The Tridimensional Navier–Stokes Equations with Almost Bidimensional Data: Stability, Uniqueness and Life Span

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Abstract. *We study the tridimensional periodic Navier–Stokes equations when the initial data is close to a bidimensional vector field: we prove a stability theorem on the tridimensional part of the solutions, as well as results regarding their life span. The stability theorem leads to a uniqueness result on the solutions of the tridimensional equations in the class of the solutions satisfying the energy estimate, when the initial data is bidimensional.*

1 Introduction

Let us consider the following tridimensional incompressible Navier–Stokes equations:

$$(NS3D)\left\{\begin{array}{rcl} \partial_t v + v\cdot\nabla v - \nu\Delta_2 v - \eta\,\partial_3^2 v & = & -\nabla p + f \quad \text{in} \quad \Omega\subset\mathbf{R}^3 \\ \operatorname{div} v & = & 0 \quad \text{in} \quad \Omega \\ v_{|t=0} & = & v_0 \end{array}\right.$$

where ν and η are two strictly positive real numbers, representing respectively the horizontal and vertical viscosities. The tridimensional vector field f corresponds to the exterior forces. We shall suppose that the initial field v_0, tridimensional and divergence free, converges to a field $\bar v_0$ which only depends on two variables.

We have written $\Delta_2 = \partial_1^2 + \partial_2^2$ and $v\cdot\nabla v = \sum_{i\leq 3} v^i\partial_i v$, where $v = (v^1, v^2, v^3)$. Let us note that, as v is divergence free, we can write $v\cdot\nabla v = \operatorname{div}(v\otimes v)$. In the following, v' will be the first two components of v, *i.e.* $v' = (v^1, v^2)$. We will also write $\nabla_2 = (\partial_1, \partial_2)$, where ∂_i represents the partial derivative in the direction x_i. Similarly we will note the divergence along the horizontal variables by $\operatorname{div}_2 v = \partial_1 v^1 + \partial_2 v^2$. The domain Ω will be the tridimensional torus $\Omega = \mathbf{T}^3$, but the results proved here also hold if $\Omega = \mathbf{R}^2_{x_1,x_2}\times\mathbf{S}^1_{x_3}$.

In the following we will say that a vector field is a solution of $(NS3D)$ if it satisfies the equation in the distribution sense.

Remark. In the whole of this text, all the proofs will be based on energy estimates; hence one can add to $(NS3D)$ any term of the type Lv where L is a skew–symmetric matrix, since such a term does not contribute to the estimates, and the results proved here are unchanged. Typically, when Lv represents the Coriolis force, this corresponds to the rotating fluid equations (see [5] for instance).

The following theorem sums up the principal and most classical results concerning $(NS3D)$, the proofs of which can be found in [7], [2] or [4] for instance:

Theorem 1.1 *If $v_0 \in L^2(\mathbf{T}^3)$ and $f \in L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^3))$, then the system $(NS3D)$ has at least one solution v in $L^\infty(\mathbf{R}^+, L^2(\mathbf{T}^3)) \cap L^2(\mathbf{R}^+, H^1(\mathbf{T}^3))$ that satisfies, for all $t > 0$,*

$$\begin{aligned}\|v(t)\|^2_{L^2(\mathbf{T}^3)} + \nu \int_0^t \|\nabla_2 v(\tau)\|^2_{L^2(\mathbf{T}^3)} d\tau + \eta \int_0^t \|\partial_3 v(\tau)\|^2_{L^2(\mathbf{T}^3)} d\tau \\ \leq \|v_0\|^2_{L^2(\mathbf{T}^3)} + \frac{C}{\min(\nu,\eta)} \|f\|^2_{L^2([0,t],H^{-1}(\mathbf{T}^3))}. \end{aligned} \tag{1}$$

If $v_0 \in H^{\frac{1}{2}}(\mathbf{T}^3)$ and $f \in L^2(\mathbf{R}^+, H^{-\frac{1}{2}}(\mathbf{T}^3))$, then there exists a unique maximal time $T^* > 0$, and a unique solution v, in $C^0([0, T^*[, H^{\frac{1}{2}}(\mathbf{T}^3)) \cap L^2_{loc}([0, T^*[, H^{\frac{3}{2}}(\mathbf{T}^3))$. Finally, if the force satisfies $\|f\|^2_{L^2(\mathbf{R}^+, H^{-\frac{1}{2}}(\mathbf{T}^3))} \leq C \min(\nu, \eta)^3$, then the time T^*, if finite, is bounded in the following way:

$$T^* \leq C \ \min(\nu,\eta)^{-5} \left(\|v_0\|^4_{L^2(\mathbf{T}^3)} + C \min(\nu,\eta)^{-2} \|f\|^4_{L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^3))} \right).$$

Let us now consider the following bidimensional Navier–Stokes equations:

$$(NS2D) \begin{cases} \partial_t u + u' \cdot \nabla_2 u - \nu \Delta_2 u &= (-\nabla_2 p, 0) + \bar{f} \quad \text{in} \quad \mathbf{T}^3 \\ \operatorname{div}_2 u' &= 0 \quad \text{in} \quad \mathbf{T}^3 \\ \partial_3 u &= 0 \quad \text{in} \quad \mathbf{T}^3 \\ u_{|t=0} &= v_0^h \, . \end{cases}$$

The initial data v_0^h is the vertical mean of v_0, $v_0^h = \displaystyle\int_{x_3} v_0 dx_3$, and $\bar{f}$ is a bidimensional force. In the following, we shall note the vertical mean of a function a by a^h, and $a^v = a - a^h$.

Compared to the classical bidimensional equations, the difference here is that u has three components instead of two. However, in the proofs of existence or uniqueness theorems, that can be found in [2] or [4] for instance, the number of variables comes alone into consideration, and not the number of components of the velocity field. Hence the results known for those equations hold here and are valid for u:

Theorem 1.2 *If $v_0^h \in L^2(\mathbf{T}^2)$ and $\bar{f} \in L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^2))$, then $(NS2D)$ has a unique solution u in $L^\infty(\mathbf{R}^+, L^2(\mathbf{T}^2)) \cap L^2(\mathbf{R}^+, H^1(\mathbf{T}^2))$, which moreover satisfies $u \in C^0(\mathbf{R}^+, L^2(\mathbf{T}^2))$, and, for all $t > 0$,*

$$\|u(t)\|^2_{L^2(\mathbf{T}^2)} + \nu \int_0^t \|\nabla_2 u(\tau)\|^2_{L^2(\mathbf{T}^2)} d\tau = \|v_0^h\|^2_{L^2(\mathbf{T}^2)} + \frac{C}{\nu} \|\bar{f}\|^2_{L^2([0,t],H^{-1}(\mathbf{T}^2))}. \tag{2}$$

Furthermore, if $v_0^h \in H^{\frac{1}{2}}(\mathbf{T}^2)$ and $\bar{f} \in L^2(\mathbf{R}^+, H^{-\frac{1}{2}}(\mathbf{T}^2))$, then there exists a unique associate solution u, and $u \in C^0(\mathbf{R}^+, H^{\frac{1}{2}}(\mathbf{T}^2)) \cap L^2(\mathbf{R}^+, H^{\frac{3}{2}}(\mathbf{T}^2))$.

Notation. In the following we will call $E_0^h(t)$ the right hand side of (2), and $E_0^h = E_0^h(+\infty)$.

We are interested here in seeing if the results concerning $(NS3D)$ recalled in Theorem 1.1 approach those concerning the bidimensional equations, from Theorem 1.2, when the initial data v_0 converges to a bidimensional field: we shall suppose that $\bar{v}_0$, the limit of v_0, satisfies $\partial_3 \bar{v}_0 = 0$, and in the same way that f converges to a bidimensional force $\bar{f}$.

We will call $\bar{u}$ the bidimensional vector field, with three components, solution of the bidimensional Navier–Stokes equations, associated with the initial data $\bar{v}_0$.

2 Statement of the results

Our aim is the study of the behaviour of the solutions of $(NS3D)$ when the initial data converges to a bidimensional field. Let us therefore define w, the tridimensional part of v, as $w = v - u$, where v and u are respectively solutions of $(NS3D)$ and $(NS2D)$, and let us study the evolution of w as a function of $w_0 = v_0 - v_0^h$.

In the first part (section 4), we shall consider the initial data v_0 in $L^2(\mathbf{T}^3)$, and study the behaviour of the associate solutions of $(NS3D)$. More precisely, we will place ourselves in the following class of solutions, which we shall call "finite energy solutions":

Definition. If $v_0 \in L^2(\mathbf{T}^3)$ and $f \in L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^3))$, then the vector field v will be said to be a solution of $(NS3D)$ of finite energy if $v \in L^\infty(\mathbf{R}^+, L^2) \cap L^2(\mathbf{R}^+, H^1)$ is a solution in the sense of distributions, that satisfies moreover the estimate

$$\|v(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla_2 v(\tau)\|_{L^2}^2 d\tau + \eta \int_0^t \|\partial_3 v(\tau)\|_{L^2}^2 d\tau \leq \|v_0\|_{L^2}^2 + \int_0^t (f \mid v)(\tau)\, d\tau.$$

Remark. In particular, any finite energy solution of $(NS3D)$ satisfies the energy inequality (1). Moreover, if u is a solution of $(NS2D)$, then in particular u is a finite energy solution in the sense of that definition; this last result is proved in section 4 (with Lemma 4.2). Finally let us note that the solutions of $(NS3D)$ obtained by Galerkin approximation, or by penalization of the divergence (see [8]), are suitable.

Theorem 2.1 *There exists a constant C such that the following property is satisfied: let v_0 be a vector field in $L^2(\mathbf{T}^3)$, of vertical mean v_0^h, and let f and $\bar{f}$ be two forces in $L^2(\mathbf{R}^+, H^{-1})$, with $\bar{f}$ bidimensional. If v and u are solutions respectively of $(NS3D)$ and $(NS2D)$, with respective initial data v_0 and v_0^h, and with v of finite energy, then the vector field w, defined by $w = v - u$, satisfies the following estimate for all $t \geq 0$:*

$$\|w(t)\|_{L^2(\mathbf{T}^3)}^2 + \nu \int_0^t \|\nabla_2 w(\tau)\|_{L^2(\mathbf{T}^3)}^2 d\tau + \eta \int_0^t \|\partial_3 w(\tau)\|_{L^2(\mathbf{T}^3)}^2 d\tau \tag{3}$$

$$\leq \left(\|w_0\|_{L^2(\mathbf{T}^3)}^2 + \frac{C}{\nu} \|f^h - \bar{f}\|_{L^2([0,t],H^{-1})}^2 + \frac{C}{\min(\nu,\eta)} \|f^v\|_{L^2([0,t],H^{-1})}^2 \right) \exp\left(\frac{C}{\nu^2} E_0^h(t) \right).$$

Corollary 2.1 *If v_0 does not depend on the third variable, $v_0 \in L^2(\mathbf{T}^2)$, and if $f = f^h$, then there exists a unique finite energy solution of $(NS3D)$ in $L^\infty(\mathbf{R}^+, L^2) \cap L^2(\mathbf{R}^+, H^1)$: it is the solution of $(NS2D)$.*

Remark. The proof of those results is founded in the method followed by W. von Wahl in [9] to prove the uniqueness of the solutions of $(NS3D)$ in certain classes of solutions: namely one procedes by energy estimates, and that is the reason why we have been led to the restriction to the finite energy solutions of $(NS3D)$, to ensure the validity of the computations.

Theorem 2.2 *If $v_0 \in L^2(\mathbf{T}^3)$ converges in L^2 to $\bar{v}_0 \in L^2(\mathbf{T}^2)$, if f converges to $\bar{f}$, a bidimensional force, in $L^2(\mathbf{R}^+, H^{-1})$, and if v is a finite energy solution of $(NS3D)$, then v converges to $\bar{u}$ in $L^\infty(\mathbf{R}^+, L^2(\mathbf{T}^2)) \cap L^2(\mathbf{R}^+, H^1(\mathbf{T}^2))$, where $\bar{u}$ is the solution of $(NS2D)$ with initial data $\bar{v}_0$ and exterior force $\bar{f}$.*

Remark. In the case when the force f is bidimensional, i.e. $f^v = 0$, one can note that in the upper bound in (3), the vertical viscosity does not appear, and in particular can be chosen as small as one likes, without that choice affecting in any way the rate of convergence of w to 0; this is not the case for the horizontal viscosity ν, which appears in a crucial way in (3). Therefore when $f = f^h$, everything takes place as if there were no dependence on the vertical variable — that will appear clearly in the course of the proofs.

In the following part, section 5, we will study the case when the data is in $H^{\frac{1}{2}}(\mathbf{T}^3)$:

Theorem 2.3 *There exists a constant C such that the following property is satisfied: let v_0 be a vector field in $H^{\frac{1}{2}}(\mathbf{T}^3)$, of vertical mean v_0^h, and let $w_0 = v_0 - v_0^h$. We consider two forces f and $\bar{f}$ in $L^2(\mathbf{R}^+, H^{-\frac{1}{2}})$, where $\bar{f}$ is bidimensional. Then if*

$$\|w_0\|^2_{H^{\frac{1}{2}}(\mathbf{T}^3)} + \frac{C}{\nu}\|f - \bar{f}\|^2_{L^2(\mathbf{R}^+, H^{-\frac{1}{2}})} \leq C\nu^2 \exp\left(-\frac{C}{\nu^2}E_0^h\right),$$

then the system $(NS3D)$, with equal horizontal and vertical viscosities, is globally well posed in $C^0(\mathbf{R}^+, H^{\frac{1}{2}}(\mathbf{T}^3)) \cap L^2(\mathbf{R}^+, H^{\frac{3}{2}}(\mathbf{T}^3))$ and we have, for all $t \geq 0$,

$$\|w\|^2_{L^\infty([0,t], H^{\frac{1}{2}})} + \nu\|w\|^2_{L^2([0,t], H^{\frac{3}{2}})} \leq \left(\|w_0\|^2_{H^{\frac{1}{2}}} + \frac{C}{\nu}\|f - \bar{f}\|^2_{L^2([0,t], H^{-\frac{1}{2}})}\right) \exp\left(\frac{C}{\nu^2}E_0^h(t)\right),$$

where $w = v - u$, and v and u are solutions respectively of $(NS3D)$ and $(NS2D)$, with respective initial data v_0 and v_0^h.

Remark. Rather than on the smallness of the initial data, the sufficient condition for there to be global existence depends on the relative size of its tridimensional part and its vertical mean. Let us note that in [6], D. Iftimie proves a more general result, computing the norm of w in spaces containing strictly $H^{\frac{1}{2}}$, called $\mathcal{H}^{\delta, \frac{1}{2}-\delta}$, with $\delta \in]0, \frac{1}{2}[$; they are defined using a Littlewood–Paley decomposition, which is non isotropic in the horizontal and vertical variables. The interesting aspect of the weaker result proved here is that it can be found in a more elementary way, without the use of those new spaces.

Corollary 2.2 *If $v_0 \in H^{\frac{1}{2}}(\mathbf{T}^3)$ satisfies $v_0^h \to \bar{v}_0$ in $L^2(\mathbf{T}^2)$ and $w_0 \to 0$ in $H^{\frac{1}{2}}(\mathbf{T}^3)$, and if $f \to \bar{f}$ in $L^2(\mathbf{R}^+, H^{-\frac{1}{2}})$, where $\bar{f}$ is bidimensional, then v converges to $\bar{u}$, solution of $(NS2D)$ with the initial data $\bar{v}_0$, and for w_0 and $f - \bar{f}$ small enough, the system $(NS3D)$ is globally well posed in $C^0(\mathbf{R}^+, H^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+, H^{\frac{3}{2}})$.*

Those various results will enable us to find an upper bound for the life span when it is finite.

Theorem 2.4 *Let us consider v_0 an element of $H^{\frac{1}{2}}(\mathbf{T}^3)$, and v_0^h its vertical mean, and let f and $\bar{f}$ be two forces in $L^2(\mathbf{R}^+, H^{-\frac{1}{2}})$, where $\bar{f}$ is bidimensional, such that*

$$\|f - \bar{f}\|^2_{L^2(\mathbf{R}^+, H^{-\frac{1}{2}})} \leq C\nu^3 \exp\left(-\frac{C}{\nu^2}E_0^h\right).$$

Then let v be a solution of $(NS3D)$ associated with v_0, when the vertical and horizontal viscosities are equal, and finally let T^* be its life span. If T^* is finite, then

$$T^* \leq \frac{C}{\nu^5}\left(\|w_0\|^4_{L^2(\mathbf{T}^3)} + \frac{C}{\nu^2}\|f - \bar{f}\|^4_{L^2(\mathbf{R}^+, H^{-1})}\right) \exp\left(\frac{C}{\nu^2}E_0^h\right).$$

That estimate is a more precise version of the one written in Theorem 1.1, as it enables one to recover the global existence when the tridimensional part w_0 is zero, and $f = f^h$.

3 Notations

In this section, we are going to define some notations, and state a few classical results concerning Sobolev spaces: we shall note $|a|_s$ the norm of any function a in the Sobolev space $H^s(\mathbf{T}^d)$, where $\mathbf{T}^d$ is the d–dimensional torus. If the a_n are the Fourier coefficients of a, then that norm is defined by

$$|a|_s^2 = \sum_{n\in\mathbf{Z}^d} (1+|n|^2)^s |a_n|^2.$$

Let us note that in order to simplify notations, we do not specify in the definition of the norm, the dimension of the space in which a is defined.

The Sobolev spaces $H^s(\mathbf{T}^d)$ can be characterized using a dyadic decomposition in the frequency space: let us recall some elements of the Littlewood–Paley theory, the details of which, along as the proofs of the results stated here, can be found in [6].

Definition. Let φ be a C^∞ function, supported in $]-1,1[$, equal to 1 near $[0,\frac{1}{2}]$. For any distribution a, we define the operators S_q and Δ_q, for $q \geq 0$, by

$$S_q a = \sum_{n\in\mathbf{Z}^d} e^{in\cdot x} a_n \varphi\left(\frac{|n|}{2^q}\right), \quad \Delta_q a = S_{q+1}a - S_q a.$$

We have written $a(x) = \sum a_n e^{in\cdot x}$.

That definition enables one to characterize $H^s(\mathbf{T}^d)$ by

$$a \in H^s(\mathbf{T}^d) \ \Leftrightarrow\ 2^{qs}|\Delta_q a|_0 \in \ell^2(\mathbf{N}).$$

Moreover we have the equivalence $|\nabla\Delta_q a|_0 \sim 2^q|\Delta_q a|_0$, where the constants of equivalence only depend on the truncation function φ chosen, and in particular depend neither on q nor on a.

That decomposition enables one to prove the following product lemma (see [2], [6]):

Lemma 3.1 *Let s and t be two real numbers such that $s,t < \frac{d}{2}$ and $s+t > 0$. If $a \in H^s(\mathbf{T}^d)$ and $b \in H^t(\mathbf{T}^d)$, then $ab \in H^{s+t-\frac{d}{2}}(\mathbf{T}^d)$, and*

$$|ab|_{s+t-\frac{d}{2}} \leq C_{s,t}|a|_s|b|_t.$$

If b is divergence free, one can take $s,t < \frac{d}{2}+1$ and $s+t>0$, and we have

$$|(\Delta_q(b\cdot\nabla a)|\Delta_q a)| \leq C'_{s,t} c_q 2^{-q(s+t-\frac{d}{2}-1)}|a|_s|b|_t|\Delta_q a|_0,$$

where $(c_q)_{q\geq 0}$ is a sequence of norm 1 in ℓ^2, and $C_{s,t}$ and $C'_{s,t}$ are constants depending on s and t.

From now on one shall note C for any universal constant.

In section 4, we shall be led to studying restrictions of tridimensional functions, to the horizontal variables. More precisely, if the horizontal variables are called x', i.e. $x' = (x_1, x_2)$, then we will note $a_{x_3}(x') = a(x_1, x_2, x_3)$.

We will also define the space $H^s_{x'}$ (resp. $L^2_{x'}$) as the Sobolev space $H^s(\mathbf{T}^3)$ (resp. $L^2(\mathbf{T}^3)$) restricted to the horizontal variables, defined by the norm $|a|_{H^s_{x'}} = |a_{x_3}|_s$. In this last equality, the H^s norm is taken in $\mathbf{T}^2$. Naturally, for a function $\bar{a}$ depending only on two variables, we have $|\bar{a}|_{H^s_{x'}} = |\bar{a}|_s$.

Finally, in the following, $(\cdot \mid \cdot)$ will be the scalar product in L^2.

4 The case when the data is in $L^2(\mathbf{T}^3)$

4.1 Proof of the stability theorem

In this section, we shall prove the estimate (3) of Theorem 2.1, by a method inspired by the one used by W. von Wahl in [9]. Let us note that in [5], a similar theorem is proved using a method requiring results on spaces of the Sobolev type, where the third variable is particularised: those spaces are presented and studied precisely by D. Iftimie in [6]. Here we will rather try to make the most of the purely L^2 setting, to present a more elementary proof, which procedes by energy estimates successively in the horizontal, followed by the vertical, variables.

Proof of Theorem 2.1. Let v and u be two solutions, respectively of $(NS3D)$ and $(NS2D)$, with v of finite energy, and let w be the divergence free vector field defined by $w = v - u$.

Let us note that $\|w\|^2_{L^\infty(\mathbf{R}^+,L^2)} \leq 2\|u\|^2_{L^\infty(\mathbf{R}^+,L^2)} + 2\|v\|^2_{L^\infty(\mathbf{R}^+,L^2)}$, therefore in particular w belongs to $L^\infty(\mathbf{R}^+, L^2(\mathbf{T}^3))$, and similarly w belongs to $L^2(\mathbf{R}^+, H^1(\mathbf{T}^3))$.

Moreover one can write, for all $t \geq 0$,

$$\frac{1}{2}|w(t)|_0^2 + \int_0^t \left(\nu|\nabla_2 w(s)|_0^2 + \eta|\partial_3 w(s)|_0^2\right)\, ds = \frac{1}{2}|u(t)|_0^2 + \nu\int_0^t |\nabla_2 u(s)|_0^2 ds$$

$$+\frac{1}{2}|v(t)|_0^2 + \int_0^t \left(\nu|\nabla_2 v(s)|_0^2 + \eta|\partial_3 v(s)|_0^2\right)\, ds - (v(t) \mid u(t)) - 2\nu\int_0^t (\nabla_2 v(s) \mid \nabla_2 u(s))\, ds. \quad (4)$$

The estimate (3) we want to prove will then result from the two following lemmas:

Lemma 4.1 *Let $t \in \mathbf{R}^+$. The application $(a, b, u) \mapsto \int_0^t (\nabla \cdot (a \otimes b) \mid u)\,(s)\, ds$ is a trilinear form, continuous in $\left(L^\infty([0,t], L^2(\mathbf{T}^3)) \cap L^2([0,t], H^1(\mathbf{T}^3))\right)^2 \times L^2([0,t], H^1(\mathbf{T}^2))$. Furthermore, there exists a constant C, independent of t, such that for any divergence free vector field a, we have*

$$\left|\int_0^t (a \cdot \nabla a | u)\,(s)\, ds\right| \leq \frac{\nu}{2}\int_0^t |\nabla_2 a(s)|_0^2\, ds + \frac{C}{\nu}\int_0^t |u(s)|_1^2 |a(s)|_0^2\, ds. \quad (5)$$

Lemma 4.2 *Let $u_0 \in L^2(\mathbf{T}^2)$ and $v_0 \in L^2(\mathbf{T}^3)$ be two divergence free vector fields, let f and $\bar{f}$ be two forces in $L^2(\mathbf{R}^+, H^{-1})$, with $\bar{f}$ bidimensional, and finally let u and v be the associate solutions of $(NS2D)$ and $(NS3D)$, in $L^\infty(\mathbf{R}^+, L^2) \cap L^2(\mathbf{R}^+, H^1)$. Then we have the following equality, for any time $t \geq 0$:*

$$
\begin{aligned}
(v(t) \mid u(t)) + 2\nu \int_0^t (\nabla_2 v(s) \mid \nabla_2 u(s))\, ds &= (u_0 \mid v_0) + \int_0^t \Big((f \mid u) + (\bar{f} \mid v) \Big)(s)\, ds \\
&- \int_0^t \Big((v \mid u' \cdot \nabla_2 u) + (u \mid v \cdot \nabla v) \Big)(s)\, ds.
\end{aligned}
$$

Proof of Lemma 4.1. We can first notice, as u only depends on two variables, that

$$
\begin{aligned}
\int_0^t |(\nabla \cdot (a \otimes b) \mid u)(s)|\, ds &= \int_0^t \left| \int_{x_3} \Big(\int_{x'} a_{x_3}\, b_{x_3} \cdot \nabla_2 u(s)\, dx' \Big)\, dx_3 \right| ds \\
&\leq C \int_0^t \int_{x_3} \|u(s)\|_{H^1_{x'}} \|a_{x_3}(s)\|_{H^{\frac{1}{2}}_{x'}} \|b_{x_3}(s)\|_{H^{\frac{1}{2}}_{x'}}\, dx_3\, ds \\
&\leq C \int_0^t |u(s)|_1 \Big(\int_{x_3} \|a_{x_3}\|^2_{H^{\frac{1}{2}}_{x'}}\, dx_3 \Big)^{\frac{1}{2}} \Big(\int_{x_3} \|b_{x_3}\|^2_{H^{\frac{1}{2}}_{x'}}\, dx_3 \Big)^{\frac{1}{2}} ds,
\end{aligned}
$$

by the rules of product in Sobolev spaces, and with the notations presented in the previous section. That estimate ensures the continuity of the trilinear application, by interpolation between $L^2_{x'}$ and $H^1_{x'}$: thus in the case of a, we have

$$
\begin{aligned}
\Big(\int_{x_3} \|a_{x_3}(s)\|^2_{H^{\frac{1}{2}}_{x'}}\, dx_3 \Big)^{\frac{1}{2}} &\leq \Big(\int_{x_3} \|a_{x_3}(s)\|_{L^2_{x'}} \|\nabla_2 a_{x_3}(s)\|_{L^2_{x'}}\, dx_3 \Big)^{\frac{1}{2}} \\
&\leq |a(s)|_0 |\nabla_2 a(s)|_0,
\end{aligned}
$$

and similarly for b, and the result follows.

The estimate (5) is then an immediate consequence of the previous computation, where one takes a divergence free, and $b = a$.

Proof of Lemma 4.2. That lemma is proved by the same method as that used by W. von Wahl in [9]. Let us note that, in addition to the hypotheses stated in the lemma, u satisfies $u \in C^0(\mathbf{R}^+, L^2(\mathbf{T}^2))$. So let us consider two divergence free vector fields u_n and v_n, elements of $C^\infty(\mathbf{R}^+ \times \mathbf{T}^2)$ and $C^\infty(\mathbf{R}^+ \times \mathbf{T}^3)$ respectively, such that

$$
u_n \quad \text{converges strongly to} \quad u \quad \text{in} \quad L^\infty(\mathbf{R}^+, L^2(\mathbf{T}^2)) \cap L^2(\mathbf{R}^+, H^1(\mathbf{T}^2)),
$$

$$
v_n \quad \text{converges strongly to} \quad v \quad \text{in} \quad L^2(\mathbf{R}^+, H^1(\mathbf{T}^3)).
$$

Since $\Delta_2 u$ and $\mathrm{div}_2(u \otimes u)$ are in $L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^2))$, then $\partial_t u$ is an element of the space $L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^2))$ as well, and we will then additionnally suppose that

$$
\partial_t u_n \quad \text{converges strongly to} \quad \partial_t u \quad \text{in} \quad L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^2)).
$$

Then as u and v are solutions of $(NS2D)$ and $(NS3D)$ respectively, we have

$$
\int_0^t \Big((\partial_s u \mid v_n) - \nu\, (\Delta_2 u \mid v_n) + (u' \cdot \nabla_2 u \mid v_n) - (\bar{f} \mid v_n) \Big)(s)\, ds = 0, \tag{6}
$$

$$\int_0^t \Big((\partial_s v \mid u_n) - \nu\,(\Delta_2 v \mid u_n) + (v\cdot\nabla v \mid u_n) - (f \mid u_n)\Big)(s)\,ds = 0. \tag{7}$$

It is now a matter of taking the limit in n: since $\Delta_2 u$, $\mathrm{div}_2(u(s)\otimes u(s))$, $\partial_t u$ and $\bar f$ are elements of the space $L^2(\mathbf{R}^+, H^{-1}(\mathbf{T}^2))$, the equality (6) has for limit the following equality:

$$\int_0^t \Big((\partial_s u \mid v) + \nu\,(\nabla_2 u \mid \nabla_2 v) + (u'\cdot\nabla_2 u \mid v) - (\bar f \mid v)\Big)(s)\,ds = 0. \tag{8}$$

Remark. An identical computation, where v_n has been replaced by u_n, enables one to prove that any solution u of $(NS2D)$ is a finite energy solution, as remarked in section 2.

As for the equality (7), one first notices that

$$\int_0^t (\partial_s v \mid u_n)\,(s)\,ds = (u_n(t) \mid v(t)) - (u_{n,0} \mid v_0) - \int_0^t (\partial_s u_n \mid v)\,(s)\,ds.$$

Since we have assumed that $\partial_s u_n$ converges to $\partial_s u$ in the space $L^2(\mathbf{R}^+, H^{-1})$, and u_n converges to u in $L^\infty(\mathbf{R}^+, L^2(\mathbf{T}^2))$, we have

$$\int_0^t (\partial_s v \mid u_n)\,(s)\,ds \quad \text{converges to} \quad (u(t) \mid v(t)) - (u_0 \mid v_0) - \int_0^t (\partial_s u \mid v)\,(s)\,ds.$$

Finally the limit of $\displaystyle\int_0^t (v\cdot\nabla v \mid u_n)\,(s)\,ds$ is given by Lemma 4.1, and the equality (7) thus has for limit

$$(u(t) \mid v(t)) - (u_0 \mid v_0) - \int_0^t \Big((\partial_s u \mid v) - \nu\,(\nabla_2 v \mid \nabla_2 u) - (v\cdot\nabla v \mid u) - (f \mid u)\Big)(s)\,ds = 0. \tag{9}$$

The sum of the two limit equalities (8) and (9) leads to the result.

Return to the proof of the theorem. Let us apply the results of Lemmas 4.1 and 4.2 to the inequality (4) found above. Since v and u are of finite energy, the right hand side is bounded by

$$\frac12 |v_0|_0^2 + \frac12 |v_0^h|_0^2 - (v_0 \mid v_0^h) + \int_0^t \Big((v(s) \mid u'(s)\cdot\nabla_2 u(s)) + (u(s) \mid v(s)\cdot\nabla v(s))\Big)\,ds$$
$$+ \int_0^t \Big((f \mid v) + (\bar f \mid u)\Big)(s)\,ds - \int_0^t \Big((\bar f \mid v) + (f \mid u)\Big)(s)\,ds.$$

But as $\displaystyle\int_0^t \Big((v(s) \mid u'(s)\cdot\nabla_2 u(s)) + (u(s) \mid v(s)\cdot\nabla v(s))\Big)\,ds = \int_0^t (w(s)\cdot\nabla w(s) \mid u(s))\,ds$, and since $|v_0|_0^2 + |v_0^h|_0^2 - 2(v_0 \mid v_0^h) = |w_0|_0^2$, the bound above can also be written

$$\frac12 |w_0|_0^2 + \int_0^t (w(s)\cdot\nabla w(s) \mid u(s))\;ds + \int_0^t (f - \bar f \mid w)(s)\,ds.$$

But we have

$$\big|(f - \bar f \mid w)\big| \le C|f - \bar f|_{-1} |\nabla w|_0. \tag{10}$$

This estimate can be modified if we split the force f into $f^h + f^v$, as we get

$$\big|(f - \bar f \mid w)\big| \le C \int_{x_3} |f^h - \bar f|_{-1} \|\nabla_2 w\|_{L^2_{x'}}\,dx_3 + C|f^v|_{-1}|\nabla w|_0.$$

Finally, using the estimate (5) with $a = w$, we find

$$|w(t)|_0^2 + \int_0^t \left(\nu\,|\nabla_2 w(s)|_0^2 + \eta\,|\partial_3 w(s)|_0^2\right)\,ds \le |w_0|_0^2 + \frac{C}{\nu}\int_0^t |u(s)|_1^2 |w(s)|_0^2\,ds$$

$$+\frac{C}{\nu}\int_0^t |(f-\bar f)(s)|_{-1}^2\,ds + \frac{C}{\min(\nu,\eta)}\int_0^t |f^v|_{-1}^2\,ds. \tag{11}$$

One then just has to conjugate that last estimate by $\exp\left(-\frac{C}{\nu}\int_0^t |u(s)|_1^2\,ds\right)$ to find finally, by standard computations,

$$|w(t)|_0^2 + \int_0^t \left(\nu\,|\nabla_2 w(s)|_0^2 + \eta\,|\partial_3 w(s)|_0^2\right)\,ds$$

$$\le \left(|w_0|_0^2 + \frac{C}{\nu}\|f^h - \bar f\|_{L^2([0,t],H^{-1})}^2 + \frac{C}{\min(\nu,\eta)}\|f^v\|_{L^2([0,t],H^{-1})}^2\right)\exp\left(\frac{C}{\nu}\int_0^t |u(s)|_1^2\,ds\right).$$

Using (2), Theorem 2.1 is proved.

Remark. In the case where $\nu = \eta$, one can rather use the estimate (10) to find

$$|w(t)|_0^2 + \int_0^t \nu\,|\nabla w(s)|_0^2 \le \left(|w_0|_0^2 + \frac{C}{\nu}\|f - \bar f\|_{L^2([0,t],H^{-1})}^2\right)\exp\left(\frac{C}{\nu}\int_0^t |u(s)|_1^2\,ds\right). \tag{12}$$

4.2 Convergence of the solutions of $(NS3D)$ to the solutions of $(NS2D)$

The proof of Theorem 2.2 is achieved by identical computations to the ones above: we define the divergence free vector field $\tilde w$ by $\tilde w = v - \bar u$, and we can then follow the proof of Theorem 2.1. The result follows directly.

5 The case when the data is in $H^{\frac{1}{2}}(\mathbf{T}^3)$

We shall write energy estimates in the space $H^{\frac{1}{2}}(\mathbf{T}^3)$, on the vector field $w = v - u$. The bidimensional character of u is used through the following lemma, the proof of which will be written at the end of this section.

Lemma 5.1 *Let s and t be two real numbers such that $s + t > 0$ and $s, t < 1$. If a and b are two distributions such that $a \in H^s(\mathbf{T}^3)$ and $b \in H^t(\mathbf{T}^2)$, then $ab \in H^{s+t-1}(\mathbf{T}^3)$, and*

$$\|ab\|_{H^{s+t-1}(\mathbf{T}^3)} \le C\|a\|_{H^s(\mathbf{T}^3)}\|b\|_{H^t(\mathbf{T}^2)}. \tag{13}$$

Moreover, if $(\Delta_q)_{q\ge 0}$ is the Littlewood–Paley decomposition in the torus $\mathbf{T}^3$, we have for any a, and for any divergence free vector field b, such that $a \in H^s(\mathbf{T}^3)$ and $b \in H^t(\mathbf{T}^2)$, and with $s, t < 2$ and $s + t > 0$,

$$|(\Delta_q(b\cdot\nabla a)|\Delta_q a)| \le C c_q 2^{-q(s+t-2)}\|a\|_{H^s(\mathbf{T}^3)}\|b\|_{H^t(\mathbf{T}^2)}\|\Delta_q a\|_{L^2}, \tag{14}$$

where $(c_q)_{q\ge 0}$ is a sequence in ℓ^2, of norm 1.

Let us note that, compared to Lemma 3.1, the dependence of a in the third variable does not play any part.

Proof of Theorem 2.3. Let us momentarily assume that lemma to be true, and apply the Littlewood–Paley decomposition to the equation satisfied by w. Defining $w_q = \Delta_q w$, we get

$$\partial_t w_q - \nu\Delta w_q + \Delta_q(w\cdot\nabla w + u\cdot\nabla w + w\cdot\nabla u) = -\Delta_q\nabla p + \Delta_q(f-\bar{f}).$$

Applying Lemma 5.1 to $w\cdot\nabla u$ and $|(\Delta_q(u\cdot\nabla w)|w_q)|$, and its tridimensional counterpart (Lemma 3.1) to $|(\Delta_q(w\cdot\nabla w)|w_q)|$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w_q|_0^2 + \nu 2^{2q}|w_q|_0^2 &\leq Cc_q|w|_1|u|_1|w_q|_0 + Cc_q 2^{\frac{q}{2}}|w|_{\frac{1}{2}}|u|_1|w_q|_0 + Cc_q 2^{\frac{q}{2}}|w|_{\frac{3}{2}}|w|_{\frac{1}{2}}|w_q|_0\\ &+ Cc_q 2^{\frac{q}{2}}|f-\bar{f}|_{-\frac{1}{2}}|w_q|_0.\end{aligned}$$

Let us then multiply that estimate by 2^q, and sum over q; we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w(t)|_{\frac{1}{2}}^2 + \nu|w(t)|_{\frac{3}{2}}^2 &\leq C|w(t)|_1^2|u(t)|_1 + C|w(t)|_{\frac{1}{2}}|w(t)|_{\frac{3}{2}}|u(t)|_1 + C|w(t)|_{\frac{1}{2}}|w(t)|_{\frac{3}{2}}^2\\ &+ C|(f-\bar{f})(t)|_{-\frac{1}{2}}|w(t)|_{\frac{3}{2}}.\end{aligned}$$

Interpolating between $H^{\frac{1}{2}}$ and $H^{\frac{3}{2}}$ yields

$$\frac{1}{2}\frac{d}{dt}|w(t)|_{\frac{1}{2}}^2 + \frac{2}{3}\nu|w(t)|_{\frac{3}{2}}^2 \leq C|w(t)|_{\frac{1}{2}}|w(t)|_{\frac{3}{2}}|u(t)|_1 + C|w(t)|_{\frac{1}{2}}|w(t)|_{\frac{3}{2}}^2 + \frac{C}{\nu}|(f-\bar{f})(t)|_{-\frac{1}{2}}^2,$$

hence

$$\frac{1}{2}\frac{d}{dt}|w(t)|_{\frac{1}{2}}^2 + \frac{\nu}{2}|w(t)|_{\frac{3}{2}}^2 \leq \frac{C}{\nu}|w(t)|_{\frac{1}{2}}^2|u(t)|_1^2 + C|w(t)|_{\frac{1}{2}}|w(t)|_{\frac{3}{2}}^2 + \frac{C}{\nu}|(f-\bar{f})(t)|_{-\frac{1}{2}}^2.$$

Then Gronwall's Lemma, along with the results recalled in (2) lead to

$$\begin{aligned}|w(t)|_{\frac{1}{2}}^2 + \nu\int_0^t |w(\tau)|_{\frac{3}{2}}^2 d\tau &\leq C\left(|w_0|_{\frac{1}{2}}^2 + \frac{C}{\nu}\|f-\bar{f}\|_{L^2([0,t],H^{-\frac{1}{2}})}^2\right)\exp\left(\frac{C}{\nu^2}E_0^h(t)\right)\\ &+ C\left(\int_0^t |w(\tau)|_{\frac{3}{2}}^2 d\tau\right)^2\exp\left(\frac{C}{\nu^2}E_0^h(t)\right).\end{aligned}$$

Then by a classical argument for that type of estimate, we obtain the global existence if

$$|w_0|_{\frac{1}{2}}^2 + \frac{C}{\nu}\|f-\bar{f}\|_{L^2(\mathbf{R}^+,H^{-\frac{1}{2}})}^2 \leq C\nu^2\exp\left(-\frac{C}{\nu^2}E_0^h\right). \qquad (15)$$

Hence Theorem 2.3 is proved.

Proof of Corollary 2.2. The result above implies that, if w_0 is sufficiently small, the system $(NS3D)$ is globally well posed. Indeed, since $v_0^h \to \bar{v}_0$ in L^2 and $w_0 \to 0$ in $H^{\frac{1}{2}}$, we have for v_0 close enough to $\bar{v}_0$,

$$|v_0^h|_0^2 \leq 2|\bar{v}_0|_0^2 \quad\text{and}\quad |w_0|_{\frac{1}{2}} \leq C\nu\exp\left(-\frac{2C}{\nu^2}|\bar{v}_0|_0^2 - \frac{C}{\nu^3}\|\bar{f}\|_{L^2(\mathbf{R}^+,H^{-\frac{1}{2}})}^2\right),$$

and since we have assumed that $f-\bar{f}$ is small enough, then (15) is satisfied, which gives the expected result.

Remark. As in the case of data in $L^2(\mathbf{T}^3)$, identical computations lead to the convergence of the solutions of $(NS3D)$ to those of $(NS2D)$ for an initial data in $H^{\frac{1}{2}}(\mathbf{T}^3)$.

Proof of Theorem 2.4. We shall suppose here that the force f is sufficiently close to a bidimensional force $\bar{f}$, that is to say more precisely that

$$\|f-\bar{f}\|^2_{L^2(\mathbf{R}^+,H^{-\frac{1}{2}})} \leq C\nu^3 \exp\left(-\frac{C}{\nu^2}E_0^h\right).$$

According to Corollary 2.2, the global existence theorem 2.3 implies that for v_0 close enough to $\bar{v}_0$, the life span is infinite. Therefore we shall rather consider the case when the initial data does not converge to some bidimensional data.

So let us suppose that the existence time T^* is finite. The classical computation below gives an upper bound for T^*: for all $T<T^*$, we have

$$meas\left\{t\in[0,T[\,|\,C'^2\nu^2\exp\left(-\frac{C}{\nu^2}E_0^h\right)\leq |w(t)|^2_{\frac{1}{2}}\right\} \leq \frac{C}{\nu^4}\exp\left(\frac{C}{\nu^2}E_0^h\right)\int_0^T |w(t)|^4_{\frac{1}{2}}\,dt,$$

by the Bienaymé–Tchebytchev inequality, where we have noted $meas\ A$ for the Lebesgue measure of A.

Interpolating between L^2 and H^1, we get

$$\begin{aligned} T^* &\leq \frac{C}{\nu^4}\exp\left(\frac{C}{\nu^2}E_0^h\right)\sup_{T<T^*}\|w\|^2_{L^\infty([0,T],L^2)}\|w\|^2_{L^2([0,T],H^1)} \\ &\leq \frac{C}{\nu^5}\left(|w_0|_0^4+\frac{C}{\nu^2}\|f-\bar{f}\|^4_{L^2(\mathbf{R}^+,H^{-1})}\right)\exp\left(\frac{C}{\nu^2}E_0^h\right), \end{aligned}$$

by the estimate (12) proved above, and Theorem 2.4 is proved.

Proof of the product lemma 5.1. The proof of that lemma is very similar to that leading to the product results of D. Iftimie in [6], where different Sobolev or Besov type spaces are studied, in which the variables are separated. However in our setting, we can write more direct computations, without using those new spaces.

To prove the result (13), we are going to follow the proof of the product lemma in Sobolev spaces, in the classical tridimensional case (see [2] for example), trying to seperate as much as possible the horizontal from the vertical variables. With that in mind, let us define the following spaces:

Definition. The space $L^{p,q}$, for $1\leq p,q\leq\infty$, is defined by the norm

$$\|f\|_{L^{p,q}} = \left\|\|f\|_{L^q_{x_3}}\right\|_{L^p_{x'}}.$$

Of course, we have in particular $L^{p,p}=L^p$ for all p.

It is well known that the connection between the L^p and L^q norms of a function whose Fourier transform is supported in a ball, is a function of the radius of that ball (see [2]). In [6] (Theorem 1.3.2), is proved an anisotropic version of that result:

Lemma 5.2 *Let u be a function defined on $\mathbf{T}^3$, whose Fourier transform is supported in the set $\mathcal{B}(0,\lambda,\mu)=\{n\in\mathbf{Z}^3 \mid |n'|\leq\lambda, |n_3|\leq\mu\}$. Let us consider a multi–index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, and four real numbers p,p',q,q' such that $1\leq p\leq p'\leq\infty$ and $1\leq q\leq q'\leq\infty$. Then*

$$\|\partial^\alpha u\|_{L^{p',q'}}\leq C\ \lambda^{\alpha_1+\alpha_2+2(\frac{1}{p}-\frac{1}{p'})}\mu^{\alpha_3+\frac{1}{q}-\frac{1}{q'}}\|u\|_{L^{p,q}}\ .$$

We can apply that lemma to the operators Δ_q, once we have noticed that if $|n|\leq C2^q$, then in particular $|n'|\leq C2^q$ and $|n_3|\leq C2^q$. Lemma 5.2, applied to $\lambda=\mu=2^q$ then leads to:

Corollary 5.1 *For any distribution a defined on $\mathbf{T}^3$, we have the estimates*

$$\|\Delta_q a\|_{L^{\infty,2}}\leq C\ 2^q\|\Delta_q a\|_{L^{2,2}}$$
$$\|\Delta_q a\|_{L^{2,2}}\leq C\ 2^q\|\Delta_q a\|_{L^{1,2}}\ .$$

To prove Lemma 5.1, we shall use J.-M. Bony's paraproduct algorithm (see [1]):

Definition. For any couple of distributions (a,b), we define the following operators:

$$T_a b=\sum_{q\geq 1}S_{q-1}a\Delta_q b,\qquad R(a,b)=\sum_q\sum_{\mu\in\{-1,0,1\}}\Delta_{q+\mu}a\Delta_q b.$$

Using that definition, we can decompose the product ab into

$$ab=T_a b+T_b a+R(a,b). \tag{16}$$

Let us now prove Lemma 5.1, by writing the computations for each of those three terms:

- The term $T_a b$:

$$\begin{aligned}
|\Delta_q T_a b|_0 &= |\Delta_q\sum_p S_{p-1}a\ \Delta_p b|_0\\
&= |\Delta_q\sum_p\sum_{r\leq p-2}\Delta_r a\ \Delta_p b|_0\\
&\leq \sum_{|p-q|\leq N_0}\sum_{r\leq p-2}\|\Delta_r a\|_{L^{\infty,2}}\|\Delta_p b\|_{L^{2,\infty}}\\
&\leq C\sum_{|p-q|\leq N_0}\sum_{r\leq p-2}2^r|\Delta_r a|_0|\Delta_p b|_0,
\end{aligned}$$

since b does not depend on the third variable, which leads to

$$2^{q(s+t-1)}|\Delta_q T_a b|_0\leq C\sum_{\substack{|p-q|\leq N_0\\ r\leq p-2}}2^{rs}|\Delta_r a|_0 2^{pt}|\Delta_p b|_0 2^{(s-1)(q-r)}2^{t(q-p)}.$$

One can then take the ℓ^2 norm in q; since $s<1$, Young's inequality leads to the expected result for the first term in (16).

The computation for $T_b a$ is identical, and we get $T_b a\in H^{s+t-1}(\mathbf{T}^3)$ if $t<1$.

- The term $R(a,b)$:

$$
\begin{aligned}
|\Delta_q R(a,b)|_0 &\leq 2^q \| \Delta_q \sum_p \sum_{\mu\in\{-1,0,1\}} \Delta_p a \Delta_{p-\mu} b \, \|_{L^{1,2}} \\
&\leq 2^q \| \sum_p \sum_{\mu\in\{-1,0,1\}} \Delta_p a \Delta_{p-\mu} b \, \|_{L^{1,2}} \\
&\leq C 2^q \sum_{\mu\in\{-1,0,1\}} \sum_{q\leq p+N_0} |\Delta_p a|_0 \|\Delta_{p-\mu} b\|_{L^{2,\infty}} \\
&\leq C 2^q \sum_{\mu\in\{-1,0,1\}} \sum_{q\leq p+N_0} |\Delta_p a|_0 |\Delta_{p-\mu} b|_0,
\end{aligned}
$$

therefore

$$
2^{q(s+t-1)}|\Delta_q R(a,b)|_0 \leq C \sum_{\substack{q\leq p+N_0\\ \mu\in\{-1,0,1\}}} 2^{ps}|\Delta_p a|_0 2^{(p-\mu)t}|\Delta_{p-\mu} b|_0 2^{s(q-p)+t(q-p)},
$$

which, by Young's inequality again, gives the result for the last term in (16); finally we have proved that $ab \in H^{s+t-1}(\mathbf{T}^3)$, and the first part of the lemma is found.

Before moving on to the proof of the second part of Lemma 5.1, let us note that the computations above lead directly to

$$
|T_{\nabla a} b|_{s+t-2} \leq C\, |\nabla a|_{s-1}|b|_t \quad \text{if} \quad s<2, \tag{17}
$$

and similarly, if b is divergence free, one can write

$$
\begin{aligned}
|\sum_j \partial_j R(a,b_j)|_{s+t-2} &\leq C\, |R(a,b)|_{s+t-1} \\
&\leq C\, |a|_s |b|_t \quad \text{if} \quad s+t>0. \qquad (18)
\end{aligned}
$$

The inequality (14) of Lemma 5.1, for $T_{\nabla a} b$ and $R(\nabla a, b)$, is therefore a consequence of (17) and (18). For the term $T_b \nabla a$ however, the previous computations only give that type of estimate for $t<1$. Therefore we are going to prove the inequality (14) directly, for $T_b\nabla a$, which will hold if $t<2$.

As above, we shall follow the proof of the tridimensional case: it is written for example in [3] in the case of functions defined on $\mathbf{R}^3$, and is identical for periodic functions, on $\mathbf{T}^3$. We will show that the dimension only appears on the low frequency term, which in our case is bidimensional. That will enable us to get the expected result.

So let us estimate the quantity $(\Delta_q T_b \nabla a | \Delta_q a)$. A computation using the fact that b is divergence free shows that

$$
\begin{aligned}
|\,(\Delta_q T_b \nabla a|\Delta_q a)\,| &\leq \sum_{j,|p-q|\leq N_0} |\,([\Delta_q, S_{p-1} b_j]\partial_j \Delta_p a | \Delta_q a)\,| \\
&+ \frac{1}{2} \sum_{j,|p-q|\leq N_0} |\,((S_q - S_p) b_j \partial_j \Delta_p a | \Delta_q a)\,|.
\end{aligned}
$$

Hence we just have to estimate the terms

$$I_q = |([\Delta_q, S_{p-1}b_j]\partial_j\Delta_p a|\Delta_q a)|, \quad \text{and} \quad J_q = \left|(\Delta_{q'}b_j\partial_j\Delta_p a|\Delta_q a)\right|,$$

where p and q' are integers, such that $|q-q'| \leq N_0$ and $|q-p| \leq N_0$.

In order to find an upper bound for I_q, one can notice that if we write $\Delta_q = h_q *$, where $*$ is the convolution operator, then, for any g and any ω,

$$[h_q*, g]\omega = \int_{\mathbf{T}^3} h_q(y)\,(g(x)-g(x-y))\,\omega(x-y)\,dy,$$

which immediately leads to

$$\|[h_q*, g]\omega\|_{L^2} \leq C\|\nabla g\|_{L^\infty}\|\omega\|_{L^2}\|xh_q\|_{L^1}.$$

Let us apply that result to the case when $\omega = \partial_j\Delta_p a$ and $g = S_{p-1}b_j$: we use the fact that

$$\begin{aligned} \|\nabla S_{p-1}b_j\|_{L^\infty} &\leq \sum_{p'\leq p-2} \|\Delta_{p'}\nabla b_j\|_{L^\infty} \\ &\leq C \sum_{p'\leq p-2} 2^{2p'}|\Delta_{p'}b_j|_0, \end{aligned}$$

using Lemma 5.2 and the fact that b is bidimensional, hence

$$\begin{aligned} \|\nabla S_{p-1}b_j\|_{L^\infty} &\leq C \sum_{p'\leq p-2} 2^{p'(2-t)}|b|_t b_p \\ &\leq C\, 2^{p(2-t)}|b|_t, \end{aligned}$$

using $t < 2$ and where b_p is a sequence of norm 1 in ℓ^2. Finally since, as in [6], one can write $\|xh_q\|_{L^1(\mathbf{T}^3)} \leq C2^{-q}$, we get

$$\begin{aligned} I_q &\leq C\, 2^{-q}\|\nabla S_{p-1}b_j\|_{L^\infty}|\partial_j\Delta_p a|_0|\Delta_q a|_0 \\ &\leq C\, 2^{-q}|b|_t 2^{p(2-t)}|a|_s 2^{-p(s-1)}a_p|\Delta_q a|_0, \end{aligned}$$

where $(a_p)_{p\geq 0}$ is a sequence of norm 1 in ℓ^2. One can notice that the dimension only appeared to bound $\|\nabla S_{p-1}b_j\|_{L^\infty}$, and it is there that we gained a half–derivative compared to the tridimensional case.

Finally the term J_q can simply be estimated by the following inequalities:

$$\begin{aligned} \|\Delta_{q'}b_j\|_{L^\infty} &\leq C\, 2^{q'}\|\Delta_q b_j\|_{L^2} \\ &\leq C\, 2^{q'(1-t)}|b|_t b_{q'}, \end{aligned}$$

and

$$\|\partial_j\Delta_p a\|_{L^2} \leq C\, 2^{-p(s-1)}|a|_s a_p,$$

where here again, the dimension appeared on the term b.

Putting together the results found for I_q and J_q we get the estimate we wanted for the term $(\Delta_q(T_b\nabla a) \mid \Delta_q a)$, and the lemma is proved.

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