu$. Indeed, in the proof of Lemma 5.15 we have showed that

$$\widetilde{F}_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)} = F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle)^{n_i}} + G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}(2) \notin J_m^\nu$$

where $\widetilde{f}_i$ denotes the part of f_i depending on f_{i-1}, i.e.,

$$\widetilde{f}_i = f_{i-1}^{n_i} + \sum_{r_i \neq 0} c_{\underline{\alpha},\underline{r}} x_1^{\alpha_1} x_2^{\alpha_2} f_0^{r_1} \cdots f_{i-1}^{r_i}.$$

Hence $\widetilde{F}_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r)} = P \cdot F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle + r)} \notin J_m^\nu$. Therefore $P \notin J_m^\nu$ and we deduce that $F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle + r)}$ appears for the first time in $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r)}$.

Now, using that $\langle \nu, \gamma_i\rangle + r = \langle \nu, n_{i-1}\gamma_{i-1}\rangle + k_{i-1}(\nu) + r$, we prove by recurrence that

$(*)$ $x_1^{(\nu_1 + k_0 + \cdots + k_{i-1} + r)}$, $z^{(\langle \nu,\gamma_1\rangle + k_0 + \cdots + k_{i-1} + r)}$ appear the first time in $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r)}$

Note that this is true also for x_2 whenever $g_1 = 0$.

Then we have proved that any $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle + r)} \in \mathcal{C}_2$ is linear with respect to at least one of the variables described in $(*)$, which appears for the first time on this equation, and with non-zero coefficient over $D(x_1^{(\nu_1)}) \cap D(x_2^{(\nu_2)})$, by Corollary 5.16.

Since any of these equations in $\mathcal{C}_2$ is *linear* in a different variable, and, by $(*)$ we have that it appears for the first time in $\mathcal{C}_2$, we deduce

$$V(J_{-1}^\nu, F_{0,\nu}^{(0)}, \ldots, F_{0,\nu}^{(\langle \nu,\gamma_1\rangle - 1)}, \mathcal{C}_2) \cap D(x_1^{(\nu_1)}) \cap D(x_2^{(\nu_2)}) \simeq \mathbb{A}^{\alpha(m,\nu)}$$

where $\alpha(m,\nu) = 3(m+1) - \nu_1 - \nu_2 - \langle \nu,\gamma_1\rangle - |A_2|$, because

$$V(J_{-1}^\nu, F_{0,\nu}^{(0)}, \ldots, F_{0,\nu}^{(\langle \nu,\gamma_1\rangle - 1)}, \mathcal{C}_2) \subseteq \mathbb{A}_m^3 \simeq \mathbb{A}^{3(m+1)}.$$

Clearly the cardinal of A_2 is $|A_2| = \sum_{m(\nu)\leq i<j,\ k_i>0}(k_i-1)+[\frac{m}{e_j}]-\langle \nu, n_j\gamma_j\rangle$. Hence

$$D_m^\nu \simeq \left(Z^{\Gamma_m^\nu} \cap D(x_1^{(\nu_1)}) \cap D(x_2^{(\nu_2)})\right) \times \mathbb{A}^{\alpha(m,\nu)}$$

The toric variety $Z^{\Gamma_m^\nu}$ is complete intersection, hence its codimension equals the cardinal of $\mathcal{C}_1$. Therefore

$$\begin{aligned}\operatorname{Codim}(C_m^\nu) &= |\mathcal{C}_1| + \nu_1 + \nu_2 + \langle \nu, \gamma_1\rangle + \textstyle\sum_{1\leq i<j,\ k_i>0}(k_i-1) + [\frac{m}{e_j}] - \langle \nu, n_j\gamma_j\rangle \\ &= \nu_1 + \nu_2 + \textstyle\sum_{0\leq i<j} k_i + [\frac{m}{e_j}] - \langle \nu, n_j\gamma_j\rangle + |\mathcal{C}_1| - |\{1\leq i<j \mid k_i>0\}|\end{aligned}$$

since $k_0 = \langle \nu, \gamma_1\rangle$. Note that $|\mathcal{C}_1| = |\{1 \leq i \leq j, k_i > 0\}| = |\{1 \leq i < j, k_i > 0\}|+1$, since $k_j > 0$ by definition of $j(m,\nu)$. Then the formula of the codimension follows.

To finish we prove the claim. For $1 \leq i \leq j(m,\nu)$ with $k_i > 0$ we have by Proposition 5.12 that $F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)} \in J_m^\nu$. Recall that in the proof of Lemma 5.15 we proved that whenever $k_i(\nu) > 0$, we have

$$-c_i {x_1^{(\nu_1)}}^{\alpha_1^{(i)}} {x_2^{(\nu_2)}}^{\alpha_2^{(i)}} {F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)}}^{r_1^{(i)}} \cdots {F_{i-2,\nu}^{(\langle \nu,\gamma_{i-1}\rangle)}}^{r_{i-1}^{(i)}} + G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}(1) \notin J_m^\nu$$

Hence we have

$${F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle)}}^{n_i} + G_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)}(2) \notin J_m^\nu$$

and since ${F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle)}}^{n_i} \notin J_m^\nu$ and $c_i {x_1^{(\nu_1)}}^{\alpha_1^{(i)}} {x_2^{(\nu_2)}}^{\alpha_2^{(i)}} {F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)}}^{r_1^{(i)}} \cdots {F_{i-2,\nu}^{(\langle \nu,\gamma_{i-1}\rangle)}}^{r_{i-1}^{(i)}} \notin J_m^\nu$ we can write

$$F_{i,\nu}^{(\langle \nu, n_i\gamma_i\rangle)} = {F_{i-1,\nu}^{(\langle \nu,\gamma_i\rangle)}}^{n_i} \cdot U_1 - c_i {x_1^{(\nu_1)}}^{\alpha_1^{(i)}} {x_2^{(\nu_2)}}^{\alpha_2^{(i)}} {F_{0,\nu}^{(\langle \nu,\gamma_1\rangle)}}^{r_1^{(i)}} \cdots {F_{i-2,\nu}^{(\langle \nu,\gamma_{i-1}\rangle)}}^{r_{i-1}^{(i)}} \cdot U_2$$

with $U_1, U_2 \neq 0$. Then we deduce that

$$V(\langle \mathcal{C}_1\rangle) \simeq V(h_i)_{1\leq i\leq j(m,\nu),\ k_i>0}$$

where $h_i = w_{i-1}^{n_i} - x_1^{\alpha_1^{(i)}} x_2^{\alpha_2^{(i)}} z^{r_1^{(i)}} w_1^{r_2^{(i)}} \cdots w_{i-2}^{r_{i-1}^{(i)}}$, with the relation $n_i\gamma_i = (\alpha_1^{(i)}, \alpha_2^{(i)}) + r_1^{(i)}\gamma_1 + \cdots + r_{i-1}^{(i)}\gamma_{i-1}$. And $V(h_i)_{1\leq i\leq j(m,\nu),\ k_i>0}$ is isomorphic to the toric variety $Z^{\Gamma_m^\nu}$. $\square$

In particular we have the following variation of the codimension of C_m^ν as m grows.

Corollary 5.26. *For $\nu \in L_m$ such that $\nu \in L_{m-1}$ we have that*

$$Codim(C_m^\nu) = \begin{cases} Codim(C_{m-1}^\nu) + 1 & \text{if } m \equiv 0 \text{ mod } e_{j(m-1,\nu)} \\ Codim(C_{m-1}^\nu) & \text{otherwise} \end{cases}$$

5.1. **Inclusions among the C_m^ν.** The collection of irreducible sets $\{C_m^\nu \mid \nu \in L_m\}$ covers $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$, but in general it is not its decomposition in irreducible components. We have to study the inclusions

$$C_m^{\nu'} \subseteq C_m^\nu \text{ for different } \nu, \nu' \in L_m.$$

We will describe a set $F_m \subset L_m$ such that $\{C_m^\nu \mid \nu \in F_m\}$ is the set of irreducible components of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$.

Proposition 5.27. *Given $m \in \mathbb{Z}_{>0}$ and $\nu, \nu' \in L_m$, if $\nu' - \nu \in \sigma_{Reg,j'(m,\nu)}$ then $C_m^{\nu'} \subseteq C_m^\nu$.*

Proof. The key point is the following observation. By (14) we deduce that for $l \in \mathbb{Z}_{>0}$ and $\nu, \nu' \in L_m$ with $\nu_k \leq \nu'_k$ for $k = 1, 2$, we have

$$F_{i,\nu}^{(l)} = F_{i,\nu'}^{(l)} + H_{i,\nu'-\nu}^{(l)}$$

where $H_{i,\nu'-\nu}^{(l)} \in \left(x_k^{(\nu_k)}, \ldots, x_k^{(\nu'_k-1)}, F_{j,\nu}^{(s_j)}\right)_{k=1,2,0\leq j<i,\langle\nu,\gamma_{j+1}\rangle\leq s_j<\langle\nu',\gamma_{j+1}\rangle}$. Note that by definition, equivalently we have that

$$H_{i,\nu'-\nu}^{(l)} \in \left(x_k^{(\nu_k)}, \ldots, x_k^{(\nu'_k-1)}, F_{j,\nu'}^{(s_j)}\right)_{k=1,2,0\leq j<i,\langle\nu,\gamma_{j+1}\rangle\leq s_j<\langle\nu',\gamma_{j+1}\rangle}.$$

When $\nu' - \nu \in \sigma_{Reg,j'(m,\nu)}$, we have that $\nu_k \leq \nu'_k$ for $k = 1, 2$. Then $\sigma_{Reg,j'(m,\nu)} \supseteq \sigma_{Reg,j'(m,\nu')}$ and then we only have to prove that $J_m^\nu \subseteq J_m^{\nu'}$. Let then $F_{i,\nu}^{(l)} \in J_m^\nu$ and let us prove that it belongs to $J_m^{\nu'}$.

Notice that $j(m,\nu') \leq j(m,\nu)$. We distinguish the following cases:

- If $j(m,\nu') = j(m,\nu)$ we have $F_{i,\nu'}^{(l)}, H_{i,\nu'-\nu}^{(l)} \in J_m^{\nu'}$.

- If $j(m,\nu') = j(m,\nu) - 1$ we have $H_{i,\nu'-\nu}^{(l)} \in J_m^{\nu'}$. Moreover $F_{i,\nu'}^{(l)} \in J_m^{\nu'}$ for $i < j(m,\nu)$. Then we have to study $F_{j(m,\nu),\nu'}^{(l)}$ with $\langle\nu, n_{j(m,\nu)}\gamma_{j(m,\nu)}\rangle \leq l \leq [\frac{m}{e_{j(m,\nu)}}]$. Note that (denoting $j(m,\nu)$ simply by j) by definition we have

$$\langle\nu, e_{j-1}\gamma_j\rangle \leq m < \langle\nu, e_j\gamma_{j+1}\rangle$$

$$\langle\nu', e_{j-2}\gamma_{j-1}\rangle \leq m < \langle\nu', e_{j-1}\gamma_j\rangle$$

Hence $\langle\nu, e_{j-1}\gamma_j\rangle \leq le_j \leq e_j[\frac{m}{e_j}] \leq m < \langle\nu', e_{j-1}\gamma_j\rangle$, which implies that $l < \langle\nu', n_j\gamma_j\rangle$ and therefore $F_{j(m,\nu),\nu'}^{(l)} = 0$

- If $j(m,\nu') < j(m,\nu) - 1$, we claim that then $\sigma_{Reg,j'(m,\nu)} = \{(0,0)\}$ and there is nothing to prove. Indeed, if $\sigma_{Reg,j'(m,\nu)} = \rho_1 \cup \rho_2$ then $j'(m,\nu) < 1$ or $j'(m,\nu) = 1 = g$ and $\gamma_1 = (1/n, 1/n)$. While if $\sigma_{Reg,j'(m,\nu)} = \rho_2$ then $1 \leq j'(m,\nu) \leq g_2$ and $\nu' = \nu + (0,\alpha)$. We have $j(m,\nu) \leq g_2 + 1$ but $j(m,\nu') \geq g_1$ because $\langle\nu,\gamma_i\rangle = \langle\nu',\gamma_i\rangle$ for $1 \leq i \leq g_1$. □

Definition 5.28. *We consider the order relation in N_0, depending on m and denoted by $\leq_m$, given by*

$$\nu \leq_m \nu' \text{ if and only if } \text{ and } \nu' - \nu \in \sigma_{Reg,j'(m,\nu)}.$$

We define the set $F_m = min_{\leq_m} L_m$.

Remark 5.29. *Notice that if $\nu \leq_m \nu'$ then in particular $\nu_i \leq \nu'_i$ for $i = 1, 2$.*

It is worth pointing out that the inclusions described in Proposition 5.27, can be explained by the fact that even though a curve may be in the singular locus of a quasi-ordinary surface, it may not be part of the singular locus of its first approximate quasi-ordinary surfaces. And as Proposition 5.14 explains, the geometry of C_m^ν is only determined by the geometry of one of its semi-roots, for m small enough. Hence, the jets which project to the singular locus of the surface but not to the singular locus of the approximate surfaces will not give rise to irreducible components of the jet schemes for m small enough, and they will be included in other components.

Now we prove that all possible inclusions among the C_m^ν are controlled by the relation defined in Definition 5.28, that is, in the set F_m.

Proposition 5.30. *Given $m \in \mathbb{Z}_{>0}$ and $\nu, \nu' \in F_m$ we have that $C_m^{\nu'} \nsubseteq C_m^\nu$.*

Proof. First notice that the claim is clear if $\nu \nleq \nu'$ (coordinate-wise). Indeed, suppose that ν and ν' are not comparable. Then we can assume that $\nu_1 < \nu'_1$ and $\nu_2 > \nu'_2$. Then, since $C_m^\nu \subseteq V(J_{-1}^\nu)$, and $C_m^{\nu'} \subseteq V(J_{-1}^{\nu'})$, it follows that

$$C_m^\nu \nsubseteq C_m^{\nu'} \text{ and } C_m^{\nu'} \nsubseteq C_m^\nu.$$

Let then ν, ν' be two different elements of F_m such that $\nu_i \leq \nu'_i$ for $i = 1, 2$. We will prove that $C_m^{\nu'} \nsubseteq C_m^\nu$ by showing that $\text{Codim}(C_m^{\nu'}) \leq \text{Codim}(C_m^\nu)$.

Notice that $\sigma_{Reg,j'(m,\nu)} \neq \rho_1 \cup \rho_2$, since otherwise $(\nu + \sigma) \cap F_m = \{\nu\}$ (recall that $\sigma = \mathbb{R}^2_{\geq 0}$). Then we deduce that $j'(m, \nu) > 0$. It is not easy to study the inequality directly by using the formula in Proposition 5.25, therefore we will prove by induction on m the inequality

$$\text{Codim}(C_m^{\nu'}) \leq \text{Codim}(C_m^\nu) \ \text{ for } \langle \nu, e_0\gamma_1 \rangle + e_1 \leq m < \langle \nu, e_{i(\nu)-1}\gamma_{i(\nu)} \rangle. \tag{17}$$

First step of induction, $m = \langle \nu, e_0\gamma_1 \rangle + e_1$ and obviously $j'(m, \nu) = 1$. We have that $\gamma_1 = (\frac{a_1}{n_1}, \frac{b_1}{n_1})$ with $a_1 > 1$ (because $\sigma_{Reg,j'(m,\nu)} \neq \rho_1 \cup \rho_2$). To study the set $(\nu + \sigma) \cap F_m$ we distinguish two cases:

- If $b_1 = 0, 1$ then $g_1 > 0$ and $\sigma_{Reg,j'(m,\nu)} = \rho_2$. Note that $\langle \nu + (1, 0), e_0\gamma_1 \rangle = \langle \nu, e_0\gamma_1 \rangle + a_1 e_1 > m$ and then $j(m, \nu + (1, 0)) = 0$. By Proposition 5.27 we have $C_m^{\nu+(0,l)} \subseteq C_m^\nu$ and $C_m^{\nu+(1,0)+(l,r)} \subseteq C_m^{\nu+(1,0)}$ and hence

$$(\nu + \sigma) \cap F_m = \{\nu, \nu + (1, 0)\}$$

By Proposition 5.25 we have that

$$\text{Codim}(C_m^{\nu+(1,0)}) = \nu_1 + \nu_2 + \langle \nu, \gamma_1 \rangle + 2 = \text{Codim}(C_m^\nu).$$

• If $b_1 > 1$ then $\sigma_{Reg,j'(m,\nu)} = \{(0,0)\}$, and as before $j(m, \nu + (1,0)) = 0$. Then by Proposition 5.27 we have $C_m^{\nu+(1,0)+(l,r)} \subseteq C_m^{\nu+(1,0)}$ for any $(l,r) \in \sigma$. Moreover $\langle \nu + (0,1), e_0\gamma_1\rangle = \langle \nu, e_0\gamma_1\rangle + e_1 b_1 > m$, and then $j(\nu + (0,1), m) = 0$. By Proposition 5.27 we have $C_m^{\nu+(0,1)+(l,r)} \subseteq C_m^{\nu+(0,1)}$. Hence

$$(\nu + \sigma) \cap F_m = \{\nu, \nu + (1,0), \nu + (0,1)\}$$

Again by Proposition 5.25 we have

$$\mathrm{Codim}(C_m^{\nu}) = \mathrm{Codim}(C_m^{\nu+(1,0)}) = \mathrm{Codim}(C_m^{\nu+(0,1)})$$

Suppose that the claim is true for $m-1$ and we prove it for m. We distinguish two cases:

(i) If $\nu' \in (\nu+\sigma)\cap F_{m-1}$, by induction hypothesis, we have that $\mathrm{Codim}(C_{m-1}^{\nu'}) \leq \mathrm{Codim}(C_{m-1}^{\nu})$. By Corollary 5.26 we know that, passing from $m-1$ to m, the codimension of C_m^{ν} grows if and only if m is divisible by $e_{j(m-1,\nu)}$, and it grows by one. But since $\nu \leq \nu'$ we have that $j(m-1,\nu') \leq j(m-1,\nu)$ and therefore, if $e_{j(m-1,\nu')}$ divides m, then $e_{j(m-1,\nu)}$ divides m, and it follows that $\mathrm{Codim}(C_m^{\nu'}) \leq \mathrm{Codim}(C_m^{\nu})$.

(ii) If $\nu' \notin (\nu + \sigma) \cap F_{m-1}$, there must exists $\widetilde{\nu} \in (\nu+\sigma)\cap F_{m-1}$ such that $\widetilde{\nu} \leq_{m-1} \nu'$ and $\widetilde{\nu} \not\leq_m \nu'$. By induction hypothesis we have that $\mathrm{Codim}(C_{m-1}^{\widetilde{\nu}}) \leq \mathrm{Codim}(C_{m-1}^{\nu})$, and again as in (i), since $\nu \lneq \widetilde{\nu}$ then $j(m,\nu) \geq j(m,\widetilde{\nu})$ and therefore $\mathrm{Codim}(C_m^{\widetilde{\nu}}) \leq \mathrm{Codim}(C_m^{\nu})$. Now we are going to prove that $\mathrm{Codim}(C_m^{\nu'}) \leq \mathrm{Codim}(C_m^{\widetilde{\nu}})$. We have two possibilities, either $\widetilde{\nu} \in L_m$ or $\widetilde{\nu} \notin L_m$.

• If $\widetilde{\nu} \in L_m$, then $j'(m-1,\widetilde{\nu}) < g_2+1$ (because $C_{m-1}^{\nu'} \subseteq C_{m-1}^{\widetilde{\nu}}$) and $j'(m,\widetilde{\nu}) \geq g_2+1$ (because $C_m^{\nu'} \nsubseteq C_m^{\widetilde{\nu}}$). Hence $m = \langle \widetilde{\nu}, e_{g_2}\gamma_{g_2+1}\rangle + e_{g_2+1}$ and

$$\pi_{m,m-1}^{-1}(D_{m-1}^{\widetilde{\nu}}) = V\big(x_1^{(0)}, \ldots, x_1^{(\widetilde{\nu}_1-1)}, x_2^{(0)}, \ldots, x_2^{(\widetilde{\nu}_2-1)}, z^{(0)}, \ldots, z^{(\langle \widetilde{\nu},\gamma_1\rangle-1)},$$

$$, F_{1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_1\gamma_1\rangle)}, \ldots, F_{g_2+1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_{g_2+1}\gamma_{g_2+1}\rangle)}, F_{g_2+1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_{g_2+1}\gamma_{g_2+1}\rangle+1)}\big) \cap D\big(x_1^{(\widetilde{\nu}_1)}\big),$$

where

$$F_{g_2+1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_{g_2+1}\gamma_{g_2+1}\rangle)} = F_{g_2,\widetilde{\nu}}^{(\langle \widetilde{\nu},\gamma_{g_2+1}\rangle)^{n_{g_2+1}}} - x_1^{(\widetilde{\nu}_1)^{\alpha_1^{(g_2+1)}}} x_2^{(\widetilde{\nu}_2)^{\alpha_2^{(g_2+1)}}} \cdots F_{g_2-1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_{g_2-1}\gamma_{g_2-1}\rangle)} + G_{g_2+1,\widetilde{\nu}},$$

with $\alpha_2^{(g_2+1)} > 1$, and

$$F_{g_2+1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_{g_2+1}\gamma_{g_2+1}\rangle+1)} = n_{g_2+1} F_{g_2,\widetilde{\nu}}^{(\langle \widetilde{\nu},\gamma_{g_2+1}\rangle)^{n_{g_2+1}-1}} F_{g_2,\widetilde{\nu}}^{(\langle \widetilde{\nu},\gamma_{g_2+1}\rangle+1)} - x_2^{(\widetilde{\nu}_2)} H,$$

where H is a polynomial in the variables

$$H(x_1^{(\widetilde{\nu}_1)}, x_1^{(\widetilde{\nu}_1+1)}, x_2^{(\widetilde{\nu}_2)}, x_2^{(\widetilde{\nu}_2+1)}, \ldots, F_{g_2-1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_{g_2-1}\gamma_{g_2-1}\rangle)}, F_{g_2-1,\widetilde{\nu}}^{(\langle \widetilde{\nu}, n_{g_2-1}\gamma_{g_2-1}\rangle+1)}).$$

Then

$$(\pi_{m,m-1}^{-1}(C_{m-1}^{\widetilde{\nu}}))_{red} = \overline{V(J_m^{\widetilde{\nu}}) \cap D(x_1^{(\widetilde{\nu}_1)}) \cap V(x_2^{(\widetilde{\nu}_2)})} \cup \overline{V(J_m^{\widetilde{\nu}}) \cap D(x_1^{(\widetilde{\nu}_1)}) \cap D(x_2^{(\widetilde{\nu}_2)})},$$

and it is not difficult to see that $(\pi^{-1}_{m,m-1}(C^{\widetilde{\nu}}_{m-1}))_{red} = C^{\nu'}_m \cup C^{\widetilde{\nu}}_m$, where $\nu' = \widetilde{\nu} + (0,\alpha)$, with

$$\alpha = \begin{cases} 1 & \text{if } g_2 = g_1 \\ \min\{n_{g_1+1}, k_{g_1+1}(\widetilde{\nu})\} & \text{otherwise} \end{cases}$$

In both cases we have, by Proposition 5.25, that $\mathrm{Codim}(C^{\nu'}_m) = \mathrm{Codim}(C^{\widetilde{\nu}}_{m-1}) + 1 = \mathrm{Codim}(C^{\widetilde{\nu}}_m)$.

• If $\widetilde{\nu} \notin L_m$, the reason is that $m = \langle \widetilde{\nu}, e_{i(\widetilde{\nu})-1}\gamma_{i(\widetilde{\nu})}\rangle$ with $i(\widetilde{\nu}) \leq g_2 + 1$, since $j'(m-1,\widetilde{\nu}) \leq g_2$. We have that $(\pi^{-1}_{m,m-1}(D^{\widetilde{\nu}}_{m-1}))_{red} = V(J^{\widetilde{\nu}}_{m-1}, F^{(\langle \widetilde{\nu}, n_{\widetilde{\nu}}\gamma_{i(\widetilde{\nu})}\rangle)}_{i(\widetilde{\nu}),\widetilde{\nu}}) \cap D(x_1^{(\widetilde{\nu}_1)})$, where

$$F^{(\langle \widetilde{\nu}, n_{i(\widetilde{\nu})}\gamma_{i(\widetilde{\nu})}\rangle)}_{i(\widetilde{\nu}),\widetilde{\nu}} = x_1^{(\widetilde{\nu}_1)^{\alpha_1^{(i(\widetilde{\nu}))}}} x_2^{(\widetilde{\nu}_2)^{\alpha_2^{(i(\widetilde{\nu}))}}} z^{(\langle \widetilde{\nu}, \gamma_1\rangle)^{r_1^{(i(\widetilde{\nu}))}}} \cdots F^{(\langle \widetilde{\nu}, n_{i(\widetilde{\nu})-2}\gamma_{i(\widetilde{\nu})-2}\rangle)^{r^{(i(\widetilde{\nu}))}_{i(\widetilde{\nu})-1}}}_{i(\widetilde{\nu})-2,\widetilde{\nu}} + G_{i(\widetilde{\nu}),\widetilde{\nu}}.$$

Therefore, by Corollary 5.16, $F^{(\langle \widetilde{\nu}, n_{i(\widetilde{\nu})}\gamma_{i(\widetilde{\nu})}\rangle)}_{i(\widetilde{\nu}),\widetilde{\nu}} = 0$ implies that $x_2^{(\widetilde{\nu}_2)} = 0$ because $i(\widetilde{\nu}) - 2 < g_2$. And, as before, if $g_2 = g_1 + 1$ and $i(\widetilde{\nu}) = g_2 + 1$ then we have that $\nu' = \widetilde{\nu} + (0,\alpha)$ with $\alpha = \min\{n_{g_1+1}, \langle \widetilde{\nu}, \gamma_{g_1+2} - n_{g_1+1}\gamma_{g_1+1}\rangle\}$. Otherwise $\nu' = \widetilde{\nu} + (0,1)$, and in both cases we have

$$(\pi^{-1}_{m,m-1}(C^{\widetilde{\nu}}_{m-1}))_{red} = C^{\nu'}_m$$

with $\mathrm{Codim}(C^{\nu'}_m) = \mathrm{Codim}(C^{\widetilde{\nu}}_{m-1}) + 1$. Since $\widetilde{\nu} \in (\nu + \sigma) \cap F_{m-1}$, it follows that $j(m-1,\nu) > j(m-1,\widetilde{\nu}) = i(\widetilde{\nu}) - 1$ and by Corollary 5.26 we have that $\mathrm{Codim}(C^{\nu}_m) = \mathrm{Codim}(C^{\nu}_{m-1}) + 1$, which finishes the proof. □

5.2. **Description of the m-jets through the singular locus.** Now we can prove the main theorem of this section.

Theorem 5.31. *For $m \in \mathbb{Z}_{>0}$ the decomposition of $\pi^{-1}_m(X_{Sing})$ in irreducible components is given by*

$$(\pi^{-1}_m(X_{Sing}))_{red} = \bigcup_{\nu \in F_m} C^{\nu}_m.$$

Proof. The irreducibility of the sets C^{ν}_m was proven in Proposition 5.25. And by Proposition 5.27 and Proposition 5.30 we have that

$$\bigcup_{\nu \in L_m} C^{\nu}_m = \bigcup_{\nu \in F_m} C^{\nu}_m.$$

Hence the result follows by Lemma 5.22. □

Remark 5.32. *When the equisingular dimension is $c = 1$ (see Definition 3.13), then $g_1 = g_2 = g$. Moreover we have the following properties for $1 \leq i \leq g$*

$$\langle \nu, e_{i-1}\gamma_i\rangle = \langle \nu + (0,r), e_{i-1}\gamma_i\rangle, \quad \textit{for all } r \in \mathbb{Z}$$

$$\textit{if } \nu \in N_i \textit{ then } \nu + (0,r) \in N_i, \quad \textit{for all } r \in \mathbb{Z}$$

Hence we deduce that for any $m \in \mathbb{Z}_{>0}$ *and* $\nu \in L_m$ *we have* $\sigma_{Reg,j'(m,\nu)} = \rho_2$*, and therefore* $F_m = L_m \cap \rho_1$*.*

The behaviour of the jet schemes is exactly as the plane curve defined by the Puiseux pairs $\lambda_1^{(1)}, \ldots, \lambda_g^{(1)}$*. In [24] the second author describes the irreducible components of jets through the origin for plane curves.*

The previous remark is the simplest evidence of the fact that the irreducible components are only affected by the topological type. This is proved in Theorem 5.35.

An alternative way to describe the irreducible components of the jet schemes through the singular locus is by representing the crucial information in a graph. To any quasi-ordinary surface singularity we can associate a weighted graph, containing information about the irreducible components of jet schemes and how they behave under truncation maps.

Definition 5.33. *The weighted graph of the jet schemes of* X *is the leveled weighted graph* Γ *defined as follows:*

- *for* $m \geq 1$ *we represent every irreducible component of* $\pi_m^{-1}(X_{Sing})$ *by a vertex* V_m*, the sub-index* m *being the level of the vertex;*
- *we join the vertices* V_{m+1} *and* V_m *if the canonical morphism* $\pi_{m+1,m}$ *induces a morphism between the corresponding irreducible components;*
- *we weight each vertex by the codimension of the corresponding irreducible component.*

We define $E\Gamma$ *to be the weighted graph that we obtain from* Γ *by adding to any vertex of* Γ *the weight given by the embedding dimensions of the corresponding irreducible component.*

Recall that if $\nu \in L_m^{(0)}$ the ideal J_m^ν is monomial, and moreover generated by hyperplane coordinates (see Remark 5.21), then we will say that C_m^ν is *hyperplane* component. Otherwise $\nu \in L_m^{(j)}$ for $j > 0$, and we will say that C_m^ν is a *lattice* component (because $\nu \in N_j$). Notice that the data of the codimension together with the embedding dimension permits to distinguish when the vertex corresponds to a hyperplane or a lattice component. Indeed, given a vertex of the graph, let e be the embedding dimension and c the codimension, then the vertex corresponds to a hyperplane component if and only if $e + c = 3(m+1)$. Therefore we can extract from $E\Gamma$ a subgraph Γ' as follows.

Definition 5.34. *We define a weighted subgraph* Γ' *of* $E\Gamma$ *by adding the condition that we join the vertices* V_m *(corresponding to a certain component, say* $C_m^{\nu'}$*) and* V_{m-1} *(corresponding to* C_{m-1}^{ν}*) only if*

- *if* $\nu \in L_{m-1}$ *with* $0 < j(m-1,\nu) \leq g_2$ *then* $\nu' = \nu + (0, \alpha)$ *with* α *minimal among the elements in* F_m*.*
- *if* $\nu \in L_{m-1}$ *with* $j(m-1,\nu) > g_2$ *then* $\nu' = \nu$*.*

The important thing about this new graph Γ' is that, with the weights, we are able to detect when we pass from a hyperplane component at level m to a lattice component at level $m+1$, as we also do in the graph $E\Gamma$, but now we can follow this component in a unique path in the graph as m grows. This will be useful to prove the following result.

Theorem 5.35. *The graph Γ' determines and it is determined by the topological type of the singularity.*

Proof. The graph is determined by the semigroup, and therefore, by [15], by the topological type. Now we prove the converse.

We prove first that we can read the number of characteristic exponents in the graph, in the following way. Any vertex V_m on the graph comes with the codimension $c(V_m)$ and the embedding dimension $e(V_m)$. Take an infinite branch (which we know that must correspond to $\nu \in N_g$), and consider the finite part that starts at

$$m_0 = \max\,\{m \mid V_{m-1} \text{ is a hyperplane component and } V_m \text{ is a lattice component}\},$$

and ends at

$$m_1 = \min\,\{r \mid c(V_m) = c(V_{m-1}) + 1 \text{ for all } m > r\}.$$

In the case $X_{Sing} = Z_1 \cup Z_2$ (which is the case with two components at level $m = 1$) we have to make sure that moreover the component corresponds to $\nu \in \overset{\circ}{\sigma}$, and this can be done by choosing a component which projects to both Z_1 and Z_2 (it always exists for m big enough). Note that then we deduce $m_0 = \langle \nu, e_0\gamma_1\rangle$ and $m_1 = \langle \nu, e_{g-1}\gamma_g\rangle$, and we can read $e_0, \ldots, e_{g-1}$ by using Corollary 5.26. Indeed, going backwards we look for the biggest m' such that $c(V_{m'}) = c(V_{m_0}) - 1$. Then $n = m_0 - m'$. Now, going from level m_0 to m_1, we know that the codimension grows by one exactly every e_1 steps at first, after every e_2 steps, and so on. Since $e_1 > e_2 > \cdots > e_g = 1$ we can read these numbers on the graph. Notice that equivalently we get $n_1, \ldots, n_g$, and in particular we have g, the number of characteristic exponents.

Suppose now that the number of generators of the semigroups is the same, say g. We will prove by induction on g that the graphs corresponding to different sets of generators, are different. We denote the vertices at level m by $V_m(c(V_m), e(V_m))$. The case $g = 1$ was treated in Theorem 4.21.

Now, suppose it is true for $g-1$ characteristic exponents, and we will prove it for g. From Proposition 5.14 we deduce that is sufficient to prove that the graphs associated to the sets $\{\gamma_1, \ldots, \gamma_{g-1}, \gamma_g\}$ and $\{\gamma_1, \ldots, \gamma_{g-1}, \gamma'_g\}$ are different, since otherwise it holds by induction hypothesis. Moreover, since we read the integers $n_1, \ldots, n_g$ in the graph, we assume that $n'_g = n_g$. As in the case $g = 1$, by looking at the singular locus (which is seen at $m = 1$) we just have to consider the case $\gamma_g^{(2)} = \gamma_g'^{(2)} = \frac{1}{n_g}$ and the case $\gamma_g^{(2)}, \gamma_g'^{(2)} > \frac{1}{n_g}$. In the first case $\gamma_g^{(1)} \neq \gamma_g'^{(1)}$ and $\gamma_i^{(2)} = \gamma_i'^{(2)} = 0$ for $1 \leq i \leq g-1$. Therefore the graphs are the same till we get to level $m = \min\,\{n_g\langle \nu, \gamma_g\rangle, n_g\langle \nu, \gamma'_g\rangle\}$, where $\nu = (\nu_1, 0) \in \sigma_{Sing} \cap N_{g-1}$ with ν_1 smallest with this property. Since $\langle \nu, \gamma_g\rangle \neq \langle \nu, \gamma'_g\rangle$ the graphs must differ at some moment. Finally, when $\gamma \neq \gamma'$ with $\gamma_g^{(2)}, \gamma_g'^{(2)} > \frac{1}{n_g}$, again by Proposition 5.14, the graphs must be the same for $\{\gamma_1, \ldots, \gamma_g\}$ and $\{\gamma_1, \ldots, \gamma_{g-1}, \gamma_g'\}$, till the last semi-root, that is, f, starts playing a role in the definition of a component, say C^ν. Since $\langle \nu, \gamma_g \rangle \neq \langle \nu, \gamma_g' \rangle$ we will see the difference on the graphs at level $m = \min\ \{n_g \langle \nu, \gamma_g \rangle, n_g \langle \nu, \gamma_g' \rangle\}$. □

5.3. **Log-canonical threshold.** In [29], Mustaţa gave a formula of the log-canonical threshold in terms of the codimension of jet schemes, which in our setting can be stated as

$$\mathrm{lct}(f) = \min_{m \geq 0} \frac{\mathrm{Codim}(X_m)}{m+1}. \tag{18}$$

Then, as an application to Theorem 5.31, we can recover, for the case of surfaces, the result in [8].

Corollary 5.36. *The log-canonical threshold of a quasi-ordinary surface singularity is given by:*

$$\mathrm{lct}_0(X, \mathbb{A}^3) = \begin{cases} \frac{1+\lambda_1^{(1)}}{e_0 \lambda_1^{(1)}} & \textit{if } \lambda_1 \neq \left(\frac{1}{n_1}, \frac{1}{n_1}\right) \\ 1 & \textit{if } \lambda_1 = \left(\frac{1}{n_1}, \frac{1}{n_1}\right) \textit{ and } g = 1 \\ \frac{n_1(1+\lambda_2^{(1)})}{e_1(n_1(1+\lambda_2^{(1)})-1)} & \textit{if } \lambda_1 = \left(\frac{1}{n_1}, \frac{1}{n_1}\right) \textit{ and } g > 1 \end{cases}$$

Moreover, the components that contribute to the log canonical threshold are

$$C^\nu_{\langle \nu, e_0\gamma_1 \rangle - 1} \quad \textit{if } \gamma_1 \neq (\tfrac{1}{n_1}, \tfrac{1}{n_1}) \textit{ or } g = 1$$

$$C^\nu_{\langle \nu, e_1\gamma_2 \rangle - 1} \quad \textit{otherwise}$$

where $\nu = (l, 0) \in N_1$ *if* $\gamma_1 \neq (\frac{1}{n_1}, \frac{1}{n_1})$ *and* $\nu = (l, 0) \in N_2$ *otherwise.*

Proof. The case $\lambda_1 = (\frac{1}{n_1}, \frac{1}{n_1})$ and $g = 1$ behaves as an A_n-singularity, and then $lct(f) = 1$. For the rest of the cases, by Corollary 5.26, the codimension of a component grows faster as m grows, for bigger $j(m, \nu)$. Therefore, the smaller codimension will be attached for $\nu \in F_m \cap \min_{\leq_m}\{\nu \in L_m \mid \sigma_{Reg,j'(m,\nu)} = \rho_1 \cup \rho_2\}$, and more concretely for $\nu \in L_m^{(0)} \cap F_m$ whenever $L_m^{(0)} \cap F_m \neq \emptyset$. If $g_1 = 0$, since $a_1 \geq b_1$, we deduce that the minimal codimension among the elements in $F_m \cap \min_{\leq_m}\{\nu \in L_m \mid \sigma_{Reg,j'(m,\nu)} = \rho_1 \cup \rho_2\}$ is attached for ν of the form $\nu = (l, 0)$, while if $g_1 > 0$ then $F_m \cap \min_{\leq_m}\{\nu \in L_m \mid \sigma_{Reg,j'(m,\nu)} = \rho_1 \cup \rho_2\}$ consists of just a point of the form $\nu = (l, 0)$.

We want to minimize not just the codimension, but the quotient $\frac{\mathrm{Codim}(X_m)}{m+1}$. That is, to find the biggest m such that ν still belongs to $F_m \cap \min_{\leq_m}\{\nu \in L_m \mid \sigma_{Reg,j'(m,\nu)} = \rho_1 \cup \rho_2\}$. Then, when $\lambda_1 \neq (\frac{1}{n_1}, \frac{1}{n_1})$, this is attached for $m = \langle \nu, e_0\gamma_1 \rangle - 1$ such that $\nu \in L_{m+1}$ with $j'(m, \nu) = 0$ and $j(m, \nu) > 0$. Then

$m = \langle (l,0), e_0\gamma_1\rangle - 1$ and $\mathrm{Codim}(C_m^\nu) = l + [\frac{m}{n}] + 1$, and since $j(m,\nu) > 0$, $(l,0) \in N_1$ and therefore $\mathrm{Codim}(C_m^\nu) = l + l\frac{a_1}{n_1}$, which implies that $\frac{\mathrm{Codim}(C_m^\nu)}{m+1} = \frac{a_1+n_1}{na_1}$.

If $\gamma_1 = (\frac{1}{n_1}, \frac{1}{n_1})$ and $g > 1$, what happens is that when $m = \langle \nu, e_0\gamma_1\rangle$ there is no subdivision of the component and $\sigma_{Reg,1} = \rho_1 \cup \rho_2$. If we denote the second exponent by $\gamma_2 = (\frac{\alpha_2}{n_1 n_2}, \frac{\beta_2}{n_1 n_2})$, we look for ν of the form $(l,0)$ such that $m+1 = \langle \nu, e_1\gamma_2\rangle$ with $\nu \in N_2$. Then $\mathrm{Codim}(C_m^\nu) = l + \langle \nu, \gamma_1\rangle + [\frac{m}{e_1}] - \langle \nu, n_1\gamma_1\rangle + 1 = l + \langle \nu, \gamma_1\rangle + \langle \nu, \gamma_2 - n_1\gamma_1\rangle$, and therefore $\frac{\mathrm{Codim}(C_m^\nu)}{m+1} = \frac{l + \frac{l}{n_1} + \frac{1}{e_1}(e_1 l \frac{\alpha_2}{n_1 n_2} - e_0\frac{l}{n_1})}{e_1 l \frac{\alpha_2}{n_1 n_2}} = \frac{1 + \frac{1}{n_1} + \frac{\alpha_2}{n_1 n_2} - 1}{e_2\frac{\alpha_2}{n_1}} = \frac{1 + \frac{\alpha_2}{n_2}}{e_1\frac{\alpha_2}{n_2}}$. This coincides with the statement since $\lambda_2 = (\frac{\alpha_2}{n_1 n_2} - \frac{n_1-1}{n_1}, \frac{\beta_2}{n_1 n_2} - \frac{n_1-1}{n_1})$. □

We now deduce a family of examples whose log canonical threshold can not be computed by a monomial valuation.

Corollary 5.37. *Let X be a quasi-ordinary surface singularity with $g > 1$ characteristic exponents, and such that $\lambda_1 = (\frac{1}{n_1}, \frac{1}{n_1})$. Then $lct(X, \mathbb{A}^3)$ can not be contributed by monomial valuations in any variables.*

Proof. It follows from Corollary 5.36 that $lct(X, \mathbb{A}^3)$ is contributed by $C^\nu_{\langle \nu, e_0\gamma_1\rangle}$, for ν as is made precise in the above statement. This is equivalent to say that the valuation

$$\mathcal{V}_{C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}} : \quad \mathbb{C}[[x_1, x_2, z]] \longrightarrow \mathbb{N}$$

$$h \longmapsto \mathrm{ord}_t(h \circ \eta)$$

where η is the generic point of $(\Psi^{\mathbb{A}^3}_{\langle \nu, e_1\gamma_2\rangle - 1})^{-1}(C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1})$ and

$$\Psi_m^{\mathbb{A}^3} : \mathbb{A}^3_\infty \longrightarrow \mathbb{A}^3_m$$

is the map induced by truncation. Note that ν can take all the values described in Corollary 5.36 but since $z^{(\langle \nu, \gamma_1\rangle)^{n_1}} - x_1^{(\nu_1)} x_2^{(\nu_2)} = 0$ is one of the defining equations of $C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}$, then $\mathcal{V}_{C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}}(z^{n_1} - x_1 x_2) > n_1 \mathcal{V}_{C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}}(z)$ and $\mathcal{V}_{C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}}(z^{n_1} - x_1 x_2) > \mathcal{V}_{C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}}(x_1) + \mathcal{V}_{C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}}(x_2)$. Therefore $\mathcal{V}_{C^\nu_{\langle \nu, e_1\gamma_2\rangle - 1}}$ is not a monomial valuation. □

5.4. **Examples.** We finish by looking at some examples, to illustrate once more the arguments we use in proving the description of $\left(\pi_m^{-1}(X_{Sing})\right)_{red}$ in irreducible components.

Example 5.38. *Let X be the q.o. surface defined by $f = (z^2 - x_1^3 x_2)^3 - x_1^{10} x_2^4$, whose generators of the semigroup are $\gamma_1 = (\frac{3}{2}, \frac{1}{2})$ and $\gamma_2 = (\frac{10}{3}, \frac{4}{3})$. We have that $g_1 = 0$ and $g_2 = 1$. The singular locus is*

$$X_{Sing} = \{z = x_1 = 0\} \cup \{z = x_2 = 0\}$$

and then $\sigma_{Sing} = \sigma \setminus \{(0,0)\}$. *Then*

$$\big(\pi_1^{-1}(X_{Sing})\big)_{red} = V\big(x_1^{(0)}, z^{(0)}\big) \cup V\big(x_2^{(0)}, z^{(0)}\big)$$

or, described with our notation, $F_1 = \{(1,0),(0,1)\}$. *Note that* $\sigma_{Sing} \cap [0,1]^2 \cap N_0 = \{(1,0),(0,1),(1,1)\}$, *and since*

$$C_1^{(1,1)} = V\big(x_1^{(0)}, x_2^{(0)}, z^{(0)}\big),$$

it is not an irreducible component, because it is contained in both $V\big(x_1^{(0)}, z^{(0)}\big)$ *and* $V\big(x_2^{(0)}, z^{(0)}\big)$.

Let us lift the component $C_1^{(0,1)}$ *to higher levels. If* $\nu = (0,1)$,

$$\big(\pi_{6,1}^{-1}(C_1^{\nu})\big)_{red} = V\big(x_2^{(0)}, x_2^{(1)}, z^{(0)}, F_{1,\nu}^{(2)}\big)$$

where $F_{1,\nu}^{(2)} = {z^{(1)}}^2 - {x_1^{(0)}}^3 x_2^{(2)}$. *We can check that* $(0,1) \in F_6$ *and that it is indeed an irreducible component of* $\big(\pi_6^{-1}(X_{Sing})\big)_{red}$.

To illustrate typical behavior of this case we have to lift the components much higher.

Straightforwardly it can be checked that

$$\big(\pi_{20}^{-1}(X_{Sing})\big)_{red} = C_{20}^{(3,0)} \cup C_{20}^{(2,1)} \cup C_{20}^{(1,3)} \cup C_{20}^{(0,6)}$$

as Theorem 5.31 claims. Let us lift the component

$$C_{20}^{(0,6)} = V\big(x_2^{(0)}, \ldots, x_2^{(5)}, z^{(0)}, z^{(1)}, z^{(2)}, F_{1,\nu}^{(6)}\big)$$

We have that, if $\nu = (0,6)$,

$$\pi_{21,20}^{-1}(C_{20}^{\nu}) = V\big(x_2^{(0)}, \ldots, x_2^{(5)}, z^{(0)}, z^{(1)}, z^{(2)}, F_{1,\nu}^{(6)}, F_{1,\nu}^{(7)}\big)$$

where

$$F_{1,\nu}^{(6)} = {z^{(3)}}^2 - {x_1^{(1)}}^3 x_2^{(3)}$$

$$F_{1,\nu}^{(7)} = 2z^{(3)}z^{(4)} - 3{x_1^{(1)}}^2 x_1^{(2)} x_2^{(3)} - {x_1^{(1)}}^3 x_2^{(4)}$$

This is not irreducible, since it decomposes as

$$V\big(x_1^{(0)}, x_1^{(1)}, x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\big) \cup$$
$$\cup \overline{V\big(x_1^{(0)}, x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, F_{1,\nu}^{(6)}, F_{1,\nu}^{(7)}\big) \cap \{x_1^{(1)} \neq 0\}}$$

We can check that $V\big(x_1^{(0)}, x_1^{(1)}, x_2^{(0)}, x_2^{(1)}, x_2^{(2)}, z^{(0)}, z^{(1)}, z^{(2)}, z^{(3)}\big)$ *does not give rise to an irreducible component, since it is contained in* $C_{21}^{(2,2)}$ *and* $(2,2) \in F_{21}$.

Example 5.39. *Consider the quasi-ordinary surface* $f = \big((z^2 - x_1^3)^2 - x_1^7 x_2^3\big)^2 - x_1^{11} x_2^5 (z^2 - x_1^3)$. *We draw the graph in Figure 4. The semigroup is generated by the vectors* $\gamma_1 = (\frac{3}{2}, 0)$, $\gamma_2 = (\frac{7}{2}, \frac{3}{2})$ *and* $\gamma_3 = (\frac{29}{4}, \frac{13}{4})$. *We have that* $g_1 = g_2 = 1$. *The singular locus is reducible, of the form*

$$X_{Sing} = \{z = x_1 = 0\} \cup \{x_2 = z^2 - x_1^3 = 0\} = Z_1 \cup Z_2.$$

Then $\sigma_{Sing} = \mathbb{R}^2_{\geq 0} \setminus \{0\}$ *and* $\sigma_{Reg,1} = \rho_2,\ \sigma_{Reg,2} = \sigma_{Reg,3} = \{(0,0)\}$.

The set F_m *describing the irreducible components is the following, for some* m*:*

$$F_m = \{(1,0),(0,1)\}, \quad \text{for } 1 \leq m < 6$$

$$F_m = \{(1,0),(0,2)\}, \quad \text{for } 6 \leq m < 12$$

$$F_{12} = \{(2,0),(0,2)\}$$

$$F_{13} = \{(2,0),(0,3)\}$$

$$F_{18} = \{(2,0),(0,4)\}$$

$$F_{26} = \{(2,0),(0,4),(0,5)\}$$

$$F_{28} = \{(3,0),(2,0),(0,4),(0,5)\}$$

and the result can be checked by lifting the components Z_1 *and* Z_2 *of the singular locus to level* m *as the following graph shows (we did not draw the weights of the vertices for clearness).*

Now we give some explanations to illustrate how Proposition 5.27 works.

For $m = 1$, *we have* $L_1^{(0)} = \{(1,0),(1,1)\}$, $L_1^{(1)} = \{(0,1)\}$ *and* $L_1^{(2)} = L_1^{(3)} = \emptyset$. *By Proposition 5.27* $C_1^{(1,1)} \subseteq C_1^{(1,0)}$, *since* $j'(1,(1,0)) = 0$ *and* $\sigma_{Reg,0} = \rho_1 \cup \rho_2$.

At level $m = 6$ *we have* $L_6^{(0)} = \{\nu \in [0,6]^2 \cap N_0 \mid \nu_1 \neq 0\}$, $L_6^{(1)} = \emptyset$ *and* $L_6^{<} = \{(0,\nu_2) \mid 2 \leq \nu_2 \leq 6\}$ *and* $L_6^{(2)} = L_6^{(3)} = \emptyset$. *Note that* $j'(6,(1,0)) = 0$, *hence* $\sigma_{Reg,j'(6,(1,0))} = \rho_1 \cup \rho_2$ *and by Proposition 5.27* $C_6^{\nu} \subseteq C_6^{(1,0)}$ *for any* $\nu \in L_6^{(0)}$. *Moreover* $j'(6,(0,2)) = 1$, *hence* $\sigma_{Reg,j'(6,(0,2))} = \rho_2$ *and by Proposition 5.27* $C_6^{\nu} \subseteq C_6^{(0,2)}$ *for any* $\nu \in L_6^{(1)}$.

Note how at this level $\nu = (0,1)$ *does no longer give rise to an irreducible component, since* $\langle (0,1), e_1\gamma_2 \rangle = 6$ *and* $(0,1) \notin N_2$. *Then we have that* $(0,2) \in F_6$ *and the vertex associated with* $C_5^{(0,1)}$ *and the one associated with* $C_6^{(0,2)}$ *are joined in the graph* Γ'.

References

1. S.S. Abhyankar, *On the ramification of algebraic functions*, Amer. J. Math., **77** (1955), 575-592.
2. M. Aprodu and D. Naie, *Enriques diagrams and log-canonical thresholds of curves on smooth surfaces*, Geom. Dedicata, **146** (2010), 43–66.
3. E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo and A. Melle Hernández, *Quasi-ordinary power series and their zeta functions*, Mem. Amer. Math. Soc., **178** (2005), no. 841, 1-85.
4. E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo and A. Melle-Hernández, *On the log-canonical threshold for germs of plane curves*. Singularities I, 1–14, Contemp. Math., **474**, Amer. Math. Soc., Providence, RI, 2008.
5. A. Assi, *Irreducibility criterion for quasi-ordinary polynomials*, J. Singul., **4** (2012), 23–34.

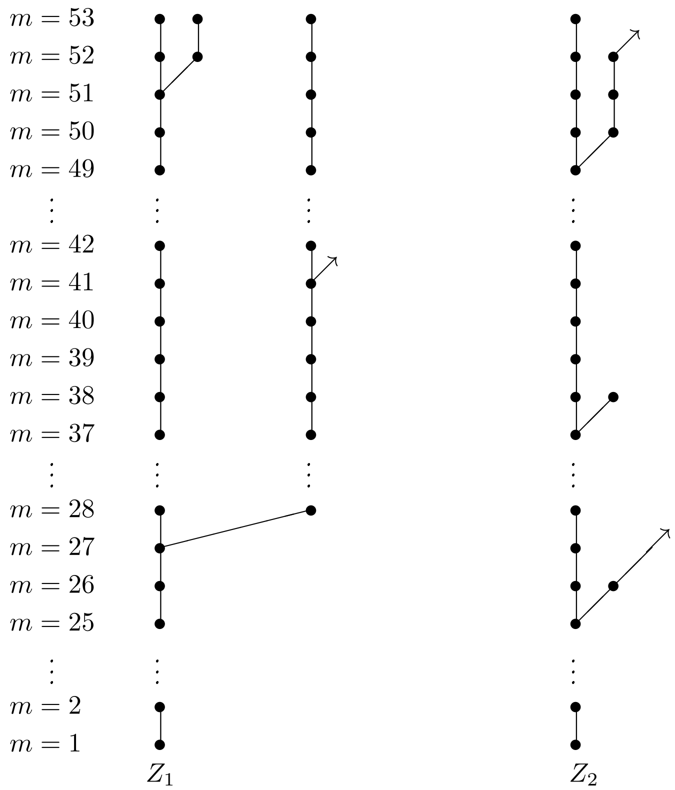

FIGURE 4. The graph of the surface defined by $f = ((z^2 - x_1^3)^2 - x_1^7 x_2^3)^2 - x_1^{11} x_2^5 (z^2 - x_1^3)$.

6. C. Ban and L. McEwan, *Canonical resolution of a quasi-ordinary surface singularity*, Canad. J. Math. **52** (2000), no. 6, 1149-1163.
7. C. Ban and L. McEwan, *Simultaneous resolution of equisingular quasi-ordinary singularities*, Singularities in algebraic and analytic geometry (San Antonio, TX, 1999), 65–75, Contemp. Math., 266, Amer. Math. Soc., Providence, RI, 2000.
8. N. Budur, P.D. González Pérez and M. González Villa, *Log-canonical thresholds of quasi-ordinary hypersurfaces singularities*, Proc. of the Amer. Math. Soc., **140** (2012), 4075-4083.
9. H. Cobo Pablos and P.D. González Pérez, *Geometric motivic Poincaré series of quasi-ordinary hypersurfaces*, Math. Proc. Camb. Phil. Soc. **149** (2010), no. 01, 49–74.
10. J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math., **135**, 1, (1999), 201-232.
11. J. Denef and F. Loeser, *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 327–348, Progr. Math., 201, Birkhäuser, Basel, 2001.
12. R. Docampo, *Arcs on determinantal varieties*, Trans. Amer. Math. Soc. **365** (2013), no. 5, 2241–2269.
13. L. Ein and M. Mustaţa, *Jet Schemes and Singularities*, Algebraic geometry-Seattle 2005, Proc. Sympos. Pure Math., 80, Part 2, Amer. Math. Soc., Providence, RI, (2009), 505-546.
14. E. García Barroso and J. Gwodziewicz, *Quasi-ordinary singularities: tree model, discriminant and irreducibility*, Int. Math. Res. Not., no. 14, (2015), 5783-5805.

15. Y-N. Gau, *Embedded Topological classification of quasi-ordinary singularities*, Mem. Amer. Math. Soc., **388** (1988).
16. P.D. González Pérez, *The semigroup of a quasi-ordinary hypersurface*, J. Inst. Math. Jussieu, **2** (3) (2003), 383-399.
17. P.D. González Pérez, *Toric embedded resolutions of quasi-ordinary hypersurface singularities*, Ann. Inst. Fourier, **53** (6) (2003), 1819-1881.
18. P.D. González Pérez and M. González Villa, *Motivic Milnor fiber of a quasi-ordinary hypersurface*, J. Reine Angew. Math., **687** (2014), 159–205.
19. S. Ishii, *Jet schemes, arc spaces and the Nash problem.*, C. R. Math. Acad. Sci. Soc. R. Can., **29** (2007), no. 1, 1–21.
20. K. Kiyek and M. Micus, *Semigroup of a quasiordinary singularity.* Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., 26, 149-156.
21. M. Lejeune-Jalabert, H. Mourtada and A. Reguera, *Jet schemes and minimal embedded desingularization of plane branches*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas: **107**, no. 1, (2013), 145-157.
22. J. Lipman, *Topological invariants of quasi-ordinary singularities*, Mem. Amer. Math. Soc., **74**, no. 388, (1988), 1-107.
23. J. Lipman, *Equisingularity and simultaneous resolution of singularities*, Resolution of singularities (Obergurgl, 1997), Progr. Math., 181, Birkhäuser, Basel, (2000), 485-505.
24. H. Mourtada, *Jet schemes of complex plane branches and equisingularity*, Ann. Inst. Fourier, **61** (2011), no. 6, 2313-2336.
25. H. Mourtada, *Jet schemes of normal toric surfaces*, Bull. SMF **145**, fascicule 2 (2017), 237-266.
26. H. Mourtada, *Jet schemes of rational double point surface singularities*, Valuation Theory in Interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., (2014), 373–388.
27. H. Mourtada, *Jet schemes and minimal generating sequences of divisorial valuations in dimension two*, Michigan Math. J., **66**, Issue 1 (2017), 155-174.
28. H. Mourtada and C. Plénat, *Jet schemes and minimal toric embedded resolutions of rational double point singularities*, Communications in Algebra, **46**, no. 3 (2018), 1314-1332.
29. M. Mustaţa, *Singularities of pairs via jets schemes*, J. Amer. Math. Soc., **15** (2002), no. 3, 599-615.
30. P. Popescu-Pampu, *On the analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity*, Duke Math. J., **124** (2004), no. 1, 67-104.
31. P. Popescu-Pampu, *Introduction to Jung's method of resolution of singularities*, Contemp. Math, **538** (2011), 401-432.
32. P. Popescu-Pampu, *From singularities to graphs.* ArXiv:1808.00378.
33. B.A. Sethuraman and K. Šivic, *Jet schemes of the commuting matrix pairs scheme*, Proc. Amer. Math. Soc., **137** (2009), no. 12, 3953–3967.
34. O. Villamayor, *On equiresolution and a question of Zariski*, Acta Math., **185** (2000), no.1, 123-159.
35. C. Yuen, *Jet schemes of determinantal varieties*, Algebra, geometry and their interactions, Contemp. Math., **448**, Amer. Math. Soc., (2007), 261–270.

E-mail address: `helenacobo@gmail.com`

Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris 7, Bâtiment Sophie Germain, 75013 Paris, France.

E-mail address: `hussein.mourtada@imj-prg.fr`