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On global infinite energy solutions to the Navier–Stokes equations in two dimensions

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Abstract

This paper studies the bidimensional Navier–Stokes equations with large initial data in the homogeneous Besov space $\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$. As long as $r, q < +\infty$, global existence and uniqueness of solutions are proved. We also prove that weak–strong uniqueness holds for the d-dimensional equations with data in $\dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $d/r + 2/q \geq 1$.

1. Introduction

We are interested in solving the 2D incompressible Navier-Stokes system in the whole space, say

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u \cdot \nabla u - \nabla p, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \; x \in \mathbb{R}^2, \; t \geq 0. \end{cases} \tag{1.1}$$

The vector field $u(t,x)$ stands for the velocity of the fluid, the scalar field p for its pressure, and $\nabla \cdot u = 0$ means that the fluid is incompressible.

Recall that global existence for large data in the energy class is well-known; that result goes back to J. Leray [19], and states that for any divergence free initial data u_0 in the space $L^2(\mathbb{R}^2)$, there is a unique, global solution u to (1.1). If $\dot{H}^1(\mathbb{R}^2)$ is the homogeneous Sobolev space then the solution u is in the energy space $C^0_b(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^2))$, where $C^0_b(\mathbb{R}^+)$ stands for the space of functions which are continuous and bounded on $\mathbb{R}^+$. Moreover, the solution u satisfies the energy equality

$$\forall t \geq 0, \quad \|u(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^2)}\, ds = \|u_0\|^2_{L^2(\mathbb{R}^2)}.$$

More recently, global existence for large data was proved for measure-valued vorticity (G.-H. Cottet [9] and Y. Giga, T. Miyakawa and H. Osada [14]); uniqueness is only known under a smallness assumption on the atomic part of the measure ([14, 16]). In this situation, the initial velocity field u_0 given by the Biot-Savart Law is known to be at least in the Lorentz space $L^{2,\infty}$, which is strictly larger than L^2; but not all $u_0 \in L^{2,\infty}$ can be paired with a measure-valued vorticity. On the other hand, global existence holds for almost every conceivable function space under a smallness assumption. The most recent and almost final result is for data which are first derivatives of BMO functions (see the work of H. Koch and D. Tataru [17]); we will call that space BMO^{-1} in the sequel.

In 3D the situation is a lot more complex, and little is known between the weak L^2 solutions (Leray's solutions, in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$, which are known to exist with no uniqueness result) and the strong small L^3 solutions (Kato's solutions [15], which exist and are unique in $C^0(\mathbb{R}^+, L^3(\mathbb{R}^3))$ (see [11] for uniqueness) for small data). One has however weak solutions for a large class of initial data: weak L^p solutions for $1 < p < \infty$ were constructed by C. Calderón in [4] and more recently, P.-G. Lemarié extended those results to "locally L^2" data ([18]). Uniqueness is of course an open problem. We refer to the work of P. Auscher and P. Tchamitchian [1] for the presentation of a large class of function spaces in which the Navier-Stokes equations can be solved uniquely and globally, for small data (or locally for large data).

On the other hand, in 2D one expects the small data existence to extend to large data, even beyond L^2 data, as long as one works with a functional space which scales like L^2. Recall that the scaling of the Navier–Stokes equations in $\mathbb{R}^d$, with $d \geq 2$, is as follows: for any real number λ, u is a solution to the Navier–Stokes equations associated with the data u_0 if the same goes for u_λ associated with $u_{0,\lambda}$, with

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) \stackrel{\text{def}}{=} \lambda u_0(\lambda x).$$

The space $L^2(\mathbb{R}^2)$ is clearly invariant under the transformation $u_0 \mapsto u_{0,\lambda}$.

In order to achieve global existence results, we will follow Calderón's procedure [4] and perform a (non-linear) interpolation between Leray's solutions and Kato's solutions (or more accurately, their extensions to Besov spaces). Hence, one expects to get any data which fits into any interpolation space between L^2 and BMO^{-1}. The Besov spaces $\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$ appear very naturally in this context, for $r > 2$, $2 < q < \infty$ (the case where $r \leq 2$ is essentially easy, as the regularity is then positive). We note that by using the different techniques developed in [18], one could get another class of initial data (roughly the density of the Schwartz class in the Morrey-Campanato space $M^1_2(\mathbb{R}^2)$), but still miss homogeneous data. We emphasize the fact that the most interesting case is for r and q large, for which $\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$ is close to BMO^{-1}. Indeed, as soon as one gets a global existence result for r, q large, it automatically implies global existence for all $r' < r$ and $q' < q$, because of the embedding $\dot{B}^{\frac{2}{r'}-1}_{r',q'} \hookrightarrow \dot{B}^{\frac{2}{r}-1}_{r,q}$.

Before stating our result we recall what Besov spaces are, through their characterizations via frequency localization (see [2] for details).

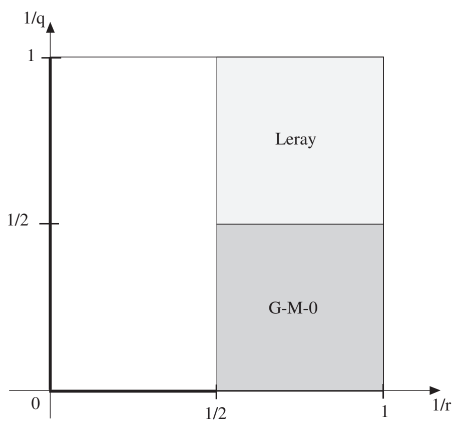
Definition 1. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\widehat{\phi}(\xi) = 1$ for $|\xi| \leq 1$ and $\widehat{\phi}(\xi) = 0$ for $|\xi| > 2$. Define, for $j \in \mathbb{Z}$, the function $\phi_j(x) \stackrel{\text{def}}{=} 2^{dj}\phi(2^j x)$, and the Littlewood–Paley operators $S_j \stackrel{\text{def}}{=} \phi_j * \cdot$ and $\Delta_j \stackrel{\text{def}}{=} S_{j+1} - S_j$. Let f be in $\mathcal{S}'(\mathbb{R}^d)$. If $s < \dfrac{d}{p}$, then f belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ if and only if

– The partial sum $\sum_{-m}^{m} \Delta_j f$ converges towards f as a tempered distribution;
– The sequence $\epsilon_j \stackrel{\text{def}}{=} 2^{js}\|\Delta_j f\|_{L^p}$ belongs to $\ell^q(\mathbb{Z})$.

Theorem 1.1 (2D global existence). *Let r and q be two real numbers such that $2 \leq r < +\infty$ and $2 < q < +\infty$. Let $u_0 \in \dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$ be a divergence free vector field. Then there exists a unique global solution to (1.1) such that $u \in C([0,\infty), \dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2))$. Moreover, if $\dfrac{2}{r} + \dfrac{2}{q} \geq 1$, then there exists a constant $C_{r,q}$ such that*

$$\forall t \geq 0, \quad \|u(t)\|_{\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)} \leq C_{r,q}\|u_0\|^{1+\frac{r+1}{2}}_{\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)}. \tag{1.2}$$

REMARK. Note that previous results recalled above, on data whose curl is a measure, do not include this situation. Indeed such results correspond heuristically to cases when $r \leq 2$ and $q = \infty$, whereas in our case the interest lies especially when r and q are close to infinity. Note moreover that examples of functions precisely in such Besov spaces can be constructed, either simply by using the definition presented above, or more explicitly for instance as in the book [22]. To get a sense of perspective, one may imagine plotting spaces in the interpolation square $(\frac{1}{r}, \frac{1}{q})$, for $1 \leq r, q \leq \infty$. Global existence was previously known only at the point $(\frac{1}{2}, \frac{1}{2})$ (J. Leray [19]) and (for a subset of) the segment $[(\frac{1}{2}, 0), (1, 0)]$ (Y. Giga, T. Miyakawa and H. Osada [14]). As the result at one point yields the result for the upper–right square, that is materialized by a "Leray-square" and a "GMO-square" on the figure below; the result proved here is the rest of the square, except the remaining part of its lower side and its left vertical boundary ($r = \infty$ or $q = \infty$).



Let us note that in the situation where $\frac{2}{r} + \frac{2}{q} > 1$, for $r > 2$ nothing prevents from choosing $q = \infty$. Indeed all estimates in this situation can be made independent of q, and thus one would recover the bottom line result between $\frac{1}{2}$ and 1. We elected not to do so, at this requires some non-trivial form of limiting procedure, in the same spirit as for measure-valued vorticities.

Let us sketch the procedure leading to the result:

1. take $u_0 \in \dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$. Split $u_0 = v_0 + w_0$ where $v_0 \in L^2(\mathbb{R}^2)$ and $w_0 \in BMO^{-1}(\mathbb{R}^2)$ with small norm (actually, taking $w_0 \in \dot{B}^{\frac{2}{\tilde{r}}-1}_{\tilde{r},\tilde{q}}$ with small norm will do, assuming $\tilde{q} > q$ and $\tilde{r} > r$).
2. construct the small data solution w to the Navier-Stokes system with initial data w_0.
3. write down the equation for $u = v + w$. This becomes a Navier-Stokes-like system for v, with additional terms containing w, which we then solve to obtain local in time $L^2(\mathbb{R}^2)$ solutions.
4. obtain an a priori bound for the energy of v. In order to do so, we need to control the additional terms containing w, by the energy of v. We then extend the local solution v to the desired global solution.
5. local existence and uniqueness are known to hold in such a Besov space $\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$, hence the solution $u = v + w$ obtained is unique.

The crucial point in the procedure described above is to obtain estimates on the additional terms in the equation on v containing w (points 4 and 5). Actually some of those estimates will turn out to be very similar to estimates useful in higher dimensions. As recalled above, if one considers an initial data u_0 in $\dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)$, for $d \geq 3$ and $2 \leq r < +\infty$, $2 < q < +\infty$, then there is a unique maximal time T^* and a unique solution u to (NS) associated with u_0, in the space $L^\infty([0,T], \dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d))$ and such that

$$t^{\frac{1}{2}(1-\frac{3}{r})}\|u\|_{L^r(\mathbb{R}^d)} \in L^\infty([0,T])$$

with $T < T^*$. Note that one has in fact continuity in time, except at time zero. A natural question one can ask is then the following: if the initial data is additionally in $L^2(\mathbb{R}^d)$, then it is not difficult to see that u is also a Leray solution associated with u_0; but does uniqueness hold in that larger class of Leray solutions? In other words, as long as one has a "strong" solution to (NS), which is also in the energy space

$$\mathcal{L} \stackrel{\text{def}}{=} L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^d)),$$

do all Leray solutions coincide with that one?

We have called "Leray solution" any weak L^2 solution v of the Navier–Stokes equations in $\mathbb{R}^d$, with $d \geq 3$, satisfying the energy estimate

$$\forall t \geq 0, \quad \|v(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla v(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \leq \|v_0\|^2_{L^2(\mathbb{R}^d)}. \tag{1.3}$$

The answer to that question is given through the following stability theorem. Before stating it, we will need the following proposition, which will be proved in the last section.

Proposition 1.1. *Consider $d \geq 2$, and let r and q be two real numbers such that $2 \leq r < +\infty, 2 < q < +\infty$. Then for any divergence free initial data $u_0 \in \dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)$, the unique solution u associated with u_0 satisfies, for all $T < T^*$ and all $p \in [q, \infty]$,*

$$u \in L^p([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{p}-1}_{r,q}(\mathbb{R}^d)).$$

Theorem 1.2 (Stability). *Consider $d \geq 2$, and let r and q be two real numbers such that $2 \leq r < +\infty, 2 < q < +\infty$. Suppose additionally that*

$$\frac{d}{r} + \frac{2}{q} > 1.$$

Let v_0 and u_0 be two divergence free vector fields in $L^2(\mathbb{R}^d)$, and suppose that u_0 is also an element of $\dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)$. Let $v \in \mathcal{L}$ be any Leray solution associated with v_0, and let u be the unique solution associated with u_0, with $u \in L^q([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)) \cap \mathcal{L}$ for some time $T > 0$. Then $w \stackrel{def}{=} v - u$ satisfies, for all times $t \leq T$,

$$\|w(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \leq \|v_0 - u_0\|^2_{L^2(\mathbb{R}^d)} \times \exp\left(C \int_0^t \|u(s)\|^q_{\dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}\, ds \right).$$

Actually, the result holds with $\dfrac{d}{r} + \dfrac{2}{q} = 1$ as well, with the Besov norm $\dot{B}^{\frac{d}{r}+\frac{2}{p}-1}_{r,q}(\mathbb{R}^d)$ inside the L^p_t norm above replaced by an $L^r(\mathbb{R}^d)$, when $r > d$. Indeed this can be seen somehow as a consequence of Serrin's criterion [23] and integrability properties of strong solutions with data u_0 in $\dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)$ with $\dfrac{d}{r} + \dfrac{2}{q} = 1$. Thus, our theorem is really of interest when $r < d$, for which one can go up to $q = \infty$. Another reason which makes the result worth stating is its proof, which divides the crucial trilinear estimate into three different pieces of which only one requires the restriction on r and q; it does not seem possible to improve on the continuity Lemma 1.1 below without using in a much deeper way the fact that not only a and b are in the Leray class $\mathcal{L}$ but also solutions of the equation.

That theorem yields in a direct way the following corollary.

Corollary 1.1 (Weak–strong uniqueness). *Let v_0 be a divergence free vector field in $L^2(\mathbb{R}^d) \cap \dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)$, with the same restrictions as in Theorem 1.2, and define the associate solution $u \in C^0([0,T], \dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)) \cap \mathcal{L}$, which is unique in $C^0([0,T], \dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d))$. Then all Leray solutions associated with v_0 coincide with u on the time $[0,T]$.*

Before going on with the proof of the results presented here, let us state the main lemma for the proof of Theorem 1.2.

Lemma 1.1. *Let $d \geq 2$ be fixed, and let r and q be two real numbers such that $2 \leq r < +\infty, 2 < q < +\infty$ and $d/r + 2/q > 1$. Then for every $T \geq 0$, the trilinear form*

$$(a,b,c) \in \mathcal{L} \times \mathcal{L} \times L^q([0,T], \dot{B}_{r,q}^{\frac{d}{r}+\frac{2}{q}-1}(\mathbb{R}^d)) \mapsto \int_0^T \int_{\mathbb{R}^d} (a \cdot \nabla b) \cdot c(t)\, dxdt$$

is continuous. In particular the following estimates hold:

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^d} (a \cdot \nabla b) \cdot c\, dxds \right| &\lesssim \|a\|_{L^\infty(\mathbb{R}^+, L^2)}^{2/q} \|\nabla a\|_{L^2(\mathbb{R}^+, L^2)}^{1-2/q} \|\nabla b\|_{L^2(\mathbb{R}^+, L^2)} \\ &\quad \times \|c\|_{L^q([0,T], \dot{B}_{r,q}^{\frac{d}{r}+\frac{2}{q}-1}(\mathbb{R}^d))} + \|\nabla a\|_{L^2(\mathbb{R}^+, L^2)} \|b\|_{L^\infty(\mathbb{R}^+, L^2)}^{2/q} \\ &\quad \times \|\nabla b\|_{L^2(\mathbb{R}^+, L^2)}^{1-2/q} |c\|_{L^q([0,T], \dot{B}_{r,q}^{\frac{d}{r}+\frac{2}{q}-1})} + \|a\|_{L^\infty(\mathbb{R}^+, L^2)}^{1/q} \|\nabla a\|_{L^2(\mathbb{R}^+, L^2)}^{1-1/q} \\ &\quad \times \|b\|_{L^\infty(\mathbb{R}^+, L^2)}^{1/q} \|\nabla b\|_{L^2(\mathbb{R}^+, L^2)}^{1-1/q} \|c\|_{L^q([0,T], \dot{B}_{r,q}^{\frac{d}{r}+\frac{2}{q}-1})}, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^d} (a \cdot \nabla a) \cdot c\, dxds \right| &\leq \|\nabla a\|_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))}^2 \\ &\quad + C \int_0^t \|a(s)\|_{L^2(\mathbb{R}^d))}^2 \|c(s)\|_{\dot{B}_{r,q}^{\frac{d}{r}+\frac{2}{q}-1}(\mathbb{R}^d)}^q \, ds. \end{aligned} \tag{1.5}$$

The rest of the paper is organized as follows. In the first section, we prove Theorem 1.2, as some of the estimates will be useful in the 2D case as well. Then in Section 3 we proceed with Theorem 1.1. The last section consists in the proof of various estimates used in the previous sections.

2. Proof of the weak–strong uniqueness result

The aim of this section is to prove Theorem 1.2. Let us recall the situation: we consider two divergence free vector fields v_0 and u_0, with

$$v_0 \in L^2(\mathbb{R}^d) \quad \text{and} \quad u_0 \in L^2 \cap \dot{B}_{r,q}^{\frac{d}{r}-1}(\mathbb{R}^d).$$

The space dimension here is $d \geq 2$, and we have chosen $2 \leq r, q < +\infty$ with

$$\frac{d}{r} + \frac{2}{q} > 1.$$

We associate with v_0 and u_0 two Leray solutions v and u, in the space $\mathcal{L}$, with additionally according to Proposition 1.1,

$$\forall p \geq q, \quad u \in C^0([0,T], \dot{B}_{r,q}^{\frac{d}{r}-1}(\mathbb{R}^d)) \cap L^q([0,T], \dot{B}_{r,q}^{\frac{d}{r}+\frac{2}{q}-1}(\mathbb{R}^d)).$$

If $w \stackrel{\text{def}}{=} v - u$, then we wish to prove that for all $p > \max(2, q)$,

$$\|w(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \leq \|v_0 - u_0\|^2_{L^2(\mathbb{R}^d)} \times \exp\left(C \int_0^t \|u(s)\|^q_{\dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}\, ds \right).$$

The method of proof of that type of stability result goes back to J. Leray [19], and can be found in the book of W. von Wahl [24] (see also the more recent works [12] and [13]). The idea is as follows: since the vector field $w \stackrel{\text{def}}{=} v - u$ is in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^d))$ by assumption, we can write

$$\|w(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)} ds = \|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^d)} ds$$

$$+ \|v(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla v(s)\|^2_{L^2(\mathbb{R}^d)} ds - 2\,(v(t)|u(t)) - 4\int_0^t (\nabla v(s)|\nabla u(s))\, ds,$$

where $(\cdot \mid \cdot)$ denotes the scalar product in $L^2(\mathbb{R}^d)$. The energy estimate (1.3) recalled in the introduction then implies that

$$\|w(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \leq \|u_0\|^2_{L^2(\mathbb{R}^d)} + \|v_0\|^2_{L^2(\mathbb{R}^d)}$$

$$-2\,(v(t)|u(t)) - 4\int_0^t (\nabla v(s)|\nabla u(s))\ ds.$$

Let us prove the following result.

Lemma 2.1. *Under the assumptions of Theorem 1.2, we have, for all times $t \leq T$,*

$$(v(t)|u(t)) + 2\int_0^t (\nabla v(s)|\nabla u(s))\ ds = (v_0|u_0) + \int_0^t (w \cdot \nabla w(s)|u(s))\ ds.$$

PROOF OF LEMMA 2.1. A formal computation yields the result with no difficulty; in order to prove it, let us consider two sequences of smooth, divergence free vector fields (v_n) and (u_n) such that

$$\lim_{n\to\infty} v_n = v \quad \text{in} \quad L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^d))$$

and

$$\lim_{n\to\infty} u_n = u \quad \text{in} \quad L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^d)) \cap L^q([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)).$$

Taking the scalar product with v_n and u_n of the Navier–Stokes equations on u and v respectively yields, after integration in time and and integration by parts in the space variables,

$$\int_0^t \Big((\partial_s u \mid v_n) + (\nabla u \mid \nabla v_n) + (u \cdot \nabla u \mid v_n) \Big)(s)\, ds = 0,$$

and

$$\int_0^t \Big((\partial_s v \mid u_n) + (\nabla v \mid \nabla u_n) + (v \cdot \nabla v \mid u_n) \Big)(s)\, ds = 0.$$

It is now a matter of taking the limit in n, and of summing the limits found. Since both ∇u_n and ∇v_n converge in $L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))$ towards ∇u and ∇v respectively, it is clear that

$$\lim_{n\to\infty} \Big(\int_0^t (\nabla u \mid \nabla v_n)\,(s)\,ds + \int_0^t (\nabla v \mid \nabla u_n)\,(s)\,ds \Big) = 2\int_0^t (\nabla u \mid \nabla v)\,(s)\,ds.$$

Then Lemma 1.1 implies that

$$\lim_{n\to\infty} \int_0^t (v \cdot \nabla v \mid u_n)\,(s)\, ds = \int_0^t (v \cdot \nabla v \mid u)\,(s)\, ds.$$

Similarly, since the divergence free condition on u yields

$$\int_0^t (u \cdot \nabla u \mid v_n)\,(s)\, ds = \int_0^t (u \cdot \nabla v_n \mid u)\,(s)\, ds,$$

we have, still by Lemma 1.1,

$$\lim_{n\to\infty} \int_0^t (u \cdot \nabla u \mid v_n)\,(s)\, ds = \int_0^t (u \cdot \nabla u \mid v)\,(s)\, ds.$$

But $\partial_s v = \Delta v - \mathbb{P}(v \cdot \nabla v)$ in $\mathcal{D}'(\mathbb{R}^d)$, where $\mathbb{P}$ stands for the Leray projector onto divergence–free vector fields, so those limits imply in particular that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^t (\partial_s v \mid u_n)\,(s)\, ds &= -\lim_{n\to\infty} \int_0^t \Big((\nabla v \mid \nabla u_n) + (v \cdot \nabla v \mid u_n) \Big)(s)\, ds \\
&= -\int_0^t \Big((\nabla v \mid \nabla u) + (v \cdot \nabla v \mid u) \Big)(s)\, ds \\
&= \int_0^t (\partial_s v \mid u)\,(s),
\end{aligned}$$

and similarly

$$\lim_{n\to\infty} \int_0^t (\partial_s u \mid v_n)\,(s)\, ds = \int_0^t (\partial_s u \mid v)\,(s)\, ds.$$

Putting everything together, we find that the limit of

$$\int_0^t \Big((\partial_s u \mid v_n) + (\nabla u \mid \nabla v_n) + (u \cdot \nabla u \mid v_n) \Big)(s)\, ds$$
$$+ \int_0^t \Big((\partial_s v \mid u_n) + (\nabla v \mid \nabla u_n) + (v \cdot \nabla v \mid u_n) \Big)(s)\, ds$$

is

$$\int_0^t \Big((\partial_s u \mid v) + (\partial_s v \mid u) + 2\,(\nabla u \mid \nabla v) + (u \cdot \nabla u \mid v) + (v \cdot \nabla v \mid u) \Big)(s)\, ds.$$

Then we just need to notice that

$$\int_0^t \Big((\partial_s u \mid v) + (\partial_s v \mid u) \Big)(s)\, ds = (u(t) \mid v(t)) - (u_0 \mid v_0)\,,$$

and on the other hand

$$\int_0^t \Big((u \cdot \nabla u \mid v)\,(s) + (v \cdot \nabla v \mid u) \Big)(s)\, ds = \int_0^t (w \cdot \nabla w \mid u)\,(s)\, ds,$$

and Lemma 2.1 is proved. □

Now let us go back to the proof of the theorem. Recall that we have obtained

$$\|w(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \le \|u_0\|^2_{L^2(\mathbb{R}^d)} + \|v_0\|^2_{L^2(\mathbb{R}^d)}$$
$$-2\,(v(t)|u(t)) - 4\int_0^t (\nabla v(s)|\nabla u(s))\;ds,$$

so with Lemma 2.1, that means that

$$\|w(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)} ds \le \|w_0\|^2_{L^2(\mathbb{R}^d)} + \left|\int_0^t (w \cdot \nabla w|u)\,(s)\, ds\right|.$$

But Lemma 1.1, and in particular estimate (1.5), then yields

$$\|w(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \le \|w_0\|^2_{L^2(\mathbb{R}^d)}$$
$$+\, C\int_0^t \|w\|^2_{L^2(\mathbb{R}^d))}\|u\|^q_{\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}(s)\, ds,$$

and since $\|u(\cdot)\|_{\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}$ is an element of $L^q([0,T])$, the result follows by a Gronwall inequality, and Theorem 1.2 is proved. □

Let us make some additional remarks on the case $\dfrac{d}{r} + \dfrac{2}{q} = 1$: from the properties of w, namely $w \in \mathcal{L}$ and $\nabla \cdot w = 0$, it is well-known that

$$w \cdot \nabla w \in L^2(\mathbb{R}^+, \mathcal{H}^1) \cap L^1(\mathbb{R}^+, L_x^{\frac{d}{d-1},2}), \tag{2.1}$$

where $\mathcal{H}^1$ is the Hardy space. The first part was proved in [8], while the second follows from (sharp) Sobolev embedding and Hölder. Hence, in order to make sense of the trilinear form in Lemma 1.1, a sufficient condition would be for the strong solution u to verify

$$u \in L^2(\mathbb{R}^+, \mathrm{BMO}) + L^\infty(\mathbb{R}^+, L_x^{d,\infty}). \tag{2.2}$$

By interpolation, one is naturally led to the (stronger) Serrin condition

$$u \in L^q(\mathbb{R}^+, L^r_x) \text{ with } \frac{d}{r} + \frac{2}{q} = 1,\ r > d. \tag{2.3}$$

Such a condition is automatically verified for data $u_0 \in \dot B^{\frac{d}{r}-1}_{r,q}$ with $\dfrac{d}{r} + \dfrac{2}{q} = 1$, see e.g. [6].

3. Global $2D$ existence

The aim of this section is to prove Theorem 1.1. In the whole of this section, all space norms will be taken over $\mathbb{R}^2$, which we will omit to specify from now on. We start by splitting the data $u_0 \in \dot{B}^{\frac{2}{r}-1}_{r,q}$ into two distinct pieces, $v_0 \in L^2$ and $w_0 \in \dot{B}^{\frac{2}{\tilde r}-1}_{\tilde r,\tilde q}$, with a small norm. We suppose that $\tilde r > r$ and $\tilde q \geq q$. This can always be achieved since our Besov space is an interpolation space between L^2 and the larger Besov space $\dot{B}^{\frac{2}{\tilde r}-1}_{\tilde r,\tilde q}$. We remark that the worst case scenario is clearly for r, q large, and thus we implicitly assume from now on that $\tilde q, \tilde r > N$ for some large N, unless explicitly mentioned.

3.1. Small solutions in the Besov space

This section corresponds to Part 2 of the procedure explained in the introduction. In order to simplify notations, we relabel $\tilde r$ and $\tilde q$ to be r and q: that should not lead to any confusion, as we will not be considering the function u any longer.

All known results apply to solve

$$\begin{cases} \dfrac{\partial w}{\partial t} = \Delta w - \mathbb{P}\nabla \cdot (w \otimes w), \\ \nabla \cdot w = 0, \\ w(x,0) = w_0(x),\, x \in \mathbb{R}^2,\, t \geq 0, \end{cases} \tag{3.1}$$

where $\mathbb{P}$ stands for the Leray projector onto divergence free vector fields. The interested reader may consult [5,7,21] for results of this type by different methods. What we will use is the following result.

Proposition 3.1. *Let $w_0 \in \dot{B}^{\frac{2}{r}-1}_{r,q}$ with small norm. Then there exists a unique global solution w of (3.1) which is such that $w \in C^0_b(\mathbb{R}^+, \dot{B}^{\frac{2}{r}-1}_{r,q})$.*

We remark that the uniqueness part does not follow from the construction of the solution and is in fact a recent result, [11]. As explained in the above references, the unique solution w satisfies many additional properties, of which the following estimate will be the most useful:

$$\sup_t t^{\frac{1}{2}-\frac{1}{\eta}+\frac{\alpha}{2}} \|\nabla^\alpha w\|_{L^\eta} \lesssim \|w_0\|_{\dot{B}^{\frac{2}{r}-1}_{r,q}} \text{ for } \alpha = 0, 1 \text{ and } r \leq \eta \leq \infty. \tag{3.2}$$

In what follows we will assume that $\|w_0\|_{\dot{B}^{\frac{2}{r}-1}_{r,q}} \leq \varepsilon_0$ is very small.

3.2. L^2 solutions to a modified Navier-Stokes system

We shall deal here with points 3 and 4 of the procedure sketched in the introduction. We aim at getting a solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - \mathbb{P}\nabla \cdot (v \otimes w) - \mathbb{P}\nabla \cdot (w \otimes v) - \mathbb{P}\nabla \cdot (v \otimes v), \\ \nabla \cdot v = 0, \\ v(x,0) = v_0(x),\, x \in \mathbb{R}^2,\, t \geq 0, \end{cases} \tag{3.3}$$

where recall that $v_0 \in L^2$, and w satisfies the estimates of Section 3.1.

For this step, many different choices are possible. One may proceed by mollifying the data and/or the equation as it is customary for the weak L^2 theory. Though this can be easily accomplished even with the addition of the w term, we proceed differently and simply get a local in time solution by fixed point on the integral equation

$$v = e^{-t\Delta} v_0 - \int_0^t e^{-(t-s)\Delta} \mathbb{P}\nabla \cdot (v \otimes w + w \otimes v + v \otimes v) ds. \tag{3.4}$$

That local solution will be made global in time by proving an energy estimate in the next section.

The result we shall prove is the following. Before stating it, note that we are going to use Lorentz spaces $L^{p,q}$, which as far as we are concerned may simply be seen as the real interpolation spaces $[L^{p^-}, L^{p^+}]_{(\theta,q)}$ (see [2]).

Proposition 3.2. *Let us define the space E_T, for $T > 0$:*

$$E_T \stackrel{def}{=} \left\{ f \in \mathcal{S}'(\mathbb{R}^2) \mid |f(x,t)\|_T < \infty \right\},$$

where

$$\begin{aligned} |f(x,t)\|_T \stackrel{def}{=} & \sup_{t<T} t^{\frac{1}{4}} \|f\|_{L^4_x} + \|\nabla f\|_{L^2([0,T],L^2_x)} \\ & + \|f\|_{L^4([0,T],L^4_x)} + \|f\|_{L^{2r,2}([0,T],L_x^{\frac{2r}{r-1}})}. \end{aligned} \tag{3.5}$$

Then there exists a time $T > 0$ and a unique solution v to (3.3) in the space E_T.

PROOF OF PROPOSITION 3.2. This follows readily by contraction in E_T (note that the choice of E_T is of course one out of many). As we are going to perform computations on Lorentz spaces, we refer to the last section for the equivalent of Hölder and Young's inequalities for those spaces, which we shall refer to as O'Neil inequalities.

We will denote by $|u\|_{T,1}$, $|u\|_{T,2}$, $|u\|_{T,3}$ and $|u\|_{T,4}$ each part of the norm defined in (3.5). Note that the introduction of Lorentz spaces will turn out useful to obtain the $L^2_t(\dot{H}^1_x)$ estimate on v.

Before proceeding with estimates, we perform the following reduction: we can replace the bilinear operator in (3.4) by its scalar counterpart, which reads

$$B(f,g) \stackrel{\text{def}}{=} \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} G\left(\frac{\cdot}{\sqrt{t-s}}\right) * fg(s)\, ds, \tag{3.6}$$

where $G \in L^1 \cap L^\infty$ (as $\widehat{G}(\xi) \approx \frac{\xi_i \xi_j \xi_k}{|\xi|^2} e^{-|\xi|^2}$). Thus every function is now a scalar function which should be understood as any coordinate of the velocity field.

We proceed now with proving contraction properties for B for all the norms of (3.5). We begin with the first one, which is Kato's weighted norm. If v_1 and v_2 stand for two successive iterates in a fixed point scheme, then we write

$$v_1 - v_2 \approx B(v_1, v_1) - B(v_2, v_2) - B(v_1 - v_2, w) - B(w, v_1 - v_2),$$

and we have

$$\|v_1 - v_2\|_{L^4}(t) \leq \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} \left\| G(\frac{\cdot}{\sqrt{t-s}}) * ((v_1 - v_2)(v_1 + v_2) - 2w(v_1 - v_2))(s) \right\|_{L^4} ds.$$

But by Young's inequality,

$$\begin{aligned} \left\|G(\frac{\cdot}{\sqrt{t-s}}) * (v_1 - v_2)(v_1 + v_2)\right\|_{L^4} &\leq \left\|G(\frac{\cdot}{\sqrt{t-s}})\right\|_{L^{4/3}} \|v_1 - v_2\|_{L^4} \\ &\quad \times (\|v_1\|_{L^4} + \|v_2\|_{L^4}) \\ &\lesssim (t-s)^{\frac{3}{4}} \|v_1 - v_2\|_{L^4} \\ &\quad \times (\|v_1\|_{L^4} + \|v_2\|_{L^4}), \end{aligned}$$

and similarly, using (3.2) for w with $\alpha = 0$ and $\eta = \infty$,

$$\begin{aligned} \|G(\frac{\cdot}{\sqrt{t-s}}) * w(v_1 - v_2)\|_{L^4} &\leq \left\|G(\frac{\cdot}{\sqrt{t-s}})\right\|_{L^1} \|v_1 - v_2\|_{L^4} \|w\|_{L^\infty} \\ &\lesssim (t-s)\|v_1 - v_2\|_{L^4} \varepsilon_0 s^{-1/2}. \end{aligned}$$

So we have obtained

$$\begin{aligned} \|v_1 - v_2\|_{L^4}(t) &\lesssim \varepsilon_0 \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|v_1 - v_2\|_{L^4}(s) ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \|v_1 - v_2\|_{L^4}(s)(\|v_1\|_{L^4} + \|v_2\|_{L^4})(s)\, ds. \end{aligned} \tag{3.7}$$

We now use the generalized Young's inequality for the second term while for the first, we use O'Neil inequalities ([20]) for Lorentz spaces, which we recall in the last section for the reader's convenience. Thus, we get

$$|v_1 - v_2\|_{T,1} \lesssim |v_1 - v_2\|_{T,1}(\varepsilon_0 + |v_1\|_{T,1} + \|v_2\|_{T,1}). \tag{3.8}$$

The control over the third and fourth parts of the norm are along the same lines: using O'Neil inequalities again, we get for any pair $1 < \gamma, \mu, \lambda < \infty$

$$\|v_1 - v_2\|_{L_t^{\gamma,\lambda}(L_x^\mu)} \lesssim \|v_1 - v_2\|_{L_t^{\gamma,\lambda}(L_x^\mu)}(\varepsilon_0 + |v_1\|_{T,1} + \|v_2\|_{T,1}). \tag{3.9}$$

We postpone dealing with the second part of the norm and recover first $v \in L_t^\infty(L_x^2)$, which follows readily from (3.7) with L_x^4 replaced by L_x^2:

$$\|v_1 - v_2\|_{L^2}(t) \lesssim \|v_1 - v_2\|_{L^2}(t)(\varepsilon_0 + |v_1\|_{T,1} + \|v_2\|_{T,1}). \tag{3.10}$$

We are left with proving $v \in L_t^2(\dot{H}^1)$. We deal with the true non-linear term first, namely $B(v) = B(v,v)$, distributing the gradient on the product: similar computations to the case of the $|\cdot\|_{T,1}$ norm yield

$$\begin{aligned} \|\nabla(B(v_1) - B(v_2))\|_{L^2}(t) &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \|\nabla(v_1 - v_2)\|_{L^2}(\|v_1\|_{L^4} + \|v_2\|_{L^4}) ds \\ &\quad + \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} (\|\nabla v_1\|_{L^2} + \|\nabla v_2\|_{L^2}) \|v_1 - v_2\|_{L^4} ds, \end{aligned}$$

for which another application of Hölder and Young allows to conclude.

Hence it remains to deal with $B(v,w)$ which as one would expect turns out to be the most annoying term (remark that we already have an $L^\infty_t(L^2)$ solution, and we are looking for the additional property on its gradient). We write

$$\nabla B(v,w) \approx B(\nabla v, w) + B(v, \nabla w),$$

of which the first term is the easiest (using (3.2) with $\alpha = 0$ and $\eta = \infty$ again): we get

$$\|B(\nabla(v_1 - v_2), w)\|_{L^2}(t) \lesssim \varepsilon_0 \int_0^t \frac{1}{(t-s)^{\frac{1}{r}+\frac{1}{2}}} \frac{\|\nabla(v_1 - v_2)\|_{L^2}(s)}{s^{\frac{1}{2}-\frac{1}{r}}} ds, \quad (3.11)$$

for which O'Neil inequalities give the desired result. Now, the fourth part $|\cdot\|_{T,4}$ of the norm is crucial, together with (3.2) for $\alpha = 1$ and $\eta = r$:

$$\|B(v_1 - v_2, \nabla w)\|_{L^2_x} \lesssim \varepsilon_0 \int_0^t \frac{1}{(t-s)^{\frac{1}{2r}+\frac{1}{2}} s^{1-\frac{1}{r}}} \|v_1 - v_2\|_{L^{\frac{2r}{r-1}}}(s) ds, \quad (3.12)$$

for which one last application of O'Neil inequalities gives the L^2_t estimate. We now have the desired contraction property for the $|\cdot\|_T$ norm. All is left is to check that the quantity $|S(t)v_0\|_T$ is finite and can be made small enough depending on T. For the first three norms it follows directly from well-known properties of the heat equation with initial data in L^2. The last one is however slightly more complicated and will be proved in the last section, Lemma 4.1. Up to that result, Proposition 3.2 is proved. □

Thus, we got a solution $v \in L^\infty([0,T], L^2) \cap L^2([0,T], \dot{H}^1)$, for some $T > 0$. Now we just need to prove it extends globally in time. That is simply due to the energy estimate we are about to prove in the next section: the solution cannot explode in finite time due to Lemma 3.1.

3.3. Energy inequality

We will prove the following lemma. We define

$$\|f(t)\|^2_{\mathcal{L}} \stackrel{\text{def}}{=} \sup_{0 \le s \le t} \|f(s)\|^2_{L^2} + \|\nabla f\|^2_{L^2([0,t],L^2)}.$$

Lemma 3.1. *There exists a time $t_0 > 0$, arbitrarily small, such that the function v defined in the previous section satisfies for all $t > t_0$*

$$\|v(t)\|^2_{\mathcal{L}} \le 2\|v(t_0)\|^2_{L^2} \left(\frac{t}{t_0}\right)^{\varepsilon_0}. \quad (3.13)$$

PROOF OF THE LEMMA. Formally, we may multiply (3.3) by v and integrate over x and t to get, using the fact that v is divergence free,

$$\|v(t)\|^2_{L^2} + 2\int_0^t \|\nabla v(s)\|^2_{L^2} \, ds + \int_0^t \int_{\mathbb{R}^2} (v \cdot \nabla) v w \, dx ds \le \|v_0\|^2_{L^2}. \quad (3.14)$$

Then, if we suppose additionally that $\frac{2}{r} + \frac{2}{q} > 1$, applying Lemma 1.1 proved in Section 4.1 yields

$$\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(s)\|_{L^2}^2 \, ds - C \int_0^t \|v(s)\|_{L^2}^2 \|w(s)\|^q_{B_{r,q}^{\frac{2}{r}+\frac{2}{q}-1}} \, ds \leq \|v_0\|_{L^2}^2, \tag{3.15}$$

which gives us a uniform bound after applying a Gronwall lemma. In particular, Lemma 3.1 is proved in that case.

However we do not wish to restrict ourselves to such (r, q), hence we need to proceed differently. One simple (though rather inelegant) way to obtain a global bound is to use the energy inequality on a time interval (t_0, T) with a small $t_0 > 0$. Indeed the previous section showed that one has a local solution up to, say, $2t_0$, and at time t_0 the small rough solution w has been smoothed out. However it does not seem straightforward to get good bounds on the norm of the solution with such a method (as one does not have good bounds on the time t_0 of local existence). Let us turn to the details: the small rough solution w verifies

$$\sup_t \sqrt{t}\|w\|_{L^\infty} < \varepsilon_0,$$

which allows to write by Hölder's inequality,

$$\left| \int_{t_0}^t \int_{\mathbb{R}^2} (v \cdot \nabla) v w \, dx ds \right| \lesssim \varepsilon_0 \left(\int_{t_0}^t \|\nabla v(s)\|_{L^2}^2 \, ds + \int_{t_0}^t \frac{\|v(s)\|_{L^2}^2}{s} \, ds \right),$$

and that yields the expected bound after applying Gronwall Lemma. We conjecture that a stronger (uniform) bound should hold for all times, but it does require significantly more work and we postpone this issue for later work.

So Lemma 3.1 is proved. Note that the formal computation (3.14) is justified since we apply the energy inequality from a time $t > t_0 > 0$, all terms are smooth and there is no difficulty in defining the various quantities.

However, it is worth noting that one can in fact write the inequality (3.14) from $t = 0$, in the case when r and q are restricted to $\frac{2}{r} + \frac{2}{q} > 1$, and we present the proof here for the sake of completeness. This can be done in a variety of ways, either by smoothing of v_0, w_0 and the associated solutions, as well as the equation through a Friedrich mollifier. We take the opportunity to proceed somewhat differently and use the localization operators Δ_j: let us consider the equation

$$\frac{\partial \Delta_j v}{\partial t} - \Delta \Delta_j v = -\Delta_j \mathbb{P}(\nabla \cdot (v \otimes w) + \nabla \cdot (w \otimes v) + \nabla \cdot (v \otimes v)), \tag{3.16}$$

with smooth initial condition $\Delta_j v_0$. We can multiply the equation by $v_{j-1} + v_j + v_{j+1}$, with $v_j \stackrel{\text{def}}{=} \Delta_j v$, and sum over $j \in \mathbb{Z}$. We have

$$\int_0^t (\partial_s v_j \mid v_{j-1} + v_j + v_{j+1}) \, ds = \int_0^t \partial_s (v_j \mid v_{j-1}) \, ds - \int_0^t (v_j \mid \partial_s v_{j-1}) \, ds$$

$$+\int_0^t \partial_s\left(v_j \mid v_j\right)\,ds - \int_0^t \left(v_j \mid \partial_s v_j\right)\,ds$$
$$+\int_0^t \partial_s\left(v_j \mid v_{j+1}\right)\,ds - \int_0^t \left(v_j \mid \partial_s v_{j+1}\right)\,ds.$$

Then we notice that

$$\sum_{j\in\mathbb{Z}}\int_0^t \Big((v_j \mid \partial_s v_{j-1}) + (v_j \mid \partial_s v_j) + (v_j \mid \partial_s v_{j+1}) \Big)\,ds$$
$$= \sum_{j\in\mathbb{Z}}\int_0^t \Big((v_{j+1} \mid \partial_s v_j) + (v_j \mid \partial_s v_j) + (v_{j-1} \mid \partial_s v_j) \Big)\,ds,$$

which implies that

$$\sum_{j\in\mathbb{Z}}\int_0^t \left(\partial_s v_j \mid v_{j-1}+v_j+v_{j+1}\right)\,ds = \frac12\sum_{j\in\mathbb{Z}}\int_0^t \partial_s\left(v_j \mid v_{j-1}+v_j+v_{j+1}\right)\,ds.$$

Finally we get

$$\int_0^t \sum_{j\in\mathbb{Z}}\left(\partial_s v_j \mid v_{j-1}+v_j+v_{j+1}\right)\,ds = \frac12\|v(t)\|^2_{L^2(\mathbb{R}^2)} - \frac12\|v_0\|^2_{L^2(\mathbb{R}^2)},$$

where we have used the fact that for any function f,

$$\sum_{j\in\mathbb{Z}}\left(f_j \mid f_{j-1}+f_j+f_{j+1}\right) = \|f\|^2_{L^2(\mathbb{R}^2)}.$$

A similar computation enables us to write

$$\int_0^t \sum_{j\in\mathbb{Z}}\left(\Delta v_j \mid v_{j-1}+v_j+v_{j+1}\right)\,ds = \int_0^t \|\nabla v(t)\|^2_{L^2(\mathbb{R}^2)}\,ds.$$

It follows that

$$\begin{aligned}&\|v(t)\|^2_{L^2(\mathbb{R}^2)} - \|v_0\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \|\nabla v(t)\|^2_{L^2(\mathbb{R}^2)}\,ds\\ &= -\sum_{j\in\mathbb{Z}}\int_0^t \left(\Delta_j\mathbb{P}(w\otimes v + v\otimes v + v\otimes w)) \mid v_{j-1}+v_j+v_{j+1}\right)\,ds.\end{aligned}\tag{3.17}$$

Now we shall estimate the right–hand side of that equality. To simplify the notation, let us define $\widetilde{v} \stackrel{\text{def}}{=} v + w$, and let us start by proving that

$$I \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}}\int_0^t \int_{\mathbb{R}^2} \Delta_j(\widetilde{v}\cdot\nabla v)\cdot(v_{j-1}+v_j+v_{j+1})\,dxds = 0.\tag{3.18}$$

Note that the operator $\mathbb{P}$ has disappeared, because $v_{j-1}+v_j+v_{j+1}$ is divergence free. By the support properties of the Littlewood–Paley operators, we have

$$\Delta_j(v_{j-1}+v_j+v_{j+1}) = v_j,$$

so shifting the operator Δ_j to $v_{j-1} + v_j + v_{j+1}$ yields

$$I = \sum_{j \in \mathbb{Z}} \int_0^t \int_{\mathbb{R}^2} (\widetilde{v} \cdot \nabla v) \cdot v_j \, dxds.$$

Now define

$$I_n \stackrel{\text{def}}{=} \int_0^t \sum_{|j|<n} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^2} (\widetilde{v} \cdot \nabla v_k) v_j \, dxds.$$

Then we have

$$I_n = \int_0^t \sum_{|j|<n} \sum_{|k|<n} \int_{\mathbb{R}^2} (\widetilde{v} \cdot \nabla v_k) v_j \, dxds \tag{3.19}$$

$$+ \int_0^t \int_{\mathbb{R}^2} (\widetilde{v} \cdot \sum_{|k| \geq n} \nabla v_k) \sum_{|j|<n} v_j \, dxds,$$

and integrating by parts in the first term, using the fact that $\widetilde{v}$ is divergence free, yields

$$I_n = -\int_0^t \sum_{|k|<n} \sum_{|j|<n} \int_{\mathbb{R}^2} (\widetilde{v} \cdot \nabla v_j) v_k \, dxds \tag{3.20}$$

$$+ \int_0^t \int_{\mathbb{R}^2} (\widetilde{v} \cdot \sum_{|k| \geq n} \nabla v_k) \sum_{|j|<n} v_j \, dxds.$$

So finally, summing (3.19) and (3.20), we get

$$2I_n = 0 + 2 \int_0^t \int_{\mathbb{R}^2} (\widetilde{v} \cdot \sum_{|k| \geq n} \nabla v_k) \sum_{|j|<n} v_j \, dxds,$$

and the last integral may be bounded using Lemma 1.1 which yields

$$|I_n| \lesssim \Big\| \sum_{|k| \geq n} v_k \Big\|_{\mathcal{L}} \|\widetilde{v}\|_{\mathcal{L}} \Big\| \sum_{|j|<n} v_j \Big\|_{L^q_t(\dot{B}^{\frac{2}{r}+\frac{2}{q}-1}_{r,q})} \to 0 \quad \text{as } n \to +\infty, \tag{3.21}$$

by Lebesgue's theorem. So we have obtained (3.18).

Now we are left with estimating the last term in (3.17), which in fact can be bounded exactly as in (3.21). The same estimate indeed implies that as n goes to infinity, the quantity $\displaystyle\int_0^t \sum_{|j|<n} (v \cdot \nabla v_j) w \, ds$ converges towards $\displaystyle\int_0^t (v \cdot \nabla v) w \, ds$, which completes the proof of Lemma 3.1. □

Now to complete the proof of Theorem 1.1, we just need to prove the result on the bound (1.2) of u in terms of u_0. But that in turn follows from an abstract interpolation result, which will be proved in Section 4.4.

That ends the proof of Theorem 1.1. □

4. Some important estimates

4.1. The trilinear estimate

The aim of this section is to prove Lemma 1.1, which was stated in the introduction and used several times in the proof of the theorems.

Let us recall that we consider a trilinear form

$$(a,b,c) \in \mathcal{L} \times \mathcal{L} \times L^q([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)) \mapsto \int_0^T \int_{\mathbb{R}^d} (a\cdot\nabla b)\cdot c(t)\, dxdt,$$

with $d \geq 2$, $2 \leq r < +\infty$, $2 < q < +\infty$ and $\dfrac{d}{r} + \dfrac{2}{q} > 1$, for which we wish to prove the continuity as well as the estimate

$$\left|\int_0^t \int_{\mathbb{R}^d} (a\cdot\nabla a)\cdot c\, dxdt\right| \leq \|\nabla a\|^2_{L^2(\mathbb{R}^+,L^2)} + C\int_0^t \|a(s)\|^2_{L^2}\|c(s)\|^q_{\dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}}\, dt.$$

We are going to separate the trilinear form in three, according to the respective size of the frequencies of each of the factors: in other words, we are going to use the paraproduct algorithm introduced by J.-M. Bony in [3].

So let us write

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} (a\cdot\nabla b)\ \cdot c\, dxdt &= \sum_{j\in\mathbb{Z}} \int_0^t \int_{\mathbb{R}^d} \Delta_j\, (a\cdot\nabla b\ \cdot c)\ dxdt\\ &= I + II + III,\end{aligned}$$

where, using the localization properties of the Littlewood–Paley operators recalled in the introduction, we have

$$I \stackrel{\text{def}}{=} \sum_{\substack{|k-k'|\leq 1\\ k\geq j-1}} \int_0^t \int_{\mathbb{R}^d} (\Delta_k a\cdot\nabla\Delta_{k'} b)\cdot\Delta_j c\, dxdt$$

$$II \stackrel{\text{def}}{=} \sum_{|j-k|\leq 1} \int_0^t \int_{\mathbb{R}^d} (\Delta_j a\cdot\nabla S_{j-1} b)\cdot\Delta_k c\, dxdt$$

$$\text{and}\quad III \stackrel{\text{def}}{=} \sum_{|j-k|\leq 1} \int_0^t \int_{\mathbb{R}^d} (S_{j-1} a\cdot\nabla\Delta_j b)\cdot\Delta_k c\, dxdt.$$

The terms II and III are paraproduct terms, whereas I is a remainder–type term.

Let us start by estimating the term I. We can write

$$I = \sum_{\substack{|k-k'|\leq 1\\ k\geq j-1}} I_{jkk'} \quad \text{with} \quad I_{jkk'} \stackrel{\text{def}}{=} \int_0^t \int_{\mathbb{R}^d} (\Delta_k a\cdot\nabla\Delta_{k'} b)\cdot\Delta_j c\, dxdt$$

and we have, by Hölder's inequality,

$$|I_{jkk'}| \lesssim \int_0^t \|\Delta_k a\|_{L^2(\mathbb{R}^d)} 2^{k'} \|\Delta_{k'} b\|_{L^2(\mathbb{R}^d)} \|\Delta_j c\|_{L^\infty(\mathbb{R}^d)}\, dt.$$

We have also used the fact that

$$\|\nabla \Delta_{k'} b\|_{L^2(\mathbb{R}^d)} \lesssim 2^{k'} \|\Delta_{k'} b\|_{L^2(\mathbb{R}^d)}.$$

But Bernstein's inequality implies that

$$\|\Delta_j c\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{dj/r} \|\Delta_j c\|_{L^r(\mathbb{R}^d)},$$

so we get

$$|I_{jkk'}| \lesssim \int_0^t \|\Delta_k a\|_{L^2(\mathbb{R}^d)} 2^{k'} \|\Delta_{k'} b\|_{L^2(\mathbb{R}^d)} 2^{dj/r} \|\Delta_j c\|_{L^r(\mathbb{R}^d)}\, dt.$$

Now by interpolation between $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d))$ and $L^2(\mathbb{R}^+, \dot H^1(\mathbb{R}^d))$, one has for every $2 \leq p \leq +\infty$,

$$a \in L^p(\mathbb{R}^+, \dot H^{2/p}(\mathbb{R}^d)),$$

with

$$\|a\|_{L^p(\mathbb{R}^+, \dot H^{2/p}(\mathbb{R}^d))} \leq \|a\|^{1-2/p}_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d))} \|a\|^{2/p}_{L^2(\mathbb{R}^+, \dot H^1(\mathbb{R}^d))},$$

and similarly for b. Recall moreover that by Proposition 1.1, we have

$$c \in L^p([0,T], \dot B^{\frac{d}{r}+\frac{2}{p}-1}_{r,q}(\mathbb{R}^d))$$

for every $q \leq p \leq \infty$. So we have

$$|I| \lesssim \sum_{|k-k'|\leq 1} \sum_{k \geq j-1} \int_0^t \|\Delta_k a\|_{L^2(\mathbb{R}^d)} 2^{k/q'} \|\Delta_{k'} b\|_{L^2(\mathbb{R}^d)} 2^{k'/q'} 2^{k'(1-2/q')}$$
$$\times 2^{j(d/r+2/q-1)} \|\Delta_j c\|_{L^r(\mathbb{R}^d)} 2^{j(1-2/q)}\, dt,$$

where $1/q + 1/q' = 1$. Here we have used the fact that $|k-k'| \leq 1$. Finally we have

$$|I| \lesssim \sum_{\substack{|k-k'|\leq 1 \\ k \geq j-1}} \int_0^t a_k(t) b_{k'}(t) 2^{(j-k)(1-2/q)} c_j(t)\, dt,$$

where a_k and $b_{k'}$ are sequences of time–dependent functions in $L^{2q'}_t(\ell^2)$, with

$$\|a_k(t)\|_{\ell^2_k} \leq \|a(t)\|^{1/q}_{L^2(\mathbb{R}^d)} \|\nabla a(t)\|^{1/q'}_{L^2(\mathbb{R}^d)}$$

and similarly for $b_{k'}(t)$, and where c_j satisfies

$$\|c_j(t)\|_{\ell^\infty_j} \leq \|c(t)\|_{\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}.$$

The result for I now simply follows by summation in j, k and k' using Young's inequality (and the fact that $q > 2$), and by integration in time: we get

$$|I| \lesssim \|a\|^{1/q}_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^d))}\|\nabla a\|^{1-1/q}_{L^2(\mathbb{R}^+,L^2(\mathbb{R}^d))}\|b\|^{1/q}_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^d))}$$
$$\times\|\nabla b\|^{1-1/q}_{L^2(\mathbb{R}^+,L^2(\mathbb{R}^d))}\|c\|_{L^q([0,T],\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}.$$

Furthermore, in the case when $a = b$, one can also write

$$|I| \lesssim \int_0^t \|a(s)\|^{1/q}_{L^2(\mathbb{R}^d)}\|\nabla a(s)\|^{2-2/q}_{L^2(\mathbb{R}^d)}\|c(s)\|_{\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}\,ds$$
$$\lesssim \|\nabla a\|^2_{L^2(\mathbb{R}^+,L^2(\mathbb{R}^d))} + C\int_0^t \|a(s)\|^2_{L^2(\mathbb{R}^d))}\|c(s)\|^q_{\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}\,ds.$$

Now let us consider the term II. The computations are quite similar to the case I: we write

$$II = \sum_{|k-j|\le 1} II_{jk}, \quad \text{with} \quad II_{jk} \stackrel{\text{def}}{=} \int_0^t\int_{\mathbb{R}^d}(\Delta_j a\cdot\nabla S_{j-1}b)\cdot\Delta_k c\,dxdt$$

and we have

$$|II_{jk}| \le \int_0^t \|S_{j-1}\nabla b\|_{L^{\overline{r}}(\mathbb{R}^d)}\|\Delta_j a\|_{L^2(\mathbb{R}^d)}\|\Delta_k c\|_{L^r(\mathbb{R}^d)}\,dt,$$

with $1/2 + 1/\overline{r} + 1/r = 1$. But we can write

$$\|S_{j-1}\nabla b\|_{L^{\overline{r}}(\mathbb{R}^d)} \le \sum_{j'\le j-1}\|\Delta_{j'}\nabla b\|_{L^{\overline{r}}(\mathbb{R}^d)}$$
$$\lesssim \sum_{j'\le j-1} 2^{dj'(1/2-1/\overline{r})}2^{j'(1-2/\overline{q})}\|\Delta_{j'}b\|_{L^2(\mathbb{R}^d)}2^{2j'/\overline{q}}$$

by Bernstein's inequality. It follows, as in the case of I, that

$$|II_{jk}| \lesssim \sum_{j'\le j-1}\int_0^t b_{j'}(t)2^{-2j'/\overline{q}}2^{dj'(1/2-1/\overline{r})+j'}a_j(t)2^{-j}\|\Delta_k c\|_{L^r(\mathbb{R}^d)}\,dt.$$

with

$$b_{j'} \in L^{\overline{q}}(\mathbb{R}^+,\ell^2) \quad \text{and} \quad a_j \in L^2(\mathbb{R}^+,\ell^2),$$

and

$$\|b_{j'}(t)\|_{\ell^2} \le \|b(t)\|^{1-2/\overline{q}}_{L^2(\mathbb{R}^d)}\|\nabla b(t)\|^{2/\overline{q}}_{L^2(\mathbb{R}^d)}, \quad \|a_j(t)\|_{\ell^2} \le \|\nabla a(t)\|_{L^2(\mathbb{R}^d)}.$$

That can also be written as

$$|II_{jk}| \lesssim C\sum_{j'\le j-1}\int_0^t b_{j'}(t)a_j(t)2^{j'(d/2-d/\overline{r}-2/\overline{q}+1)}2^{k(-d/r-2/q)}$$
$$\times 2^{k(-1+d/r+2/q)}\|\Delta_k c\|_{L^r(\mathbb{R}^d)}\,dt,$$

using the fact that $|j-k| \leq 1$. The result follows by summation, since

$$d/2 - d/\overline{r} - 2/\overline{q} + 1 = d/r + 2/q > 0.$$

We get

$$|II| \lesssim \|b\|^{1-2/\overline{q}}_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d))} \|\nabla b\|^{2/\overline{q}}_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))} \times \|\nabla a\|_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))} \|c\|_{L^q([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d))}.$$

Furthermore, in the case when $a = b$, one can also write

$$\begin{aligned} |II| &\lesssim \int_0^t \|a(s)\|^{1-2/\overline{q}}_{L^2(\mathbb{R}^d)} \|\nabla a(s)\|^{1+2/\overline{q}}_{L^2(\mathbb{R}^d)} \|c(s)\|_{\dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)} \, ds \\ &\lesssim \|\nabla a\|^2_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))} + C \int_0^t \|a(s)\|^2_{L^2(\mathbb{R}^d))} \|c(s)\|^q_{\dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)} \, ds. \end{aligned}$$

So we have found the result for II. As claimed in the introduction, note that to estimate the integrals I and II, we have not used the restriction on q and r stated in the proposition. That restriction is going to appear now: let us consider the term III. We have

$$III = \sum_{|k-j|\leq 1} III_{jk}, \quad \text{with} \quad III_{jk} \stackrel{\text{def}}{=} \int_0^t \int_{\mathbb{R}^d} (S_{j-1} a \cdot \nabla \Delta_j b) \cdot \Delta_k c \, dx dt,$$

so Hölder's inequality yields

$$|III_{jk}| \leq \int_0^t \|S_{j-1} a\|_{L^{\overline{r}}(\mathbb{R}^d)} \|\Delta_j \nabla b\|_{L^2(\mathbb{R}^d)} \|\Delta_k c\|_{L^r(\mathbb{R}^d)} \, dt.$$

But clearly by Bernstein's inequality,

$$\begin{aligned} \|S_{j-1} a\|_{L^{\overline{r}}(\mathbb{R}^d)} &\leq \sum_{j' \leq j} \|\Delta_{j'} a\|_{L^2(\mathbb{R}^d)} 2^{dj'(1/2-1/\overline{r})} \\ &\leq \sum_{j' \leq j} \|\Delta_{j'} a\|_{L^2(\mathbb{R}^d)} 2^{2j'/\overline{q}} 2^{j'(d/2-d/\overline{r}-2/\overline{q})}. \end{aligned}$$

Then writing $a_{j'}(t)$ for a function in $L^{\overline{q}}(\mathbb{R}^+, \ell^2)$, with

$$\|a_{j'}(t)\|_{\ell^2} \leq \|a(t)\|^{1-2/\overline{q}}_{L^2(\mathbb{R}^d)} \|\nabla a(t)\|^{2/\overline{q}}_{L^2(\mathbb{R}^d)},$$

we get

$$|III_{jk}| \leq \sum_{j' \leq j} \int_0^t a_{j'}(t) 2^{j'(d/2-d/\overline{r}-2/\overline{q})} \|\Delta_j \nabla b\|_{L^2(\mathbb{R}^d) 2^{-k(d/r+2/q-1)}} \times \|\Delta_k c\|_{L^r(\mathbb{R}^d)} 2^{k(d/r+2/q-1)}.$$

Now to conclude, one just needs to notice that

$$d/2 - d/\overline{r} - 2/\overline{q} = d/r + 2/q - 1 > 0,$$

and the result follows.

So the lemma is proved. □

4.2. A Lorentz spaces estimate for the heat flow

We aim at proving the following result, which we had postponed in Section 3.2.

Lemma 4.1. *Let* $2 < p < \infty$, $u_0 \in L^2(\mathbb{R}^d)$ *and* $u = S(t)u_0$. *Then*

$$\|u\|_{L_t^{p,2}(L_x^q)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}. \tag{4.1}$$

with $\dfrac{2}{p} + \dfrac{d}{q} = \dfrac{d}{2}$.

The proof is based on real interpolation: let $\varepsilon > 0$ be fixed, and let us consider an initial data $u_0 \in \dot{B}^{\pm\varepsilon}_{2,2}(\mathbb{R}^d)$. Then by interpolation, we have for each dyadic bloc,

$$2^{js^\pm} \|\Delta_j S(t)u_0\|_{L_t^{p^\mp}(L_x^2)} \lesssim 2^{\pm\varepsilon j} \|\Delta_j u\|^{1-\frac{2}{p^\mp}}_{L_t^\infty(L_x^2)} \|\nabla \Delta_j u\|^{\frac{2}{p^\mp}}_{L^2_{t,x}} \tag{4.2}$$

with $s^\pm - \frac{2}{p^\mp} = \pm\varepsilon$, for all $1 \leq p^\mp \leq \infty$. Hence further restricting $p^\mp \geq 2$ we can sum up the (square of the) dyadic blocs, use Minkowski for the left–hand side and Hölder for the right–hand side, to get

$$\|S(t)u_0\|^2_{L_t^{p^\mp}(\dot{B}^{s^\pm}_{2,2})} \lesssim \|u\|^{\frac{2}{(p^\mp)'}}_{L_t^\infty(\dot{B}^{\pm\varepsilon}_{2,2})} \||\nabla|^{1\pm\varepsilon} u\|^{\frac{2}{p^\mp}}_{L^2_{t,x}} \stackrel{\text{def}}{=} E_{p^\mp}(u). \tag{4.3}$$

Next, we choose $s^- = s^+$ and then perform real interpolation, to get

$$\|S(t)u_0\|_{L_t^{p,2}(\dot{B}^s_{2,2})} \lesssim \|u_0\|_{L_x^2}, \tag{4.4}$$

which gives the desired result through Sobolev embedding. Remark there is no need to compute all indices since scaling ties them up. In addition, we point out that the real interpolation step can be performed only because on the left the space inside the Lebesgue norms in t is the same at both endpoints ([10]). Finally, keeping the $E_p(u)$ quantities all along allows good control over the time norm when T goes to zero. □

4.3. Proof of Proposition 1.1

This section is devoted to the proof of Proposition 1.1. Let u be the solution of the Navier–Stokes equations in d space dimensions, associated with an initial data in $\dot{B}^{d/r-1}_{r,q}(\mathbb{R}^d)$. As recalled in Section 3, we can write

$$u(t) = e^{t\Delta}u_0 + B(u,u)(t), \tag{4.5}$$

with as stated in (3.6),

$$B(u,u)(t) = \int_0^t e^{(t-s)\Delta} P\text{div}(u \otimes u)(s)\, ds,$$

and we shall estimate both terms of (4.5) separately.

The heat flow satisfies classically the following estimate:

$$\forall t \geq 0, \quad \|\Delta_j e^{t\Delta} u_0\|_{L^r(\mathbb{R}^d)} \lesssim e^{-t2^{2j}} \|\Delta_j u_0\|_{L^r(\mathbb{R}^d)},$$

which implies by an immediate computation that for every $p \geq 1$ and every $T \geq 0$,

$$\|\Delta_j e^{t\Delta} u_0\|_{L^p([0,T],L^r(\mathbb{R}^d))} \leq 2^{-2j/p} \|\Delta_j u_0\|_{L^r(\mathbb{R}^d)}.$$

Taking the ℓ^q_j norm, Minkowski's inequality yields the result for $q \leq p \leq +\infty$.

Hence all we are left to deal with is the bilinear term. We shall start by proving the result in the case when

$$d/r + 2/q > 1.$$

In that case, the result follows from Littlewood-Paley type estimates. Note that the restriction above holds in fact throughout this article, due to the trilinear estimate (1.4) which we were only able to prove in that case. However Proposition 1.1 is true with no restriction on r and q, and the proof of the other case is given below for the sake of completeness. Moreover, it is enough to prove the result for $p = q$, as the others are obtained by interpolation with the known result when $p = \infty$.

We have

$$\|\Delta_j B(f,g)(t)\|_{L^r(\mathbb{R}^d)} \lesssim \int_0^t e^{-(t-s)2^{2j}} 2^j \|\Delta_j (f \otimes g)\|_{L^r(\mathbb{R}^d)}\, ds.$$

But Bony's paraproduct algorithm implies that

$$\begin{aligned}\Delta_j(f \otimes g) = \Delta_j \sum_{|j'-j|\leq 1} (S_{j'-1} f \otimes \Delta_{j'} g) + \Delta_j \sum_{|j'-j|\leq 1} (S_{j'-1} g \otimes \Delta_{j'} f) \\ + \Delta_j \sum_{\substack{|k'-k|\leq 1 \\ k \geq j}} (\Delta_k f \Delta_{k'} g),\end{aligned}$$

which we shall note

$$\Delta_j(f \otimes g) \stackrel{\text{def}}{=} \Delta_j(I + II + III).$$

Let us estimate all three contributions separately: one can write

$$\begin{aligned}2^{j(d/r+2/q-1)} \|\Delta_j B(f,g)(t)\|_{L^r(\mathbb{R}^d)} &\lesssim \int_0^t e^{-(t-s)2^{2j}} 2^{j(d/r+2/q)} \\ &\quad \times \Big(\|\Delta_j I\|_{L^r(\mathbb{R}^d)} + \|\Delta_j II\|_{L^r(\mathbb{R}^d)} + \|\Delta_j III\|_{L^r(\mathbb{R}^d)} \Big)\, ds \\ \stackrel{\text{def}}{=} 2^{j(d/r+2/q-1)} (\|\Delta_j B_I\|_{L^r(\mathbb{R}^d)} &+ \|\Delta_j B_{II}\|_{L^r(\mathbb{R}^d)} + \|\Delta_j B_{III}\|_{L^r(\mathbb{R}^d)}).\end{aligned}$$

In the first case, note that

$$\begin{aligned}\|\Delta_j I\|_{L^r(\mathbb{R}^d)} &\lesssim \sum_{|j'-j|\leq 1} \|S_{j'-1} f\|_{L^\infty(\mathbb{R}^d)} \|\Delta_{j'} g\|_{L^r(\mathbb{R}^d)} \\ &\lesssim \sum_{\substack{|j'-j|\leq 1 \\ j'' \leq j'-1}} \|\Delta_{j''} f\|_{L^r(\mathbb{R}^d)} 2^{dj''/r} \|\Delta_{j'} g\|_{L^r(\mathbb{R}^d)},\end{aligned}$$

using Bernstein's inequality. It follows that

$$2^{j(d/r+2/q-1)}\|\Delta_j B_I\|_{L^r(\mathbb{R}^d)} \lesssim \int_0^t e^{-(t-s)2^{2j}} \sum_{\substack{|j'-j|\le 1\\ j''\le j'-1}} \|\Delta_{j''} f\|_{L^r(\mathbb{R}^d)} \times 2^{j''(d/r-1)}2^{j''}\|\Delta_{j'} g\|_{L^r(\mathbb{R}^d)}2^{j'(d/r+2/q-1)}2^{j'}\,ds,$$

where we have used the fact that $|j'-j|\le 1$.

Now let us define

$$f_{j''} \stackrel{\mathrm{def}}{=} \|\Delta_{j''} f\|_{L^r(\mathbb{R}^d)}2^{j''(d/r-1)} \in L^\infty([0,T],\ell^\infty)$$

$$\text{and}\quad g_{j'} \stackrel{\mathrm{def}}{=} \|\Delta_{j'} g\|_{L^r(\mathbb{R}^d)}2^{j'(d/r+2/p-1)} \in L^q([0,T],\ell^q).$$

Then taking the L^p norm in time yields, by Young's inequality,

$$2^{j(d/r+2/q-1)}\|\Delta_j B_I\|_{L^q([0,T],L^r(\mathbb{R}^d))} \lesssim \sum_{\substack{|j'-j|\le 1\\ j''\le j'-1}} \|f_{j''}\|_{L^\infty([0,T])} \times 2^{j''+j'-2j}\|g_{j'}\|_{L^q([0,T])}.$$

Then the result follows from Young's inequality in the summations.

The second case is proved symmetrically, exchanging f and g, so let us now deal with the third case. One can write

$$\begin{aligned}\|\Delta_j III\|_{L^r(\mathbb{R}^d)} &\lesssim 2^{jd/r}\sum_{\substack{|k'-k|\le 1\\ k\ge j}} \|\Delta_k f\Delta_{k'} g\|_{L^{r/2}(\mathbb{R}^d)}\\ &\lesssim 2^{jd/r}\sum_{\substack{|k'-k|\le 1\\ k\ge j}} \|\Delta_k f\|_{L^r(\mathbb{R}^d)}\|\Delta_{k'} g\|_{L^r(\mathbb{R}^d)}.\end{aligned}$$

Then one just needs to notice that

$$\|\Delta_k f(s)\|_{L^r(\mathbb{R}^d)} \le 2^{-k(d/r+2/q-1)} f_k(s),$$

where

$$f_k(s) \stackrel{\mathrm{def}}{=} 2^{k(d/r+2/q-1)}\|\Delta_k f\|_{L^r(\mathbb{R}^d)} \in L^q([0,T],\ell^q),$$

and similarly for g. So we get

$$2^{j(d/r+2/q-1)}\|\Delta_j B_{III}\|_{L^r(\mathbb{R}^d)} \lesssim \int_0^t e^{-(t-s)2^{2j}}2^{j(2d/r+2/q)} \sum_{\substack{|k'-k|\le 1\\ k\ge j}} f_k(s) \times 2^{-2k(d/r+2/q-1)}g_{k'}(s)\,ds.$$

Then we use the fact that both f_k and $g_{k'}$ are in $L^q([0,T],\ell^q)$, so the product $f_k g_{k'}(s)$ is in $L^{q/2}$ in time; we will note $h_{k,k'}$ the result of that product after time integration, with

$$h_{k,k'} \in \ell^q_k \ell^q_{k'}.$$

Now getting the L^q norm in time for $\|\Delta_j B_{III}\|_{L^r(\mathbb{R}^d)}$ requires taking an $L^{q/(q-1)}$ norm in time for $e^{-(t-s)2^{2j}}$. So finally

$$2^{j(d/r+2/q-1)}\|\Delta_j B_{III}\|_{L^q([0,T],L^r(\mathbb{R}^d))} \lesssim \sum_{\substack{|k'-k|\leq 1\\ k\geq j}} h_{k,k'} 2^{-2k(d/r+2/q-1)} \times 2^{j(2d/r+2/q)} 2^{-2j(1-1/q)},$$

and the result is proved by Young's inequality, under the condition that

$$d/r + 2/q > 1.$$

Now let us prove the result in the other cases. In fact one shall only need to suppose that $d/r - 1 < 0$. Recall that using a continuous characterization of Besov spaces, one has (for $s < 0$)

$$\int_0^\infty \left(t^{-\frac{s}{2}}\|S(t)u_0\|_{L^r}\right)^q \frac{dt}{t} \lesssim \|u_0\|^q_{\dot{B}^s_{r,q}}. \tag{4.6}$$

Thus, it makes sense to ask whether a solution u satisfies (recall $s = \dfrac{d}{r} - 1 < 0$)

$$\int_0^T \left(t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}}\|u\|_{L^r}\right)^q dt \lesssim \|u_0\|^q_{\dot{B}^{\frac{d}{r}-1}_{r,q}}. \tag{4.7}$$

In order to achieve this, we simply need to check a continuity property for the nonlinear part. We are going to use the notation (3.6) presented in Section 3. So what we need to check is that if f and g are two scalar functions satisfying

$$t^{1/2}\|f(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \in L^\infty([0,T]), \quad \text{and} \quad t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}}\|g\|_{L^r(\mathbb{R}^d)} \in L^q([0,T]),$$

then

$$B(f,g)(t) \stackrel{\text{def}}{=} \int_0^t \frac{1}{(t-s)^{\frac{d+1}{2}}} G\left(\frac{\cdot}{\sqrt{t-s}}\right) * fg(s)\, ds \ \in \ L^q([0,T]).$$

In order to do so, let us write

$$\|B(f,g)(t)\|_{L^r(\mathbb{R}^d)} \leq \int_0^t \frac{1}{(t-s)^{\frac{d+1}{2}}} \left\|G\left(\frac{\cdot}{\sqrt{t-s}}\right)\right\|_{L^1(\mathbb{R}^d)} \|fg(s)\|_{L^r(\mathbb{R}^d)}\, ds.$$

Then we get

$$t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}}\|B(f,g)(t)\|_{L^r(\mathbb{R}^d)} \lesssim t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{1-\frac{d}{2r}-\frac{1}{q}}} \varphi(s)\, ds,$$

where $\varphi(t) \stackrel{\text{def}}{=} t^{1-\frac{d}{2r}-\frac{1}{q}}\|fg(s)\|_{L^r(\mathbb{R}^d)}$ is a function of $L^q([0,T])$, due to the assumptions on f and g. Setting $s = t\theta$, we have

$$t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}}\|B(f,g)(t)\|_{L^r(\mathbb{R}^d)} \lesssim t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}} \int_0^1 \frac{\varphi(t\theta)}{t^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}}(t\theta)^{1-\frac{d}{2r}-\frac{1}{q}}}\, td\theta,$$

so we come up with

$$t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}}\|B(f,g)(t)\|_{L^r(\mathbb{R}^d)} \lesssim \int_0^1 \frac{\varphi(t\theta)}{(1-\theta)^{\frac{1}{2}}\theta^{1-\frac{d}{2r}-\frac{1}{q}}}\, d\theta.$$

Using the fact that $\|\varphi(\theta\cdot)\|_{L^q([0,T])} \lesssim \theta^{-\frac{1}{q}}$, we get finally

$$t^{\frac{1}{2}-\frac{d}{2r}-\frac{1}{q}}\|B(f,g)(t)\|_{L^r(\mathbb{R}^d)} \lesssim \int_0^1 \frac{d\theta}{(1-\theta)^{\frac{1}{2}}\theta^{1-\frac{d}{2r}}} \leq C.$$

So estimate (4.7) is proved.

Finally, in order to prove Proposition 1.1, we shall estimate

$$\||\nabla|^{\frac{d}{r}+\frac{2}{q}-1}B(f,g)\|_{L^r_x} \lesssim \int_0^t \frac{1}{(t-s)^{\frac{d}{r}+\frac{1}{q}}}\|f\|_{L^r}\|g\|_{L^r}ds, \tag{4.8}$$

where we took advantage of the power of $|\nabla|$ to substitute to G a function H such that $\widehat{H}(\xi) = |\xi|^{\frac{d}{r}+\frac{2}{q}}e^{-|\xi|^2}$. Subsequently using (3.2) for f and (4.7) for g gives the desired estimate using O'Neil inequalities again. Actually we get better since we have

$$B(f,g) \in L^q_t(\dot{H}_r^{\frac{d}{r}+\frac{2}{q}-1}). \tag{4.9}$$

With a little extra work one may prove that the same holds for the linear part as well, and even replace the Sobolev space by $\dot{B}_{r,1}^{\frac{d}{r}+\frac{2}{q}-1}$, but we will not pursue this matter here. The result is proved. □

4.4. An abstract interpolation result

The aim of this section is to prove an interpolation result, which will yield the a priori estimate on u given in Theorem 1, which we have not proved yet. Note that in the case of small initial data u_0 (say, smaller than some constant ε_0), the result is well known, so we shall suppose in the following that $\|u_0\|_{\dot{B}_{r,q}^{\frac{d}{r}-1}}$ is larger than ε_0.

Let us start by recalling a basic definition of the real interpolation method (we refer for instance to [2] for an extensive presentation). Let E, E_1 and E_2 be three Banach spaces, with

$$E = [E_1, E_2]_{\theta,q}, \quad \text{with} \quad \theta \in [0,1] \quad \text{and} \quad 1 \leq q \leq \infty.$$

Then for any $f \in E$, we have

$$\|f\|_E = \left(\sum_{j \in \mathbb{Z}} 2^{jq\theta} K(f,j)^q \right)^{1/q}, \tag{4.10}$$

with

$$K(f,j) \stackrel{\text{def}}{=} \inf_{g \in E_2} \left(\|f-g\|_{E_1} + 2^{-j} \|g\|_{E_2} \right).$$

In our case, in the context of Theorem 1, the spaces E_1, E and E_2 stand respectively for $\dot{B}^{\frac{2}{\tilde{r}}-1}_{\tilde{r},\tilde{q}}(\mathbb{R}^2)$, $\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$. In particular we have the following embedding, which will be very useful:

$$E_2 \hookrightarrow E \hookrightarrow E_1. \tag{4.11}$$

We shall keep that abstract notation in the following study, as the result stands independently of the underlying spaces: the only properties from the nonlinear setting which will be used are the bounds (4.12) below.

Our goal will be to prove the following statement: if $u_0 \in E$ is written as

$$u_0 = v_0 + w_0, \quad v_0 \in E_2, \quad w_0 \in E_1, \quad \text{and} \quad \|w_0\|_{E_1} \le \varepsilon_0,$$

then the associate solution $u = v + w$, where the functions v and w were constructed in Section 3, satisfies the a priori estimate stated in Theorem 1. Note that the key point is the two uniform estimates

$$\|w(t)\|_{E_1} \le 2\|w_0\|_{E_1} \quad \text{and} \quad \|v(t)\|_{E_2} \le C(\varepsilon_0)\|v_0\|_{E_2}. \tag{4.12}$$

As a prerequisite we prove a well-known property of the interpolation norm, when we have (4.11).

Lemma 4.2. *With the notation presented above, there exists a constant $C(\theta,q)$ such that for any integer $j_0 \ge 1$ and any function f in E, the following equivalence holds:*

$$\left(\sum_{j \ge j_0} 2^{jq\theta} K(f,j)^q \right)^{1/q} \le \|f\|_E \le C(\theta,q) 2^{j_0} \left(\sum_{j \ge j_0} 2^{jq\theta} K(f,j)^q \right)^{1/q}.$$

PROOF OF THE LEMMA. The first inequality is obvious due to (4.10), so let us prove the second one. We claim that for any $j \in \mathbb{Z}$ and any $j_0 \ge 1$, we have

$$\forall f \in E, \quad K(f,j) \lesssim 2^{j_0} K(f,j_0). \tag{4.13}$$

That result is due to the following easy sequence of computations: taking $g = 0$ in the definition of K presented above enables us to write, using the fact that E is embedded in E_1,

$$K(f,j) \le \|f\|_{E_1}.$$

Then for any function g in $E_2 \subset E_1$, we have

$$K(f,j) \leq \|f-g\|_{E_1} + \|g\|_{E_1},$$

hence, using $E_2 \subset E_1$,

$$K(f,j) \lesssim 2^{j_0}(\|f-g\|_{E_1} + 2^{-j_0}\|g\|_{E_2}).$$

The result (4.13) follows directly by taking the minimum over $g \in E_2$.

Now let us finish the proof of the lemma. Recall that

$$\|f\|_E^q = \sum_{j\in\mathbb{Z}} 2^{jq\theta} K(f,j)^q,$$

so separating the sum according to the relative size of j and j_0, we get

$$\|f\|_E^q \lesssim \sum_{j\leq j_0} 2^{j\theta q} 2^{j_0 q} K(f,j_0)^q + \sum_{j>j_0} 2^{j\theta q} K(f,j)^q,$$

where we have used (4.13). Then one finally has

$$\begin{aligned}\|f\|_E^q &\lesssim 2^{j_0 q} 2^{j_0\theta q} K(f,j_0)^q + \sum_{j>j_0} 2^{j\theta q} K(f,j)^q\\ &\lesssim 2^{j_0 q} \sum_{j\geq j_0} 2^{j\theta q} K(f,j)^q,\end{aligned}$$

and the lemma follows. □

To obtain the final result, we note that the left inequality of the equivalence above implies that

$$\|u_0\|_E \geq \left\| 2^{j\theta}(\|w_0^j\|_{E_1} + 2^{-j}\|v_0^j\|_{E_2})\right\|_{\ell^q(j\geq j_0)}, \tag{4.14}$$

where we have noted w_0^j and v_0^j the functions of E_1 and E_2 respectively realizing the minimum for $K(f,j)$. In particular we get

$$\forall j \geq j_0, \quad \|w_0^j\|_{E_1} \leq 2^{-j_0\theta}\|u_0\|_E,$$

and we choose for the remaining computations j_0 such that

$$2^{-j_0\theta} = \frac{\varepsilon_0}{\|u_0\|_E}\cdot \tag{4.15}$$

Recall that we have supposed the initial data u_0 to be large enough in E.

Finally we construct v^j and w^j associated with v_0^j and w_0^j as in Section 3 above, which satisfy (4.12) with initial data w_0^j and v_0^j; combining this with (4.14), we get

$$\|u_0\|_E \geq \frac{1}{C}\left\|2^{j\theta}(\|w^j\|_{E_1} + 2^{-j}\|v^j\|_{E_2})\right\|_{\ell^q(j\geq j_0)}.$$

Since we have $u = v^j + w^j$, we infer that

$$\|u_0\|_E \geq \frac{1}{C}\|2^{j\theta}K(u,j)\|_{\ell^q(j\geq j_0)},$$

and the upper bound in Lemma 4.2 yields finally, with (4.15),

$$\|u\|_E \leq C(\theta, q)\|u_0\|^{1+\frac{1}{\theta}}. \tag{4.16}$$

One may easily check the best possible θ to be $\frac{2}{r}$, which would be the interpolation parameter were we allowed to use interpolation between L^2 and BMO^{-1}. Since we chose to use a smaller Besov space instead of BMO^{-1}, we lose an epsilon, getting $\frac{1}{\theta} = \frac{r}{2} + \varepsilon$ instead. Since the constant $C(\theta, q)$ blows up when ε goes to zero, we elected to state the theorem with $\varepsilon = 1/2$. One should remark anyway that various constants blow up when r gets close to ∞, unless one is using Koch and Tataru's result for existence in the given Besov class to provide (small) bounds on the w part independently of r.

4.5. O'Neil inequalities

Let $f \in L^{p_1,q_1}$ and $g \in L^{p_2,q_2}$. Then Hölder inequality generalizes to

$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \tag{4.17}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, while Young's inequality becomes

$$\|f * g\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}, \tag{4.18}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, provided we avoid $p = 1$ or $p = \infty$ except when $p = q$. Their direct proof may be found in O'Neil ([20]) using rearrangements, while a more modern proof would proceed by bilinear interpolation between the usual inequalities for Lebesgue spaces.

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