Construction of real algebraic varieties

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February 26, 2002

1 Hilbert's 16-th problem

The first part of Hilbert's 16-th problem (see [23]) contains several questions concerning the topology of real algebraic varieties, and more precisely, the topology of plane algebraic curves and spatial algebraic surfaces.

Let us start from the definitions. A *real algebraic curve* of degree m in the real projective plane $\mathbb{R}P^2$ is a real homogeneous polynomial of degree m in three variables considered up to a constant factor. Let A be such a polynomial. Then, the set of points defined by the equation $A(x_0, x_1, x_2) = 0$ in the real projective plane $\mathbb{R}P^2$ is called the *real point set* of the curve A and is denoted by $\mathbb{R}A$. We consider only *nonsingular* curves, which means that corresponding polynomials do not have critical points in $\mathbb{R}^3 \setminus \{0\}$.

In a similar way one defines a *real algebraic hypersurface* of degree m in a real projective space $\mathbb{R}P^n$ as a real homogeneous polynomial of degree m in $n+1$ variables considered up to a constant factor.

The real point set $\mathbb{R}A$ of a nonsingular projective real algebraic curve A is a union of disjoint circles smoothly embedded in $\mathbb{R}P^2$. A circle can be positioned in $\mathbb{R}P^2$ either one-sidedly or two-sidedly. A two-sidedly positioned circle is called an *oval*. An oval divides $\mathbb{R}P^2$ into two parts. The part homeomorphic to a disk is called the *interior* of the oval. All the connected components of the real point set of a nonsingular curve of an even degree in $\mathbb{R}P^2$ are ovals. The real point set of a nonsingular curve of an odd degree in $\mathbb{R}P^2$ contains exactly one one-sidedly positioned connected component. The topological type of the pair $(\mathbb{R}P^2, \mathbb{R}A)$ is defined by the scheme of disposition of the connected components of $\mathbb{R}A$. This scheme is called the *real scheme* of the curve A.

In the 16-th problem D. Hilbert asked, in particular, to describe all the real schemes of nonsingular curves of a given degree in $\mathbb{R}P^2$. For small degrees (less

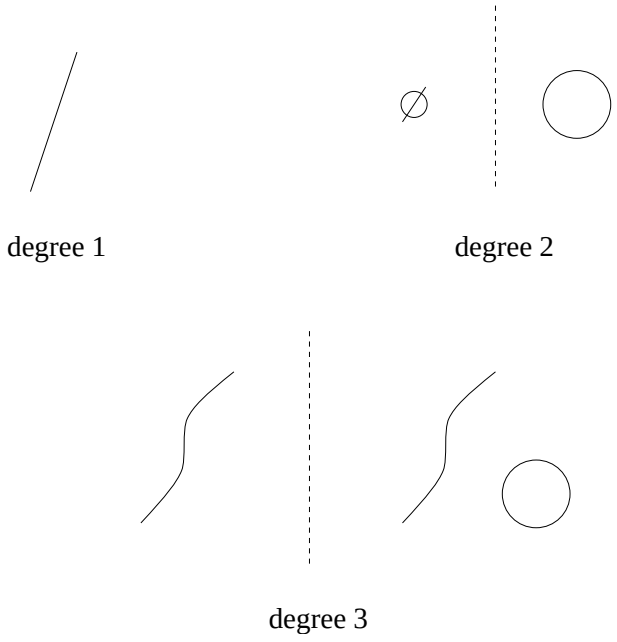


Figure 1: Real schemes of nonsingular curves of degrees 1, 2 and 3 in $\mathbb{R}P^2$

than 6) the answer is shown in Figures 1, 2 and 3. This answer was known to Hilbert and can be obtained using the Harnack inequality (see below) and the fact that the number of intersection points of a curve of degree m and a line is at most m (if this number of intersection points is finite). The latter statement is a particular case of the Bézout theorem.

Theorem 1.1 (Harnack inequality, see [19]) *The number of connected components of the real point set of a curve of degree m in $\mathbb{R}P^2$ is at most $\frac{(m-1)(m-2)}{2}+1$.*

The curves of degree m in $\mathbb{R}P^2$ whose real point set has $\frac{(m-1)(m-2)}{2}+1$ connected components are called *maximal* (or *M-curves*).

To prove that any real scheme of Figures 1-3 is realizable by a curve of the corresponding degree one can take an appropriate collection of lines and ellipses intersecting transversally and slightly perturb their union (*i.e.*, slightly change the coefficients of the product of the polynomials of degrees 1 and 2 defining these lines and ellipses) in order to obtain the desired picture.

For example, to obtain a maximal curve of degree 4 in $\mathbb{R}P^2$, we can take two ellipses intersecting in 4 points and consider a polynomial $E_1 \cdot E_2 + \varepsilon(x_0^4 + x_1^4 + x_2^4)$, where E_1 and E_2 are polynomials defining two ellipses, and ε is a sufficiently small positive number. If the signs of E_1 and E_2 are appropriately chosen, then this polynomial defines a curve whose real point set consists of 4 ovals (see Figure 4).

A classification of real schemes of nonsingular curves of degree 6 in $\mathbb{R}P^2$ is more difficult to obtain. This classification was completed by D. A. Gudkov (see [18]) in 1969. Only 3 real schemes are realizable by maximal curves of degree 6 in $\mathbb{R}P^2$. These schemes are shown in Figure 5.

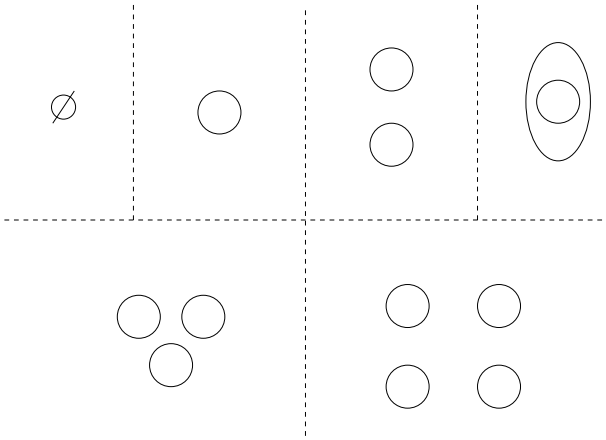


Figure 2: Real schemes of nonsingular curves of degree 4 in $\mathbb{R}P^2$

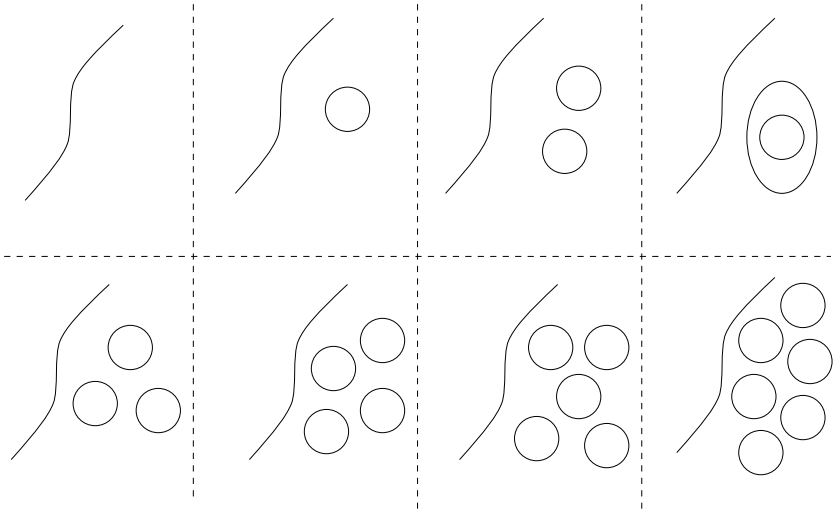


Figure 3: Real schemes of nonsingular curves of degree 5 in $\mathbb{R}P^2$

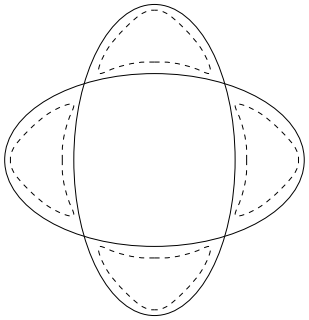


Figure 4: Maximal curve of degree 4 in $\mathbb{R}P^2$

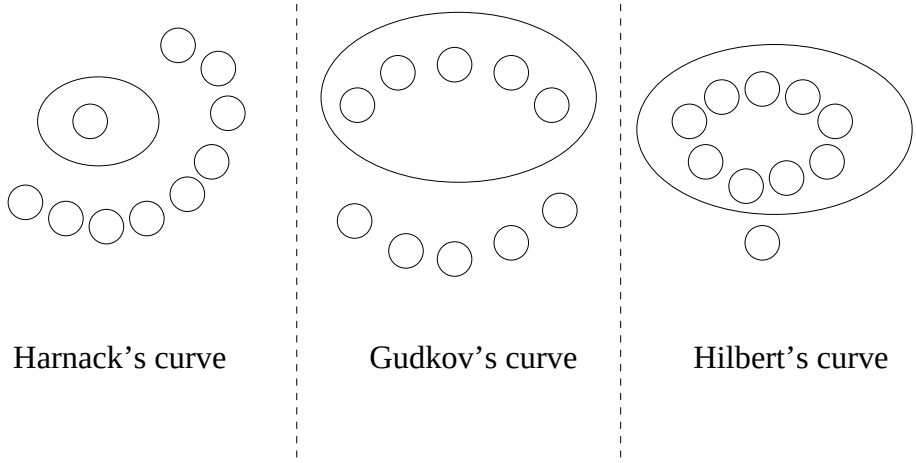


Figure 5: Real schemes of maximal curves of degree 6 in $\mathbb{R}P^2$

In 1979 O. Viro [70] obtained a classification of real schemes of nonsingular curves of degree 7 in $\mathbb{R}P^2$. Despite of the efforts to classify the real schemes of nonsingular curves of degree 8 (see, for example, [7, 42, 46, 50, 51, 58, 59, 60, 70, 72]), the classification is still not completed.

A question similar to the first part of Hilbert's 16-th problem concerns a classification of real schemes realized by real pseudo-holomorphic curves of a given degree. A Riemann surface M embedded in $\mathbb{C}P^2$ is a *real pseudo-holomorphic curve*, if it is a J-holomorphic curve in some tame almost complex structure J on X (see [14]) such that $\mathrm{Conj}_* \circ J = J^{-1} \circ \mathrm{Conj}_*$ and $\mathrm{Conj}(M) = M$, where $\mathrm{Conj} : \mathbb{C}P^2 \to \mathbb{C}P^2$ is the involution of complex conjugation. The fixed point set $\mathbb{R}M \subset \mathbb{R}X$ of Conj restricted to a real pseudo-holomorphic curve M is called the *real point set* of M. A real pseudo-holomorphic curve M in $\mathbb{C}P^2$ is *of degree* m if M realizes the class $m[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$. It is interesting to compare isotopy classifications of real point sets of real algebraic curves and of real pseudo-holomorphic curves of the same degree. The classifications coincide for the curves of degree ≤ 7. Recently, S. Orevkov [47] obtained an isotopy classification of the real point sets of maximal real pseudo-holomorphic curves of degree 8 (exactly as in the algebraic case, a real pseudo-holomorphic curve of degree m is called *maximal* if its real point set has $\frac{(m-1)(m-2)}{2} + 1$ connected components). As it was already mentioned above, the corresponding classification of real algebraic curves is not yet known.

The question on a topological classification (*i.e.*, a classification of the real point sets up to homeomorphism) of nonsingular surfaces of a given degree in $\mathbb{R}P^3$ was also included by Hilbert in the first part of the 16-th problem. At the end of the XIX-th century the answer was known for surfaces of degrees ≤ 3. The topological classification of surfaces of degree 4 in $\mathbb{R}P^3$ was completed by V. Kharlamov [38] in 1976. Kharlamov also obtained finer classifications of surfaces of degree 4 (see [39] and [41]). A topological classification of surfaces of degree ≥ 5 in $\mathbb{R}P^3$ is unknown.

The last three decades of the XX-th century were the years of an intensive progress in topology of real algebraic varieties. Many important restrictions on the topology

of real algebraic varieties were obtained using the methods of modern topology of manifolds and complex algebraic geometry (see [1, 54, 55, 56, 34, 35, 36, 37, 11, 44], and also the surveys [16, 77, 2, 74, 10]).

We mention here one general restriction on the topology of real algebraic curves which was obtained in the 1970-s: the Gudkov-Rokhlin congruence. The congruence was conjectured by D. A. Gudkov [15] on the base of the classification of curves of degree 6 and the available examples of curves of higher degrees, and was proved by V. A. Rokhlin [54]. To formulate the congruence, introduce additional definitions and notations. An oval of a nonsingular curve of an even degree in $\mathbb{R}P^2$ is called *even* (resp. *odd*), if it lies inside of even (resp. odd) number of other ovals of the curve. Denote by p (resp., n) the number of even (resp., odd) ovals of a nonsingular curve of an even degree in $\mathbb{R}P^2$.

Theorem 1.2 (Gudkov-Rokhlin congruence, see [54]) *For an M-curve of degree $2k$ in $\mathbb{R}P^2$, on has the congruence*

$$p - n \equiv k^2 \mod 8.$$

Note that the Gudkov-Rokhlin congruence and the Bézout theorem show, in particular, that an M-curve of degree 6 in $\mathbb{R}P^2$ should have one of the real schemes of Figure 5.

The above mentioned progress in topology of real algebraic varieties concerns also the methods of constructions. Until the latest 1970-s, all the constructions of real algebraic curves were based on the method of small perturbations of a product of curves which have smaller degrees and intersect transversally. One can mention here the classical methods of construction proposed by Harnack [19], Hilbert [22], Brusotti [6] and Wiman [78]. To construct several curves of degree 6 in $\mathbb{R}P^2$ Gudkov (see [18]) extended a little bit the classical scheme of construction of curves: he combined the method of small perturbations with the use of quadratic transformations of $\mathbb{R}P^2$.

In 1979, O. Viro proposed a principally new method of construction of real algebraic varieties (see [70, 71, 73, 75, 76] and also [53]). It provides a nice interaction of real algebraic geometry, toric geometry and combinatorics, and gives rise to various generalizations and applications [3, 7, 30, 46, 61, 62, 63, 64, 67]. Almost all the constructions in topology of real algebraic varieties since 1979 use the Viro method.

It is impossible to give here a complete description of the Viro method or a detailed survey of its applications. We restrict ourselves by a discussion of a particular case of the Viro method (the *combinatorial patchworking*) and some of its applications found during the last ten years. The combinatorial patchworking is a powerful construction which gives a possibility to build real algebraic hypersurfaces in a simple combinatorial fashion: one can patchwork them from pieces which essentially are hyperplanes.

2 Combinatorial patchworking

Let m be a positive integer number, and T the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(m,0)$ and $(0,m)$. Take a triangulation τ of T with vertices having integer coordinates. Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates (i,j) is denoted by $\delta_{i,j}$.

Take the copies

$$T_x = s_x(T),\ \ T_y = s_y(T),\ \ T_{xy} = s_x \circ s_y(T)$$

of T, where $s_x,\ s_y$ are reflections with respect to the coordinate axes. Denote by T_* the square $T \cup T_x \cup T_y \cup T_{xy}$. Extend the triangulation τ to a symmetric triangulation of T_*, and the distribution of signs $\delta_{i,j}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate axis we preserve its sign if the distance from the vertex to the axis is even, and change the sign if the distance is odd.

If a triangle of the triangulation of T_* has vertices of different signs, select a midline joining the middle points of the edges having endpoints of opposite signs. Denote by L the union of the selected segments. It is a piecewise-linear curve contained in T_*. Glue by $s_x \circ s_y$ the opposite sides of T_*. The resulting quotient space $\tilde{T}$ is homeomorphic to $\mathbb{R}P^2$. Denote by $\tilde{L}$ the image of L in $\tilde{T}$.

Let us introduce an additional assumption: the triangulation τ of T is *convex*. This means that there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbb{R}$ whose domains of linearity coincide with the triangles of τ. Sometimes, such triangulations are also called coherent (see [13]) or regular (see [79]).

Theorem 2.1 (O. Viro) *If the triangulation τ of T is convex, then there exist a nonsingular real algebraic curve A of degree m in $\mathbb{R}P^2$ and a homeomorphism $\mathbb{R}P^2 \to \tilde{T}$ mapping the set of real points $\mathbb{R}A$ of A onto $\tilde{L}$.*

The described construction is called the *combinatorial patchworking*. An example of a combinatorial patchworking of a curve of degree 3 in $\mathbb{R}P^2$ is shown in Figure 6.

A curve A obtained by the combinatorial patchworking is called a *T-curve*. A polynomial representing A can be written down explicitly: if t is positive and sufficiently small, the polynomial $\sum_{(i,j)\in V} \delta_{i,j} x_0^i x_1^j x_2^{m-i-j} t^{\nu(i,j)}$ (where V is the set of vertices of τ, and ν is a convex function certifying the convexity of τ) defines a curve with the properties described in Theorem 2.1.

The combinatorial patchworking can be naturally generalized to higher dimensions. The construction starts with the simplex T in $\mathbb{R}^n$ having the vertices $(0,0,0,\ldots,0,0)$, $(m,0,0,\ldots,0,0)$, $(0,m,0,\ldots,0,0)$, $\ldots$, $(0,0,0,\ldots,0,m)$. One chooses a convex triangulation τ of T such that all the vertices of τ have integer coordinates, and a distribution of signs $\delta : V \to \{+,-\}$ at the set V of vertices of τ. Take the union T_* of 2^n symmetric copies of T under the compositions of reflections with respect to coordinate hyperplanes. Extend τ to a symmetric triangulation τ_* of T_*, and δ to a distribution of signs at the vertices of τ_* using the same

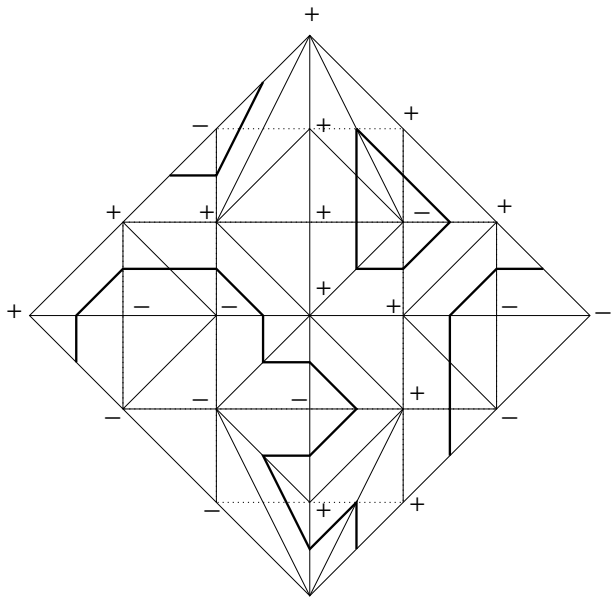


Figure 6: Combinatorial patchworking of a curve of degree 3 in $\mathbb{R}P^2$

rule as above (replacing the words *axis* by the words *hyperplane*). If an n-simplex σ of τ_* has vertices of different signs, select the convex hull of the middles of edges of σ which have vertices of opposite signs. Let L be the union of all the selected hyperplane pieces, $\tilde{T}$ the quotient space (homeomorphic to $\mathbb{R}P^n$) of T_* under the identification of the points on the boundary which are symmetric with respect to the origin, and $\tilde{L}$ the image of L in $\tilde{T}$.

Theorem 2.2 (O. Viro) *If the triangulation τ of T is convex, then there exist a nonsingular hypersurface A of degree m in $\mathbb{R}P^n$ and a homeomorphism $\mathbb{R}P^n \to \tilde{T}$ mapping the set of real points $\mathbb{R}A$ of A onto $\tilde{L}$.*

Such a hypersurface A in $\mathbb{R}P^n$ is called a *T-hypersurface*.

3 Applications of combinatorial patchworking

3.1 Counterexamples to Ragsdale conjecture

In 1906, V. Ragsdale [52] proposed the following conjecture.

Conjecture 3.1 (Ragsdale conjecture) *For a nonsingular curve of degree $2k$ in $\mathbb{R}P^2$, one has*

$$p \leq \frac{3k(k-1)}{2} + 1, \quad n \leq \frac{3k(k-1)}{2}.$$

Ragsdale also formulated weaker conjectures. About 30 years later, I. G. Petrovsky [48, 49] proved one of them.

Theorem 3.2 (Petrovsky inequalities, see [49]) *For a nonsingular curve of degree $2k$ in $\mathbb{R}P^2$, one has the inequalities*

$$p - n \leq \frac{3k(k-1)}{2} + 1, \quad n - p \leq \frac{3k(k-1)}{2}.$$

It is clear from [48] and [49] that Petrovsky was not familiar with Ragsdale's paper. He also formulated conjectures on the upper bounds for p and n. His conjecture about n was more cautious (by 1) than Ragsdale's one.

Both the Ragsdale conjecture and Petrovsky's one are wrong. The Ragsdale conjecture for n was disproved by O. Viro [70] in 1979: in his counterexamples $n = 3k(k-1)/2 + 1$. So, these curves do not contradict to Petrovsky's conjecture.

The first counterexamples to the Ragsdale conjecture for p and to Petrovsky's conjecture for n were constructed in [25] (see also [26] and [32]). In these counterexamples, the difference between p (or n) and $3k(k-1)/2 + 1$ is a quadratic function of k: they are curves of degree $2k$ in $\mathbb{R}P^2$ with

$$p = \frac{3k(k-1)}{2} + 1 + \left[\frac{(k-3)^2 + 4}{8}\right],$$

and curves of degree $2k$ in $\mathbb{R}P^2$ with

$$n = \frac{3k(k-1)}{2} + \left[\frac{(k-3)^2 + 4}{8}\right].$$

These counterexamples were improved by B. Haas [20] who constructed curves of degree $2k$ in $\mathbb{R}P^2$ with

$$p = \frac{3k(k-1)}{2} + 1 + \left[\frac{k^2 - 7k + 16}{6}\right].$$

In the best counterexamples (see [28]) to the Ragsdale conjecture that are currently known, the difference between p and $3k(k-1)/2 + 1$ (resp., between n and $3k(k-1)/2 + 1$) is asymptotically equivalent of $9k^2/48$.

The counterexamples to the Ragsdale conjecture are constructed using the combinatorial patchworking, *i.e.*, they are T-curves. The constructions of [25, 26, 32, 28] and [20] are based on the two main observations which are formulated in Lemmas 3.3 and 3.4.

One can also mention here the construction proposed by F. Santos [57]. This construction produces, via the combinatorial patchworking, piecewise-linear curves with more even ovals than in the examples above. However, the triangulation used in Santos' construction is not convex, so one cannot affirm the existence of corresponding algebraic curves. It follows from the results of [31] that the real schemes of Santos' piecewise-linear curves can be realized by real pseudo-holomorphic curves.

A distribution of signs at the integer points of T is called a *Harnack* one if

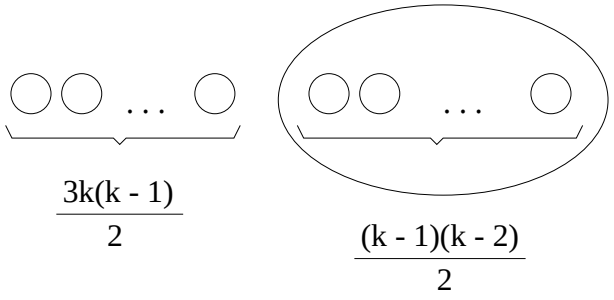


Figure 7: Real scheme of the simplest Harnack curve of degree $2k$ in $\mathbb{R}P^2$

- any two points (i_1, j_1) and (i_2, j_2) such that

$$i_1 \equiv i_2 \mod 2, \quad j_1 \equiv j_2 \mod 2$$

  have the same sign;

- the product of signs at the points $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ is negative.

A triangulation τ of T is called *primitive* if all the triangles of τ are of area $1/2$.

Lemma 3.3 (see [25]) *Let $m = 2k \geq 2$. Then, the combinatorial patchworking applied to an arbitrary primitive convex triangulation of T and a Harnack distribution of signs produces an M-curve with the real scheme shown in Figure 7.*

Proof. An integer point (i, j) of T is called *interior* if it lies strongly inside T, *even* if both i and j are even, and *odd* if at least one of its coordinates i and j is odd. Permuting the quadrants and switching if necessary all the signs to the opposite ones (pluses for minuses and minuses for pluses), we can suppose that the distribution of signs is as follows:

all the even points get "$-$", and all the odd points get "$+$".

Any even interior point (i, j) of T gives rise to an oval of the curve L. This oval encircles (i, j) and is contained in the star of (i, j). Any odd interior point (i, j) of T also gives rise to an oval of L. This oval encircles the symmetric copy $\sigma(i, j)$ of (i, j) equipped with "+", and is contained in the star of $\sigma(i, j)$. We have found $\frac{(m-1)(m-2)}{2}$ ovals and, thus, the curve L can have only one oval more. This oval does exist, because L intersects the coordinate axes. To finish the proof it remains to notice that the union of the segments

$$\{i - j = -m,\ i \leq 0,\ j \geq 0\} \cup$$
$$\{-m \leq i \leq 0,\ j = 0\} \cup \{i = 0,\ -m \leq j \leq 0\}$$

contains only minuses, and its image in $\tilde{T}$ is not contractible. This means that $3k(k-1)/2$ ovals corresponding to the odd interior points of T (and encircling pluses) are situated outside of the non-empty oval. $\square$

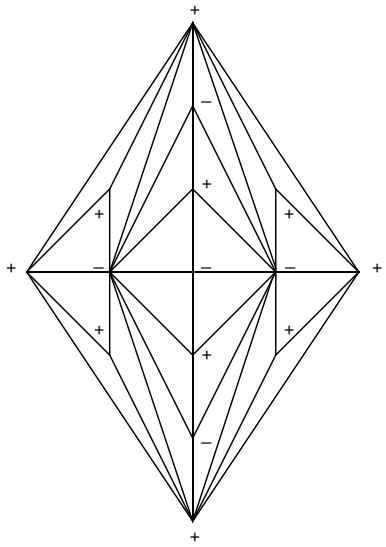


Figure 8: Diamond S

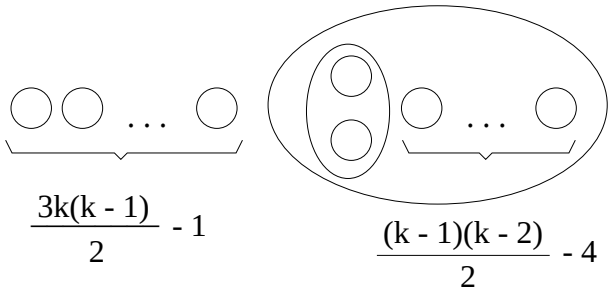


Figure 9: Real scheme of a curve of degree $2k$ in $\mathbb{R}P^2$ with $p = 3k(k-1)/2 + 2$

The distribution of signs mentioned in the proof of Lemma 3.3 (all the even points get "$-$", and all the odd points get "$+$") is called the *standard Harnack distribution.*

Let S be a diamond equipped with a primitive triangulation and a distribution of signs shown in Figure 8.

Lemma 3.4 (see [25]) *Let $m = 2k$, and suppose that the diamond S is placed inside T in such a way that the center of S has both the coordinates odd. Extend the triangulation of S to a primitive convex triangulation of T. At the integer points of $T \setminus S$ take the standard Harnack distribution of signs. Then the combinatorial patchworking applied to the described triangulation and distribution of signs produces a curve with the real scheme shown in Figure 9. The number of even ovals of this curve is equal to $3k(k-1)/2 + 2$.*

Proof. Straightforward. □

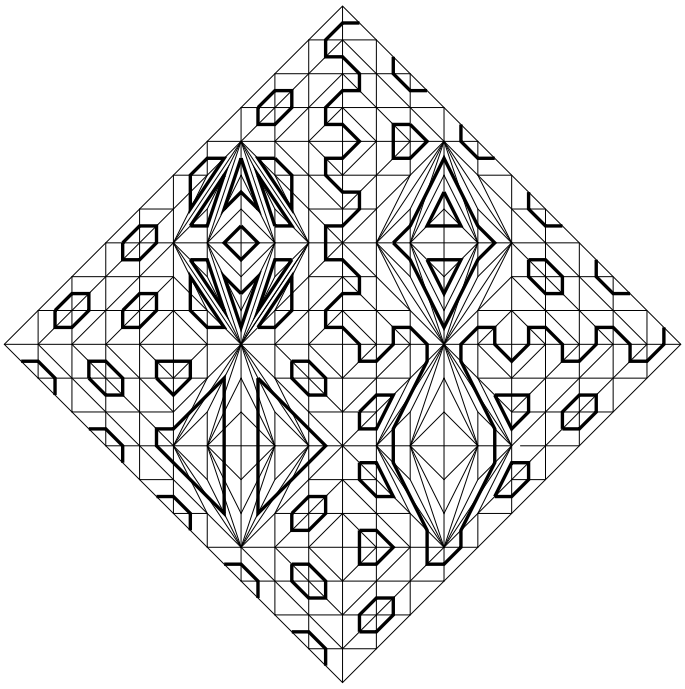


Figure 10: Combinatorial patchworking of a counterexample to the Ragsdale conjecture in degree 10 with $p = 32$

The combinatorial patchworking of a counterexample to Ragsdale conjecture in degree 10 (for the degrees ≤ 8 the Ragsdale conjecture is true) is shown in Figure 10.

Placing N copies of S in T in such a way that the center of each copy has both the coordinates odd, we can construct a curve of degree $2k$ with $p = 3k(k-1)/2+1+N$. For example, to obtain the counterexamples described in [25], it remains to note that we can place $\left[\frac{(k-3)^2+4}{8}\right]$ copies of S respecting the condition on the coordinates of the centers.

An important open question is to find a sharp upper bound for the number of even (resp., odd) ovals of a curve of degree $2k$ in $\mathbb{R}P^2$. For example, the Harnack and Petrovsky inequalities give the bounds $p \leq 7k^2/4 - 9k/4 + 3/2$ and $n \leq 7k^2/4 - 9k/4 + 1$, but, probably, these bounds are not sharp. Another question is whether the Ragsdale conjecture is true for M-curves (all the known counterexamples are not M-curves; for maximal T-curves the inequality $p \leq 3k(k-1)/2 + 1$ was proved by Haas [21]). These questions are directly related to the questions on possible values of Betti numbers of real algebraic surfaces (see Section 3.2).

3.2 Betti numbers of real algebraic surfaces

Let X be a complex algebraic variety, and $c : X \to X$ an antiholomorphic involution. The pair (X, c) is called a *real algebraic variety*. The fixed point set $\mathbb{R}X$ of c is called the *real part* of (X, c). An important question in topology of real algebraic varieties is to determine, for a given complex algebraic variety X, the possible values of Betti

numbers of $\mathbb{R}X$. A similar question concerns the possible values of Betti numbers of the real point set of a hypersurface of a given degree in $\mathbb{R}P^n$. For any topological space Y denote by $b_i(Y)$ the dimension of the homology group $H_i(Y;\mathbb{Z}/2)$, and put $b_*(Y) = \sum_i^\infty b_i(Y)$. The first information on the possible values of Betti numbers of $\mathbb{R}X$ is given by the generalized Harnack inequality, generalized Petrovsky-Oleinik inequalities, and Rokhlin and Kharlamov-Gudkov-Krakhnov congruences.

Theorem 3.5 (Generalized Harnack inequality, see [66, 12, 68, 34, 54]) *For a real algebraic variety (X, c), one has the inequality $b_*(\mathbb{R}X) \leq b_*(X)$.*

In the case of real algebraic curves, the inequality of Theorem 3.5 becomes the Harnack inequality. A real algebraic variety (X, c) such that $b_*(\mathbb{R}X) = b_*(X)$ is called *maximal* (or an *M-variety*). Note that for any real algebraic variety (X, c) one has $b_*(\mathbb{R}X) \equiv b_*(X) \mod 2$. If $b_*(\mathbb{R}X) = b_*(X) - 2a$, then the real algebraic variety (X, c) is called an *$(M - a)$-variety*.

Theorem 3.6 (Generalized Petrovsky-Oleinik inequalities, see [36]) *For a compact complex Kähler manifold of dimension $2k$ with an antiholomorphic involution c, one has the inequalities $2 - h^{k,k}(X) \leq \chi(\mathbb{R}X) \leq h^{k,k}(X)$, where $\chi(\mathbb{R}X)$ is the Euler characteristic of the fixed point set $\mathbb{R}X$ of c, and $h^{k,k}(X)$ is the middle Hodge number of X.*

The generalized Petrovsky-Oleinik inequalities incorporate the Petrovsky inequalities (the case of double planes), the Comessatti inequalities (the case of real algebraic surfaces, see [8]) and Petrovsky-Oleinik inequalities (the case of projective hypersurfaces, see [43]).

Theorem 3.7 (Rokhlin congruence, see [54]) *For a maximal real algebraic variety (X, c) of even (complex) dimension, one has the congruence $\chi(\mathbb{R}X) \equiv \sigma(X) \mod 16$, where $\sigma(X)$ is the signature of X.*

The Rokhlin congruence in the case of double planes gives the Gudkov-Rokhlin congruence for real algebraic curves.

Theorem 3.8 (Kharlamov-Gudkov-Krakhnov congruence, see [35] and [17]) *For a real algebraic $(M - 1)$-variety (X, c) of even (complex) dimension, one has the congruence $\chi(\mathbb{R}X) \equiv \sigma(X) \pm 2 \mod 16$.*

We give here a brief survey of known results on the Betti numbers of real algebraic surfaces in 3-dimensional projective space (for a detailed information on general real algebraic surfaces see [9, 10, 40, 65, 77]). For real algebraic surfaces the question on the individual Betti numbers of the real part reduces to that on the number of connected components and the number of handles. It turns out that almost all recent results concerning the possible values of Betti numbers of real algebraic surfaces in $\mathbb{R}P^3$ are constructions of surfaces with record values of individual Betti numbers.

Let us start with the discussion of the following question: *what is the possible values of $b_0(\mathbb{R}X)$, where X is a nonsingular surface of degree m in $\mathbb{R}P^3$?* The answer

is known for $m \leq 4$, *i.e.*, exactly in the cases when the topological classification is available. It follows from the generalized Harnack and Comessatti inequalities, that the number of connected components of the real point set of a nonsingular surface of degree m in $\mathbb{R}P^3$ is at most $5m^3/12 - 3m^2/2 + 25m/12$. In particular, $b_0(\mathbb{R}X) \leq 25$ for a nonsingular surface X of degree 5 in $\mathbb{R}P^3$. No nonsingular surface of degree 5 in $\mathbb{R}P^3$ with 24 or 25 connected components is known. A nonsingular surface of degree 5 with 23 connected components was constructed by S. Orevkov (see [45]). The construction presented in [45] is a modification of the construction of a nonsingular surface of degree 5 with 22 connected components proposed in [29]. For algebraic surfaces of higher degrees in $\mathbb{R}P^3$, the asymptotic difference between the number of connected components in the best examples currently known and the estimate $5m^3/12 - 3m^2/2 + 25m/12$ is big enough. F. Bihan [3] proved that for any given $\epsilon > 0$, there exist a family $\{X_m,\ m \in \mathbb{N}\}$, where X_m is a nonsingular surface of degree m in $\mathbb{R}P^3$, and a constant C such that

$$b_0(\mathbb{R}X_m) > (\frac{13}{36} - \epsilon) \cdot m^3 - C.$$

Consider now the similar question on the first Betti number: *what is the possible values of $b_1(\mathbb{R}X)$, where X is a nonsingular surface of degree m in $\mathbb{R}P^3$?* Again the answer is known only for $m \leq 4$. The generalized Harnack and Comessatti inequalities imply that the first Betti number of the real point set of a nonsingular surface of degree m in $\mathbb{R}P^3$ is at most $5m^3/6 - 3m^2 + 25m/6 - 1$. In particular, $b_1(\mathbb{R}X) \leq 49$ for a nonsingular surface X of degree 5 in $\mathbb{R}P^3$. A surface of degree 5 with the first Betti number of the real part equal to 49 would be an M-surface and would contradict to the Rokhlin congruence. Hence, for a nonsingular surface X of degree 5 in $\mathbb{R}P^3$, one has $b_1(\mathbb{R}X) \leq 47$.

O. Viro [74] proposed the following conjecture: *let X be a nonsingular simply connected compact complex surface with an antiholomorphic involution $c : X \to X$; then $b_1(\mathbb{R}X) \leq h^{1,1}(X)$, where $\mathbb{R}X$ is the fixed point set of c.* This conjecture would imply that the first Betti number of the real point set of a nonsingular surface of degree m in $\mathbb{R}P^3$ is at most $2m^3/3 - 2m^2 + 7m/3$, and in particular, that $b_1(\mathbb{R}X) \leq 45$ for a nonsingular surface X of degree 5 in $\mathbb{R}P^3$.

Any nonsingular curve of degree $2k$ in $\mathbb{R}P^2$ with $p > \frac{3k^2-3k+2}{2}$ produces a counterexample to Viro's conjecture: one can take a double plane ramified along the complex point set of a counterexample to the Ragsdale conjecture with an appropriate choice of a lifting of the involution of complex conjugation. Thus, the counterexamples to Ragsdale conjecture obtained in [25] show that Viro's conjecture is not true. Counterexamples to Viro's conjecture among algebraic surfaces in $\mathbb{R}P^3$ where constructed in [27] and [4] (the counterexamples of [27] are of degrees ≥ 10; the counterexample of [4] is of degree 6). Recall that a counterexample in degree 5 would have the first Betti number of the real part equal to 47. Bihan [5] constructed a real numerical quintic surface X with $b_1(\mathbb{R}X) = 47$, and, hence, showed that the maximal possible first Betti number of the real part of a real numerical quintic is 47. It is not known whether a surface of degree 5 in $\mathbb{R}P^3$ can have the first Betti number of the real part equal to 47.

For algebraic surfaces of higher degrees in $\mathbb{R}P^3$, again the asymptotic difference between the first Betti number in the best examples currently known and the estimate $5m^3/6 - 3m^2 + 25m/6 - 1$ is big enough. Bihan [3] proved that for any given $\epsilon > 0$ there exist a family $\{X_m,\ m \in \mathbb{N}\}$, where X_m is a nonsingular surface of degree m in $\mathbb{R}P^3$, and a constant C such that

$$b_1(\mathbb{R}X_m) > (\frac{13}{18} - \epsilon) \cdot m^3 - C.$$

Note that any improvement of counterexamples to the Ragsdale conjecture gives automatically, via Bihan's construction [3], a new asymptotic lower bounds for the maximal number of connected components and the maximal first Betti number of a surface of a given degree in $\mathbb{R}P^3$.

The constructions of real algebraic surfaces with record Betti numbers mentioned in this section are, in a sense, based on the construction of counterexamples to the Ragsdale conjecture [25, 26, 32]. All these constructions use combinatorial patchworking. Some of them ([29, 45, 5]) combine the combinatorial patchworking with a construction of quintic surfaces due to E. Horikawa (see [24]). Two Bihan's constructions (see [3] and [4]) combine the combinatorial patchworking with the doubling construction (the latter produces surfaces of degree $2k$ in $\mathbb{R}P^3$ given by polynomials of the form $P(x_0, x_1, x_2, x_3) = P_k(x_0, x_1, x_2, x_3)^2 + tP_{2k}(x_0, x_1, x_2, x_3)$, where P_k and P_{2k} are homogeneous polynomials of degrees k and $2k$, respectively, and $t > 0$ is a small parameter). The main fragment used in the record constructions is either the diamond S (see Figure 8) or a half of S (as in the constructions of [29] and [45]).

Another important application of the combinatorial patchworking concerning the Betti numbers of real algebraic varieties is a construction, for any natural m and n, of a maximal hypersurface of degree m in $\mathbb{R}P^n$ (see the forthcoming paper [33]).

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