Asymptotics and stability for global solutions to the Navier-Stokes equations

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Abstract

We consider an a priori global strong solution to the Navier-Stokes equations. We prove it behaves like a small solution for large time. Combining this asymptotics with uniqueness and averaging in time properties, we obtain the stability of such a global solution.

Introduction

We consider the incompressible Navier-Stokes equations in $\mathbb{R}^3$,

$$\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t} & = & \Delta u - \nabla\cdot(u\otimes u) - \nabla\pi, \\ \nabla\cdot u & = & 0, \\ u(x,0) & = & u_0(x). \end{array}\right. \tag{1}$$

There exist essentially two different kinds of results on the Cauchy problem for these equations. In the pioneering work [15], Jean Leray introduced the concept of weak solutions and proved global existence for datum $u_0 \in L^2$. However, their uniqueness (or propagation/breakdown of regularity for smooth data) has remained an open problem. In [11], H. Fujita and T. Kato obtained solutions for datum $u_0 \in \dot{H}^{\frac{1}{2}}$ by semi-group methods. These solutions are unique ([8]) but only local in time: $u \in C([0,T^\star),\dot{H}^{\frac{1}{2}})$, unless one is willing to make a smallness assumption on the datum. This line of work has been subsequently extended by many authors, see [14] for a bibliography. The most recent result states global well-posedness for small datum in BMO^{-1}, [13]. On the other hand, in an attempt to bridge the gap between weak L^2 solutions and strong L^3 (or $\dot{H}^{1/2}$) solutions, C. Calderòn proved existence of global weak solutions with datum in L^p, $2 \le p \le 3$, [3]. These results were later recovered independently by P.-G. Lemarié and extended to uniformly locally L^2 datum [14].

The theory of weak solutions is intimately tied to the specific structure of the Navier-Stokes equations, and in particular to the energy inequality. On the other hand, Kato's approach is more general and it can be applied to many parabolic (or dispersive) semilinear equations, as it does not use in any way the special form of the Navier-Stokes equations. One can relate the use of different spaces for the datum to scaling considerations: the energy inequality involves the L^2 norm, which is below the scale-invariant norm for the equations, namely L^3. Hence the Navier-Stokes equations can be said to be "supercritical" with respect to scaling. This seems to preclude any attempt to use the energy inequality to derive some information for strong solutions. In [3], the weak and strong theories are blended together to provide infinite energy weak solutions, including the case of L^3 datum. In [10], we used a similar approach in 2D: the weak and strong theories coincide to provide global strong L^2 solutions, and we extended global existence of strong solutions to large datum between L^2 and BMO^{-1}. In the present work, we develop this approach in 3D and study a priori global strong solutions. Let us consider a particular case of our results: take a strong solution $u \in C_t(L^3)$. Such a solution admits a maximal time of existence $T^\star$. Let us suppose that $T^\star = +\infty$ (which is only proved for small data or special cases like axisymetric data). In a first step, we prove there cannot be blow-up at infinity. In fact, we prove a stronger result, which is decay to zero of the L^3 norm for large time. This smallness at infinity can be combined with persistence of local in time averages to prove that various mixed space-time norms are globally defined for such a global solution. Then, we proceed to prove the stability of this solution. Hence, the set of initial data in L^3 for which one has global existence is open. We note that the key step is to obtain decay at infinity: under such an assumption, combined with local space-time integrability, stability in L^3 was obtained in [12].

Our approach relies on frequency localization and paradifferential calculus, combined with smoothing properties of the heat kernel (namely, regularity gains through time averaging). The reader may consult [6] for a very nice presentation in the context of the Navier–Stokes equations. Hence the natural framework becomes data in Besov spaces, and we will indeed prove our results for datum $u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$, with $1 \leq p, q < \infty$. Remark we miss the end-point BMO^{-1}, for which our techniques are known to break down. One has instead to rely on pointwise decay estimates for the parabolic linearized equations. Estimates of this type have been developped by P. Tchamitchian ([20]), and allow to recover the L^3 stability result under a smallness at infinity condition: such a condition has been derived in [14] as a by-product of the construction of locally L^2 weak solutions. We believe that the BMO^{-1} stability can be treated by combining our approach of the asymptotics (which can be made independent of the technical tools at use in the present work) with different estimates on the parabolic flow (after completion of the present work, we were informed that this was indeed carried out [1]).

Let us in this introduction state a theorem which does not require any Besov spaces, at least in its statement. It states the L^3 case (covered later by Theorem 3.2), and is the counterpart of the $\dot{H}^{\frac{1}{2}}$ case stated in [9].

Theorem 0.1 (Stability in L^3)
Let $u \in C_t(L^3)$ be an a priori global solution to (1). Then,

- *this solution tends to zero at infinity in L^3,*

$$\lim_{t \to +\infty} \|u(\cdot, t)\|_3 = 0,$$

- *this solution is stable: there exists $\varepsilon(u)$ such that if $\|v_0 - u_0\|_3 < \varepsilon(u)$, the local solution $v \in C_t(L^3)$ to (1) is global, with*

$$\sup_{t\geq 0} \|v(\cdot,t) - u(\cdot,t)\|_3 < C(u)\|v_0 - u_0\|_3.$$

We refer to Theorems 1.1, 2.1, 3.1 and 3.2 for precise statements in the more general Besov setting.

The rest of the paper is organized as follows. In the first section we study blow-up and persistence of various space-time norms which appear naturally in constructing the solution. The second section addresses the behavior at large time of an a priori global solution, and the third section deals with the stability of such a solution. The fourth and last section is devoted to a priori estimates for a parabolic equation with lower order terms, which are of constant use in the previous sections. In the appendix, we recall several known results on existence and properties of solutions in Besov spaces. Some of them can be easily derived from the estimates in the fourth section, and this allows the presentation to be essentially self-contained.

1 On the blow-up of strong solutions

For the convenience of the reader, we recall the usual definition of Besov spaces.

Definition 1.1
Let ϕ be a function in $\mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\phi} = 1$ for $|\xi| \leq 1$ and $\widehat{\phi} = 0$ for $|\xi| > 2$, and define $\phi_j(x) = 2^{nj}\phi(2^j x)$. Then the frequency localization operators are defined by

$$S_j = \phi_j * \cdot, \quad \Delta_j = S_{j+1} - S_j.$$

Let f be in $\mathcal{S}'(\mathbb{R}^n)$. We say f belongs to $\dot{B}^s_{p,q}$ if and only if

- *The partial sum $\sum_{-m}^{m} \Delta_j(f)$ converges to f as a tempered distribution if $s < \frac{n}{p}$ and after taking the quotient with polynomials if not.*
- *The sequence $\epsilon_j = 2^{js}\|\Delta_j(f)\|_{L^p}$ belongs to l^q.*

We will also need a slight modification of those spaces, taking into account the time variable; we refer to [7] for the introduction of that type of space in the context of the Navier–Stokes equations.

Definition 1.2
Let $u(x,t) \in \mathcal{S}'(\mathbb{R}^{n+1})$ and let Δ_j be a frequency localization with respect to the x variable. We will say that $u \in \widetilde{L^\rho}((a,b), \dot{B}^s_{p,q})$ if and only if

$$2^{js}\|\Delta_j u\|_{L^\rho((a,b),L^p_x)} = \varepsilon_j \in l^q,$$

and other requirements are the same as in the previous definition. We define

$$\|u\|_{\widetilde{L^\rho}((a,b),\dot{B}^s_{p,q})} \stackrel{\text{def}}{=} \|2^{js}\|\Delta_j u\|_{L^\rho((a,b),L^p_x)}\|_{\ell^q},$$

and we will note $\widetilde{L^\rho_T}(\dot{B}^s_{p,q}) \stackrel{\text{def}}{=} \widetilde{L^\rho}((0,T),\dot{B}^s_{p,q})$.

Remark 1.1
In the case when $\rho \geq q$ *one has of course* $\widetilde{L^\rho}((a,b),\dot{B}^s_{p,q}) \subset L^\rho((a,b),\dot{B}^s_{p,q})$. *Most notably,* $\widetilde{L^2}((0,T),\dot{H}^1) = L^2((0,T),\dot{H}^1)$ *and* $\widetilde{L^\infty_{loc}}(\dot{B}^s_{2,2}) \subset L^\infty_{loc}(\dot{H}^s) \subset L^2_{loc}(\dot{H}^s)$.

In the following we shall note

$$s_p \stackrel{\text{def}}{=} -1 + \frac{3}{p}\cdot$$

The results we prove in the sequel hold for p and q in $[1,\infty)$. We shall actually suppose in all proofs of this paper that $p \geq 3$ since the other cases can be deduced from that one.

It is well-known that if we consider the Navier-Stokes equations with an initial data u_0 in $\dot{B}^{s_p}_{p,q}$, then there exists a unique local strong solution (see Theorem A.1). The aim of this section is to prove that as long as this solution is continuous in time with values in $\dot{B}^{s_p}_{p,q}$, blow-up does not occur. More precisely, we will prove the following theorem.

Theorem 1.1
Let $u_0 \in \dot{B}^{s_p}_{p,q}$ *be a divergence free vector-field. Let* u *be the local strong solution associated to* u_0 *and* $T^\star$ *the blow-up time, i.e.*

$$u \in C\big([0,T^\star);\dot{B}^{s_p}_{p,q}\big) \cap \widetilde{L^r_t}\dot{B}^{s_p+\frac{2}{r}}_{p,q} \quad \forall r \in [1,\infty], \quad \forall t < T^\star$$

and $T^\star$ *is maximal such that the above relation holds. Then, if* $T^\star < \infty$, *we must necessarily have that*

$$u \notin C\big([0,T^\star];\dot{B}^{s_p}_{p,q}\big).$$

Moreover, if $T^\star = \infty$ *and* $\lim_{t\to\infty} \|u(t)\|_{\dot{B}^{s_p}_{p,q}} = 0$ *then*

$$u \in \widetilde{L^r_\infty}\dot{B}^{s_p+\frac{2}{r}}_{p,q} \quad \forall r \in [1,\infty].$$

Remark 1.2
The blow-up time $T^\star$ *is well defined. Indeed, according to Remark A.2 and to the continuity in time of the norms we consider, the maximal time* $T^\star$ *may be chosen to be the blow-up time of one of the norms* $\widetilde{L^r}\dot{B}^s_{p,q}$ *with* $s \in (0,\frac{3}{p})$.

Remark 1.3
A well-known continuation (or blow-up) criterion is the reverse statement (see [5] for the $p=2$ case). If $u \in \widetilde{L^r_T}\dot{B}^{s_p+\frac{2}{r}}_{p,q}$, then it can be extended past T, or equivalently, if T is the maximal time of existence then $\|u\|_{\widetilde{L^r_t}\dot{B}^{s_p+\frac{2}{r}}_{p,q}}$ blows up at time $t=T$. However, this quantity could possibly blow-up at time T while the solution could extend in $C\big([0,T^\star);\dot{B}^{s_p}_{p,q}\big)$ for $T^\star > T$. Theorem 1.1 proves that this cannot happen.

Proof of Theorem 1.1.

We first consider the case $T^\star < \infty$. It will be equivalent to prove that if

$$u \in C\big([0,T^\star];\dot{B}^{s_p}_{p,q}\big)$$

then

$$u \in \widetilde{L^r_{T^\star}}\dot{B}^{s}_{p,q} \quad \forall r\in[1,\infty], \quad \text{with} \quad s = s_p + \frac{2}{r}\cdot$$

By Remark A.2, it is sufficient to consider the case $s > -s_p$ ($\Leftrightarrow r < -\frac{1}{s_p} = \frac{p}{p-3}$). Next, by density of smooth functions in $C\big([0,T^\star];\dot{B}^{s_p}_{p,q}\big)$, there exists some decomposition

$$u = u_1 + u_2$$

such that for all time,

$$\|u_1\|_{\dot{B}^{s_p}_{p,q}} \le \frac{1}{2K_1} \tag{2}$$

where K_1 is the constant of relation (3) below and u_2 is as smooth as we want. We now apply the estimate given in Proposition 4.1 to find

$$\|u\|_{\widetilde{L^r_t}\dot{B}^s_{p,q}} \le K_1\|u_0\|_{\dot{B}^{s_p}_{p,q}} + K_1\|u_1\|_{\widetilde{L^\infty_t}\dot{B}^{s_p}_{p,\infty}}\|u\|_{\widetilde{L^r_t}\dot{B}^s_{p,q}} + K_1\|u\|_{\widetilde{L^\infty_t}\dot{B}^{s_p}_{p,\infty}}\|u_2\|_{\widetilde{L^r_t}\dot{B}^s_{p,q}} \tag{3}$$

for some constant K_1. According to relation (2) we have that

$$\|u_1\|_{\widetilde{L^\infty_t}\dot{B}^{s_p}_{p,\infty}} = \sup_{[0,t]}\|u_1\|_{\dot{B}^{s_p}_{p,\infty}} \le \sup_{[0,t]}\|u_1\|_{\dot{B}^{s_p}_{p,q}} \le \frac{1}{2K_1}$$

which, in turn, implies that

$$\|u\|_{\widetilde{L^r_t}\dot{B}^s_{p,q}} \le K_1\|u_0\|_{\dot{B}^{s_p}_{p,q}} + \frac{1}{2}\|u\|_{\widetilde{L^r_t}\dot{B}^s_{p,q}} + K_1\|u\|_{\widetilde{L^\infty_t}\dot{B}^{s_p}_{p,\infty}}\|u_2\|_{\widetilde{L^r_t}\dot{B}^s_{p,q}}$$

that is

$$\begin{aligned}\|u\|_{\widetilde{L^r_t}\dot B^s_{p,q}} &\leq 2K_1\|u_0\|_{\dot B^{s_p}_{p,q}} + 2K_1\|u\|_{\widetilde{L^\infty_t}\dot B^{s_p}_{p,\infty}}\|u_2\|_{\widetilde{L^r_t}\dot B^s_{p,q}}\\ &\leq 2K_1\|u_0\|_{\dot B^{s_p}_{p,q}} + 2K_1\sup_{[0,t]}\|u\|_{\dot B^{s_p}_{p,q}}\|u_2\|_{\widetilde{L^r_t}\dot B^s_{p,q}}.\end{aligned}$$

Passing to the limit $t \to T^\star$ we finally get

$$\|u\|_{\widetilde{L^r_{T^\star}}\dot B^s_{p,q}} \leq 2K_1\|u_0\|_{\dot B^{s_p}_{p,q}} + 2K_1\sup_{[0,T^\star]}\|u\|_{\dot B^{s_p}_{p,q}}\|u_2\|_{\widetilde{L^r_{T^\star}}\dot B^s_{p,q}}$$

which completes the proof in the case $s \in (-s_p, s_p+2)$. The general case $s \in [s_p, s_p+2]$ follows from Remark A.2. This completes the proof in the case $T^\star < \infty$.

The case $T^\star = \infty$ follows immediately from the case $T^\star < \infty$. Indeed, since we know by hypothesis that $\lim\limits_{t\to\infty}\|u(t)\|_{\dot B^{s_p}_{p,q}} = 0$, there exists T such that

$$\|u(T)\|_{\dot B^{s_p}_{p,q}} \leq K_0,$$

where K_0 is the constant of Theorem A.1. According to that theorem, we have that

$$u \in \widetilde{L^\rho}((T,\infty), \dot B^{s_p+\frac{2}{r}}_{p,q}) \qquad \forall r \in [1,\infty].$$

The previous case implies that

$$u \in \widetilde{L^r_T}\dot B^{s_p+\frac{2}{r}}_{p,q} \qquad \forall r \in [1,\infty].$$

We infer that

$$u \in \widetilde{L^r_\infty}\dot B^{s_p+\frac{2}{r}}_{p,q} \qquad \forall r \in [1,\infty]$$

and this completes the proof of the theorem. □

It was pointed out to us by J.-Y. Chemin that one could derive Theorem 1.1 using an abstract argument. Indeed (see also [6]), as it can be seen from the local existence proof in the Appendix, the local existence time T is uniform for a compact set of initial data. Applying this result to the family $v_{0,t} = u(t)$ which is compact as $u([0,T])$, one immediately gets Theorem 1.1. It is however worth noting that such a property is true in a general setting, independent of the Navier-Stokes equation, and we therefore frame it in a rather generic way. Consider a semilinear equation

$$\partial_t u + Au = G(u), \qquad u(x,0) = u_0 \in E, \tag{4}$$

and suppose that if

$$\int_0^{T^\star} \|\exp(tA)u_0\|^p_F < \varepsilon_0,$$

then the time of existence of (4) is at least $T^\star$.

Suppose moreover that for all $v \in E$,

$$\int_0^{+\infty} \|\exp(tA)v\|_F^p \lesssim \|v\|_E^p, \tag{5}$$

with E, F Banach spaces and $p < \infty$ given. Then we have the following result.

Proposition 1.1
Let C be a compact set of initial data in E. Then there exists a uniform time of existence T such that all solutions to (4) with initial data in C exist up to time T.

Proof of Proposition 1.1
Let us consider an initial data u_0. There exists T such that

$$\int_0^T \|\exp(tA)u_0\|_F^p < \varepsilon_0/2,$$

and therefore the solution u to (4) exists up to T. Let us take v_0 such that $\|u_0 - v_0\|_E^p \lesssim \varepsilon_0/2$, from the definition of T and (5), we have

$$\int_0^T \|\exp(tA)v_0\|_F^p < \varepsilon_0,$$

which guarantees that the solution v to (4) with initial data v_0 exists up to T. Hence the local time of existence T is uniform in a small ball around u_0, with a radius which is moreover independent of u_0. A compact set can by definition be covered by a finite number of such balls, which ends the proof. $\square$

Remark 1.4
In our setting, one has to deal with $\widetilde{L}_t^p$ spaces rather than just L_t^p, but the same argument applies as well.

2 Large time behaviour of global solutions

We consider a global in time solution, and are interested by its asymptotics when $t \to +\infty$. For small data, we know (A.2) that the solution vanishes at infinity. On the other hand, for a weak L^2 solution, the energy is known to decrease to zero at large time. Both phenomenons are linked to the dissipative nature of the equation, and one is tempted to expect the same behaviour for any global solution, be it of finite energy or not. The next theorem proves this is indeed true for any a priori global solution.

To give the reader a sense of perspective, let us consider an a priori global solution in $C_t(L^3)$, but which moreover has an $L^3 \cap L^2$ datum. By weak-strong uniqueness ([22]), such a global solution coincides with the (unique in this case) Leray solution. Hence, it has finite energy, and by interpolation,

$$(\int_0^T \|u\|_3^4 ds)^{\frac{1}{2}} \lesssim \|u\|_{L_T^\infty(L^2)} \|\nabla u\|_{L_T^2(L^2)} \lesssim E_0.$$

Therefore the L^3 norm of u is necessarily small at some large time T, and one can conclude from the small data result. Thus, we have proved that for a global solution which is both a weak and a strong solution (with a large data), both norms go to zero. This result was already proven in [12], with an argument which is akin to ours. The proof of the following theorem relies on a suitable modification of this observation.

Theorem 2.1

Let $u_0 \in \dot{B}^{s_p}_{p,q}$, with $s_p \stackrel{def}{=} -1 + \dfrac{3}{p}$. Suppose its associate solution is global, $u \in C^0(\mathbb{R}^+, \dot{B}^{s_p}_{p,q})$, and unique (uniqueness is guaranteed for instance as soon as $u \in \widetilde{L^r}_{loc}(\mathbb{R}^+, \dot{B}^{s_p+\frac{2}{r}}_{p,q})$, for some $r \in (2, 2/(1-3/p))$, or as in Proposition A.1). Then

$$\lim_{t\to\infty} \|u(t)\|_{\dot{B}^{s_p}_{p,q}} = 0, \tag{6}$$

and $u \in \widetilde{L^r}(\mathbb{R}^+, \dot{B}^{s_p+\frac{2}{r}}_{p,q})$ for all $r \in [1, +\infty]$.

Remark 2.1

When the regularity s_p is positive, uniqueness in known to hold without any extra condition, in $C^0(\mathbb{R}^+, \dot{B}^{s_p}_{p,q})$, see [8]. However, when $s_p \leq 0$, one needs some additional condition to even define the nonlinear term in the equation. The condition $u \in \widetilde{L^r}_{loc}(\mathbb{R}^+, \dot{B}^{s_p+\frac{2}{r}}_{p,q})$ is one out of many which give uniqueness (from Theorem 1.1, $\widetilde{L^r}((0,\varepsilon),\cdot)$ is actually enough). Proposition A.1 gives a different meaning to uniqueness, by somehow renormalizing the solution to reduce to positive regularity. Of course the unique solution verifies $u \in \widetilde{L^r}_{loc}(\mathbb{R}^+, \dot{B}^{s_p+\frac{2}{r}}_{p,q})$.

Proof of Theorem 2.1.

Assuming the result (6), the second part of the theorem follows directly from Theorem 1.1. For the sake of simplicity, all "small" universal constants will be denoted by ε_0, which if necessary can be made smaller from one line to the other.

So all we need is to prove the result (6). We shall use the method introduced by C. Calderon in [3] to prove results on weak solutions in L^p spaces, and used in [10] in the context of 2D Navier–Stokes equations: we split the initial data into two parts,

$$u_0 = v_0 + w_0, \quad \text{with} \quad v_0 \in L^2 \quad \text{and} \quad \|w_0\|_{\dot{B}^{s_p}_{p,q}} \leq \varepsilon_0.$$

By the small data theory (Theorem A.1) we know that there is a unique solution w to the Navier–Stokes equations with data w_0, with

$$\forall r \in [1, +\infty], \quad \|w\|_{\widetilde{L^r}([0,T], \dot{B}^{s_p+\frac{2}{r}}_{p,q})} \leq \varepsilon_0,$$

and we recall moreover (Proposition A.2) that

$$\sup_{t \geq 0} \sqrt{t} \|w(t)\|_{L^\infty} \leq \varepsilon_0.$$

Now let us define $v \stackrel{\text{def}}{=} u - w$, which satisfies the following system.

$$\begin{cases} \partial_t v - \Delta v + u \cdot \nabla v + v \cdot \nabla w &= -\nabla \pi, \\ \nabla \cdot v &= 0, \\ v(x,0) &= v_0(x). \end{cases}$$

We know that $v \in C^0(\mathbb{R}^+, \dot{B}^{s_p}_{p,q}) \cap \widetilde{L^r}_{loc}(\mathbb{R}^+, \dot{B}^{s_p+\frac{2}{r}}_{p,q})$ since that result holds for both u and w. We claim that there exists a time $T^\star > 0$ such that

$$v \in \widetilde{L^\infty}([0,T^\star], L^2) \cap L^2([0,T^\star], \dot{H}^1). \tag{7}$$

The proof of the local regularity $\widetilde{L^\infty}([0,T^\star], L^2)$ is a direct application of Lemma A.2 and Remark 4.1. The regularity $L^2([0,T^\star], \dot{H}^1)$ follows from a repeated application of Proposition 4.2.

Since we know from (7) that v stays locally in L^2 let us now write an energy estimate in L^2, starting at some time $t_0 \in (0, T^\star)$. Recall that such an estimate can be entirely derived from the localized energy estimates on $\Delta_j v$, as written with great details in [10], Section 3. In our case the situation is even simpler as all the functions considered in the equation are in $C^0(\mathbb{R}^+, \dot{B}^{s_p}_{p,q}) \cap \widetilde{L^r}_{loc}(\mathbb{R}^+, \dot{B}^{s_p+\frac{2}{r}}_{p,q})$, and the computation below estimating the integral containing $v \cdot \nabla w$ justifies the fact that the other integral (containing $u \cdot \nabla v$) is zero.

We get

$$\|v(t)\|^2_{L^2} + 2\int_{t_0}^t \|\nabla v(s)\|^2_{L^2}\, ds = \|v(t_0)\|^2_{L^2} - 2\int_{\mathbb{R}^3}\int_{t_0}^t (v \cdot \nabla w) \cdot v \, ds dx. \tag{8}$$

But

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\int_{t_0}^t (v\cdot\nabla w)\cdot v\,dsdx\right| &\leq \int_{t_0}^t \|v(s)\|_{L^2}\|\nabla v(s)\|_{L^2}\|w(s)\|_{L^\infty}\,ds\\ &\leq \int_{t_0}^t \|v(s)\|_{L^2}\|\nabla v(s)\|_{L^2}\sqrt{s}\|w(s)\|_{L^\infty}\,\frac{ds}{\sqrt{s}},\end{aligned}$$

which, using the fact that $\sqrt{s}\|w(s)\|_{L^\infty}$ is uniformly bounded by ε_0 implies that

$$\left|\int_{\mathbb{R}^3}\int_{t_0}^t (v\cdot\nabla w)\cdot v\,dsdx\right| \leq \frac{1}{2}\int_{t_0}^t \|\nabla v(s)\|_{L^2}^2\,ds+\frac{\varepsilon_0^2}{2}\int_{t_0}^t \|v(s)\|_{L^2}^2\,\frac{ds}{s}\cdot$$

So we get, plugging that estimate into (8),

$$\|v(t)\|_{L^2}^2+\int_{t_0}^t \|\nabla v(s)\|_{L^2}^2\,ds \leq \|v(t_0)\|_{L^2}^2+\varepsilon_0^2\int_{t_0}^t \|v(s)\|_{L^2}^2\,\frac{ds}{s}\cdot$$

We now use Gronwall's lemma, which yields

$$\|v(t)\|_{L^2}^2+\int_{t_0}^t \|\nabla v(s)\|_{L^2}^2\,ds \leq \|v(t_0)\|_{L^2}^2\left(\frac{t}{t_0}\right)^{\varepsilon_0^2}.$$

Now by Sobolev embedding and interpolation we have

$$\int_{t_0}^t \|v(s)\|_{\dot B^{s_p}_{p,q}}^4\,ds \lesssim \int_{t_0}^t \|v(s)\|_{\dot H^{1/2}}^4\,ds \leq \int_{t_0}^t \|v(s)\|_{L^2}^2\|\nabla v(s)\|_{L^2}^2\,ds,$$

which by the above estimate yields

$$(t-t_0)\inf_{[t_0,t]}\|v(s)\|_{\dot B^{s_p}_{p,q}}^4 \lesssim \|v(t_0)\|_{L^2}^4\left(\frac{t}{t_0}\right)^{\varepsilon_0^2}.$$

Finally we have obtained

$$\inf_{[t_0,t]}\|v(s)\|_{\dot B^{s_p}_{p,q}} \lesssim \|v(t_0)\|_{L^2}\left(\frac{t}{t_0}\right)^{\varepsilon_0^2}(t-t_0)^{-1/4}.$$

In particular we can write, for all $t\geq t_0+1$,

$$\inf_{[t_0,t]}\|v(s)\|_{\dot B^{s_p}_{p,q}} \leq C(t_0)\|v(t_0)\|_{L^2}t^{\varepsilon_0^2-1/4},$$

which can be made arbitrarily small for $\varepsilon_0<\frac{1}{2}$ and t large enough.

It follows that one can find a time τ_0 such that,

$$\|v(\tau_0)\|_{\dot{B}^{s_p}_{p,q}} \leq \varepsilon_0$$

and since $\|w\|_{L^\infty(\mathbb{R}_+,\dot{B}^{s_p}_{p,q})} \leq \varepsilon_0$ we infer that $\|u(\tau_0)\|_{\dot{B}^{s_p}_{p,q}} \leq 2\varepsilon_0$. We conclude by the small data theory (Proposition A.2):

$$\lim_{t\to\infty} \|u(t)\|_{\dot{B}^{s_p}_{p,q}} = 0$$

and Theorem 2.1 is proved. □

3 Stability of global solutions

We are now in a position to address the stability of an a priori global solution. We prove that the flow associated to the Navier-Stokes equation is Lipschitz: perturbating a global solution gives again a global solution, which moreover stays close to the given one. Note that the first part of that statement guarantees the set of initial conditions for which a global solution exists to be open. The second part, which is stability, is actually a stronger property.

THEOREM 3.1
Let $u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$ be a divergence free vector field, generating a global solution u, continuous in $\dot{B}^{\frac{3}{p}-1}_{p,q}$. Suppose that this is (the extension of) the solution obtained by fixed point on a small time interval, or that this solution is the unique solution in the sense of Proposition A.1.

Then there is an η_0 (depending on $p, q, \|u\|_{\widetilde{L^r}(\mathbb{R}^+,\dot{B}^{\frac{3}{p}+\frac{2}{r}-1}_{p,q})}$ for some $2 < r < 2/(1-3/p)$), such that for any divergence free vector field $v_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$ satisfying $\|v_0 - u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leq \eta_0$, its associate solution v satisfies, for $1 \leq r \leq +\infty$

$$v \in C^0(\mathbb{R}^+, \dot{B}^{\frac{3}{p}-1}_{p,q}) \cap \widetilde{L^r}(\mathbb{R}^+, \dot{B}^{\frac{3}{p}+\frac{2}{r}-1}_{p,q}),$$

with

$$\sup_{t\geq 0} \|v(t)-u(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} + \|v-u\|_{\widetilde{L^r}(\mathbb{R}^+,\dot{B}^{\frac{3}{p}+\frac{2}{r}-1}_{p,q})} < C_u \|v_0-u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}, \tag{9}$$

where C_u depends on p, q and $\|u\|_{\widetilde{L^r}(\mathbb{R}^+,\dot{B}^{\frac{3}{p}+\frac{2}{r}-1}_{p,q})}$.

REMARK 3.1
In [19], stability is proved for H^1 datum, under an additional assumption, namely it is required that $\int_0^\infty \|u\|^4_{H^1} dt < \infty$. For L^3 datum, stability is derived in [12], under the assumption $u \in C_0(\mathbb{R}^+, L^3)$, where the C_0 notation refers to continuous functions u such that $\lim_{t\to\infty} \|u\|_3 = 0$. Here, we dispense ourselves with such a priori assumptions, using the results of Section 2. Moreover, we work in scaling invariant norms, superceding both type of previous results.

Exactly as in the small data theory, one can dissociate the norm in which the smallness assumption is made and the regularity of the data. We will only state a case which is of particular interest, but the proof can be taylored to replace L^3 by virtually any space in the scales $\dot{B}^s_{p,q}$ or $\dot{F}^s_{p,q}$ (the Triebel-Lizorkin spaces), not necessarily scale-invariant with respect to the Navier-Stokes equations (heuristically it can be regarded as a byproduct of the "propagation of regularity" lemma from the appendix).

THEOREM 3.2
Let u and v be as in Theorem 3.1. Suppose moreover that $u_0, v_0 \in L^3$. Then both solutions are global in L^3, and

$$\sup_{t\geq 0} \|v(t) - u(t)\|_3 < \tilde{C}_u \|v_0 - u_0\|_3, \tag{10}$$

where $\tilde{C}_u$ depends on $\|u_0\|_3$ and C_u from (9).

PROOF OF THEOREM 3.1.

Consider a divergence free vector field $u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$ generating a global solution $u \in C^0(\mathbb{R}^+, \dot{B}^{\frac{3}{p}-1}_{p,q})$. Under the assumptions of the theorem, we know that

$$u \in \widetilde{L^r}(\mathbb{R}^+, \dot{B}^{\frac{3}{p}+\frac{2}{r}-1}_{p,q})$$

for all $r \in [1, +\infty]$ (see Theorem 2.1).

Now let $v_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$ be another divergence free vector field. Its associate solution, which a priori only has a finite life span, is called v, and we have

$$v \in \widetilde{L^r}([0,T], \dot{B}^{\frac{3}{p}+\frac{2}{r}-1}_{p,q}),$$

for all $r \in [1, +\infty]$ and for some time $T > 0$.

We fix for the rest of the proof some $r \in (2, 2/(1-3/p))$ so that $s = s_p + \frac{2}{r} \in (0, \frac{3}{p})$.

If w is defined by $w \stackrel{\text{def}}{=} v - u$, then it is enough to prove that for $\|w_{|t=0}\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ small enough,

$$w \in C^0(\mathbb{R}^+, \dot{B}^{\frac{3}{p}-1}_{p,q}) \cap \widetilde{L^r}(\mathbb{R}^+, \dot{B}^{\frac{3}{p}+\frac{2}{r}-1}_{p,q}).$$

The function w satisfies the following system:

$$\left\{ \begin{array}{rcl} \partial_t w - \Delta w + w \cdot \nabla w + u \cdot \nabla w + w \cdot \nabla u & = & -\nabla \pi, \\ \nabla \cdot w & = & 0, \\ w(x,0) & = & u_0(x) - v_0(x). \end{array} \right.$$

We deduce from Proposition 4.1 that w satisfies the following estimate:

$$(11)\quad \|w\|_{\widetilde{L^{r'}}([\alpha,\beta],\dot B^{s'}_{p,q})} \leq K_2\|w(\alpha)\|_{\dot B^{s_p}_{p,q}} + K_2\|w\|^2_{\widetilde{L^r}([\alpha,\beta],\dot B^{s}_{p,q})} + K_2\|w\|_{\widetilde{L^r}([\alpha,\beta],\dot B^{s}_{p,q})}\|u\|_{\widetilde{L^r}([\alpha,\beta],\dot B^{s}_{p,q})}$$

for some constant $K_2 > 1$ and all times α and β. The constant s' is arbitrary in $[s_p, s]$ and r' is determined by $s' = s_p + \frac{2}{r'}$.

We claim that there exist N real numbers $(T_i)_{1\leq i\leq N}$ such that $T_1 = 0$ and $T_N = +\infty$, satisfying

$$(12)\qquad \mathbb{R}_+ = \bigcup_{i=1}^{N-1}[T_i, T_{i+1}] \quad\text{and}\quad \|u\|_{\widetilde{L^r}([T_i,T_{i+1}],\dot B^{s}_{p,q})} \leq \frac{1}{4K_2} \quad \forall i\in\{1,\dots,N-1\}.$$

Let us prove that statement. Recall that since $q < +\infty$, we have

$$\|u\|_{\widetilde{L^r}([\alpha,\beta],\dot B^{s}_{p,q})} = \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j u\|^q_{L^r([\alpha,\beta],L^p)}\Big)^{\frac{1}{q}},$$

so there exists some integer M such that $u^M \stackrel{\text{def}}{=} \sum_{|j|\geq M}\Delta_j u$ satisfies

$$\|u^M\|_{\widetilde{L^r}(\mathbb{R}^+,\dot B^{s}_{p,q})} \leq \frac{1}{8K_2}.$$

Then to obtain the desired time decomposition for $u - u^M$, we use the fact that $r < +\infty$ and that one is only summing over a finite number of j's. The result (12) follows.

Now let us go back to the proof of the theorem. Suppose that

$$(13)\qquad \|w_0\|_{\dot B^{s_p}_{p,q}} \leq \frac{1}{8K_2 N(2K_2)^N}.$$

By time continuity we can define a maximal time $T\in\mathbb{R}^+\cup\{\infty\}$ such that

$$(14)\qquad \|w\|_{\widetilde{L^r}([0,T],\dot B^{s}_{p,q})} \leq \frac{1}{4K_2}.$$

If $T = \infty$ then the theorem is proved as a consequence of Remark A.2. Suppose now that $T < +\infty$. Then we can define an integer $k\in\{1,\dots,N-1\}$ such that

$$T_k \leq T < T_{k+1},$$

and plugging (12) and (14) into (11) with $s' = s$ we get for any $i \leq k-1$

$$\|w\|_{\widetilde{L^r}([T_i,T_{i+1}],\dot B^{s}_{p,q})} \leq K_2\|w(T_i)\|_{\dot B^{s_p}_{p,q}} + \frac{1}{4}\|w\|_{\widetilde{L^r}([T_i,T_{i+1}],\dot B^{s}_{p,q})} + \frac{1}{4}\|w\|_{\widetilde{L^r}([T_i,T_{i+1}],\dot B^{s}_{p,q})},$$

so finally

$$\|w\|_{\widetilde{L^r}([T_i,T_{i+1}],\dot B^s_{p,q})} \le 2K_2 \|w(T_i)\|_{\dot B^{s_p}_{p,q}}. \tag{15}$$

$>$From relation (11) with $s' = s_p$ we also get

$$\|w\|_{\widetilde{L^\infty}([T_i,T_{i+1}],\dot B^{s_p}_{p,q})} \le K_2 \|w(T_i)\|_{\dot B^{s_p}_{p,q}} + 2K_2 \frac{1}{4K_2} 2K_2 \|w(T_i)\|_{\dot B^{s_p}_{p,q}}$$

that is

$$\|w\|_{\widetilde{L^\infty}([T_i,T_{i+1}],\dot B^{s_p}_{p,q})} \le 2K_2 \|w(T_i)\|_{\dot B^{s_p}_{p,q}}. \tag{16}$$

Since $\widetilde{L^\infty}([T_i,T_{i+1}],\dot B^{s_p}_{p,q}) \subset L^\infty([T_i,T_{i+1}],\dot B^{s_p}_{p,q})$, we further infer that

$$\|w(T_{i+1})\|_{\dot B^{s_p}_{p,q}} \le 2K_2 \|w(T_i)\|_{\dot B^{s_p}_{p,q}}.$$

A trivial induction now shows that

$$\|w(T_i)\|_{\dot B^{s_p}_{p,q}} \le (2K_2)^{i-1} \|w_0\|_{\dot B^{s_p}_{p,q}} \qquad \forall i \in \{1,\dots,k-1\}.$$

We conclude from (15) and (16) that

$$\|w\|_{\widetilde{L^r}([T_i,T_{i+1}],\dot B^s_{p,q})} \le (2K_2)^i \|w_0\|_{\dot B^{s_p}_{p,q}}$$

and

$$\|w\|_{\widetilde{L^\infty}([T_i,T_{i+1}],\dot B^{s_p}_{p,q})} \le (2K_2)^i \|w_0\|_{\dot B^{s_p}_{p,q}}$$

for all $i \le k-1$. The same arguments as above also apply on the interval $[T_k,T]$ and yield

$$\|w\|_{\widetilde{L^r}([T_k,T],\dot B^s_{p,q})} \le (2K_2)^k \|w_0\|_{\dot B^{s_p}_{p,q}} \le (2K_2)^N \|w_0\|_{\dot B^{s_p}_{p,q}}$$

and

$$\|w\|_{\widetilde{L^\infty}([T_k,T],\dot B^{s_p}_{p,q})} \le (2K_2)^k \|w_0\|_{\dot B^{s_p}_{p,q}} \le (2K_2)^N \|w_0\|_{\dot B^{s_p}_{p,q}}.$$

Next, denoting $w_j(t) = 2^{js}\|\Delta_j w(t)\|_p$ one has

$$\begin{aligned}
\|w\|_{\widetilde{L^r}([0,T],\dot B^s_{p,q})} &= \Big\| \Big(\int_{T_1}^{T_2} w_j^r + \cdots + \int_{T_k}^{T} w_j^r \Big)^{\frac1r} \Big\|_{\ell^q} \\
&\le \Big\| \Big(\int_{T_1}^{T_2} w_j^r \Big)^{\frac1r} + \cdots + \Big(\int_{T_k}^{T} w_j^r \Big)^{\frac1r} \Big\|_{\ell^q} \\
&\le \|w\|_{\widetilde{L^r}([T_1,T_2],\dot B^s_{p,q})} + \cdots + \|w\|_{\widetilde{L^r}([T_k,T],\dot B^s_{p,q})} \\
&\le N(2K_2)^N \|w_0\|_{\dot B^{s_p}_{p,q}}.
\end{aligned}$$

Under assumption (13) this contradicts the maximality of T as defined in (14). So the theorem is proved. $\square$

Note that one has moreover an estimate of the type

$$\|w\|_{\widetilde{L^\infty}(\mathbb{R}^+,\dot B^{s_p}_{p,q})} \lesssim \|w_0\|_{\dot B^{s_p}_{p,q}} \exp\left(C\|u\|_{\widetilde{L^r}(\mathbb{R}^+,\dot B^{s_p+\frac{2}{r}}_{p,r})}\right),$$

for $r \in (2, 2/(1-3/p))$. This follows from the remark that in the case $\|u\|_{\widetilde{L^r}(\mathbb{R}^+,\dot B^{s_p+\frac{2}{r}}_{p,r})} \geq 1$, the integer N can be chosen of size equivalent to $\|u\|_{\widetilde{L^r}(\mathbb{R}^+,\dot B^{s_p+\frac{2}{r}}_{p,r})}$. Similar estimates hold for the other $\widetilde{L^r}(\mathbb{R}^+, \dot B^{s_p+\frac{2}{r}}_{p,r})$ norms of w.

Remark 3.2
In the case when $u_0 \in \dot B^{s_p}_{p,q}$ with $q = +\infty$, the result still holds under the condition that u_0 is in the closure in $\mathcal{S}'(\mathbb{R}^3)$ of smooth functions for the $\dot B^{s_p}_{p,\infty}$ norm. All the computations above indeed hold for $q = +\infty$, up to the proof of (12), for which that additional assumption is needed.

Proof of Theorem 3.2
One cannot apply directly Theorem 3.1 to L^3 datum. Indeed, one needs to use the embedding $L^3 \hookrightarrow \dot B^0_{3,3}$, prove stability there, and then recover an L^3 estimate, as explained in [18]. Here we proceed exactly in the same way, taking advantage of the estimates developped in the Appendix. Once our solution v has been proved to be stable in $\dot B^{s_p}_{p,q}$, it can be decomposed as $v = S(t)v_0 + \sum_l B_l(S(t)v_0) + r_v(x,t)$, where $r_v \in C_t(L^3)$ and $S(t)v_0$ is the linear flow. Since we moreover assumed $v_0 \in L^3$, taking the difference between this decomposition for v and the similar one for u, we deduce (10), since in all multilinear operators involving the linear parts we can choose all entries to be in $\widetilde{L^r_t}(\dot B^{s_p+2/r}_{p,q})$ except for the $S(t)(u_0 - v_0)$ which we see as an L^3 function. The difference between remainders is bounded by the difference of the data in the large Besov space, and hence by the difference in L^3 by Sobolev embedding. $\square$

4 Estimates for parabolic equations

In the previous sections, we were brought to writing a priori estimates for the Navier–Stokes equation as well as for the linearized equation around a given vector field. The aim of this section is to prove such estimates in the following setting.

$$(17) \qquad \partial_t w - \Delta w + u \cdot \nabla w + \overline{v} \cdot \nabla v = -\nabla \pi, \qquad \operatorname{div} w = \operatorname{div} u = \operatorname{div} v = \operatorname{div} \overline{v} = 0.$$

Let us note right away that in the non blow-up result we set $v = 0$ and $u = w = \overline{v} = u_1 + u_2$ was some decomposition of w, while for the stability theorem we required $\overline{v} = w$ and $u = w + U$, where U was a solution of the Navier-Stokes equations.

4.1 A priori estimates

The main result of this section is the following proposition. We only elected to state the results we needed previously, but it should be clear from the proof that we have not stated all possible estimates in their greatest generality. Let us note that estimates in L^p–type spaces have been proved by M. Vishik in [21], in the context of the Euler equations.

Proposition 4.1
Let s_p be the regularity of the initial data

$$s_p = -1 + \frac{3}{p} \tag{18}$$

and consider $s \in [s_p, s_p + 2]$ and $r \in [1, \infty]$ such that

$$s = s_p + \frac{2}{r} \cdot \tag{19}$$

Let w be a solution of system (17). *Then the following relation holds:*

$$\|w\|_{\widetilde{L_T^r}\dot{B}^s_{p,q}} \lesssim \|w_0\|_{\dot{B}^{s_p}_{p,q}} + \|w\|_{\widetilde{L_T^{r_1}}\dot{B}^{s_1}_{p,q_1}} \|u\|_{\widetilde{L_T^{r_2}}\dot{B}^{s_2}_{p,q_2}} + \|\overline{v}\|_{\widetilde{L_T^{r'_1}}\dot{B}^{s'_1}_{p,q'_1}} \|v\|_{\widetilde{L_T^{r'_2}}\dot{B}^{s'_2}_{p,q'_2}}, \tag{20}$$

under the following assumptions

$$s_1, s_2 \in [s_p, s_p + 2), \qquad s_1 + s_2 \in (0, 2 + 2s_p], \qquad s \in [s_p, s_1 + s_2 - s_p] \tag{21}$$

and

$$s_i = s_p + \frac{2}{r_i}, \; i = 1, 2, \qquad \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2} \cdot \tag{22}$$

Also, relations similar to (21) *and* (22) *with primes on s_i, r_i and q_i must hold plus the additional restriction*

$$s'_1 < s_p + 1. \tag{23}$$

We also used in Section 2 some estimates in "energy spaces". These are included in the next proposition.

Proposition 4.2
We set again $s_p = -1 + \frac{3}{p}\cdot$ *Let* $s \in [0,2]$ *and* $r \in [1,\infty]$ *be such that* $s = \frac{2}{r}\cdot$ *Let* w *be a solution of system* (17)*. Then the following relation holds:*

$$\|w\|_{\widetilde{L^r_T}\dot H^s} \lesssim \|w_0\|_{L^2} + \|w\|_{\widetilde{L^{r_1}_T}\dot H^{s_1}}\|u\|_{\widetilde{L^{r_2}_T}\dot B^{s_2}_{p,q}} + \|\overline{v}\|_{\widetilde{L^{r'_1}_T}\dot H^{s'_1}}\|v\|_{\widetilde{L^{r'_2}_T}\dot B^{s'_2}_{p,q}},$$

under the following assumptions

$$s_1 \in [0, s_p+2), \quad s_2 \in [s_p, s_p+2), \quad s_1+s_2 \in (0, 2+s_p], \quad s \in [0, s_1+s_2-s_p] \tag{24}$$

and

$$s_1 = \frac{2}{r_1}, \qquad s_2 = s_p + \frac{2}{r_2}\cdot \tag{25}$$

Also, relations similar to (24) *and* (25) *with primes on* s_i *and* r_i *must hold plus the additional restriction*

$$s'_1 < s_p + 1.$$

Proof of Proposition 4.1.

Let us start by giving an idea of the proof. An estimate in L^p would consist simply in multiplying the equation by $|w|^{p-2}w$ and integrating. Then there is only the term $\overline{v}\cdot\nabla v$ to consider, and one notes that the derivative is on v. In the case of Besov spaces of course one first has to localize in frequency space using the operators Δ_j, and the $u\cdot\nabla w$ term no longer disappears after integration. Most of the work in the following consists in proving that one can nevertheless shift the derivative on u, using commutator estimates (which will appear in the following section).

We apply the operator $\mathbb{P}$, the orthogonal projection on divergence free vector fields, to the equation. Then

$$\partial_t w - \Delta w + \mathbb{P}(u\cdot\nabla w + \overline{v}\cdot\nabla v) = 0.$$

Now let $i \in \{1,2,3\}$ be given, and define $f^{(i)}$ as the i-th coordinate of any vector field f. Let Δ_j be the usual Littlewood–Paley operator and define

$$w_j \stackrel{\text{def}}{=} \Delta_j w$$

Then by the "modified Poincaré lemma" ([17]) we have

$$\int_{\mathbb{R}^3} (\partial_t w_j^{(i)} - \Delta w_j^{(i)})|w_j^{(i)}|^{p-2} w_j^{(i)}\,dx \geq \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|w_j^{(i)}\|_p^p + c_p 2^{2j}\|w_j^{(i)}\|_p^p,$$

where we have defined $\|f\|_p \stackrel{\text{def}}{=} \|f\|_{L^p}$. It follows that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|w_j^{(i)}(t)\|_p^p + c_p 2^{2j}\|w_j^{(i)}(t)\|_p^p \leq -\int_{\mathbb{R}^3} (\Delta_j \mathbb{P}(u\cdot\nabla w + \overline{v}\cdot\nabla v))^{(i)}\, |w_j^{(i)}|^{p-2} w_j^{(i)}\,dx.$$

Summing on i we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_j\|_p^p + c_p 2^{2j}\|w_j\|_p^p \lesssim \|w_j\|_p^{p-1} F_j(t)$$

where F_j is such that

$$\Big|\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\Delta_j\mathbb{P}(u\cdot\nabla w+\overline{v}\cdot\nabla v))^{(i)}|w_j^{(i)}|^{p-2}w_j^{(i)}\,dx\Big| = \|w_j\|_p^{p-1}F_j(t). \tag{26}$$

We deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_j\|_p + c_p 2^{2j}\|w_j\|_p \lesssim F_j(t).$$

Gronwall's lemma now gives

$$\|w_j(t)\|_p \lesssim \|w_j(0)\|_p e^{-tc_p 2^{2j}} + e^{-tc_p 2^{2j}} * F_j,$$

where the sign $*$ denotes the convolution of functions defined on $\mathbb{R}_+$. We now multiply by 2^{js} and take the $L^r(0,T)$ norm to obtain

$$\begin{aligned} 2^{js}\|\|w_j\|_p\|_{L^r(0,T)} &\lesssim 2^{js}\|w_j(0)\|_p\,\|e^{-tc_p 2^{2j}}\|_{L^r(0,T)} + 2^{js}\|e^{-tc_p 2^{2j}} * F_j\|_{L^r(0,T)} \\ &\lesssim 2^{js}\|w_j(0)\|_p\,\|e^{-tc_p 2^{2j}}\|_{L^r(0,T)} + 2^{js}\|e^{-tc_p 2^{2j}}\|_{L^{r'}(0,T)}\|F_j\|_{L^{r''}(0,T)} \end{aligned} \tag{27}$$

where we have used Young's inequality and

$$1+\frac{1}{r} = \frac{1}{r'}+\frac{1}{r''}\cdot$$

We now use that

$$\|e^{-tc_p 2^{2j}}\|_{L^a(0,T)} = (ac_p)^{-\frac{1}{a}}2^{-\frac{2j}{a}}\big(1-e^{-aTc_p 2^{2j}}\big)^{\frac{1}{a}} \lesssim 2^{-\frac{2j}{a}} \tag{28}$$

to obtain

$$2^{js}\|\|w_j\|_p\|_{L^r(0,T)} \lesssim \|w_j(0)\|_p 2^{j(s-\frac{2}{r})} + 2^{j(s-\frac{2}{r'})}\|F_j\|_{L^{r''}(0,T)}. \tag{29}$$

According to (18) and (19), taking the ℓ^q norm in (29) yields

$$\|w\|_{\widetilde{L^r_T}\dot{B}^s_{p,q}} \lesssim \|w_0\|_{\dot{B}^{s_p}_{p,q}} + \Big\|2^{j(s-\frac{2}{r'})}\|F_j\|_{L^{r''}(0,T)}\Big\|_{\ell^q}. \tag{30}$$

Applying Lemma 4.1 with $\overline{p}=p$, we infer that

$$\Big\|2^{j(s-\frac{2}{r'})}\|F_j\|_{L^{r''}(0,T)}\Big\|_{\ell^q} \lesssim \|w\|_{\widetilde{L^{r_1}_T}\dot{B}^{s_1}_{p,q_1}}\|u\|_{\widetilde{L^{r_2}_T}\dot{B}^{s_2}_{p,q_2}} + \|\overline{v}\|_{\widetilde{L^{r'_1}_T}\dot{B}^{s'_1}_{p,q'_1}}\|v\|_{\widetilde{L^{r'_2}_T}\dot{B}^{s'_2}_{p,q'_2}} \tag{31}$$

provided that

$$\frac{1}{r''}=\frac{1}{r_1}+\frac{1}{r_2},\qquad \frac{1}{q}\le\frac{1}{q_1}+\frac{1}{q_2},\qquad s_1+s_2-1-\frac{3}{p}=s-\frac{2}{r'},\qquad s_i=s_p+\frac{2}{r_i},\ i=1,2,$$
$$s_1+s_2>0,\qquad s_1,s_2<s_p+2$$

and that similar relations with primes on s_i, r_i and q_i hold plus the additional restriction

$$s_1'<s_p+1.$$

Taking into account the fact that r' and r'' must belong to $[1,\infty]$ we finally end up with the announced restrictions (21), (22) and (23). Relation (20) follows from relations (30) and (31). This completes the proof of Proposition 4.1. □

Proof of Proposition 4.2.

This proof follows closely the previous one, up to some obvious changes. We will therefore omit the details and just point out the main steps.

First, we rewrite relation (27) for $p=2$:

$$2^{js}\|\|w_j\|_2\|_{L^r(0,T)}\lesssim 2^{js}\|w_j(0)\|_2\,\|e^{-tc2^{2j}}\|_{L^r(0,T)}+2^{js}\|e^{-tc2^{2j}}\|_{L^{r'}(0,T)}\|F_j\|_{L^{r''}(0,T)}\tag{32}$$

where

$$1+\frac{1}{r}=\frac{1}{r'}+\frac{1}{r''}$$

and F_j is defined by (26) with $p=2$. Explicitly computing the two norms of exponential functions above and taking the ℓ^2 norm in (32) yields

$$\|w\|_{\widetilde{L^r_T}\dot H^s}\lesssim\|w_0\|_{L^2}+\left\|2^{j(s-\frac{2}{r'})}\|F_j\|_{L^{r''}(0,T)}\right\|_{\ell^2}.$$

Applying Lemma 4.1 with $\overline{p}=q_2=q_2'=2$ and $q_1=q_1'=q$ finally gives

$$\|w\|_{\widetilde{L^r_T}\dot H^s}\lesssim\|w_0\|_{L^2}+\|w\|_{\widetilde{L^{r_1}_T}\dot H^{s_1}}\|u\|_{\widetilde{L^{r_2}_T}\dot B^{s_2}_{p,q}}+\|\overline{v}\|_{\widetilde{L^{r_1'}_T}\dot H^{s_1'}}\|v\|_{\widetilde{L^{r_2'}_T}\dot B^{s_2'}_{p,q}}$$

provided that

$$\frac{1}{r''}=\frac{1}{r_1}+\frac{1}{r_2},\qquad s_1+s_2-1-\frac{3}{p}=s-\frac{2}{r'},\qquad s_1=\frac{2}{r_1},\qquad s_2=s_p+\frac{2}{r_2},$$
$$s_1+s_2>0,\qquad s_1,s_2<s_p+2$$

and that similar relations with primes on s_i and r_i hold plus the additional restriction

$$s_1'<s_p+1.$$

As in the previous proof, the fact that r' and r'' must belong to $[1,\infty]$ implies the restrictions announced in the statement. This completes the proof. □

Remark 4.1
As a byproduct of Propositions 4.1 and 4.2 we get the following estimates for the standard bilinear term of the Navier-Stokes equations. Denoting by $S(t)$ the semigroup associated to the heat equation, the bilinear operator

$$B(\overline{v},v)=\int_0^t S(t-s)\mathbb{P}\nabla\cdot(\overline{v}\otimes v)(s)ds$$

satisfies the estimates

$$\|B(\overline{v},v)\|_{\widetilde{L^r_T}\dot{B}^s_{p,q}}\lesssim\|\overline{v}\|_{\widetilde{L^r_T}\dot{B}^s_{p,q}}\|v\|_{\widetilde{L^r_T}\dot{B}^s_{p,q}}$$

and

$$\|B(\overline{v},v)\|_{\widetilde{L^{r'}_T}\dot{H}^{s'}}\lesssim\|\overline{v}\|_{\widetilde{L^{r'}_T}\dot{H}^{s'}}\|v\|_{\widetilde{L^r_T}\dot{B}^s_{p,q}}$$

where $s=s_p+\frac{2}{r}\in(0,\frac{3}{p})$ and $s'=\frac{2}{r'}\in[0,\frac{3}{p})$.

The proof of this remark follows immediately from Propositions 4.1 and 4.2 once we have noticed that $w=B(\overline{v},v)$ verifies the system

$$\partial_t w-\Delta w-\mathbb{P}(\overline{v}\cdot\nabla v)=0,\qquad w(0,x)=0.$$

4.2 Estimate of convective terms

In this section, we shall state and prove a lemma enabling one to estimate the convective terms in (17). The main difficulty in the statement of Proposition 4.1 is to obtain the extended range s_2 close s_p+2. This corresponds to the first term below, and one immediately observes the difference between this term and the second one: in the first term the derivative does not fall on a, and we will have to take advantage of the frequency localization as well as of the structure to reduce this "bad" term to a "good" one, like the second one. Indeed this second term can be directly estimated by standard product rules and Hölder, estimating in effect the L^1 norm of the function under the integral sign.

Lemma 4.1
Let a, $\overline{a}$ and b be divergence free vector fields. The following relations hold true:

$$\Big|\sum_{i=1,2,3}\int_{\mathbb{R}^3}(\mathbb{P}\Delta_j\big(a\cdot\nabla\big)b)^{(i)}|\Delta_j b^{(i)}|^{\overline{p}-2}\Delta_j b^{(i)}\,dx\Big|=\|\Delta_j b\|_{L^{\overline{p}}}^{\overline{p}-1}F_j(t)$$

$$\Big|\sum_{i=1,2,3}\int_{\mathbb{R}^3}(\mathbb{P}\Delta_j\big(\overline{a}\cdot\nabla\big)a)^{(i)}|\Delta_j b^{(i)}|^{\overline{p}-2}\Delta_j b^{(i)}\,dx\Big|=\|\Delta_j b\|_{L^{\overline{p}}}^{\overline{p}-1}F'_j(t)$$

with
$$\left\|2^{j(s_1+s_2-1-\frac{3}{p})}\|F_j\|_{L^r(0,T)}\right\|_{\ell^q}\lesssim\|a\|_{\widetilde{L^{r_1}_T}\dot B^{s_1}_{p,q_1}}\|b\|_{\widetilde{L^{r_2}_T}\dot B^{s_2}_{\overline{p},q_2}},$$
and
$$\left\|2^{j(s'_1+s'_2-1-\frac{3}{p})}\|F'_j\|_{L^r(0,T)}\right\|_{\ell^q}\lesssim\|a\|_{\widetilde{L^{r'_1}_T}\dot B^{s'_1}_{p,q'_1}}\|\overline{a}\|_{\widetilde{L^{r'_2}_T}\dot B^{s'_2}_{\overline{p},q'_2}},$$
where p, $\overline{p}$, r, r_i, q and q_i are elements of $[1,\infty]$ and s and s_i are real numbers satisfying
$$p\geq\overline{p}\geq 2,\qquad \frac{1}{q}\leq\frac{1}{q_1}+\frac{1}{q_2},\qquad \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2},$$
$$s_1+s_2>0,\qquad s_1,s_2<1+\frac{3}{p}\cdot$$
Similar relations must hold with primes on s_i, r_i, q_i and the additional relation must also be true:
$$s'_2<\frac{3}{p}\cdot$$

Proof of Lemma 4.1. It turns out that the first term is the most difficult to estimate, and the other will be obtained by using parts of the computations of the first one. Here, ideally one would like the derivative to be on the low frequency term, $S_{j'-1}a$. If one ignores $\mathbb{P}\Delta_j$ and set $j=j'$, then one can integrate by parts to achieve this. In effect, combining this remark with a commutator estimate, we will be able to do so.

Let us start by estimating the term
$$I\stackrel{\text{def}}{=}\int_{\mathbb{R}^3}\big(\mathbb{P}\Delta_j(a\cdot\nabla)b\big)\cdot B_j^{\overline{p}-1}\,dx,$$
where $B_j^{\overline{p}-1}$ denotes the vector field with coordinates $|\Delta_j b^{(i)}|^{\overline{p}-2}\Delta_j b^{(i)}$. According to J.-M. Bony's paraproduct algorithm [2], we have
$$\Delta_j(a\cdot\nabla)b=\sum_{|j'-j|\leq 2}\Delta_j\Big((S_{j'-1}a\cdot\nabla)\Delta_{j'}b+(\Delta_{j'}a\cdot\nabla)S_{j'-1}b\Big)$$
$$+\Delta_j\sum_{\substack{k\geq j-1\\|k-k'|\leq 1}}(\Delta_k a\cdot\nabla)\Delta_{k'}b.$$

So we can decompose I into three parts, defining
$$I.1\stackrel{\text{def}}{=}\sum_{|j'-j|\leq 2}\int_{\mathbb{R}^3}\Big(\mathbb{P}\Delta_j\left(S_{j'-1}a\cdot\nabla\right)\Delta_{j'}b\Big)\cdot B_j^{\overline{p}-1}\,dx,$$
$$I.2\stackrel{\text{def}}{=}\sum_{|j'-j|\leq 2}\int_{\mathbb{R}^3}\Big(\mathbb{P}\Delta_j\left(\Delta_{j'}a\cdot\nabla\right)S_{j'-1}b\Big)\cdot B_j^{\overline{p}-1}\,dx$$

and

$$I.3 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \Big(\mathbb{P}\Delta_j \sum_{\substack{k \geq j-1 \\ |k-k'| \leq 1}} (\Delta_k a \cdot \nabla)\Delta_{k'} b \Big) \cdot B_j^{\overline{p}-1}\, dx$$

and we shall estimate those terms successively. The most tricky one is again the first one. Let us denote by $\sum_l f^{(l)}\partial_l = f \cdot \nabla$, Id is the identity matrix, and recall that $\mathbb{P}$ is a matrix operator such that $\mathbb{P}b = b$:

$$I.1 = \underbrace{\sum_{|j'-j|\leq 2} \sum_{l} \int_{\mathbb{R}^3} [\mathbb{P}\Delta_j, S_{j'-1} a^\ell Id]\, \partial_\ell \Delta_{j'} b \cdot B_j^{\overline{p}-1}\, dx}_{I.1.1} + \underbrace{\int_{\mathbb{R}^3} (S_j a \cdot \nabla)\Delta_j b \cdot B_j^{\overline{p}-1}\, dx}_{I.1.2}$$
$$+ \underbrace{\sum_{|j'-j|\leq 2} \int_{\mathbb{R}^3} ((S_{j'-1} - S_j) a \cdot \nabla)\Delta_{j'}\Delta_j b \cdot B_j^{\overline{p}-1}\, dx}_{I.1.3}$$

We used above that $\sum_{j'=j-2}^{j+2} \Delta_j \Delta_{j'} b = \Delta_j b$. The notation $[A,B]$ denotes the commutator of the operators A et B. To estimate $I.1.1$, we need a commutator estimate. One can forget about the matrices and perform a scalar estimate. Hence we need an estimate on the scalar commutator $[\widetilde{\Delta}_j, S_{j-2}a^{(l)}]$, where $\widetilde{\Delta}_j$ stands for Δ_j or $R_\mu R_\nu \Delta_j$ depending on which entry in the $\mathbb{P}$ matrix we pick. In either case, $\widetilde{\Delta}_j$ is the convolution by a smooth function of the form $2^{nj}\phi(2^j x)$. We then use the following lemma.

Lemma 4.2
Let $f \in L^{\overline{p}}$, $\nabla g \in L^\infty$, then

$$\|[\widetilde{\Delta}_j, g] f\|_{\overline{p}} \lesssim 2^{-j} \|\nabla g\|_\infty \|f\|_{\overline{p}}.$$

Proof of Lemma 4.2.
Since $\widetilde{\Delta}_j f = 2^{nj}\phi(2^j x) * f$, one writes

$$[\widetilde{\Delta}_j, g] f(x) = \int 2^{nj} \phi(2^j(x-y))(g(y) - g(x)) f(y) dy.$$

Then

$$\begin{aligned} |[\widetilde{\Delta}_j, g] f(x)| &\leq \int 2^{nj} |\phi|(2^j(x-y)) |x-y| \|\nabla g\|_\infty |f|(y) dy, \\ &\leq 2^{-j} \|\nabla g\|_\infty \int 2^{nj} \psi(2^j(x-y)) |f|(y) dy, \end{aligned}$$

and we conclude by Young's inequality that

$$\|[\widetilde{\Delta}_j, g]f\|_{\overline{p}} \leq 2^{-j}\|\nabla g\|_\infty \|\psi\|_1 \|f\|_{\overline{p}},$$

since $\psi(x) = |x||\phi|(x) \in L^1$. This ends the proof of Lemma 4.2. $\square$

Coming back to the term $I.1.1$, we apply Hölder's inequality and Lemma 4.2: using the fact that $|j - j'| \leq 2$, we essentially have to prove that

$$|I.1.1| \lesssim \|\nabla S_j a\|_\infty 2^{-j} \|\nabla \Delta_j b\|_{\overline{p}} \|\Delta_j b\|_{\overline{p}}^{\overline{p}-1}. \tag{33}$$

So to prove the desired result, all we need to do is to check that

$$F_j^1(t) \stackrel{\text{def}}{=} \|\nabla S_j a\|_\infty 2^{-j} \|\nabla \Delta_j b\|_{\overline{p}} \tag{34}$$

can be estimated in the following way:

$$\|F_j^1\|_{L_T^r} \lesssim a_j 2^{-j(s_1+s_2-1-\frac{3}{p})} \|a\|_{\widetilde{L_T^{r_1}}(\dot{B}^{s_1}_{p,q_1})} \|b\|_{\widetilde{L_T^{r_2}}(\dot{B}^{s_2}_{\overline{p},q_2})},$$

with a_j a sequence of ℓ^q of norm 1. But we have, by Hölder's inequality,

$$\|F_j^1\|_{L_T^r} \lesssim \|\nabla S_j a\|_{L_T^{r_1}(L^\infty)} \|\Delta_j b\|_{L_T^{r_2}(L^{\overline{p}})},$$

so it follows by Bernstein's inequality that

$$\|F_j^1\|_{L_T^r} \lesssim \sum_{k \leq j} 2^{ks_1} \|\Delta_k a\|_{L_T^{r_1}(L^p)} 2^{k(1-s_1+\frac{3}{p})} \|\Delta_j b\|_{L_T^{r_2}(L^{\overline{p}})}.$$

Then under the condition

$$1 - s_1 + \frac{3}{p} > 0,$$

we can apply Young inequality to get

$$\|F_j^1\|_{L_T^r} \lesssim \alpha_j \beta_j \|a\|_{\widetilde{L_T^{r_1}}(\dot{B}^{s_1}_{p,q_1})} \|b\|_{\widetilde{L_T^{r_2}}(\dot{B}^{s_2}_{\overline{p},q_2})} 2^{-j(s_1+s_2-1-\frac{3}{p})}$$

where α_j, β_j are normalized l^{q_1}, l^{q_2} sequences. We conclude for the term $I.1.1$ by Hölder ; the sequence a_j is in fact in $\ell^{q_{12}}$ with $q_{12} = q_1 q_2/(q_1 + q_2)$, hence in ℓ^q for any $q \geq q_{12}$.

For $I.1.2$, we notice that since a is divergence free, an integration by parts implies that $I.1.2 = 0$ (remark however that should we forget the divergence free condition on this term, integration by parts would lead to a "good" term, with the derivative on the low frequencies).

Finally we consider the term $I.1.3$, for which we just need to notice that since $|j' - j| \leq 2$, the term $(S_{j'-1} - S_j)a$ is a sum of $\Delta_{j''}a$, with $j'' \approx j$. So the term $I.1.3$ can be added to $I.3$ and we do so implicitly. So finally we have proved the result for $I.1$.

Now let us estimate the term $I.2$. Hölder's inequality together with the continuity of $\mathbb{P}$ enables us to write, since $|j-j'|\leq 2$,

$$|I.2| \lesssim \sum_{|j'-j|\leq 2} \|(\Delta_{j'}a\cdot\nabla)S_{j'-1}b\|_{\overline{p}}\|\Delta_j b\|_{\overline{p}}^{\overline{p}-1} \lesssim \sum_{|j'-j|\leq 2} \|\Delta_{j'}a\|_p\|\nabla S_{j+1}b\|_{p_1}\|\Delta_j b\|_{\overline{p}}^{\overline{p}-1},$$

where $\dfrac{1}{\overline{p}} = \dfrac{1}{p}+\dfrac{1}{p_1}\cdot$ Without loss of generality one can only consider the term $j=j'$, so we just need to check that

$$F_j^2 \stackrel{\mathrm{def}}{=} \|\Delta_j a\|_p\|\nabla S_{j+1}b\|_{p_1}$$

satisfies

$$\|F_j^2\|_{L^r_T} \lesssim a_j 2^{-j(s_1+s_2-1-\frac{3}{p})}\|a\|_{\widetilde{L^{r_1}_T}(\dot{B}^{s_1}_{p,q_1})}\|b\|_{\widetilde{L^{r_2}_T}(\dot{B}^{s_2}_{\overline{p},q_2})}.$$

By Hölder's and Bernstein's inequality,

$$\begin{aligned}\|F_j^2\|_{L^r_T} &\lesssim \|\Delta_j a\|_{L^{r_1}_T(L^p)}\sum_{k\leq j}\|\Delta_k b\|_{L^{r_2}_T(L^{\overline{p}})}2^{k(\frac{3}{p}+1)}\\ &\lesssim \|\Delta_j a\|_{L^{r_1}_T(L^p)}2^{js_1}\sum_{k\leq j}\|\Delta_k b\|_{L^{r_2}_T(L^{\overline{p}})}2^{ks_2}2^{(k-j)(\frac{3}{p}+1-s_2)}2^{-j(s_1+s_2-\frac{3}{p}-1)}.\end{aligned}$$

So the result follows for $I.2$ by Young's inequality.

Finally let us estimate $I.3$. Since a is divergence free, we have

$$I.3 = \int_{\mathbb{R}^3}\Big(\mathbb{P}\Delta_j \operatorname{div}\sum_{\substack{k\geq j-1\\ |k-k'|\leq 1}}\Delta_k a\otimes\Delta_{k'}b\Big)\cdot B_j^{\overline{p}-1}\,dx \tag{35}$$

so by Bernstein's inequality, continuity of $\mathbb{P}$ and Hölder's inequality we get

$$\begin{aligned}I.3 &\lesssim 2^j\Big\|\Delta_j\sum_{\substack{k\geq j-1\\ |k-k'|\leq 1}}\Delta_k a\otimes\Delta_{k'}b\Big\|_{\overline{p}}\|\Delta_j b\|_{\overline{p}}^{\overline{p}-1}\\ &\lesssim 2^{j(1+\frac{3}{p})}\Big\|\Delta_j\sum_{\substack{k\geq j-1\\ |k-k'|\leq 1}}\Delta_k a\otimes\Delta_{k'}b\Big\|_{p_2}\|\Delta_j b\|_{\overline{p}}^{\overline{p}-1}\\ &\lesssim 2^{j(1+\frac{3}{p})}\sum_{\substack{k\geq j-1\\ |k-k'|\leq 1}}\|\Delta_k a\|_p\|\Delta_{k'}b\|_{\overline{p}}\|\Delta_j b\|_{\overline{p}}^{\overline{p}-1}\end{aligned}$$

where $\dfrac{1}{p_2}=\dfrac{1}{p}+\dfrac{1}{\overline{p}}\cdot$ It follows, since $|j-j'|\leq 2$, that we essentially need to prove that

$$F_j^3 \stackrel{\text{def}}{=} 2^{j(1+\frac{3}{p})}\sum_{k\geq j}\|\Delta_k a\|_p\|\Delta_k b\|_{\overline{p}} \tag{36}$$

satisfies

$$\|F_j^3\|_{L^r_T}\lesssim a_j 2^{-j(s_1+s_2-1-\frac{3}{p})}\|a\|_{\widetilde{L^{r_1}_T}(\dot{B}^{s_1}_{p,q_1})}\|b\|_{\widetilde{L^{r_2}_T}(\dot{B}^{s_2}_{\overline{p},q_2})}.$$

But we have

$$\begin{aligned}\|F_j^3\|_{L^r_T}&\lesssim 2^{j(1+\frac{3}{p})}\sum_{k\geq j}\|\Delta_k a\|_{L^{r_1}_T(L^p)}\|\Delta_k b\|_{L^{r_2}_T(L^{\overline{p}})}\\&\lesssim 2^{j(1+\frac{3}{p}-s_1-s_2)}\sum_{k\geq j}2^{ks_1}\|\Delta_k a\|_{L^{r_1}_T(L^p)}2^{ks_2}\|\Delta_k b\|_{L^{r_2}_T(L^{\overline{p}})}2^{(j-k)(s_1+s_2)},\end{aligned}$$

and the result follows by Young's inequality under the condition that $s_1+s_2>0$.

Finally we obtained the expected estimate for I. Now we consider the second term to estimate,

$$II\stackrel{\text{def}}{=}\Big|\sum_{i=1,2,3}\int_{\mathbb{R}^3}(\mathbb{P}\Delta_j\big(\overline{a}\cdot\nabla\big)a)^{(i)}|\Delta_j b^{(i)}|^{\overline{p}-2}\Delta_j b^{(i)}\,dx\Big|.$$

We shall show that II can be estimated along the same lines as I, in fact in a much simpler way. One can forget about the vector structure, and perform scalar estimates. More precisely, we use the continuity of the projector $\mathbb{P}$ and Hölder inequality to obtain

$$II\lesssim\|\Delta_j(\overline{a}\cdot\nabla a)\|_{\overline{p}}\|\Delta_j b\|_{\overline{p}}^{\overline{p}-1},$$

so it is enough to estimate the L^r_T norm of $\|\Delta_j(b\cdot\nabla a)\|_{\overline{p}}$. More precisely we shall prove that

$$\|\Delta_j(\overline{a}\cdot\nabla a)\|_{L^r_T L^{\overline{p}}}\lesssim a_j 2^{-j(s_1'+s_2'-\frac{3}{p}-1)}\|\overline{a}\|_{\widetilde{L^{r_2}_T}(\dot{B}^{s_2'}_{\overline{p},q_2})}\|a\|_{\widetilde{L^{r_1}_T}(\dot{B}^{s_1'}_{p,q_1})}.$$

As usual we decompose the product into three parts, and we need to estimate

$$II.1\stackrel{\text{def}}{=}\|\Delta_j\sum_{|j-j'|\leq 2}S_{j'-1}\overline{a}\cdot\nabla\Delta_{j'}a\|_{L^r_T L^{\overline{p}}},\quad II.2\stackrel{\text{def}}{=}\|\Delta_j\sum_{|j-j'|\leq 2}\Delta_{j'}\overline{a}\cdot\nabla S_{j'-1}a\|_{L^r_T L^{\overline{p}}},$$

$$\text{and}\quad II.3\stackrel{\text{def}}{=}\|\Delta_j\sum_{k\geq j-1}\sum_{|k-k'|\leq 2}\nabla\cdot(\Delta_k a\otimes\Delta_{k'}\overline{a})\|_{L^r_T L^{\overline{p}}}.$$

Note that in the remainder term, we have taken advantage of the fact that $\overline{a}$ is divergence free to put the space derivative outside the product. The first paraproduct term $II.1$ is the only one we really have to deal with, as the two others were studied previously: we have indeed

$$\|\Delta_j \sum_{|j-j'|\leq 2} \Delta_{j'}\overline{a} \cdot \nabla S_{j'-1}a\|_{\overline{p}} \lesssim \sum_{j'=j-2}^{j+2} \|\nabla S_{j'-1}a\|_\infty 2^{-j} \|\nabla \Delta'_j \overline{a}\|_{\overline{p}}$$

so the result for $II.2$ is found using the previous estimate (34) on F_j^1. Similarly we have

$$\|\Delta_j \sum_{k\geq j-1} \sum_{|k-k'|\leq 2} \nabla \cdot (\Delta_k a \otimes \Delta_{k'}\overline{a})\|_{L^{\overline{p}}} \lesssim \sum_{k\geq j-1} \sum_{|k-k'|\leq 2} 2^{j(1+\frac{3}{p})} \|\Delta_k a\|_p \|\Delta_{k'}\overline{a}\|_{\overline{p}}$$

so one estimates $II.3$ using the estimate for F_j^3 defined in (36).

Finally let us estimate $II.1$. We simply write

$$\|\Delta_j \sum_{|j-j'|\leq 2} S_{j'-1}\overline{a} \cdot \nabla \Delta_{j'}a\|_{\overline{p}} \lesssim \sum_{|j-j'|\leq 2} \|S_{j'-1}\overline{a}\|_{p_1} 2^j \|\Delta_{j'}a\|_p$$

with $\dfrac{1}{p_1} + \dfrac{1}{p} = \dfrac{1}{\overline{p}}$, which implies by Bernstein's inequality that

$$\|\Delta_j \sum_{|j-j'|\leq 2} S_{j'-1}\overline{a} \cdot \nabla \Delta_{j'}a\|_{L^{\overline{p}}} \lesssim \sum_{k\leq j+1} \|\Delta_k \overline{a}\|_{\overline{p}} 2^{3k/p} 2^j \|\Delta_j a\|_p$$

so finally

$$\|\Delta_j \sum_{|j-j'|\leq 2} S_{j'-1}\overline{a} \cdot \nabla \Delta_{j'}a\|_{\overline{p}} \lesssim \sum_{k\leq j+1} \|\Delta_k \overline{a}\|_{\overline{p}} 2^{ks'_2} 2^{(k-j)(\frac{3}{p}-s'_2)} 2^{js'_1} \|\Delta_j a\|_p 2^{j(1-s'_1-s'_2+\frac{3}{p})}$$

which proves the result under the condition $s'_2 < \dfrac{3}{p}\cdot$

This completes the proof of Lemma 4.1. □

Appendix

For the sake of completeness, we give in this appendix proofs of several results which can all be found, though perhaps under different formulations, in the litterature on Navier-Stokes. As a matter of fact, we reproduce here the proof of the existence in $\dot{B}^{s_p}_{p,q}$ (Theorem A.1) which can be found in [6]. The only new result is Proposition A.1, but the ideas behind it have been used extensively before ([16, 8, 4]). We feel however that there exists no reference providing these results in a unified framework, hence the short recollection given here might be of help to the reader.

Theorem A.1
Let $u_0 \in \dot{B}_{p,q}^{-1+\frac{3}{p}}$ be a divergence free vector field. There exists a unique local in time solution to (1) such that

$$u \in C\big([0,T);\dot{B}_{p,q}^{s_p}\big) \cap \widetilde{L_T^r}\dot{B}_{p,q}^{s_p+\frac{2}{r}} \quad \forall r \in [1,\infty].$$

Moreover, there exists a constant K_0 such that if $\|u_0\|_{\dot{B}_{p,q}^{s_p}} \leq K_0$, then we can choose $T = +\infty$.

Remark A.1
In (1), we have set the viscosity $\nu = 1$. Scaling allows to recover all values of ν. As usual, we could track the dependence on ν of the constant K_0 to see that K_0 can be chosen of the form $K_0 = \widetilde{K}_0\nu$, with $\widetilde{K}_0$ independent of ν.

Remark A.2
To prove Theorem A.1, it is sufficient to control one norm $\widetilde{L_T^{r'}}\dot{B}_{p,q}^{s'}$ ($s' = s_p + \frac{2}{r'}$) with $s' \in (0,\frac{3}{p})$ in order to control all those norms for $s' \in [s_p, s_p+2]$. Moreover, the control of the norms $\widetilde{L_T^\infty}\dot{B}_{p,q}^{s_p}$ and $\widetilde{L_T^{r'}}\dot{B}_{p,q}^{s'}$ with some $s' > \frac{3}{p}$ implies the control of all those norms for $s' \in [s_p, s_p+2]$

Proof of Remark A.2.
Consider first the case $s' \in (0,\frac{3}{p})$. One can use the estimate of Proposition 4.1 to get an estimate for the norms $\widetilde{L_T^r}\dot{B}_{p,q}^{s}$ for all $s \in [s_p, 2s'-s_p]$. Reapplying Proposition 4.1 with the new set of indices $[s_p, 2s'-s_p]$ and so on yields the result.

The case $s' > \frac{3}{p}$ follows immediately from the case $s' \in (0,\frac{3}{p})$ by interpolating the spaces $\widetilde{L_T^\infty}\dot{B}_{p,q}^{s_p}$ and $\widetilde{L_T^{r'}}\dot{B}_{p,q}^{s'}$. $\square$

Proof of Theorem A.1.
Such a theorem can be proved with a priori estimates, regularizing and taking the limit. Or one can set up a fixed point in an appropriate Banach space. We take the opportunity to recall this approach and cast it in an abstract setting. In doing so we reduce ourselves to proving a priori estimates on the nonlinear term. We state two lemmas:

Lemma A.1 (existence and uniqueness)
Let X be a Banach space, L a linear operator from $X \to X$ such that a constant $\lambda < 1$ exists such that

$$\forall x \in X, \quad \|L(x)\|_X \leq \lambda \|x\|_X,$$

B a bilinear operator such that

$$\forall (x,y) \in X^2, \quad \|B(x,y)\|_X \leq \gamma \|x\|_X \|y\|_X,$$

then, for all $x_1 \in X$ such that

$$4\gamma\|x_1\|_X < (1-\lambda)^2$$

the sequence defined by

$$\begin{cases} x^{(0)} &= 0 \\ x^{(n+1)} &= x_1 + L(x^{(n)}) + B(x^{(n)}, x^{(n)}) \end{cases}$$

converges in X towards the unique solution of

$$x = x_1 + L(x) + B(x,x)$$

such that

$$2\gamma\|x\|_X < (1-\lambda).$$

The proof is an elementary exercise: first prove the sequence $\|x_n\|_X$ to be uniformly bounded, then prove the convergence of the telescopic series $\sum \|x_{n+1} - x_n\|_X$, which gives $x = \sum(x_{n+1} - x_n)$.

The next lemma allows to propagate additional information on the datum, provided the operators B, L behave nicely with respect to the norm encoding this information.

LEMMA A.2 (PROPAGATION OF REGULARITY)
Let x be the solution from the previous lemma, and assume moreover that $x_1 \in E$ for some Banach space E, and L, B are such that

$$\|L(z)\|_E < \mu\|z\|_E, \quad \forall z \in E$$
$$\|B(z,y)\|_E < \eta\|z\|_E\|y\|_X \quad \text{and} \quad \|B(y,z)\|_E < \eta\|z\|_E\|y\|_X \quad \forall z \in E, y \in X,$$

with $\mu < 1$ and $\eta(1-\lambda) < (1-\mu)\gamma$. Then the solution x belongs to E, and

$$\|x\|_E \lesssim \|x_1\|_E.$$

The proof of this second lemma is again elementary: prove the boundedness and then the convergence of the sequence x_n in E.

Let go back to the proof of Theorem A.1, and let us prove the global existence for small initial data. We have to solve the equation $x = S(t)u_0 + B(x,x)$ (see Remark 4.1 for the notation). We choose some $s = s_p + \frac{2}{r} \in (0, s_p+1)$ and set $X = \widetilde{L^r_\infty}\dot{B}^s_{p,q}$ and $L = 0$. Bicontinuity for the operator B in the space X follows from Remark 4.1. Next, we check that $x_1 = S(t)u_0 \in X$, which is a direct consequence of (20) (with $u = v = 0$), namely

$$\|x_1\|_{\widetilde{L^r_\infty}\dot{B}^s_{p,q}} \lesssim \left\| 2^{js_p}\|\Delta_j u_0\|_p \right\|_{\ell^q}. \tag{37}$$

>From Lemma A.1 this gives existence (and uniqueness in a ball) for small data. We then deduce from Remark A.2 that all other norms $\widetilde{L^{\tilde r}_T}\dot B^{\tilde s}_{p,q}$ are also bounded.

Local existence follows from the remark that relation (37) can actually be improved to

$$\|x_1\|_{\widetilde{L^r_t}\dot B^s_{p,q}} \leq K_2 \Big\| \big(1-e^{-rtc_p 2^{2j}}\big)^{\frac{1}{r}} 2^{js_p} \|\Delta_j u_0\|_p \Big\|_{\ell^q}.$$

This improved estimate can be easily deduced from relations (27) and (28). Next, an application of Lebesgue's dominated convergence theorem shows that

$$\lim_{t\to 0} \Big\| \big(1-e^{-rtc_p 2^{2j}}\big)^{\frac{1}{r}} 2^{js_p} \|\Delta_j u_0\|_p \Big\|_{\ell^q} = 0.$$

We deduce that for given ε_0 a time $T = T(\varepsilon_0)$ exists such that

$$\Big\| \big(1-e^{-rtc_p 2^{2j}}\big)^{\frac{1}{r}} 2^{js_p} \|\Delta_j u_0\|_p \Big\|_{\ell^q} < \varepsilon_0 \quad \forall t \in [0,T].$$

>From this point we can continue as in the proof of the small initial data case and infer that for ε_0 small enough the local strong solution exists at least up to the time $T(\varepsilon_0)$.

We are now left with the proof of uniqueness. Fix again $s \in (0, s_p+1)$, let u and $\widetilde{u}$ be two strong solutions with the same initial data and set $w = u - \widetilde{u}$. Applying again the basic estimate (20) we get that

$$\|w\|_{\widetilde{L^r_t}\dot B^s_{p,q}} \leq K_3 \|w\|^2_{\widetilde{L^r_t}\dot B^s_{p,q}} + K_3 \|w\|_{\widetilde{L^r_t}\dot B^s_{p,q}} \|u\|_{\widetilde{L^r_t}\dot B^s_{p,q}}$$

for some constant K_3. We infer that

$$\|w\|_{\widetilde{L^r_t}\dot B^s_{p,q}} \big(K_3 \|w\|_{\widetilde{L^r_t}\dot B^s_{p,q}} + K_3 \|u\|_{\widetilde{L^r_t}\dot B^s_{p,q}} - 1\big) \geq 0. \tag{38}$$

By continuity of the norm of $\widetilde{L^r_t}\dot B^s_{p,q}$ with respect to the time, there exists T such that

$$K_3 \|w\|_{\widetilde{L^r_t}\dot B^s_{p,q}} + K_3 \|u\|_{\widetilde{L^r_t}\dot B^s_{p,q}} - 1 < 0 \quad \forall t \in [0,T].$$

Therefore, for $t \in [0,T]$ relation (38) can hold only if $\|w\|_{\widetilde{L^r_t}\dot B^s_{p,q}}$ vanishes, that is if w vanishes on $[0,T]$. This proves local uniqueness and, by continuity, global uniqueness too. $\square$

As pointed out earlier, when $s_p < 0$, there is no easy way to define uniqueness in $C_t(\dot B^{s_p}_{p,q})$, since u^2 may not be defined even in the distributional sense. One can however take advantage of the regularizing effect, to obtain the following result:

Proposition A.1
Let $u_0 \in \dot{B}^{s_p}_{p,q}$. Then there exist $N \in \mathbb{N}$ and N multilinear operators B_l such that the local solution to (1) can be decomposed as

$$u(x,t) = S(t)u_0 + \sum_{l=2}^{N} B_l(S(t)u_0) + r_{N+1}(x,t), \tag{39}$$

where B_l is of order l and $r_{N+1}(x,t) \in C_t(L^3)$ is unique.

Remark A.3
Such an expansion could be refined, to obtain a remainder in $\dot{B}^2_{1,1}$ (or even better if one is willing to consider $p, q < 1$).

Proof of Proposition A.1.

Recall that if we write the equation in integral form (which, in our setting, can be done after frequency localisation, to have an infinite set of equations with smooth solutions), we have $u = S(t)u_0 + B(u,u)$. To make notations simpler, let us call $u_1 = S(t)u_0$, and $B_2 = B$. One can replace u in $B_2(u,u)$ by $u_1 + B_2(u,u)$, and iterate. Obviously one obtains a multilinear expension, with multilinear operators of increasing order. For example, $B_3 = B_2(u_1, B_2(u_1,u_1)) + B_2(B_2(u_1,u_1),u_1)$. If we order these multilinear operators in increasing order up to order N, with N arbitrary, we get

$$u = u_1 + \sum_{l=2}^{N} B_l(u_1) + r_{N+1},$$

where B_i is multilinear of order i, and r itself is a finite sum of N multilinear operators of order $N+1$, with M entries being u_1 and $N+1-M$ being u, for $0 \le M \le N$. We will denote these operators generically as B_{N+1}. One could of course write such a decomposition explicitly, but it does not provide any useful information, as one needs only to remember that for all $2 \le i \le N+1$, B_i is of order i and is a iterate of B_2. Now, recall that $u, u_1 \in \widetilde{L^\infty_t}(\dot{B}^{s_p}_{p,q}) \cap \widetilde{L^1_t}(\dot{B}^{s_p+2}_{p,q}) = F_1$. Consider $B_2(x,y)$, where x, y are indifferently u, u_1. From product rules, using the L^∞_t information for the low frequencies (as $s_p < 0$) and the L^1_t information for high frequencies (and as $s_p + s_p + 2 > 0$, the high-high frequencies interactions are dealt with in the same way), we obtain that $x \otimes y \in \widetilde{L^1_t}(\dot{B}^{2s_p+2}_{p/2,q/2})$, with

$$\|x \otimes y\|_{\widetilde{L^1_t}(\dot{B}^{2s_p+2}_{p/2,q/2})} \lesssim \|x\|_{F_1} \|y\|_{F_1}.$$

Then, using the bicontinuity of the operator B as proved in Proposition 4.1 (see also Remark 4.1), we get

$$B_2(x,y) \in \widetilde{L^\infty_t}(\dot{B}^{2s_p+1}_{p/2,q/2}) \cap \widetilde{L^1_t}(\dot{B}^{2s_p+3}_{p/2,q/2}) = F_2.$$

One can iterate these estimates, to obtain

$$B_N(x_1,...,x_N) \in \widetilde{L_t^\infty}(\dot{B}^{Ns_p+N-1}_{p/N,q/N}) \cap \widetilde{L_t^1}(\dot{B}^{Ns_p+N+1}_{p/N,q/N}) = F_N.$$

At this point, to avoid any unpleasantness with indices less than one, we suppose $p = q$. There is no loss in generality, as one can consider $P = \sup(p,q)$ and simply use $\dot{B}^{s_p}_{p,q} \hookrightarrow \dot{B}^{s_P}_{P,P}$. Then we stop the expansion as soon as $1 < p/(N+1) \le 2$. This ensures that $r_{N+1} \in F_{N+1}$, and by embedding, $r_{N+1} \in C_t(\dot{H}^{\frac{1}{2}}) \hookrightarrow C_t(L^3)$. From there, uniqueness follows in the standard way, from

$$\|B(x,y)\|_{C_t(\dot{B}^1_{(\frac{3}{2},\infty),\infty})} \lesssim \|x\|_{C_t(L^{3,\infty})}\|y\|_{C_t(L^{3,\infty})}.$$

Here $L^{3,\infty}$ denotes weak L^3, and $\dot{B}^1_{(\frac{3}{2},\infty),\infty}$ is the Besov space constructed on weak $L^{3/2}$. Since $\dot{B}^1_{(\frac{3}{2},\infty),\infty} \hookrightarrow \dot{B}^{1/2}_{2,\infty} \hookrightarrow L^{3,\infty}$, after taking differences between two solutions with remainders r_{N+1} and $\tilde{r}_{N+1}$ one concludes $\|r_{N+1} - \tilde{r}_{N+1}\|_{L^{3,\infty}} = 0$. We ignored entries in the remainders which have at least one u_1 factor, since they are easier to treat (just use an $\widetilde{L_t^r}$ on such a factor, which will guarantee its smallness for a small time interval). This ends the proof. □

For the sake of completeness, we now proceed to prove two other useful properties which have been used in Section 3.

PROPOSITION A.2
Let u be the small solution constructed by Theorem A.1. Then,

- *u is such that $\sqrt{t}\|u\|_\infty \lesssim \|u_0\|_{\dot{B}^{s_p}_{p,q}}$,*
- *and moreover $\lim_{t\to\infty} \|u\|_{\dot{B}^{s_p}_{p,q}} = 0$.*

We remark that we could have actually encoded both informations in the definition of our space X where the fixed point is proved. However, we feel it is of interest to see they can be recovered afterward.

PROOF OF PROPOSITION A.2.

Let us start with the first one. We first prove an intermediate step, defining a Banach space G_p with norm

$$\|u\|_{G_p} = \sup_t t^{\frac{1}{2}-\frac{3}{2p}}\|u\|_p$$

and prove

$$\|u\|_{G_p} \lesssim \|u_0\|_{\dot{B}^{s_p}_{p,q}}.$$

One can prove, using the spatial decay of the convolution kernel appearing in B_2 ([16]),

$$\|B_2(x,y)\|_{G_p} \lesssim \|x\|_{G_p}\|y\|_{G_p},$$

which ensures that in the decomposition (39) all B_i for $2 \leq i \leq N$ belong to G_p, since $\|u_1\|_{G_p} \sim \|u_0\|_{\dot{B}^{s_p}_{p,q}}$. One has then to consider the remainder r_{N+1}. For this term, we simply use

$$\|B_2(x,y)\|_{G_p} + \|B_2(y,x)\|_{G_p} \lesssim \|x\|_{G_p}\|y\|_{L^\infty(L^3)},$$

which follows again from the decay of the kernel. Remark that in order to avoid a $1/s$ weight in these last two estimates on B_2 we are forced to prove the intermediate result for $p < \infty$. Hence, we have $u = u_p + r_{N+1}$ where $u_p \in G_p$ and $r_{N+1} \in L^\infty(L^3)$. Therefore,

$$r_{N+1} = u_1 - u_p + B(u_p, u_p) + B(u_p, r_{N+1}) + B(r_{N+1}, u_p) + B(r_{N+1}, r_{N+1}),$$

and combining the two previous bicontinuity estimates and Lemma A.2, with $L = B(u_p, \cdot) + B(\cdot, u_p)$, we obtain the desired result, $r_{N+1} \in G_p$, provided we take the constant K_0 small enough (possibly smaller than in the proof of Theorem A.1, but we never need a precise estimate on the size of K_0 throughout the rest of the paper). One then recovers $u \in G_\infty$ since

$$\|B(x,y)\|_{G_\infty} + \|B(y,x)\|_{G_\infty} \lesssim \|x\|_{G_p}\|y\|_{G_p}.$$

We are left with the limit when $t \to \infty$. This property is true for the linear term, as

$$S(t)u_0 = \sum_{-N<j<N} S(t)\Delta_j u_0 + \sum_{|j|\geq N} S(t)\Delta_j u_0,$$

and for N large the remainder can be bounded uniformly in time in $\dot{B}^{s_p}_{p,q}$ by the reminder, $\sum_{|j|\geq N}(2^{js_p}\|\Delta_j u_0\|_p)^q$, which goes to zero with N; then at fixed N the first sum goes to zero for large time as a finite sum. Now, consider the nonlinear term, $B(u,u) = \int_0^t S(t-s)f(s)ds$, with $f = \mathbb{P}\nabla \cdot (u \otimes u)$. Split the integral in two,

$$B(u,u) = \int_0^T S(t-s)f(s)ds + \int_T^t S(t-s)f(s)ds.$$

Since we have an $\widetilde{L^1_t}$ information on f, the second part will obviously be as small as we want for large T. Then the first part can be rewritten as $S(t-T)\int_0^T S(T-s)f(s)ds$, for which one can proceed as for the linear part. $\square$

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