On the influence of the Earth's rotation on geophysical flows

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Introduction

The aim of this survey is to describe the influence of the Earth's rotation on geophysical flows, both from a physical and a mathematical point of view.

In the first chapter, we gather from the physical literature the main pieces of information concerning the physical understanding of oceanic and atmospheric flows. For the scales considered, i.e., on domains extending over many thousands of kilometers, the forces with dominating influence are the gravity and the Coriolis force. The question is therefore to understand how they counterbalance each other to impose the so-called geostrophic constraint on the mean motion, and to describe the oscillations which are generated around this geostrophic equilibrium. The main equations are then introduced, along with the approximations commonly used by Physicists. The rest of the survey is devoted to the mathematical study of those equations.

At mid-latitudes, on "small" geographical zones, the variations of the Coriolis force due to the curvature of the Earth are usually neglected, which leads to a problem of singular perturbation with constant coefficients. The study of that problem is the object of Chapter 2, which consists in the recollection of rather classical mathematical results and the methods leading to them. We are therefore interested in the wellposedness of the three dimensional Navier-Stokes system, penalized by a constant-coefficient Coriolis force, as well as in the asymptotics of the solutions as the amplitude of the force becomes large. We focus on two types of boundary conditions, which lead to two very different types of convergence results. In the case when the equations are set in $\mathbf{R}^3$, we exhibit an interesting dispersive behaviour for the Coriolis operator which enables one to deduce a strong convergence result towards a vector field satisfying the two dimensional Navier-Stokes system. In the periodic case, dispersion cannot hold; it is replaced by a highly oscillatory behaviour, where the oscillations are linked to the eigenvalues of the Coriolis operator. Once those oscillations have been filtered out, a strong convergence result can also be proved. In both situations (the whole space case and the periodic case), the global existence of smooth solutions for a large enough rotation is also proved, using the special structure of the limiting system in each case. References to more general, constant coefficient situations are given at the end of Chapter 2.

A first step in order to a get a more realistic description, is to take into account the geometry of the Earth (variations of the local vertical component of the Earth rotation). Chapter 3 is therefore devoted to the study of the three dimensional Navier-Stokes system with a variable

Coriolis force. We assume that the direction of the force is constant (taking into account only the vertical component of the Earth's rotation), and that its amplitude depends on the latitude only (and does not vanish). The price to pay is that the analysis can no longer be as precise as in the constant case, and in particular we have no way in general of describing precisely the waves generated by such a variable-coefficient rotation. As in the previous chapter, the questions of the uniform existence of weak or strong solutions are addressed, and we study their asymptotic behaviour as the amplitude of the rotation goes to infinity.

In the last chapter we focus on equatorial, oceanic flows. In view of the typical horizontal and vertical length scales, it is relevant to consider in a first approximation a two dimensional model with free surface, known as the shallow-water model, supplemented with the Coriolis force. In such an approximation all the vertical oscillations are neglected; this (unjustified) simplification seems to be nevertheless consistent with experimental measures. The question here is then to understand the combination of the effects due to the free surface, and of the effects due to the variations of the Coriolis force. Contrary to Chapter 3, the particularity of such flows is that the Coriolis force vanishes at the equator. Note that, for the sake of simplicity, we will not discuss the effects of the interaction with the boundaries, describing neither the vertical boundary layers, known as Ekman layers, nor the lateral boundary layers, known as Munk and Stommel layers. We indeed consider a purely horizontal model, assume periodicity with respect to the longitude (omitting the stopping conditions on the continents) and infinite domain for the latitude (using the exponential decay of the equatorial waves to neglect the boundary). As in the previous chapters, the questions addressed are first to solve this system, and then to understand the asymptotic behaviour of the solutions. Using the betaplane approximation of the Coriolis force, we are able to carry out computations further than in the abstract case studied in the previous chapter. In particular we recover rigorously the well-known trapping of the equatorial waves.

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Contents

Chapter 1

Modelling geophysical flows

The first chapter of this survey is essentially descriptive, it aims at familiarizing the reader with the basic notions of geophysics, both from the experimental and the theoretical points of view.

In the first part, we collect from the books of J. Pedlosky [50] and A. E. Gill [27] the main pieces of information concerning the physical understanding of the oceanic and atmospheric flows. This understanding is based upon a comparison between the orders of magnitude of the various measurable physical parameters. A heuristic study allows then to separate the mean flows on large time scales (which obey some strong constraint, called geostrophic equilibrium) from the deviations consisting of fast oscillations which can be classified.

In the second part of the chapter we introduce the fundamental mathematical models which should allow in the sequel to describe systematically the observed qualitative features of the geophysical flows. This formalism lies essentially on the classical fluid mechanics theory. The main points to be considered are the occurence of the Coriolis force, and the determination of relevant boundary conditions. We will also introduce simplified models (which are expected to provide a good approximation of the fundamental ones under some conditions) to be used to analyze mathematically some precise phenomenon.

1.1 Physical background

In a first approximation the atmosphere and oceans rotate with the earth with small but significant deviations which we, also rotating with the earth, identify as winds and currents. It is useful to recognize explicitly that the interesting motions are small departures from solid-body rotation by describing the motions in a rotating coordinate frame which kinematically eliminates the rigid rotation. Since such a rotating frame is an accelerating rather than an inertial frame, certain well-known forces will be sensed, namely the centrifugal force and the subtle and important Coriolis force.

Before discussing further the effects of rotation, let us introduce some basic notation. Both in the case of the atmosphere and of oceans, the situation to be considered is that of a thin layer of fluid close to the earth's surface. It appears therefore that the direction which is orthogonal to the earth's surface, i.e. radial in the spherical approximation, is somewhat special. In the sequel, it is called "vertical", and is denoted x_3. In this direction, the length scales are characterized by the parameter D. Conversely, we call "horizontal" and denote by the subscript h the vector components parallel to the earth's surface. More precisely, we use generally the notations x_1 and x_2 respectively for the eastward and northward directions. The corresponding length scales are characterized by L. The coordinates considered here and

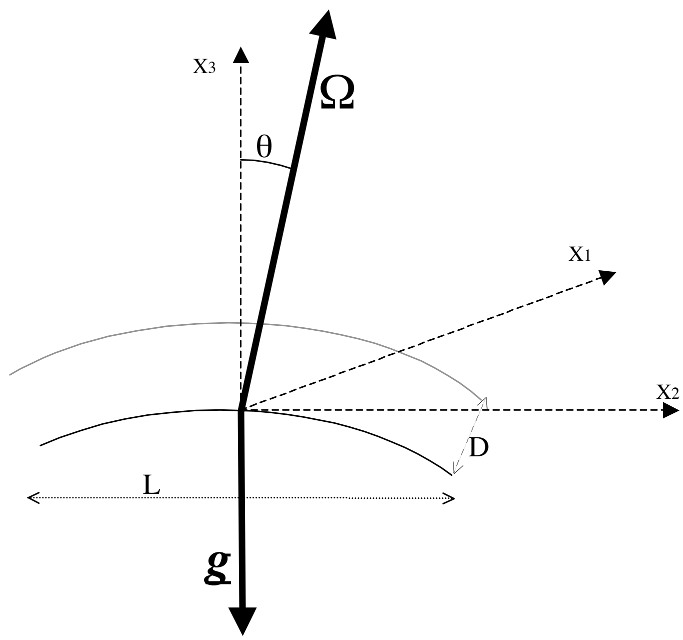

Figure 1.1: Description of a thin spherical layer of fluid

depicted on Figure 1.1 are therefore

(i) neither associated with an inertial frame because of the rotation of the earth;

(ii) nor cartesian coordinates because of the curvature of the earth.

These facts have of course important repercussions on the dynamics that are naturally taken into account in the heuristic description and will be discussed in a more formal way in the second part of this chapter.

1.1.1 Geostrophic and Hydrostatic approximations

The gravitational force

A first force with dominating influence is gravity. In the absence of relative motion, it must be balanced by the pressure p, so that the pressure is given by the *hydrostatic law* :

$$\rho = \rho_0(x_3), \quad p = p_0(x_3), \text{ with } \frac{\partial p_0}{\partial x_3} = -\rho_0 g,$$

where ρ is the density of the fluid and g the gravitational acceleration. Note that we consider in this text atmospheric or oceanic flows, that are motions occuring in a thin layer of fluid close to the surface of the earth, so that we can assume that the gravitational acceleration is a constant

$$g = 9.8 \text{ ms}^{-2}.$$

It actually comes out that the vertical distribution of density $\rho_0(x_3)$ in both the atmosphere and the oceans is almost always gravitationally stable, meaning that heavy fluid underlies lighter fluid. Such a stable stratification implies in particular that motion parallel to the local direction of gravity is inhibited and this constraint tends to produce large scale motions which are nearly horizontal.

A measure of this stratification is given by the *Burger number*

$$S = g\frac{\Delta\rho}{\rho}\frac{D}{4\Omega^2 L^2}\,, \tag{1.1.1}$$

where $\Delta\rho/\rho$ is a characteristic density-difference ratio for the fluid over its vertical scale of motion D, while L is its horizontal scale and Ω is the angular speed of rotation of the earth. The nondimensional parameter S may be written in terms of the ratio of length scales,

$$S = \left(\frac{L_D}{L}\right)^2,$$

where the length L_D is called the Rossby deformation radius.

Figure 1.2 shows a typical height profile of density in the atmosphere : the density decrease indicates gravitational stability of vertically displaced elements even if the compressibility of air weakens this stability. Major atmospheric phenomena have a characteristic vertical scale $D \sim 10km$, while $L \sim 1000km$. For such phenomena, the Burger number is $S \sim 1$.

Figure 1.3 shows a similar depth density profile for the ocean. The depth of the ocean rarely exceeds six kilometers, and the vertical extent D of major current systems is usually much less than that. Yet the horizontal scale L is hundreds of kilometers. For such currents, the Burger number is $S \sim 0.1$.

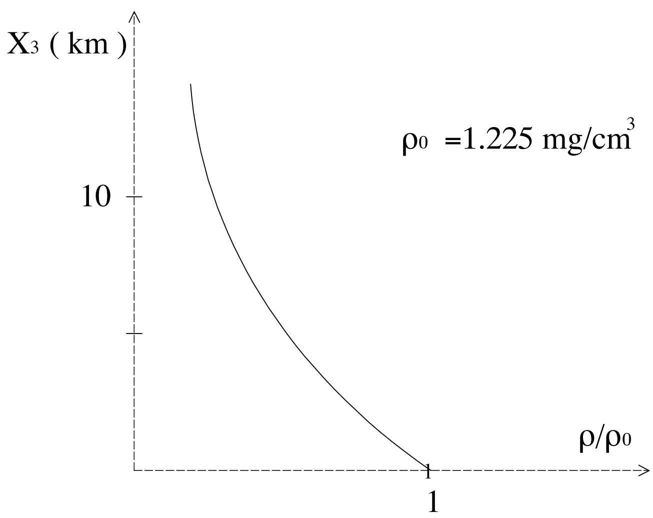

Figure 1.2: Distribution of density with height in the atmosphere *(from NASA, 1962)*

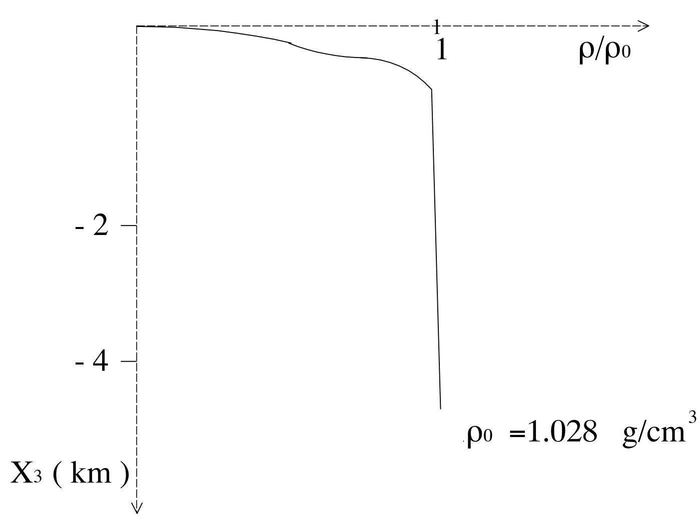

Figure 1.3: Distribution of density with depth in the ocean (*from Pedlosky [50])*

Remark 1.1 *Note that in both situations and more generally for almost all large-scale geophysical flows, there is an important disparity between horizontal and vertical scales of motion, which is measured by the aspect ratio*

$$\delta = \frac{D}{L}.$$

The Coriolis force

When considering winds or currents, i.e., relative motions of the oceans or atmosphere, because the reference frame is rotating, another force has to be taken into account, namely the Coriolis force. An important measure of the Coriolis force, i.e. of the significance of rotation for a particular phenomenon is the *Rossby number*, which is defined as follows. Let L be a characteristic horizontal length scale of the motion under consideration, or in other words a length scale that characterizes the horizontal spatial variations of the dynamic fields (for instance the distance between a pressure peak and a succeeding trough). Similarly let U be a horizontal velocity scale characteristic of the motion. The time it takes a fluid element moving with speed U to cross the distance L is L/U. If that period of time is much less than the period of rotation $|\Omega|^{-1}$ of the earth, the fluid can scarcely sense the earth's rotation over the time scale of motion. For rotation to be important, then, we anticipate that $\frac{L}{U} \geq |\Omega|^{-1}$, or equivalently we expect the Rossby number to be small

$$\varepsilon = \frac{U}{2|\Omega|L} \leq 1. \tag{1.1.2}$$

For the purpose of this text we will only consider large-scale motions, namely those which are significantly influenced by the earth's rotation :

$$|\Omega| = 7.3 \times 10^{-5} s^{-1}.$$

Note that the smaller the characteristic velocity U is, the smaller L can be and yet still qualify for a large-scale flow.

For the troposphere, the characteristic length scales are $D \sim 10km$ and $L \sim 1000km$. The distribution of wind speed along latitude circles (called zonal wind) given in Figure 1.4 shows that $U \sim 20ms^{-1}$. The Rossby number is therefore $\varepsilon = 0.137$ and we can expect the earth's rotation to be important.

The Gulf Stream has velocities of order $U \sim 1ms^{-1}$. Although its characteristic horizontal scale as shown in Figure 1.5 is only $L \sim 100km$, the associated Rossby number is $\varepsilon = 0.07$. Although the use of the local normal component of the earth's rotation would double this value at a latitude of 30^o, it is still clear that such currents meet the criterion of large-scale motion.

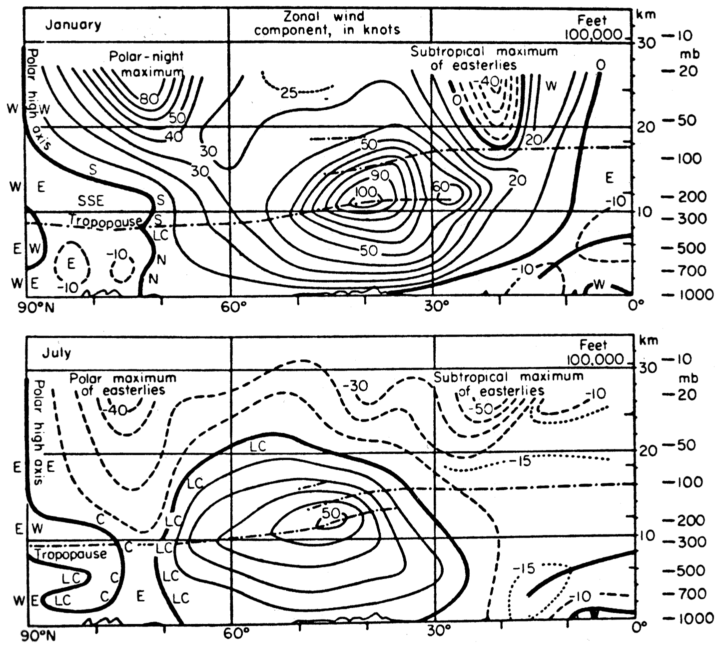

Figure 1.4: Distribution of wind speed along latitude circles, (*from Palmén & Newton [49]*)

Balance between gravity and rotation

• General considerations on rotating fluids allow to determine some constraints in order that motions with time scales long compared to the rotation period and with relative vorticity ω small with respect to $2|\Omega|$ can persist.

- In the absence of friction the production of vorticity due to the pressure must indeed cancel the production of relative vorticity by the stretching and twisting terms. This constraint can be written

$$(\Omega \cdot \nabla)u - \Omega \nabla \cdot u = -\frac{(\nabla \rho \wedge \nabla p)}{2\rho^2},$$

where u denotes the local velocity of the fluid, ρ its density and p its pressure.

- If the relative motion has a small aspect ratio δ, which is generally satisfied by currents and winds, only the local vertical component of the earth's rotation $f = |\Omega| \sin \theta$ where θ denotes the latitude, is dynamically significant (the horizontal Coriolis acceleration due to the vertical motion and the vertical Coriolis acceleration due to the horizontal motion are both small terms when compared to the pressure gradients in their respective equations). The constraint states therefore

$$\begin{aligned} (fe_3 \cdot \nabla)u_h &= -\frac{(\nabla \rho \wedge \nabla p)_h}{2\rho^2}, \\ fe_3 \nabla \cdot u_h &= 0, \end{aligned} \qquad (1.1.3)$$

where f is the *Coriolis parameter* or *inertial frequency* defined as the local component of the planetary vorticity normal to the earth's surface. Since the density variations are commonly

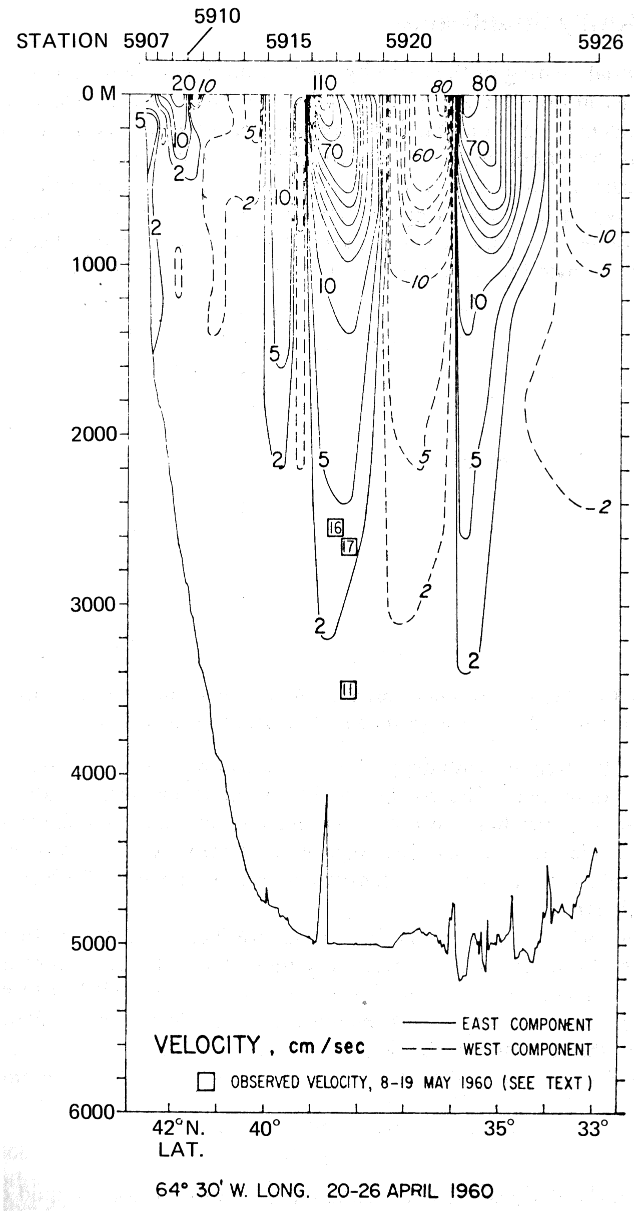

Figure 1.5: Structure of the current velocity through the Gulf Stream (*from Fuglister [19]*)

connected with temperature variations, the winds or currents satisfying the first equation relating the variation of the horizontal velocity to the density variation are called the *thermal wind*.

- If in addition the fluid is barotropic, meaning that the pressure p is a function of the density ρ, then

$$(fe_3 \cdot \nabla)u_h = 0,$$

which implies that a material line once parallel to Ω must always remain so.

- If the fluid is essentially incompressible, the incompressibility constraint implies further that

$$(fe_3 \cdot \nabla)u_3 = 0,$$

so that all three components of the relative velocity are independent of the vertical coordinate. This constraint is called the *Taylor-Proudman theorem*. If the vertical component of the velocity is zero at some level, for example at a rigid surface, the motion is then completely two dimensional and can be pictured as moving in columns parallel to the rotation axis referred to as *Taylor columns*. The simplest situation in which such motions can occur is in the slow relative motion of a homogeneous fluid.

• Specifying conservative forces leads to a more explicit constraint, expressing the balance between gravity, pressure and the Coriolis force (in the absence of friction) :

$$\rho 2\Omega \wedge u = -\nabla p - \rho g e_3. \tag{1.1.4}$$

In the absence of relative motion, such a constraint reduces to the Archimedian principle for a static fluid.

- If the relative motion has a small aspect ratio δ, we have seen that only the local vertical component of the earth's rotation fe_3 is dynamically significant. Furthermore the pressure and density are small departures from their basic states, the magnitude of which is of the order of

$$\varepsilon \frac{\Omega^2 L^2}{gD}$$

with the previous notations. Then (1.1.4) can be approximated by

$$\begin{aligned} u_h &= \frac{1}{f\rho_0} e_3 \wedge \nabla p, \\ \rho g &= -\partial_3 p. \end{aligned} \tag{1.1.5}$$

The first relation is the *geostrophic approximation* expressing the balance between the horizontal pressure gradient and the horizontal component of the Coriolis acceleration. It gives no direct information about the vertical velocity (without further assumption on the thermodynamic properties of the fluid). The other equation does not involve the velocity at all, it is just the *hydrostatic approximation* describing a balance between the vertical pressure gradient and gravity.

• The geostrophic approximation is very useful to predict the motion of geophysical flows : once the pressure field is known, the horizontal velocities, their vertical shear and the vertical component of the vorticity are immediately determined. Nevertheless

(i) the approximation fails in the vicinity of the equator since f cancels. A more complicated dynamical framework is then required in the equatorial regions.

(ii) even at higher latitudes, the geostrophic relations do not allow to calculate the pressure field nor predict its evolution with time. Consideration of small departures from complete geostrophy is then required to complete the dynamical determination of the motion. These small departures involve either the relative acceleration terms, of the order of the Rossby number, or the frictional forces.

1.1.2 Departures from geostrophy

Waves arising in the case of shallow water

In order to determine the corrections to the geostrophic motion, we first consider the case of a shallow rotating layer of homogeneous, incompressible and inviscid fluid. Such a fluid is described by its height H which is assumed to be a fluctuation η around a reference height H_0, and by its purely horizontal velocity u.

The specification of incompressibility and constant density immediately decouples the dynamics from the thermodynamics, and imposes a condition of non divergence on the velocity u.

The shallow-water assumption, based on the smallness of the aspect ratio $\delta << 1$ consists in ignoring stratification and considering only the two-dimensional motion of the fluid. Such a simple case contains some of the important dynamical features of the atmosphere and ocean. Of course it does not allow to catch physical phenomena which depend in a crucial way on stratification.

In this framework, the geostrophic approximation reduces to

$$u \cdot \nabla \left(\frac{f}{H_0} \right) = 0,$$

where H_0 is the reference depth (in absence of relative motion), meaning that streamlines are the isobaths. Of course real motions are not precisely geostrophic and we now consider what happens when the constraint of steadiness is relaxed.

Perturbations (η, u) to this geostrophic approximation satisfy

$$\begin{aligned}
\partial_t \left((\partial^2_{tt} + f^2)\eta + \nabla \cdot (C_0^2 \nabla \eta) \right) - gf \left(\partial_1 H_0 \partial_2 \eta - \partial_2 H_0 \partial_1 \eta \right) = 0, \\
(\partial^2_{tt} + f^2) u_1 = -g \left(\partial^2_{1t} \eta + f \partial_2 \eta \right), \\
(\partial^2_{tt} + f^2) u_2 = -g \left(\partial^2_{2t} \eta - f \partial_1 \eta \right)
\end{aligned}$$

where f denotes the Coriolis parameter and $C_0 = \sqrt{gH_0}$ is the classical shallow-water phase speed.

Wave solutions which are periodic in x and t can be sought in the form

$$\exp(i(\sigma t + k_1 x_1 + k_2 x_2))$$

where σ is the wave frequency and k is the wave vector (which is possibly quantized if the domain under consideration is a partially bounded region). These free oscillations can actually be classified into three types, planetary waves, gravity waves and non-rotating waves, as shown in Figure 1.6.

• Gravity waves, also known as *Poincaré waves*, satisfy the dispersion relation

$$\sigma^2 = f^2 + C_0^2 k^2, \tag{1.1.6}$$

They depend crucially neither on the geometry of the domain, nor on the variations of f.

The presence of rotation increases the wave speed. Indeed it is clear that all these waves have frequencies σ which exceed f, i.e. have periods less than half a rotation period and consequently are at frequencies considerably in excess of those of large-scale, slow atmospheric and oceanic flows. In particular these waves are far from being in geostrophic balance.

For instance, in a channel of width L oriented parallel to the x_1-axis, the boundary conditions constrain k_2 to take discrete values, namely $n\pi/L$ with $n \in \mathbf{Z}$, and the corresponding modes are given by

$$\begin{aligned}
\eta &= \eta_0 \left(\cos\left(\frac{n\pi x_2}{L}\right) - \frac{fL}{n\pi}\frac{k_1}{\sigma} \sin\left(\frac{n\pi x_2}{L}\right) \right) \cos(k_1 x_1 - \sigma t + \phi),\\
u_1 &= \frac{\eta_0}{H_0} \left(\frac{C_0^2 k_1}{\sigma} \cos\left(\frac{n\pi x_2}{L}\right) - \frac{fL}{n\pi} \sin\left(\frac{n\pi x_2}{L}\right) \right) \cos(k_1 x_1 - \sigma t + \phi),\\
u_2 &= -\frac{\eta_0}{H_0}\frac{L}{\sigma n\pi} \left(f^2 + \frac{C_0^2 n^2 \pi^2}{L^2} \right) \sin\left(\frac{n\pi x_2}{L}\right) \sin(k_1 x_1 - \sigma t + \phi),
\end{aligned}$$

meaning that the fluid flow is primarily in the direction of the pressure gradient. Note however that the Poincaré wave corresponding to the value $k_2 = 0$ is not physically relevant for a rotating fluid (boundary conditions cannot be taken into account).

• In this last case (i.e. when the domain has at least one internal boundary), the set of Poincaré modes is supplemented by the so-called *Kelvin waves*. They satisfy the dispersion relation

$$\sigma^2 = C_0^2 k_1^2, \tag{1.1.7}$$

which is also the dispersion relation for gravity waves in a non-rotating fluid.

The corresponding modes are given by

$$\begin{aligned}
\eta &= \eta_0 \exp\left(\pm \frac{f x_2}{C_0} \right) \cos(k_1(x_1 \pm C_0 t) + \phi),\\
u_1 &= \frac{\eta_0 C_0}{H_0} \exp\left(\pm \frac{f x_2}{C_0} \right) \cos(k_1(x_1 \pm C_0 t) + \phi),\\
u_2 &= 0.
\end{aligned}$$

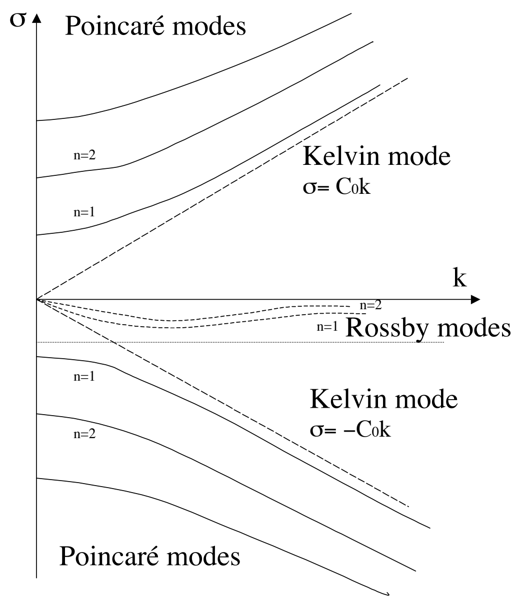

Figure 1.6: Dispersion diagram for shallow water in a channel (*from Pedlosky [50])*

There are several extraordinary features to note. The cross-channel velocity u_2 is identically zero, whereas the flow in the x_1-direction is in precise geostrophic balance even though the frequency is not, in general, small with respect to f. More precisely, the Coriolis acceleration is balanced by a free surface slope. This cross-channel slope is exponential, with intrinsic length scale $R = C_0/f$ which is independent of any property of the wave field. This intrinsic length scale is linked to the Rossby deformation radius. Note that $R \to \infty$ as $f \to 0$, so that the Kelvin waves become in that limit the missing gravest modes of the Poincaré set.

These two types of waves give a complete picture of the departures from geostrophy in the simplest case, when the Coriolis parameter can be considered as a constant. Such an approximation is relevant at *mid-latitudes* for small geographical zone such as lakes or small portions of the oceans.

• When considering more extended domains, the variations of the Coriolis parameter has to be taken into account and a third family of waves appear. The planetary waves, also called *Rossby waves*, whose existence requires both f and ∇f to be nonzero, have a very different dynamical structure.

They are low-frequency oscillations, in the sense that their periods are greater than a rotation period, or in other words that $\sigma/f << 1$. To lowest order in σ/f, the velocity fields, though changing with time, remain continuously in geostrophic balance with the pressure field. Thus the motion is *quasigeostrophic* and it is the very small cross-isobath flow, which is a nongeostrophic effect, which produces the oscillation.

Another characteristic property of the Rossby waves is that, for high wave numbers, the frequency decreases as the wave number increases, in contradistinction to both the Poincaré and Kelvin waves. For instance, if the Coriolis parameter depends linearly on the northward coordinate x_2

$$f(x_2) = f_0 + \beta x_2 \text{ with } f_0 = f(x_2^0) \text{ and } \beta = \partial_2 f(x_2^0)$$

(which is locally a good approximation, known as beta-plane approximation), the dispersion relation for the Rossby waves states

$$\sigma = \frac{\beta k_1 C_0^2}{f^2 + C_0^2 k^2}. \tag{1.1.8}$$

The last feature of the Rossby waves we would like to mention here is that their phase speeds in the x_1-direction are always negative, as shown by the dispersion relation (1.1.8).

In the particular case of a channel of width L oriented parallel to the x_1-axis, the explicit formulas for the Rossby modes to lowest order are

$$\begin{aligned} \eta &= \eta_0 \sin\left(\frac{n\pi x_2}{L}\right)\cos(k_1 x_1 - \sigma t + \phi) + O\left(\frac{\beta L}{f}\right), \\ u_1 &= -\frac{g}{f}\frac{n\pi}{L}\eta_0 \cos\left(\frac{n\pi x_2}{L}\right)\cos(k_1 x_1 - \sigma t + \phi) + O\left(\frac{\beta L}{f}\right), \\ u_2 &= -\frac{g}{f}k\eta_0 \sin\left(\frac{n\pi x_2}{L}\right)\sin(k_1 x_1 - \sigma t + \phi) + O\left(\frac{\beta L}{f}\right). \end{aligned}$$

Equatorial trapping

As mentioned in the first section of this chapter, the adjustment processes are expected to be somewhat special in the vicinity of the equator when the Coriolis acceleration vanishes. A very important property of the equatorial zone is that it acts as a *waveguide*, i.e., disturbances are trapped in the vicinity of the equator. The waveguide effect is due entirely to the variation of the Coriolis parameter with the latitude.

• The simplest wave that illustrates this property is the equatorial Kelvin wave. As for the usual Kelvin waves, the motion is unidirectional, being everywhere parallel to the equator. At each fixed latitude, the motion is exactly the same as that in a non-rotating fluid. Nevertheless, because of the variations (and the cancellation) of the Coriolis parameter

$$f(x_2) \sim \beta_e x_2 \text{ with } \beta_e = \partial_2 f(0) = \frac{2\Omega}{R},$$

rotation effects do not allow the motion at each latitude to be independent : a geostrophic balance is required between the eastward velocity and the north-south pressure gradient. The equatorial Kelvin wave shows therefore an exponential decay in a distance of order a_e, where a_e is given by

$$a_e = \left(\frac{C_0}{2\beta_e} \right)^{1/2} \tag{1.1.9}$$

and is called the *equatorial deformation radius* because of its relationship with the decay scale for the usual Kelvin waves.

• In addition to the Kelvin wave, there is an infinite set of other equatorially trapped waves, with trapping scale of the same order that for Kelvin waves, namely, the equatorial deformation radius defined by (1.1.9). Note that another important effect of the waveguide is the separation into a discrete set of modes $n = 0, 1, 2, ...$ as occurs in a channel. The dispersion curves for equatorial waves are given in Figure 1.7.

- For $n \geq 1$, the waves subdivide into two classes. For the upper branches, the appropriate dispersion relation has the same form as that for Poincaré waves, approximately

$$\sigma^2 \sim (2n+1)\beta C_0 + k_1^2 C_0^2, \tag{1.1.10}$$

and so these waves are called *equatorially trapped Poincaré waves.*

On the lower branches of the curves, the dispersion curves are given approximately by

$$\sigma \sim \frac{\beta C_0 k_1}{C_0 k_1^2 + (2n+1)\beta}. \tag{1.1.11}$$

The corresponding waves are called *equatorially trapped Rossby waves.*

Note that there is a large gap between the minimum gravity wave frequency and the maximum planetary wave frequency, so these waves are easily distinguished. The frequency gap for wave n involves a factor of $2(2n + 1)$, which is equal to 6 for the lowest value $n = 1$.

- For $n = 0$, the solution is somewhat special. The dispersion curve, given by

$$\frac{\sigma}{C_0} + k_1 - \frac{\beta}{\sigma} = 0, \tag{1.1.12}$$

is unique in that for large positive k_1 it behaves like a gravity wave, whereas for large negative k_1 it behaves like a planetary wave. For this reason it is called a *mixed Rossby-gravity wave.* The phase velocity can be to the east or west, but the group velocity is always eastward, being a maximum for short waves with eastward group velocity (gravity waves). Particles follow *anticyclonic orbits* everywhere.

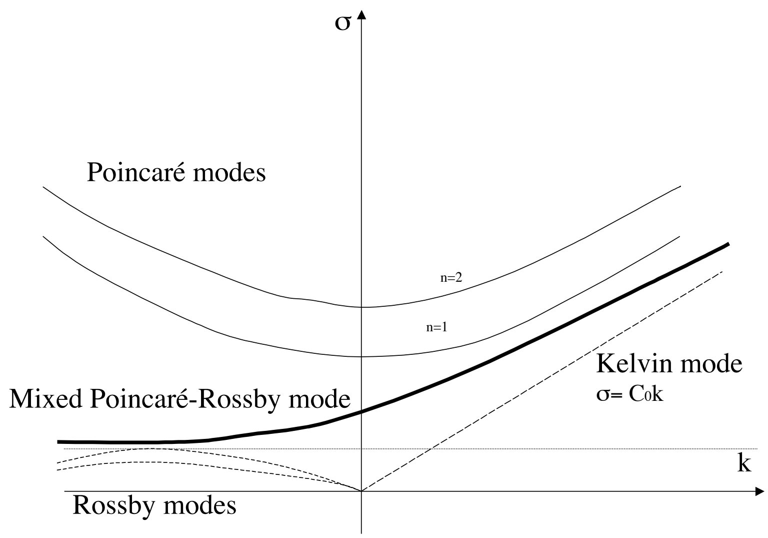

Figure 1.7: Dispersion diagram for shallow water in the equatorial waveguide (*from Gill [27]*)

Effects of stratification

The large-scale field of vertical motion in the atmosphere is of great importance because strong upward motion is associated with the development of severe weather conditions. Note that the vertical motion cannot be easily measured (due to its smallness compared to horizontal scales), but deductions can be made from properties of the pressure field. We therefore have to study the adjustment processes for continuously stratified fluids, i.e., fluids with continuously varying density.

The fluids to be considered will be actually restricted to a class such that the density depends only on entropy and on composition. The motion that takes place is assumed to be isentropic and without change of phase, so that ρ is constant for a material element. Such a fluid is therefore incompressible. The equilibrium state to be perturbed is the state of rest, so the distribution of density and pressure is the hydrostatic equilibrium given by :

$$\rho = \rho_0(x_3), \quad p = p_0(x_3) \text{ with } \partial_3 p_0 = \rho_0 g.$$

For such an incompressible stratified fluid, free oscillations exhibit different behaviours according to the frequency regime to be considered.

A first relation between the vertical velocity u_3 and the pressure perturbation p' is associated with the vertical part of the motion, and thus is unaltered by rotation effects :

$$\partial^2_{tt} u_3 + N^2 u_3 = -\rho_0^{-1} \partial_{t3} p',$$

where $N(x_3)$ is a quantity of fundamental importance to this problem, defined by

$$N^2 = -g\rho_0^{-1} \frac{d}{dx_3} \rho_0. \tag{1.1.13}$$

N has the dimensions of a frequency, and is known as the *buoyancy frequency* since it is the frequency of oscillation for purely vertical motion. The restoring force that produces the oscillation is the *buoyancy force*.

The other equation relating u_3 and p' is provided by the horizontal part of the motion, and more precisely combining the equation for the vertical vorticity and the incompressibility constraint :

$$\partial^2_{tt}\partial_3 u_3 + f^2 \partial_3 u_3 = \rho_0^{-1}\partial_t \Delta_h p',$$

which involves the inertial frequency f but not the buoyancy frequency N.

• The dispersion relation for internal Poincaré waves in a rotating fluid with uniform buoyancy frequency N is therefore

$$\sigma^2 = \frac{f^2 k_3^2 + N^2 k_h^2}{k^2}. \tag{1.1.14}$$

In the atmosphere and ocean, N usually exceeds f by a large factor, typically of order 100, so the contribution of the Coriolis parameter in (1.1.14) is essentially negligible, and the dispersion curves will not look any different because of rotation, except that the vertical axis would have to be labeled $\sigma/N = 0.01$ instead of zero.

More precisely, when N/f is large, different regimes appear according to the value of σ/f as shown in Figure 1.8.

Frequency σ	0.1f	f		10f		N	10N
Period (typical value)	1 week	1 day	6 hr	1hr	10 min		1min
Regime	Quasi-geostrophic	Rotating		Hydrostatic nonrotating			Potential flow
Vertical structure	Evanescent	Evane-scent	Wave	Wave	Wave	Evane-scent	Evanescent

Figure 1.8: Effects of stratification on a rotating fluid (*from Gill, 1982)*

- The *nonhydrostatic wave regime* is defined as the range of frequencies for which σ is of order N but $\sigma \leq N$. In this range the dispersion relation is approximated by

$$\sigma^2 \sim \frac{N\delta^2}{1+\delta^2},$$

which is the relation obtained when rotation effects are ignored.

- The *hydrostatic "non-rotating" wave regime* is defined as the range of frequencies for which $f << \sigma << N$. In this range the dispersion relation is approximated by

$$\sigma^2 \sim N^2\delta^2.$$

Rotation effects do not appear to this order of approximation, which is the reason for calling this a "non-rotating" regime, although it must be remembered that rotation does have an effect at the next order of approximation, and it is sometimes important to consider this.

- The *rotating wave regime* is defined as the range of frequencies for which σ is of order f but $\sigma \geq f$. Since f/N is small, α is small and the hydrostatic approximation applies. The approximate dispersion relation reads

$$\sigma^2 \sim f^2 + N^2\delta^2,$$

which is effectively the dispersion relation for Poincaré waves.

• If the variations of Coriolis parameter are taken into account

$$f(x_2) \sim f_0 + \beta x_2 \text{ with } f_0 = f(x_2^0) \text{ and } \beta = \partial_2 f(x_2^0),$$

we have to consider furthermore the vertical propagation of planetary waves.

As previously, in the case of a uniformly stratified incompressible fluid, the dispersion relation for vertically propagating waves is the same as that for a single mode, but with the wave speed C_0 replaced by N/k_3. In other words, for vertically propagating Rossby waves, it reads

$$\sigma = \frac{\beta k_1}{k_h^2 + f_0^2 k_3^2/N^2}. \quad (1.1.15)$$

Such upward-propagating waves have a very particular structure, with phase lines tilting toward the west with height, meaning that warm air is carried poleward and cold air equatorward, so that there is an apparent net poleward transport of heat. The corresponding buoyancy fluxes play an important role in the atmosphere, in the phenomenon known as a sudden stratospheric warming, which occurs in winter.

The classification of vertically propagating waves begun previously can now be carried to larger scale k_1^{-1} that correspond to lower encounter frequencies

$$\sigma = Uk_1$$

for an observer traveling with the mean flow at speed U. If this flow is uniform, the disturbances are trapped (evanescent) at scales k_1^{-1} larger than that given by $Uk_1 \sim f$, i.e., for scales greater than about 100km. This is because gravity waves (also called Poincaré waves and defined by (1.1.14)) are negligible at such frequencies (see Figure 1.8).

If, however, the scale k_1^{-1} is further increased, thereby reducing the encounter frequency to levels at which variations with latitude of the Coriolis parameter become important in the dynamics, the situation is changed once again because planetary waves may now be possible.

Frequency σ	$(U\beta)^{1/2}$	$0.1f$
Period (typical value)	1 month	1 week
Regime	β-plane quasi-geostrophic	f-plane quasi-geostrophic
Vertical structure	Evanescent	Evanescent

Figure 1.9: Effects of stratification on an inhomogeneous rotating fluid (*from Gill, 1982)*

It is then clear that

- the *f-plane quasi-geostrophic regime* occupies the spectral gap defined by $|U/f| << k_1^{-1} << |U/\beta|^{1/2}$,

- the so-called *β-plane quasi-geostrophic regime* is a new regime to be considered for k_1^{-1} of order $|U/\beta|^{1/2}$. This is about 1000km for the atmosphere, i.e., the scale of the major topographic features of the earth's surface, so the response to these features falls within this regime. The corresponding scale for the ocean is 30 to 100km. Note that in this new regime, there is a major asymmetry between eastward and westward directions of the undisturbed flow. Westward currents are in the same direction as the phase propagation of planetary waves, so stationary waves are not possible : disturbances remain evanescent no matter how small the wavenumber.

1.1.3 Prediction of the observed motion

Contribution of small scales

Although a single wave of arbitrary amplitude is an exact solution of the quasigeostrophic equation, a superposition of waves will not be. The nonlinear interaction between the waves, by which the velocity field of one advects the vorticity of another, leads to a nonlinear coupling and energy transfer between the waves.

When the Rossby number $\varepsilon << 1$, the characteristic period of the waves describing the departures from geostrophy is much less than the advective time : the nonlinear coupling term can be therefore considered as a perturbation of the linear equation governing the waves. In particular, on can try to proceed by successive approximations and to characterize the resulting motion as a perturbation of the linear superposition of waves. The interaction of the mth and nth waves produces then a forcing term in the problem for the first correction which oscillates with the sum and difference of their two phases, i.e., a forcing term with wave vector

$$K_{mn} = K_m \pm K_n$$

and frequency

$$\sigma_{mn} = \sigma_m \pm \sigma_n.$$

The problem for the first correction is a linear, forced problem, and therefore the response to each forcing term can be considered separately and the results summed. If the forcing frequency σ_{mn} is not equal to the natural frequency of oscillation of a free wave with the wave number K_{mn} of the forcing, such a process converges : these interactions merely produce a small-amplitude background jangle of forced waves whose amplitudes are small. Otherwise a resonance occurs, that is, two waves then combine to force a third wave with a wave number and frequency appropriate to a free, linear oscillation. A simple example is the case of the Kelvin waves. Such interactions, called *resonant interactions* are of great interest because of the slow growth of the first correction on the nonlinear advective time : the approximation process is then clearly invalid. This means that, filtering the high frequency waves, one obtains a motion on the advective time-scale which is nonzero.

Note that to the lowest order the filtered motion conserves both the energy (defined as half the average of the square of the velocity) and the enstrophy (defined as half the average of the square of the vorticity), or in other words that the resonant interaction is an energy and enstrophy preserving mechanism.

In naturally occuring situations, there is usually a whole spectrum of waves, i.e., a superposition of waves with wavenumbers varying continuously over some range of values. In such cases, wave interactions occur in the same way as they do when a small number of waves is present, and provided that the wave amplitude is not too large, the transfer of energy is dominated by those waves that are associated with the resonant triads (if such are present). The phases of the different wavenumber components in the spectrum are often assumed to be distributed randomly and this assumption can be used to calculate the evolution of the spectrum with time. This behaviour can be largely understood by considering the following three mechanisms :

- *Induced Diffusion* occurs when two nearly identical waves interact with another wave of much lower frequency and much smaller wavenumber. The shear of the latter wave acts to diffuse wave action (wave energy divided by frequency) among vertical wavenumbers.

- *Elastic Scattering* occurs when two waves with wavenumbers that are almost mirror images in the horizontal plane interact with a wave of much slower frequency and double the vertical wavenumber. The latter wave tends to equalize the energy between upward- and downward-propagating waves. The conditions for elastic scattering to occur are satisfied only for waves with frequency substantially greater that f, so near-inertial frequency waves are little affected.

- *Parametric Subharmonic Instability* occurs when two waves of nearly opposite wavenumber interact with a wave of much smaller wavenumber and of twice the frequency. The process transfers energy from low-wavenumber energetic waves to high-wavenumber waves of half frequency, and so tends to produce inertial frequency waves with high vertical wavenumber.

These processes have a strong influence on the internal wave spectrum, and one result is that the spectrum has a shape that varies rather little.

Dissipation coming from viscosity

The observed persistence over several days of large-scale waves in the atmosphere, and the oceans shows that frictional forces are weak, almost everywhere, when compared with the Coriolis acceleration and the pressure gradient. Friction rarely upsets the geostrophic balance to lowest order.

Nevertheless friction, and the dissipation of mechanical energy it implies, cannot be ignored. For the time-averaged flow, i.e., for the general circulation of both the atmosphere and the oceans, the fluid motions respond to a variety of essentially steady external forcing. The atmosphere, for example, is set in motion by the persistent but spatially nonuniform solar heating. This input of energy produces a mechanical response, namely kinetic energy of the large-scale motion, and eventually this must be dissipated if a steady state - or at least a statistically stable average state of motion - is to be maintained.

Finally, even though friction may be weak compared with other forces, its dissipative nature, qualitatively distinct from the conservative nature of the inertial forces, require its consideration if questions of decay of free motions are to be studied.

• To estimate the frictional force a representation of $\mathcal{F}$ must be specified. Considering the dissipation due to the interactions at the microscopic level, this force is proportional to the spatial derivative of the stress tensor, with a coefficient depending in principle of the thermodynamic state variables, the so-called molecular viscosity. Then

$$\frac{\mathcal{F}}{\rho} \sim \frac{\nu U}{L^2}$$

where ρ is the local density, L the length scale characterizing the variations of the velocity field, and ν is the order of magnitude of the molecular viscosity. The ratio of the frictional force per unit mass to the Coriolis force acceleration is a nondimensional parameter, called the *Ekman number*, E :

$$E = \frac{\nu U/L^2}{2\Omega U} = \frac{\nu}{2\Omega L^2}. \tag{1.1.16}$$

If ν is the molecular kinematic viscosity of water, for example, a straightforward estimate for E for oceanic motions, would be, for $L = 1000km$, $\nu = 10^{-6}m^2s^{-1}$,

$$E \sim 10^{-14}.$$

This is a terribly small number, and such frictional forces are clearly negligible for large scale motions.

• The important issue is whether this representation of $\mathcal{F}$ is adequate if the state variables are to describe only the large-scale motions. The previous paragraph shows indeed that motions on one spatial scale interact with motions on other scales. There is therefore an a priori possibility that small scale motions, which are not the focus of our interest, may yet influence the large-scale motions. One common but not very precise notion is that small-scale motions, which appear sporadic or on longer time scales, act to smooth and mix properties on the larger scales by processes analogous to molecular, diffusive transports.

For the present purposes it is only necessary to note that one way to estimate the dissipative influence of smaller-scale motions is to retain the same representation of the frictional force but replace ν by a *turbulent viscosity*, of much larger magnitude than the molecular value, supposedly because of the greater efficiency of momentum transport by macroscopic chunks of fluid. This is, of course, an empirical concept very hard to quantify.

Influence of boundary conditions

Rotation effects have thus far been studied in the absence of boundaries. If now a boundary is inserted that crosses the isobars, further adjustment would have to take place because no flow is possible accross the boundary. This indicates that the adjustment process is strongly affected by the presence of boundaries, at least in the neighborhood of those boundaries.

• Consider first the action of a stress at the horizontal surface. For instance, on the ocean surface, this stress is due to the action of the wind. It produces a direct response called the *Ekman transport*, which is principally confined to a thin layer near the ocean surface. In fact, the Ekman transport is thought to be usually found within the upper mixed layer of the ocean, which is mostly between 10 and 100m deep. A sudden change of wind can cause oscillations in the Ekman transport of inertial period, or can reduce the amplitude of preexisting oscillations.

If the wind stress were spatially uniform, the ocean below the mixed layer would be little affected by the wind, which would produce a time-varying Ekman transport that is confined to the near-surface region. However spatial variations in the wind (which of course occur) cause spatial variations in Ekman transport. In other words, the Ekman flow will cause mass to flow horizontally into some regions and out of others. This results in vertical motion. For instance, if the horizontal flow is converging in a particular region, vertical motion away from the boundary is required in order to conserve mass. The vertical velocity just outside the boundary layer which is so produced is called the *Ekman pumping* velocity. It is this velocity in the ocean that distorts the density field of the ocean and thereby causes the wind-driven currents.

The stress at the underlying ocean (or land) surface, from the atmospheric point of view, is a frictional drag whose magnitude is dependent on the strength of the wind, usually called *bottom friction*. With the stress is associated an Ekman transport in the atmosphere whose horizontal mass flux is opposite to that in the underlying ocean. Consequently variations in Ekman transport produce Ekman pumping with a vertical mass flux that is the same in the atmosphere as in the ocean.

Such a bottom friction exists also for the ocean. The boundary layer at the bottom of the ocean (the benthic boundary layer) is much thinner than is the atmospheric boundary layer, typically in the range 2 to 10m, which affects the relative importance of topographic effects. Detailed modeling of the velocity structure of the boundary layer is therefore particularly difficult.

The important feature of this process, called *spin-down*, is that the presence of (turbulent) friction in general tends to reduce motion and make the system tend toward a state of rest.

• The second mechanism to be understood is the adjustment process in presence of side boundaries. In fact, the presence of such a boundary implies that the longshore component of the Coriolis acceleration vanishes at the boundary so that the mutual adjustment of the longshore velocity field and the pressure field along the boundary is more like in a non-rotating fluid than like in a rotating one.

This is certainly true in the extreme case in which there are two boundaries close together, as in a narrow gulf or estuary. The rotation effects can be neglected at the first approximation because the motion is mainly along the gulf and the component of the Coriolis acceleration in this direction is negligible. At the next order of approximation, rotation modifies the flow in two ways. One is to give a cross-channel pressure gradient in order to geostrophically balance the longshore flow. The other is to produce a shear whenever the surface elevation departs from its equilibrium level, this being required in order that potential vorticity be conserved.

The narrow channel approximation can be applied with success to studies of tides and seiches in gulfs, estuaries, and lakes, and even to tides in the Atlantic Ocean.

When the two sides of a channel are not close together, the question arises as how far from the shore the longshore component of the Coriolis force can be neglected. The answer is a distance of the order of the Rossby deformation radius, so channels must have width small compared with this scale for the narrow-channel approximation to be valid. For wide channels, there is a special form of adjustment near the boundary by means of a Kelvin wave whose peculiarity is that it can travel along the coast in one direction only, and whose amplitude is only significant within a distance of the order of the Rossby deformation radius from the boundary. Note that the presence of boundaries also affects the Poincaré waves, but effects of the end of the channel can be quite difficult to work out. Of course details are strongly influenced by the complicated shape of the world's oceans.

1.2 Mathematical modelling

The starting point of geophysical fluid dynamics is the premise that the dynamics of meteorological and oceanographic motions are determined by the systematic application of the fluid continuum equations of motion. The dynamic variables generally required to describe the motion are the density ρ, the vector velocity u, and certain further thermodynamic variables like the temperature T or the internal energy per unit mass e.

1.2.1 Introducing a general mathematical framework

In the absence of sources or sinks of mass within the fluid, the condition of mass conservation is expressed by the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0. \tag{1.2.1}$$

Newton's law of motion written for a fluid continuum takes the form

$$\rho(\partial_t + u \cdot \nabla)u = -\nabla p + \rho \nabla \phi + \mathcal{F}(u), \tag{1.2.2}$$

meaning that the mass per unit volume times the acceleration is equal to the sum of the pressure gradient force, the body force $\rho\nabla\phi$ where ϕ is the potential by which conservative force such as gravity can be represented, and the frictional force $\mathcal{F}$.

Unless the density is considered a constant, the momentum and continuity equation are insufficient to close the dynamical system. The first law of thermodynamics must be considered; it can be written as

$$\rho(\partial_t + u \cdot \nabla)e = -p\rho(\partial_t + u \cdot \nabla)\rho^{-1} + k\nabla^2 T + \chi + \rho Q, \tag{1.2.3}$$

where k is the thermal conductivity, T is the temperature, Q is the rate of heat addition per unit mass by internal heat sources, and χ is the addition of heat due to viscous dissipation - which is negligible in all situations to be discussed.

To complete the system, further thermodynamic state relations expressing the physical nature of the fluid are required.

Mathematical features of geophysical fluids

For example, in the atmosphere the state relation for dry air is well-represented by the ideal-gas law

$$\rho = \frac{p}{RT}, \tag{1.2.4}$$

where R is the gas constant for dry air. The local conservation of energy then becomes

$$(\partial_t + u \cdot \nabla)\theta = \frac{\theta}{C_p T}\left\{\frac{k}{\rho}\nabla^2 T + Q\right\},$$

where the potential temperature θ is defined by

$$\theta = T\left(\frac{p_0}{p}\right)^{R/C_p},$$

for some reference pressure p_0. We have denoted by C_p the specific heat at constant pressure.Note that in the absence of conductive and internal heating θ is a conserved quantity for each fluid element.

For the oceans, density differences are so slight that they have a negligible effect on the mass balance, so that the local conservation of mass can be approximated by

$$\nabla \cdot u = 0, \tag{1.2.5}$$

which is the incompressibility relation. Note that the incompressibility constraint does not imply that the fluid is homogeneous, meaning that $(\partial_t + u \cdot \nabla)\rho$ is generally not assumed to vanish.

The Navier-Stokes model with Coriolis force

We noted earlier that the most natural frame for which to describe atmospheric and oceanic motions is one which rotates with the planetary angular velocity Ω.

Let r be the position vector of an arbitrary fluid element. We have

$$\left(\frac{dr}{dt}\right)_I = \left(\frac{dr}{dt}\right)_R + \Omega \wedge r,$$

where the subscript I denotes rates of change as seen by the observer in the non-rotating inertial frame. The velocity seen in the non-rotating frame u_I is therefore equal to the velocity observed in the rotating frame augmented by the velocity imparted to the fluid element by the solid-body rotation $\Omega \wedge r$. We may write this as

$$u_I = u_R + \Omega \wedge r,$$

where u_R is called the relative velocity. As Newton's law of motion equates the applied forces per unit mass to the acceleration in inertial space, we have then to express this acceleration in terms of quantities which are directly observed in the rotating frame :

$$\begin{aligned}\left(\frac{du_I}{dt}\right)_I &= \left(\frac{du_I}{dt}\right)_R + \Omega \wedge u_I \\ &= \left(\frac{du_R}{dt}\right)_R + 2\Omega \wedge u_R + \Omega \wedge (\Omega \wedge u_R) + \frac{d\Omega}{dt} \wedge r.\end{aligned}$$

The discrepancy between the accelerations perceived in the different frames is equal to the three additional terms on the right-hand side. They are the Coriolis acceleration $2\Omega \wedge u_R$, the centripetal acceleration $\Omega \wedge (\Omega \wedge r)$ and the acceleration due to variations in the rotation rate itself, which can be neglected for most oceanographic or atmospheric phenomena. Since the centrifugal force can be written as a potential

$$\Omega \wedge (\Omega \wedge r) = -\nabla \frac{|\Omega \wedge r|^2}{2},$$

it is included with the force potential. The Coriolis acceleration $2\Omega \wedge r$ is therefore the only new term which explicitly involves the fluid velocity, and it is responsible for the structural change of the momentum equation.

If we note that spatial gradients are perceived identically in rotating and non-rotating coordinate frames, the momentum equation becomes

$$\rho\left((\partial_t + u \cdot \nabla)u + 2\Omega \wedge u\right) = -\nabla p + \rho \nabla \phi + \mathcal{F}. \tag{1.2.6}$$

It is important to note that the total time rate change of any scalar such as the temperature is the same in rotating as in non-rotating frames. Thus the equation of conservation of mass and the various thermodynamic equations are unaffected by the choice of coordinate frame.

Boundary conditions

In most cases of interest the (turbulent) Ekman number E is sufficiently small that it might appear that friction could be neglected. However, the viscosity ν is the coefficient of the highest spatial derivatives and thus the fact that it is nonzero is quite important as regards the mathematical structure of the equations of motion, and the number of boundary conditions to be imposed.

If the surface $\partial_3\mathcal{O}$ of the fluid layer is in contact with a solid surface, for instance in the case of the bottom boundary of the ocean, the natural condition to be considered is a *no-slip condition* :

$$u_{|\partial_3\mathcal{O}} \equiv 0.$$

If the surface $\partial_3\mathcal{O}$ of the fluid layer is free rather than in contact with a solid surface, for instance in the case of the interface between the ocean and the atmosphere, the appropriate boundary condition is continuity of pressure and continuity of frictional stress accross the fluid surface

$$u_{|\partial_3\mathcal{O}} \cdot n = 0, \quad (\nu(\nabla u + (\nabla u)^T) - p\, Id)_{|\partial_3\mathcal{O}} \cdot n = \text{constraint}.$$

Then, for models of atmospheric phenomena the *Ekman layer* or some more elaborate model of the friction layer at the lower boundary usually suffices to represent the frictional interaction of the fluid and the boundary.

Models of oceanic (or lake) dynamics which explicitly recognize the fact that the water is gathered together in basins have to be supplemented by a no-slip boundary condition at the lateral boundaries :

$$u_{|\partial_h\mathcal{O}} \equiv 0.$$

In such a framewok one has generally to introduce *side-wall friction layers* whose structure differ considerably from that of the Ekman layer.

1.2.2 Taking into account the geometry of the earth

The situation to be described is schematically depicted in Figure 1.10.

We consider motions on a sphere of radius r_0, meaning that we will ignore ab initio the slight departures of the figure of the earth from sphericity. The characteristic vertical scale of the motion, D, is in all cases of interest small compared to r_0 so that the effective gravitational acceleration g can be considered constant over the depth of the fluid. The horizontal scale motion L is large in the sense described in the first section (i.e., L is large enough so that the Rossby number is small), but in the sequel we will focus our attention on the situation where L is considerably smaller than r_0.

The equations of motion in spherical coordinates

The coordinate system to be used in the spherical system is shown in Figure 1.11. The position of any point in the fluid is fixed by r, θ and ϕ, which are the distance from the earth's centre, the latitude and the longitude respectively. The velocities in the eastward, northward, and vertical directions are u_ϕ, u_θ and u_r, as shown.

The equation for conservation of mass (1.2.1) in these spherical coordinates is

$$\frac{d}{dt}\rho + \rho\left(\partial_r u_r + \frac{2u_r}{r} + \frac{\partial_\theta(u_\theta\cos\theta)}{r\cos\theta} + \frac{\partial_\phi u_\phi}{r\cos\theta}\right) = 0, \tag{1.2.7}$$

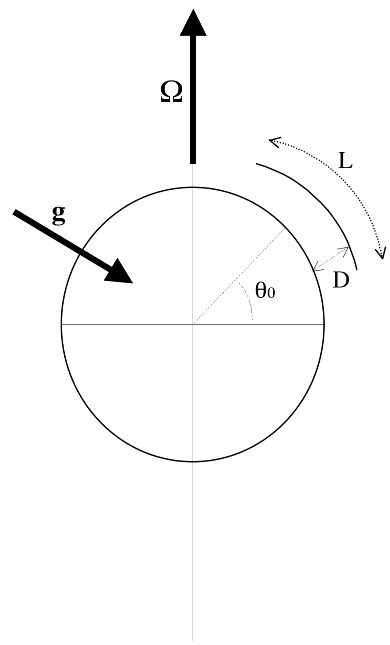

Figure 1.10: Characteristic length scales for geophysical flows

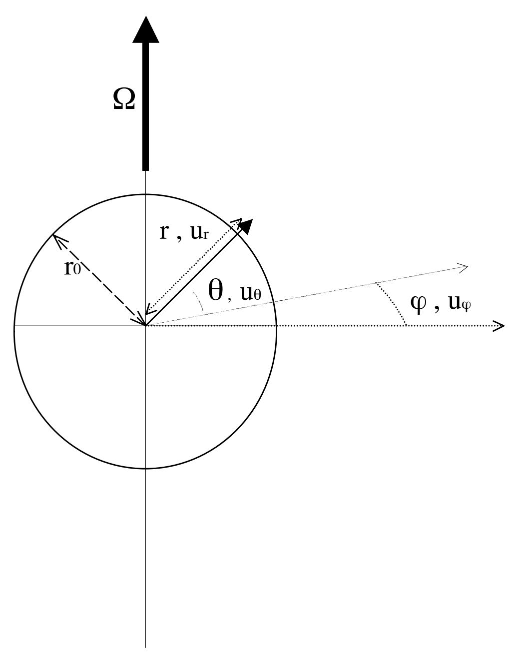

Figure 1.11: Spherical coordinates for the description of geophysical flows

where the total time derivative is defined by

$$\frac{d}{dt} = \partial_t + \frac{u_\phi}{r\cos\theta}\partial_\phi + \frac{u_\theta}{r}\partial_\theta + u_r\partial_r.$$

The momentum equations are

$$\begin{aligned} \frac{d}{dt}u_\phi + \frac{u_\phi(u_r - u_\theta\tan\theta)}{r} - 2\Omega\sin\theta u_\theta + 2\Omega\cos\theta u_r &= -\frac{1}{\rho r\cos\theta}\partial_\phi p + \frac{\mathcal{F}_\phi}{\rho}, \\ \frac{d}{dt}u_\theta + \frac{u_r u_\theta + u_\phi^2\tan\theta}{r} + 2\Omega\sin\theta u_\phi &= -\frac{1}{\rho r}\partial_\theta p + \frac{\mathcal{F}_\theta}{\rho}, \\ \frac{d}{dt}u_r - \frac{u_\phi^2 + u_\theta^2}{r} - 2\Omega\cos\theta u_\phi &= -\frac{1}{\rho r}\partial_r p - g + \frac{\mathcal{F}_r}{\rho}, \end{aligned} \tag{1.2.8}$$

where $\mathcal{F}_\phi, \mathcal{F}_\theta, \mathcal{F}_r$ are the three components of the frictional forces acting on the fluid. The equations of motion must as previously be completed with the addition of a thermodynamic equation, for example the incompressibility constraint

$$\partial_r u_r + \frac{2u_r}{r} + \frac{\partial_\theta(u_\theta\cos\theta)}{r\cos\theta} + \frac{\partial_\phi u_\phi}{r\cos\theta} = 0. \tag{1.2.9}$$

Consider now the description of a motion, in either the ocean or the atmosphere, whose horizontal spatial scale of variation is given by the length scale L and whose horizontal velocities are characterized by the velocity scale U. Geometrical considerations imply that if the vertical scale of motion is D, the corresponding slope of a fluid element's trajectory will not exceed D/L, so that appropriate scaling for the vertical velocity is DU/L (note that the actual scale of the vertical velocity may be less than DU/L if other dynamical constraints act to reduce the vertical motion). The scaling of the pressure and density is more subtle. For small Rossby number, the relative velocities are small and the pressure is expected to be only slightly disturbed from the value it would have in the absence of motion, whereas the horizontal pressure gradients should be of the same order as the Coriolis acceleration. Similarly we may anticipate that the buoyancy force per unit mass will be of the same order as the vertical pressure gradient, since an observed feature of large-scale motions is the excellence of the hydrostatic approximation. Such considerations allow to scale the equations so that the relative order of each term is clearly measured by the nondimensional parameter multiplying it. It is then possible to systematically exploit the smallness of the parameters ε (Rossby number), δ (aspect ratio), L/r_0 and $F = (2\Omega\sin\theta_0)^2L^2/gD$. Note that these parameters are all independent, and their relative orders will vary from phenomenon to phenomenon. The nature of the approximations will depend on these relative orders. Below we present the shallow water approximation, and refer for instance to [42] or [56] for other simplified models.

Some geometrical approximations

• If the motion occurs in a mid-latitude region, distant from the equator, around a central latitude θ_0, it becomes convenient to introduce new longitude and latitude coordinates as follows.

Define x_1 and x_2 by

$$x_1 = \phi \frac{r_0}{L} \cos\theta_0, \quad x_2 = (\theta - \theta_0)\frac{r_0}{L}. \tag{1.2.10}$$

They are however measures of eastward and northward distance only at the earth's surface ($r = r_0$) and at the central latitude θ_0. Although x_1 and x_2 are in principle simply new longitude and latitude coordinates in terms of which the equations of motion may be rewritten without approximation, they are obviously introduced in the expectation that for small L/r_0 and D/r_0 they will be the Cartesian coordinates of the β-plane approximation as introduced page 18. It is also convenient to introduce

$$x_3 = \frac{1}{D}(r - r_0),$$

so that

$$\partial_\phi = \frac{r_0}{L}\cos\theta_0 \partial_1, \quad \partial_\theta = \frac{r_0}{L}\partial_2, \quad \partial_r = \frac{1}{D}\partial_3.$$

To this point no approximation has been made.

As we focus our attention on the situation where L is considerably smaller than r_0, the trigonometric functions can be expanded about the latitude θ_0 :

$$\begin{aligned}
\sin\theta &= \sin\theta_0 + \frac{L}{r_0}x_2\cos\theta_0 - \left(\frac{L}{r_0}\right)^2 \frac{x_2^2}{2}\sin\theta_0 + ...,\\
\cos\theta &= \cos\theta_0 - \frac{L}{r_0}x_2\sin\theta_0 - \left(\frac{L}{r_0}\right)^2 \frac{x_2^2}{2}\cos\theta_0 + ...,\\
\tan\theta &= \tan\theta_0 + \frac{L}{r_0}x_2(\cos\theta_0)^{-2} - \left(\frac{L}{r_0}\right)^2 \frac{x_2^2}{2}\tan\theta_0(\cos\theta_0)^{-2} + ...
\end{aligned}$$

This allows to simplify system (1.2.7)(1.2.8) and (1.2.9) in the following way (where the Coriolis force has not yet been approximated):

$$\begin{aligned}
\frac{d}{dt}\rho &= 0,\\
\frac{d}{dt}u_1 - 2\Omega\sin\left(\theta_0 + \frac{Lx_2}{r_0}\right)u_2 + 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)u_3 &= \frac{1}{\rho L}\partial_1 p + \frac{\mathcal{F}_1}{\rho},\\
\frac{d}{dt}u_2 + 2\Omega\sin\left(\theta_0 + \frac{Lx_2}{r_0}\right)u_1 + 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)u_3 &= \frac{1}{\rho L}\partial_2 p + \frac{\mathcal{F}_2}{\rho},\\
\frac{d}{dt}u_3 - 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)u_1 + 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)u_2 &= \frac{1}{\rho D}\partial_3 p - g + \frac{\mathcal{F}_3}{\rho},\\
\frac{1}{L}(\partial_1 u_1 + \partial_2 u_2) + \frac{1}{D}\partial_3 u_3 &= 0,
\end{aligned} \tag{1.2.11}$$

where

$$\frac{d}{dt} = \partial_t + \frac{1}{L}(u_1\partial_1 + u_2\partial_2) + \frac{1}{D}u_3\partial_3.$$

We then introduce

$$f_0 = 2\Omega \sin\theta_0, \quad \beta_0 = \frac{2\Omega L}{r_0}\cos\theta_0 = \left(\frac{L}{r_0}\frac{d}{d\theta}f\right)_{\theta=\theta_0}$$

as the reference Coriolis acceleration and northward gradient of the Coriolis parameter at the latitude θ_0. Note that

$$\frac{\beta_0/f_0}{\varepsilon} \sim \frac{L}{r_0\varepsilon}.$$

Thus while ε measures the ratio of the relative vorticity and the planetary vorticity normal to the sphere at θ_0, the magnitude of the relative-vorticity gradient and the planetary vorticity gradient is measured by the parameter $\varepsilon r_0/L$. While ε may be small, $\varepsilon r_0/L$ may be large, order one, or small, and each of these possibilities gives rise to a quite different quasigeostrophic dynamical system.

- If the geographical zone to be considered is small, meaning that $\varepsilon r_0/L >> 1$, we will neglect the variations of the Coriolis parameter and use the *f-plane approximation* :

$$\sin\theta \sim \sin\theta_0.$$

Most of the mathematical studies on geophysical flows deal with this framework. As the Rossby operator has constant coefficients, one can make use of a powerful mathematical tool to study the asymptotic behaviour of the fluid as the rotation rate tends to infinity : the Fourier transform allows indeed to carry out explicit computations and to establish qualitative properties of the Poincaré waves (dispersion, resonances...). Thereby, the rotating fluid equations in the f-plane approximation have been the object of a number of mathematical works in the past decade, and the second chapter of this survey aims at giving an overview of the main results as well as the methods of proof.

- If the geographical zone to be considered is more extended, meaning that $L/\varepsilon r_0 = O(1)$, more subtle adjustment processes due to the variations of the Coriolis parameter, and characterized by time scales large compared with f_0^{-1} have to be taken into account, which is done using the *mid-latitude β-plane approximation* :

$$\sin\theta \sim \sin\theta_0 + \beta_0 x_2.$$

This situation is much more complicated to study from a mathematical point of view than the previous one, since the techniques based on the Fourier transform can no longer be used. The works devoted to this study are presented in the third chapter, they essentially allow to determine the mean motion of the fluid in the absence of boundaries : in particular we do not get any description of the boundary layers. Concerning the waves, we obtain some informations about the oscillating modes (which are the eigenmodes of the Rossby operator), but nothing on their shape equations.

• For motions near the equator, the approximations

$$\sin\theta \sim \theta, \quad \cos\theta \sim 1$$

may be used, giving what is called the *equatorial β-plane approximation* :

$$f \sim \beta_0 x_2 \text{ with } \beta_0 = \frac{2\Omega L}{r_0} = 2.3 \times 10^{-11} m^{-1} s^{-1} \tag{1.2.12}$$

Note that half of the earth's surface lies at latitudes of less then 30^o and the maximum percentage error in the above approximation in that range of latitudes is only 14 percent. In particular, this approximation can usefully be applied over the whole of the tropics.

The shallow-water approximation

- Assuming that the aspect ratio is very small $\delta << 1$, vertical motion can be neglected in view of the scalings imposed by the incompressibility constraint. Indeed it is natural to consider the non dimensional unknowns

$$\tilde{u}_1 = \frac{\tau}{L} u_1, \quad \tilde{u}_2 = \frac{\tau}{L} u_2 \text{ and } \tilde{u}_3 = \frac{\tau}{D} u_3,$$

where τ is the order of the times to be considered. Rescaling time and plugging the previous formulas in (1.2.11) leads to

$$\begin{aligned}
(\partial_t + \tilde{u}\cdot\nabla)\rho = 0,\\
(\partial_t + \tilde{u}\cdot\nabla)\tilde{u}_1 - 2\Omega\sin\left(\theta_0 + \frac{Lx_2}{r_0}\right)\tilde{u}_2 + 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)\delta\tilde{u}_3 = \frac{1}{\rho}\partial_1\tilde{p} + \frac{\mathcal{F}_1}{\rho},\\
(\partial_t + \tilde{u}\cdot\nabla)\tilde{u}_2 + 2\Omega\sin\left(\theta_0 + \frac{Lx_2}{r_0}\right)\tilde{u}_1 + 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)\delta\tilde{u}_3 = \frac{1}{\rho}\partial_2\tilde{p} + \frac{\mathcal{F}_2}{\rho},\\
(\partial_t + \tilde{u}\cdot)\delta\tilde{u}_3 - 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)\tilde{u}_1 + 2\Omega\cos\left(\theta_0 + \frac{Lx_2}{r_0}\right)\tilde{u}_2 = \frac{1}{\rho\delta}\partial_3\tilde{p} - \frac{\tau^2}{L}g + \frac{\mathcal{F}_3}{\rho},\\
\nabla\cdot\tilde{u} = 0,
\end{aligned}$$

In particular, if the vertical viscosity is strong enough (for instance independent on δ), we expect u to be asymptotically independent on the vertical variable. Thus taking formally limits as $\delta \to 0$ we obtain the horizontal momentum equations

$$\begin{aligned}
(\partial_t + \tilde{u}_h\cdot\nabla_h)\tilde{u}_1 - 2\Omega\sin\left(\theta_0 + \frac{Lx_2}{r_0}\right)\tilde{u}_2 = \frac{1}{\rho}\partial_1\tilde{p} + \frac{\tilde{\mathcal{F}}_1}{\rho},\\
(\partial_t + \tilde{u}_h\cdot\nabla_h)\tilde{u}_2 + 2\Omega\sin\left(\theta_0 + \frac{Lx_2}{r_0}\right)\tilde{u}_1 = \frac{1}{\rho}\partial_2\tilde{p} + \frac{\tilde{\mathcal{F}}_2}{\rho},\\
\partial_3 u_h = 0.
\end{aligned} \tag{1.2.13}$$

Note that this accounts for the fact that only the vertical component of the rotation of the Earth $f = 2\Omega\sin\theta$ is considered.

- If we suppose moreover that the Rossby deformation radius is very small $S << 1$ or in other words that the fluid is almost homogeneous $\rho \sim \rho_0$, the pressure is given at leading order by the hydrostatic law

$$\tilde{p} = \rho_0 \tilde{g} \eta,$$

where $\tilde{g}$ is the non dimensional gravity constant, η is the depth variation due to the free surface, and the continuity equation is, taking into account the form for the divergence operator,

$$\partial_t \eta + \nabla_h \cdot ((D + \eta)\tilde{u}_h) = 0 \qquad (1.2.14)$$

These sets of equations are the ones derived by Laplace, but with the tide-generating terms omitted. Because a shallow layer is considered, r can be taken as a constant equal to the radius of the earth.

Note that, from a theoretical point of view, it is not clear that the use of the shallow water approximation is relevant in this context since the Coriolis force is known to generate vertical oscillations which are completly neglected in such an approach. Indeed the components of the Coriolis acceleration that are associated with the horizontal component of the rotation vector are not everywhere small compared with the terms retained.

That very particular case is the matter of the mathematical works presented in the last chapter. The results obtained are close to that of the third chapter, but because of the equatorial trapping, the waves - in particular the Rossby waves - have decay properties which allow to get a more precise strong convergence result. That is due to the fact that explicit compputations can be written in that framework.

Chapter 2

A simplified model for midlatitudes

2.1 Introduction

In this chapter we intend to study a model for the movement of the ocean at midlatitudes. As explained in the introduction, at such latitudes the Coriolis acceleration can in a crude approximation be considered as a constant, which makes the analysis much simpler than in the case of the full model. This chapter is therefore devoted to the analysis of the so-called "rotating fluid equations", consisting in the three-dimensional Navier-Stokes system in which a constant coefficient penalization operator has been added to account for the Earth rotation. The model is the following:

$$(RF_\varepsilon) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \dfrac{1}{\varepsilon} u^\perp = -\dfrac{1}{\varepsilon} \nabla p \\ \operatorname{div} u = 0, \end{cases}$$

where $u^\perp = (u_2, -u_1, 0)$. We will be interested in the wellposedness of this system for a fixed ε, as well as in the asymptotics of the solutions as ε goes to zero. We will by no means be exhaustive in the presentation, neither in the various results that can be found in the literature nor in the proofs. The aim of this chapter is rather to give an insight to the questions usually addressed when dealing with this type of system, and to the methods commonly used to answer them. Those methods will be used in the coming chapters in more realistic situations (the Coriolis force will no longer be constant), and we feel it can be useful to present them first in this easier, though unrealistic model.

The question of the wellposedness of this system can be dealt with quite easily, considering the skew-symmetry of the rotation operator. This is explained in Paragraph 2.3 below. More interesting is the question of the asymptotic behaviour of the solutions as ε goes to zero. As noted in the introduction, we expect by the Taylor-Proudman theorem a two dimensional behaviour at the limit. We show in Paragraph 2.4 that this is indeed the case, as long as weak limits are considered, rather than strong. Paragraph 2.5 is devoted to strong asymptotics, where we will see that it all depends on the boundary conditions imposed on the system. We will mainly focus on two types of boundary conditions, which lead to two very different types of convergence results. In Section 2.5.1 we consider the case when the equations are set in $\mathbf{R}^3$. This is highly unrealistic, but the fact that the rotation is constant allows to write explicit calculations in Fourier space, and in particular the formulas found for the eigenvalues of the Coriolis operator enable us to exhibit an interesting dispersive behaviour for the Coriolis operator; thus we are able to deduce a strong convergence result towards a vector fields satisfying the two dimensional Navier-Stokes system. Section 2.5.2 is devoted to the periodic case: the three variables are supposed to be periodic, and in that case dispersion cannot hold; it is replaced by a highly oscillatory behaviour, where the oscillations are linked to the eigenvalues of the Coriolis operator; once again those can be explicitly computed, due to the absence of boundary conditions and to the fact that the rotation is constant. It is only once those oscillations have been filtered out that a strong convergence result can also be proved. In both situations (the whole space case and the periodic case), the global existence of smooth solutions for a large enough rotation is also proved, using the special structure of the limiting system in each case. A word on more general domains is said in Section 2.5.3, while references can be found in Section 2.6. Finally the main results of this chapter are stated in the next section.

2.2 Statement of the main results

As explained in the introduction of this chapter, we are interested in the uniform existence of solutions to (RF_ε), as well as in the asymptotic behaviour of the solutions in the limit of a fast rotation, that is, as ε goes to zero. To simplify the presentation, we will restrict our attention to the case when the equations are set in a domain with no boundary. We will call Ω such a domain, and we will denote by Ω_h the space of horizontal coordinates $x_h = (x_1, x_2)$. Then Ω_h will be indifferently the space $\mathbf{R}^2$ or $\mathbf{T}^2$, and Ω_3, defined by $\Omega = \Omega_h \times \Omega_3$, will be indifferently $\mathbf{R}$ or $\mathbf{T}$, unless specified otherwise. Let us start by stating the uniform existence theorem, which will be easily proved in Section 2.3 below.

Theorem 2.1 *Let u^0 be a divergence free vector field in $L^2(\Omega)$. Then there is a solution u (in the sense of distributions) to (RF_ε) with $u_{|t=0} = u^0$, and which satisfies the following energy estimate, uniformly in ε:*

$$\forall t \geq 0, \quad \|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2}^2 \, dt' \leq \|u^0\|_{L^2}^2.$$

In particular u is bounded in $L^2_{loc}(\mathbf{R}^+, L^q(\Omega))$ for any $q \in [2, 6]$.

Furthermore if $u^0 \in H^{\frac{1}{2}}(\Omega)$, then there is a time $T > 0$ independent of ε such that u belongs to $C([0,T], H^{\frac{1}{2}}(\Omega)) \cap L^2([0,T], H^{\frac{3}{2}}(\Omega))$, with a norm independent of ε, and all solutions associated with u^0 coincide with u on $[0,T]$.

Remark 2.1 *As usual the pressure is not considered as an unknown in this system, since once u is known, p is retrieved through the formula*

$$-\Delta p = div\,(u \cdot \nabla u) + \frac{1}{\varepsilon} div\, u^\perp.$$

The important point to notice in that statement is the fact that all bounds are uniform in ε. It therefore makes sense to inquire on the limiting behaviour of the solution as ε goes to zero. In particular can one describe the dependence of the life span T on ε? Can one find a limit to the system as ε goes to zero? In the following we will emphasize the dependence on ε of the solutions given by Theorem 2.1 by denoting them u_ε. They will therefore be seen as a bounded (in L^2) family of divergence free vector fields, whose asymptotics as ε goes to zero we want to explore. We will start by studying the weak asymptotics, and recover the Taylor-Proudman theorem, stating that as rotation increases, the mean flow becomes two dimensional. The proof of the following result can be found in Section 2.4 below. We have noted $\nabla_h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$, $\mathrm{div}_h \stackrel{\text{def}}{=} \nabla_h \cdot$, and $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$. Moreover for any vector field $u = (u_1, u_2, u_3)$ we define $u_h = (u_1, u_2)$. In the next theorem we have defined $|\Omega|$ as the measure of the set Ω if it is bounded, and $|\Omega|^{-1} = 0$ otherwise.

Theorem 2.2 *Let u^0 be any divergence free vector field in L^2, and let u_ε be any weak solution of (RF_ε). Then u_ε converges weakly in $L^2_{loc}(\mathbf{R}^+ \times \Omega)$ to a limit u which if $\Omega_3 = \mathbf{R}$ is zero, and if $\Omega_3 = \mathbf{T}$ is the solution of the two dimensional Navier–Stokes equations in Ω_h*

$$(NS2D) \qquad \begin{cases} \partial_t u - \Delta_h u + u_h \cdot \nabla_h u = (-\nabla_h p, 0) \\ div_h u_h = 0 \\ u_{|t=0} = \displaystyle\int_{\mathbf{T}} u^0(x_h, x_3)\, dx_3 - \frac{1}{|\Omega|} \int_\Omega (u_h^0(x), 0)\, dx. \end{cases}$$

Remark 2.2 *We recall that J. Leray proved in [37] that a unique, global solution to the two dimensional Navier-Stokes equations exists, as soon as the initial data is in L^2.*

Once the mean flow has been described, it is natural to address the question of the strong convergence of solutions. In fact the answer to that question depends strongly on the boundary conditions. We will be mainly interested in two very different situations here: the case when the equations are posed in the whole space, and the periodic case. Let us state the theorem concerning each situation, starting by the whole space case which is studied in Section 2.5.1.

Theorem 2.3 *Let $\overline{u}^0$ and w^0 be two divergence free vector fields, respectively in $L^2(\mathbf{R}^2)$ and in $L^2(\mathbf{R}^3)$. Let $\overline{u}$ be the unique solution of the two dimensional Navier-Stokes equations associated with $\overline{u}^0$, and let u_ε be any weak solution to (RF_ε) associated with $\overline{u}^0 + w^0$ (such a solution may be constructed as in Theorem 2.1 above). Then for any $q \in]2, 6[$ and for any time T, we have*

$$\lim_{\varepsilon \to 0} \int_0^T \|u_\varepsilon(t) - \overline{u}(t)\|^2_{L^q(\mathbf{R}^3)}\, dt = 0.$$

Remark 2.3 *Theorem 2.3 shows that the weak convergence result stated in Theorem 2.2 is in fact strong. All x_3-dependent vector fields converge strongly to zero as ε goes to zero, and at the limit remains only the two- dimensional behaviour — note that the presence of $\overline{u}^0$ in the initial data enables one to understand precisely that two-dimensional behaviour; if the initial data is purely three-dimensional (that is, if $\overline{u}^0 = 0$), then Theorem 2.3 states that all weak solutions u_ε converge strongly to zero with ε.*

The main ingredient in the proof of that result is a dispersive estimate, implying that the eigenvectors corresponding to the oscillatory modes created by the Coriolis operator converge strongly to zero. That fact, when applied to strong solutions, will enable us to prove the global wellposedness of (RF_ε), despite its likeness to the 3D Navier-Stokes equations for which such a result is unknown. We state the result in an unprecise way here, and refer to Theorem 2.7 page 55 for a precise statement.

Theorem 2.4 *Let $\overline{u}^0$ and w^0 be two divergence free vector fields, respectively in $L^2(\mathbf{R}^2)$ and $H^{\frac{1}{2}}(\mathbf{R}^3)$. Then a positive ε_0 exists such that for all $\varepsilon \leq \varepsilon_0$, there is a unique global solution u_ε to the system (RF_ε) associated with $\overline{u}^0 + w^0$.*

We will also be interested in the periodic situation. In that case the equations are set in a periodic box $\mathbf{T}^3 \stackrel{\text{def}}{=} (\mathbf{R}_{/\mathbf{Z}})^3$, and we will also be able to prove the global existence of strong solutions; however the asymptotic behaviour of the solutions is less easy to describe: due to the absence of dispersion, we need to filter out the oscillatory modes before taking the strong limit. In the next theorem we have defined the operator $\mathcal{L}(t) = e^{tL}$ where L is the Coriolis operator

$$L : u \in L^2 \mapsto \mathbf{P}(u^\perp) \in L^2 \tag{2.2.1}$$

and $\mathbf{P}$ denotes the Leray projection from $L^2(\Omega)$ onto its subspace of divergence-free vector fields. In that statement, a limit system is also referred to, which will be studied in Section 2.5.2. That system is presented page 58, and the main steps of the result are described in Section 2.5.2.

Theorem 2.5 *Let u^0 be a divergence free vector field in $H^{\frac{1}{2}}(\mathbf{T}^3)$. Then a positive ε_0 exists such that for all $\varepsilon \leq \varepsilon_0$, there is a unique global solution to the system (RF_ε) in $C_b(\mathbf{R}^+; H^{\frac{1}{2}}(\mathbf{T}^3)) \cap L^2(\mathbf{R}^+; H^{\frac{3}{2}}(\mathbf{T}^3))$ associated with u^0. Moreover we have*

$$\limsup_{\varepsilon \to 0} \left(u_\varepsilon - \mathcal{L}(\frac{t}{\varepsilon}) u \right) = 0 \quad in \quad L^\infty(\mathbf{R}^+; H^{\frac{1}{2}}(\mathbf{T}^3)) \cap L^2(\mathbf{R}^+; H^{\frac{3}{2}}(\mathbf{T}^3)),$$

where u is the unique, global solution of the limit system (RFL) page 58 associated with u^0.

Remark 2.4 *Let us compare this theorem with Theorem 2.4 stated above. As far as the life span of the solutions is concerned, those two theorems state essentially the same result: for any initial data, if the rotation is large enough, then the rotating fluid equations are globally wellposed, although they are very like the 3D Navier-Stokes equations, for which that is an open question. In other words, the rotation term has a stabilizing effect. In the case of the whole space $\mathbf{R}^3$ this global wellposedness for small enough ε is due to the fact that the Rossby waves go to infinity immediately; that is a dispersive effect. In the case of the torus, there is of course no such dispersive effect (at least for uniform time intervals). The global wellposedness comes in a totally different way: it is a consequence of the analysis of resonances of Poincaré waves in the non linear term $v \cdot \nabla v$, using again the explicit formulation of the eigenvalues of the Coriolis operator, in Fourier variables. As far as the asymptotics are concerned, the statements of Theorems 2.3 and 2.5 are very different since in the whole space case, there is no trace of the rotation at the limit whereas in the periodic case, the limit system includes spectral information on the rotation operator.*

The rest of this chapter is devoted to the proof of those results.

2.3 Uniform existence

In this short section we address the question of the wellposedness of system (RF_ε) and we prove Theorem 2.1. This system is very like the three dimensional Navier-Stokes system, for which it is well known that global (possibly not unique) weak solutions exist if the initial data is of finite energy (meaning it belongs to L^2). Furthermore local in time, unique solutions exist if the initial data is smooth enough (say in the Sobolev space $H^{\frac{1}{2}}$). The proof of both those results relies on energy estimates, the main ingredient consisting (formally) in the first case in multiplying scalarly the system by u, and by ∇u in the second — of course there is much more to the proofs than that calculation, and we refer to [36] and [20] for the original proofs, and for instance to [13] for a more recent presentation (as well as the application to (RF_ε)). Since the Coriolis operator is skew-symmetric in every Sobolev space, in the sense that for any $s \in \mathbf{R}$,

$$(u^\perp \mid u)_{H^s} = 0,$$

the previous proofs go unchanged if we add the rotation term to the Navier-Stokes equations. Theorem 2.1 follows therefore immediately, once the corresponding proofs for the three dimensional Navier-Stokes system are known. □

2.4 Weak asymptotics

In this section we are going to describe the weak limiting behaviour of u_ε, and prove Theorem 2.2. Let us start by making some general comments on the asymptotics of u_ε as ε goes to zero. As the family $(u_\varepsilon)_{\varepsilon>0}$ is bounded in the energy space, up to the extraction of a subsequence it has a weak limit point u. Formally taking the limit in the equation satisfied by u_ε allows to expect the weak limit points u to satisfy

$$\operatorname{div} u = 0 \quad \text{and} \quad u^\perp = -\nabla p$$

for some function p. It is easy to see that, in the absence of nonvanishing boundary conditions, this is equivalent to the fact that $\partial_3 u = 0$ and $\operatorname{div}_h u_h = 0$. We therefore formally recover the Taylor-Proudman theorem: the mean motion at the limit is governed by a two-dimensional, divergence free vector field. Let us now find rigorously the nature of the weak limit points of u_ε. Below $\dot H^1(\Omega)$ denotes the homogeneous Sobolev space of order one, made of the distributions f such that ∇f belongs to $L^2(\Omega)$.

Proposition 2.5 *Let u^0 be any divergence free vector field in $L^2(\Omega)$. Denote by $(u_\varepsilon)_{\varepsilon>0}$ a family of weak solutions of (RF_ε), and by u any of its limit points. Then u is a three component, divergence free vector field satisfying*

$$u \in L^\infty(\mathbf{R}^+; L^2(\Omega_h) \cap L^2(\Omega)) \cap L^2(\mathbf{R}^+; \dot H^1(\Omega_h)).$$

Moreover we have $\int_{\Omega_h} u_h(t, x_h)\, dx_h = 0.$

Remark 2.6 *If* $\Omega_3 = \mathbf{R}$ *then the only possible limit point is 0. Indeed there are no vector fields other than 0 which are in* $L^2(\Omega_h \times \mathbf{R})$ *and do not depend on the vertical variable.*

Proof of Proposition 2.5. The proof simply consists in multiplying (RF_ε) by a divergence-free test function $\varepsilon\chi$, where $\chi \in \mathcal{D}(\mathbf{R}^+ \times \Omega)$. Integrating with respect to t and x gives directly, using the bounds coming from the energy estimate, that $u \in \mathrm{Ker}(L)$, where L was defined in (2.2.1). Then it is just a matter of determining the kernel of L, and an easy computation gives the following proposition. We omit the proof here (for the interested reader, it is written in Chapter 3 in a more general case, see Proposition 3.3). □

Proposition 2.7 *If* u *is a divergence free vector field in* $L^2(\Omega)$ *belonging to* $\mathrm{Ker}(L)$, *then* u *is in* $L^2(\Omega_h)$ *and satisfies the following properties:*

$$\mathrm{div}_h\, u_h = 0 \quad \text{and} \quad \int_{\Omega_h} u_h\, dx_h = 0.$$

The next question consists in finding the evolution equation satisfied by u. Due to Remark 2.6, we shall now consider only the case when $\Omega_3 = \mathbf{T}$. Moreover to simplify we normalize $\mathbf{T}$ in the following so that $\int_{\mathbf{T}} dx_3 = 1$.

The idea to find the limit equation is to take weak limits in (RF_ε), the difficulty coming of course from the nonlinear terms. The first step of the analysis consists in proving the compactness of the vertical average of u_ε. The second step then consists in proving a compensated-compactness type result to show that there are no constructive interferences of x_3-dependent vector fields.

2.4.1 Compactness of vertical averages

Let us start by proving the following proposition, which shows that the defect of compactness of the sequence of solutions u_ε can only be due to functions depending on the vertical variable.

Proposition 2.8 *Let* u^0 *be any divergence free vector field in* L^2. *For all* $\varepsilon > 0$, *denote by* u_ε *a weak solution of* (RF_ε), *and define*

$$\overline{u}_\varepsilon(x_h) \stackrel{\mathrm{def}}{=} \int_{\mathbf{T}} u_\varepsilon(x)\, dx_3 \quad \text{and} \quad \overline{\overline{u}}_\varepsilon \stackrel{\mathrm{def}}{=} \frac{1}{|\Omega_h|} \int_{\Omega_h \times \mathbf{T}} (u_{\varepsilon,h}(x), 0)\, dx.$$

Then the sequence $(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon)_{\varepsilon>0}$ *is strongly compact in* $L^2([0,T] \times \Omega_h)$, *for all times* T.

Proof of Proposition 2.8. Let us take the vertical average of (RF_ε). Since horizontal mean free, x_3-independent vector fields are in the kernel of L due to Proposition 2.7, we have

$$\int_{\mathbf{T}} \mathbf{P}(u_\varepsilon^\perp)\, dx_3 - \frac{1}{|\Omega_h|} \int_{\Omega_h \times \mathbf{T}} \mathbf{P}(u_\varepsilon^\perp)\, dx = \mathbf{P}\left((\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon)^\perp \right) = 0.$$

It follows that

$$\partial_t(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon) - \Delta_h \overline{u}_\varepsilon + \mathbf{P} \int_{\mathbf{T}} u_\varepsilon(x) \cdot \nabla u_\varepsilon(x)\, dx_3 = 0. \tag{2.4.1}$$

Regularity with respect to space variables follows from the energy estimate, since u_ε is uniformly bounded in $L^2([0,T], H^1(\Omega))$ for all times T. Regularity with respect to time is obtained classically by finding an a priori bound on $\partial_t(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon)$. It is indeed easy to see that $u_\varepsilon \cdot \nabla u_\varepsilon$ is bounded in $L^2(\mathbf{R}^+; H^{-3/2}(\Omega))$, and that Δu_ε is bounded in $L^2(\mathbf{R}^+; \dot{H}^{-1}(\Omega)) \subset L^2(\mathbf{R}^+; H^{-3/2}(\Omega))$, so $\partial_t \overline{u}_\varepsilon$ is uniformly bounded in $L^2(\mathbf{R}^+; H^{-3/2}(\Omega_h))$. We can therefore infer by interpolation (using Aubin's lemma for instance) that $(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon)_{\varepsilon>0}$ is strongly compact in $L^2_{loc}(\mathbf{R}^+; L^2(\Omega_h))$, which proves the proposition. □

We infer from that result that x_3-dependent vector fields are the only obstacles to taking the limit in the non linear term. We are going to see that such vector fields do not interfere constructively in the non linear term of the equation.

2.4.2 The weak limit of the nonlinear term

This section is the main step of the analysis of the weak limiting behaviour of (RF_ε), since it consists in proving that when taking the limit of the nonlinear term, there are no constructive interferences of oscillations. We will start by giving the general strategy of the proof, before going into the details.

Strategy of the proof

Our aim is to prove that the limit of the nonlinear term only involves the nonlinear interaction of the weak limit. More precisely we want to prove that as ε goes to zero, we have

$$\mathbf{P} \int_{\mathbf{T}} u_\varepsilon \cdot \nabla u_\varepsilon dx_3 \to \mathbf{P}(u \cdot \nabla u),$$

where u is a weak limit of u_ε. Of course that convergence must be made more precise, in particular we need to determine in what function space it holds. In fact since we are dealing with nonlinear quantities involving weak solutions to our system, it will be convenient to start by regularizing the family u_ε by introducing a smooth vector field u_ε^δ which converges uniformly towards u_ε in $L^2_{loc}(\mathbf{R}^+; L^2(\Omega))$ as δ goes to zero. That is possible due to the additional smoothness of u_ε given by the Laplacian. Then we will be able to carry out computations on nonlinear quantities involving u_ε^δ without worrying about regularity issues (only at the very end of the argument will we let δ go to zero). Those computations consist in writing out the nonlinear term $u_\varepsilon^\delta \cdot \nabla u_\varepsilon^\delta$ as the expected limit $u \cdot \nabla u$, to which one needs to add error terms. Those error terms naturally involve functions which are oscillatory in time, and using the algebraic properties of the wave equations associated with those oscillatory functions (see Lemma 2.9), it is possible to prove that they contribute to negligible quantities, up to gradient terms. The precise statement is given in Proposition 2.10 below.

In the following we will therefore start by writing out those wave equations, applied to smoothened vector fields. It turns out that the computations are best carried out on the vorticity formulation of the equation (since the vector fields are smooth for a fixed δ, that does not create additional regularity problems). Using those equations given in Lemma 2.9, we are then able to prove the expected convergence of the quadratic term (Proposition 2.10).

Convergence of the quadratic term

The proof of that result requires some preparation, and we will start this study by rewriting the equations in a convenient way for future algebraic computations. Let us start by taking the rotational of the equation, by defining

$$\omega_\varepsilon \stackrel{\text{def}}{=} \partial_1 u_{\varepsilon,2} - \partial_2 u_{\varepsilon,1} \quad \text{and} \quad \partial_3 \widetilde{\Omega}_{\varepsilon,h} \stackrel{\text{def}}{=} (\text{rot}\, \widetilde{u}_\varepsilon)_h = \nabla_h^\perp \widetilde{u}_{\varepsilon,3} - \partial_3 \widetilde{u}_{\varepsilon,h}^\perp,$$

with $\int_{\mathbf{T}} \widetilde{\Omega}_{\varepsilon,h}(x)\, dx_3 = 0$. We write, for any vector field a,

$$\overline{a}(x_h) = \int_{\mathbf{T}} a(x_h, x_3)\, dx_3 \quad \text{and} \quad \widetilde{a} = a - \overline{a}, \quad \text{with} \int_{\mathbf{T}} \widetilde{a}(x_h, x_3)\, dx_3 = 0.$$

In particular we have $\widetilde{a} = \partial_3 \widetilde{A}$, with $\int_{\mathbf{T}} \widetilde{A}(x_h, x_3)\, dx_3 = 0$. Equation (2.4.1) derived in the previous section implies that

$$\varepsilon \partial_t \overline{\omega}_\varepsilon = \varepsilon(\partial_1 \overline{F}_{\varepsilon,2} - \partial_2 \overline{F}_{\varepsilon,1})$$

where F_ε denotes the flux term

$$F_\varepsilon \stackrel{\text{def}}{=} \Delta_h u_\varepsilon - \mathbf{P} \nabla \cdot (u_\varepsilon \otimes u_\varepsilon).$$

It is easy to see that $(\partial_1 F_{\varepsilon,2} - \partial_2 F_{\varepsilon,1})$ is bounded in $L^2(\mathbf{R}^+; H^{-5/2}(\Omega))$ (see for instance the proof of Proposition 2.8), so we can write

$$\varepsilon \partial_t \overline{\omega}_\varepsilon = \varepsilon \overline{r}_\varepsilon, \quad \text{where } \overline{r}_\varepsilon \text{ is uniformly bounded in } L^2(\mathbf{R}^+; H^{-5/2}(\Omega)).$$

Similarly an easy computation yields the following equation for $\widetilde{\omega}_\varepsilon$:

$$\varepsilon \partial_t \widetilde{\omega}_\varepsilon - \text{div}_h \widetilde{u}_{\varepsilon,h} = \varepsilon \widetilde{r}_\varepsilon, \quad \text{where } \widetilde{r}_\varepsilon \text{ is uniformly bounded in } L^2(\mathbf{R}^+; H^{-5/2}(\Omega)).$$

For the other components of the vorticity, the computations are similar: since $\nabla \wedge u_\varepsilon^\perp = \partial_3 u_\varepsilon$, we find after integration in the vertical variable

$$\varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h} + \widetilde{u}_{\varepsilon,h} = \varepsilon \widetilde{R}_\varepsilon, \quad \text{where } \widetilde{R}_\varepsilon \text{ is uniformly bounded in } L^2(\mathbf{R}^+; H^{-5/2}(\Omega)).$$

Now let us proceed with the regularization: let $\kappa \in C_c^\infty(\mathbf{R}^3; \mathbf{R}^+)$ such that $\kappa(x) = 0$ if $|x| \geq 1$ and $\int_\Omega \kappa dx = 1$. We define

$$\kappa_\delta : x \mapsto \frac{1}{\delta^3} \kappa\left(\frac{\cdot}{\delta}\right) \tag{2.4.2}$$

as well as

$$\omega_\varepsilon^\delta \stackrel{\text{def}}{=} \omega_\varepsilon * \kappa_\delta = \overline{\omega}_\varepsilon^\delta + \widetilde{\omega}_\varepsilon^\delta, \quad \text{and} \quad \widetilde{\Omega}_\varepsilon^\delta \stackrel{\text{def}}{=} \widetilde{\Omega}_\varepsilon * \kappa_\delta.$$

It is not difficult to see that the following result holds. We leave the details to the reader (see [24]).

Lemma 2.9 *Let u^0 be any divergence free vector field in L^2. For all $\varepsilon > 0$, denote by u_ε a weak solution of (RF_ε). Then, for all $\varepsilon > 0$, there is a family $(u_\varepsilon^\delta)_{\delta>0}$ of smooth vector fields in $L^2(\mathbf{R}^+; \cap_s H^s(\Omega))$ such that*

$$\lim_{\delta\to 0} u_\varepsilon^\delta = u_\varepsilon \quad \text{in} \quad L^2_{loc}(\mathbf{R}^+; L^p(\Omega)) \quad \text{for all } p \in [2,6[, \quad \text{uniformly in } \varepsilon,$$

and such that the functions

$$\omega_\varepsilon^\delta \stackrel{\text{def}}{=} \partial_1 u^\delta_{\varepsilon,2} - \partial_2 u^\delta_{\varepsilon,1} \quad \text{and} \quad \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h} \stackrel{\text{def}}{=} \left(rot\, \widetilde{u}_\varepsilon^\delta\right)_h, \text{ with } \quad \int_{\mathbf{T}} \widetilde{\Omega}^\delta_{\varepsilon,h}(x)\, dx_3 = 0$$

satisfy the following equations:

$$\begin{aligned} \varepsilon\partial_t \overline{\omega}_\varepsilon^\delta &= \varepsilon \overline{r}_\varepsilon^\delta, \\ \varepsilon\partial_t \widetilde{\omega}_\varepsilon^\delta - \mathrm{div}_h \widetilde{u}^\delta_{\varepsilon,h} &= \varepsilon \widetilde{r}_\varepsilon^\delta, \\ \text{and} \quad \varepsilon\partial_t \widetilde{\Omega}^\delta_{\varepsilon,h} + \widetilde{u}^\delta_{\varepsilon,h} &= \varepsilon \widetilde{R}^\delta_{\varepsilon,h} \end{aligned}$$

where for all $\delta > 0$, the functions $\overline{r}_\varepsilon^\delta$ and $\widetilde{r}_\varepsilon^\delta$, as well as the vector field $\widetilde{R}^\delta_{\varepsilon,h}$ are uniformly bounded in ε in the space $L^2(\mathbf{R}^+; L^2(\Omega))$.

With that lemma we are ready to study the limit of the non linear term. Let us give the main steps of the proof of the following result.

Proposition 2.10 *Let u^0 be any divergence free vector field in L^2. For all $\varepsilon > 0$, denote by u_ε a weak solution of (RF_ε), and by $(u_\varepsilon^\delta)_{\delta>0}$ the approximate family of Lemma 2.9. Then for any $\varepsilon > 0$ and any $\delta > 0$, we have*

$$\int_{\mathbf{T}} \left(u_\varepsilon^\delta \cdot \nabla u_\varepsilon^\delta\right)_h dx_3 = -\overline{\omega}^\delta_{\varepsilon,h} (\overline{u}^\delta_{\varepsilon,h})^\perp + \nabla_h \int_{\mathbf{T}} \frac{|u_\varepsilon^\delta|^2}{2} dx_3 - \nabla_h \frac{|\overline{u}^\delta_{\varepsilon,3}|^2}{2} + \varepsilon\partial_t \int_{\mathbf{T}} \widetilde{\omega}_\varepsilon^\delta (\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp dx_3 + \varepsilon \rho^\delta_{\varepsilon,h},$$

and

$$\int_{\mathbf{T}} \left(u_\varepsilon^\delta \cdot \nabla u_\varepsilon^\delta\right)_3 dx_3 = \mathrm{div}_h (\overline{u}^\delta_{\varepsilon,3} \overline{u}^\delta_{\varepsilon,h}) + \frac{1}{2}\varepsilon\partial_t \int_{\mathbf{T}} (\widetilde{\Omega}^\delta_{\varepsilon,h} \cdot \partial_3 (\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp)\, dx_3 + \varepsilon \rho^\delta_{\varepsilon,3},$$

where the vector field ρ_ε^δ satisfies

$$\forall \delta > 0, \quad \forall T > 0, \quad \sup_{\varepsilon > 0} \|\rho_\varepsilon^\delta\|_{L^1([0,T]; L^{6/5}(\Omega))} < +\infty.$$

Proof of Proposition 2.10. Since u_ε^δ is divergence free, we have

$$u_\varepsilon^\delta \cdot \nabla u_\varepsilon^\delta = \nabla \cdot (u_\varepsilon^\delta \otimes u_\varepsilon^\delta) = \nabla \frac{|u_\varepsilon^\delta|^2}{2} - u_\varepsilon^\delta \wedge (\nabla \wedge u_\varepsilon^\delta), \tag{2.4.3}$$

so we shall now restrict our attention to the term $u_\varepsilon^\delta \wedge (\nabla \wedge u_\varepsilon^\delta)$. We have of course

$$\int_{\mathbf{T}} u_\varepsilon^\delta \wedge (\nabla \wedge u_\varepsilon^\delta)\, dx_3 = \overline{u}_\varepsilon^\delta \wedge (\nabla \wedge \overline{u}_\varepsilon^\delta) + \int_{\mathbf{T}} \widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta)\, dx_3. \tag{2.4.4}$$

Let us start by considering the first term in the right-hand side of (2.4.4). A direct computation gives

$$\overline{u}_\varepsilon^\delta \wedge (\nabla \wedge \overline{u}_\varepsilon^\delta) = \frac{1}{2}\nabla|\overline{u}_{\varepsilon,3}^\delta|^2 + \overline{\omega}_{\varepsilon,h}^\delta(\overline{u}_{\varepsilon,h}^\delta)^\perp - \operatorname{div}_h(\overline{u}_{\varepsilon,3}^\delta \overline{u}_{\varepsilon,h}^\delta)e_3.$$

To compute the second term in the right-hand side of (2.4.4), we will use the equations derived in Lemma 2.9. To simplify the presentation we shall set to zero all remainder terms appearing in that lemma. We have

$$\widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta) = \begin{pmatrix} (\widetilde{u}_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta - \partial_3(\widetilde{u}_{\varepsilon,3}^\delta(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp) - \operatorname{div}_h \widetilde{u}_{\varepsilon,h}^\delta(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \\ -(\widetilde{u}_{\varepsilon,h}^\delta)^\perp \cdot \partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta \end{pmatrix}. \tag{2.4.5}$$

Let us study first the horizontal components in (2.4.5): by Lemma 2.9, neglecting all remainder terms, we have

$$(\widetilde{u}_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta - \operatorname{div}_h \widetilde{u}_{\varepsilon,h}^\delta(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp = -\varepsilon\partial_t(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta - \operatorname{div}_h \widetilde{u}_{\varepsilon,h}^\delta(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp.$$

But by Lemma 2.9 again, we have

$$-\operatorname{div}_h \widetilde{u}_{\varepsilon,h}^\delta = -\varepsilon\partial_t \widetilde{\omega}_\varepsilon^\delta,$$

so

$$\int_{\mathbf{T}} \Big(\widetilde{u}_{\varepsilon,h}^{\delta,\perp} \widetilde{\omega}_\varepsilon^\delta - \operatorname{div}_h \widetilde{u}_{\varepsilon,h}^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \Big) \, dx_3 = -\varepsilon\partial_t \int_{\mathbf{T}} (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta \, dx_3.$$

Now we are left with the last term in (2.4.5), which is the third component: we can write, by Lemma 2.9,

$$\widetilde{u}_{\varepsilon,h}^\delta = -\varepsilon\partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta,$$

so

$$(\widetilde{u}_{\varepsilon,h}^\delta)^\perp \cdot \partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta = -\varepsilon\partial_t(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \cdot \partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta.$$

Then we just need to notice that

$$\varepsilon\partial_t(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \cdot \partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta = -\frac{1}{2}\varepsilon\partial_t \left(\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \right) + \frac{1}{2}\partial_3 \left(\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot (\varepsilon\partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \right).$$

Putting those computations together yields finally the proposition. Note that the regularization procedure is useful here, as the (omitted) remainder terms go to zero in the expected functional space as ε goes to zero, for all $\delta > 0$. The parameter δ will go to zero at the very end of the argument leading to the theorem. □

That result enables us easily to infer the following corollary.

Corollary 2.11 *Let u^0 be any divergence free vector field in L^2. For all $\varepsilon > 0$, denote by u_ε a weak solution of (RF_ε). Then for any vector field $\phi \in H^1 \cap \operatorname{Ker}(L)$, we have the following limit in $W^{-1,1}([0,T])$ for any $T > 0$:*

$$\lim_{\varepsilon\to 0} \left(\int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi(x_h) \, dx - \int_{\Omega_h} \nabla_h \cdot (\overline{u}_\varepsilon \otimes \overline{u}_\varepsilon) \cdot \phi(x_h) \, dx_h \right) = 0.$$

Now we are ready to prove the convergence theorem.

Proof of Theorem 2.2. The proof of Theorem 2.2 follows quite easily from the previous results: we have seen that u_ε converges weakly in $L^2([0,T]\times\Omega)$ towards a vector field u depending only on the horizontal variable and mean free, due to Proposition 2.5. Then we proved that $\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon$ is strongly compact, therefore converges strongly towards u in the space $L^2([0,T]\times\Omega)$ (since $u=\overline{u}$ and $\overline{\overline{u}}=0$). Finding the equation satisfied by the limit is therefore a matter of computing the limit of Equation (2.4.1). The linear terms converge in the sense of distributions of course, and to find the limit of the nonlinear term we use Proposition 2.11 as well as the following weak-strong limit argument: we have

$$\nabla\cdot(\overline{u}_\varepsilon\otimes\overline{u}_\varepsilon)=\nabla\cdot(\overline{\overline{u}}_\varepsilon\otimes(\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon))+\nabla\cdot((\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)\otimes\overline{\overline{u}}_\varepsilon)))+\nabla\cdot((\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)\otimes(\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)).$$

The two first terms converge towards zero in $\mathcal{D}'(\Omega)$ since $\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon$ is compact and $\overline{\overline{u}}_\varepsilon$ converges weakly to zero, whereas the last term satisfies

$$\nabla\cdot((\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)\otimes(\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon))\to\nabla\cdot(u\otimes u)\quad\text{in}\quad\mathcal{D}'(\Omega).$$

That gives the expected result: the limit u satisfies the two dimensional Navier-Stokes equation

$$\partial_t u-\Delta_h u+\mathbf{P}\nabla_h\cdot(u\otimes u)=0.$$

Theorem 2.2 is proved. □

We therefore recover as expected the Taylor-Proudman theorem. Now the question consists in making that convergence result more precise, by describing more finely the oscillations of u_ε. We have seen that they do not contribute to the limiting behaviour of the system, but it remains to understand if they are actually an obstruction to the strong convergence or not. The answer to that question depends on the boundary conditions, as shown in the following section where the case of the whole space $\mathbf{R}^3$ and the periodic case are studied. As pointed out in the introduction, the fact that the rotation is constant will enable us to describe very precisely the oscillations, by use of the Fourier transform.

2.5 Strong asymptotics

In this section we are going to prove precised versions of Theorem 2.2, by analyzing the strong asymptotics of u_ε. We will mainly focus on two situations, first the case when the equations are set in the whole space $\mathbf{R}^3$ (proving Theorems 2.3 and 2.4), and then the periodic case (proving Theorem 2.5). Comments on more general boundary conditions can be found in Paragraph 2.5.3.

2.5.1 The whole space case

Let us suppose here that the equations are set in $\mathbf{R}^3$, and let us consider again a family of weak solutions to (RF_ε). Due to the result proved in the previous section, we know that the weak limit of any such family is necessarily zero, and the question is to know whether it converges strongly or not. The answer is given in Theorem 2.3, where in order to give a more general statement we have considered the case when the initial data is the sum of a purely two-dimensional vector field (with possibly three components) and a three dimensional vector field.

Proof of Theorem 2.3. We shall leave as an exercise to the reader the proof of the existence of a solution to (RF_ε) with initial data $\overline{u}^0 + w^0$ in $L^2(\mathbf{R}^2) + L^2(\mathbf{R}^3)$, which is an easy adaptation of Proposition 2.3. The main ingredient of the proof of the strong convergence result is a so-called "Strichartz estimate" on the Coriolis operator, which we will write now. Let us consider the linearized equations

$$(LR_\varepsilon) \qquad \begin{cases} \partial_t v - \Delta v + \dfrac{v^\perp}{\varepsilon} + \nabla p = f \\ \operatorname{div} v = 0 \\ v_{|t=0} = v^0, \end{cases}$$

which yields in Fourier variables $\xi \in \mathbf{R}^3$

$$\begin{cases} \partial_t \widehat{v} + |\xi|^2 \widehat{v} + \dfrac{\xi_3 \widehat{v} \wedge \xi}{\varepsilon |\xi|^2} = \widehat{f} \\ \widehat{v}_{|t=0} = \widehat{v}^0. \end{cases}$$

We will denote by $\widehat{f}$ or $\mathcal{F}f$ the Fourier transform of any function or vector field f, defined by

$$\mathcal{F}f(\xi) = \int_{\mathbf{R}^3} e^{-ix\cdot\xi} f(x)\, dx.$$

The matrix $Mv \stackrel{\text{def}}{=} \dfrac{\xi_3 v \wedge \xi}{|\xi|^2}$ has three eigenvalues, 0 and $\pm i \dfrac{\xi_3}{|\xi|}\cdot$ The associate eigenvectors are

$$e^0(\xi) = {}^t(0,0,1) \quad \text{and}$$
$$e^\pm(\xi) = \frac{1}{\sqrt{2}|\xi||\xi_h|}\, {}^t\left(\xi_1\xi_3 \mp i\xi_2|\xi|,\ \xi_2\xi_3 \pm i\xi_1|\xi|, -|\xi_h|^2\right).$$

The precise value of those vectors will not be necessary for our study; all we shall need to know is that the last two are divergence free, in the sense that $\xi \cdot e^\pm(\xi) = 0$. Furthermore they are orthogonal, in the sense that if v belongs to $L^2(\mathbf{R}^3)$ then

$$\|v\|_{L^2}^2 = \|v^+\|_{L^2}^2 + \|v^-\|_{L^2}^2 \quad \text{where} \quad v^\pm \stackrel{\text{def}}{=} \mathcal{F}^{-1}\left((\widehat{v}(\xi)\cdot e^\pm(\xi))e^\pm(\xi)\right).$$

We are now led to studying the application

$$\mathcal{G}^{\varepsilon,\pm}(\tau)\colon\ g \mapsto \int_{\mathbf{R}^3_\xi} \widehat{g}(\xi) e^{\pm i\tau\frac{\xi_3}{|\xi|} - \tau\varepsilon|\xi|^2 + ix\cdot\xi}\, d\xi = \int_{\mathbf{R}^3_\xi \times \mathbf{R}^3_y} g(y) e^{\pm i\tau\frac{\xi_3}{|\xi|} - \tau\varepsilon|\xi|^2 + i(x-y)\cdot\xi}\, d\xi dy,$$

and we will start by considering the case when $\widehat{g}$ is supported in $\mathcal{C}_{r,R}$ for some $r < R$, where

$$\mathcal{C}_{r,R} = \{\xi \in \mathbf{R}^3 \ / \ |\xi_3| \geq r \ \text{ and } \ |\xi| \leq R\}. \tag{2.5.1}$$

In that situation we can multiply $\widehat{g}(\xi)$ in the previous formula by a function ψ in $\mathcal{D}(\mathbf{R}^3 \setminus \{0\})$, such that $\psi \equiv 1$ in a neighborhood of $\mathcal{C}_{r,R}$, and which is radial with respect to the horizontal variable $\xi_h = (\xi_1, \xi_2)$. For instance we suppose that ψ is supported in the set $\mathcal{C}_{r/2,2R}$. We are now led to studying the following function:

$$K^{\pm}(t,\tau,z) \stackrel{\text{def}}{=} \int_{\mathbf{R}^3_\xi} \psi(\xi) e^{\pm i\tau a(\xi) + iz\cdot\xi - t|\xi|^2}\, d\xi, \quad \text{where} \quad a(\xi) \stackrel{\text{def}}{=} \frac{\xi_3}{|\xi|}.$$

The following result is the main step of the proof of Strichartz estimates; it is a dispersion estimate.

Lemma 2.12 *For any (r,R) such that $0 < r < R$, a constant $C_{r,R}$ exists such that $\forall z \in \mathbf{R}^3$,*

$$|K^{\pm}(t,\tau,z)| \leq C_{r,R} \min\{1, \tau^{-\frac{1}{2}}\} e^{-\frac{1}{2}r^2 t}.$$

Proof of Lemma 2.12. For the sake of simplicity we will only consider K^+, the term K^- being dealt with exactly in the same way. This proof is very like the proof of the more usual dispersive estimate for the wave equation. First using the rotation invariance in ξ_h, we restrict ourselves to the case when $z_2 = 0$. Next, denoting $\alpha(\xi) \stackrel{\text{def}}{=} -\partial_{\xi_2} a(\xi) = \xi_2\xi_3/|\xi|^3$, we introduce the following differential operator:

$$\mathcal{X} \stackrel{\text{def}}{=} \frac{1}{1+\tau\alpha^2(\xi)} \left(1 + i\alpha(\xi)\partial_{\xi_2}\right),$$

which satisfies $\mathcal{X}(e^{i\tau a}) = e^{i\tau a}$. Integrating by parts, we obtain

$$K^+(t,\tau,z) = \int_{\mathbf{R}^3} e^{i\tau a(\xi) + iz_1\xi_1 + iz_3\xi_3} \left({}^t\mathcal{X}(\psi(\xi)e^{-t|\xi|^2})\right) d\xi.$$

Easy computations yield

$${}^t\mathcal{X}\left(\psi(\xi)e^{-t|\xi|^2}\right) = \left(\frac{1}{1+\tau\alpha^2} - i(\partial_{\xi_2}\alpha)\frac{1-\tau\alpha^2}{(1+\tau\alpha^2)^2}\right)\psi(\xi)e^{-t|\xi|^2} - \frac{i\alpha}{1+\tau\alpha^2}\partial_{\xi_2}\left(e^{-t|\xi|^2}\psi(\xi)\right).$$

As ξ belongs to the support of ψ, which is supposed to be included in $\mathcal{C}_{r/2,2R}$ as defined in (2.5.1), we can prove that

$$\left|{}^t\mathcal{X}\left(\psi(\xi)e^{-t|\xi|^2}\right)\right| \leq \frac{C_{r,R}}{1+\tau\xi_2^2} e^{-\frac{1}{2}tr^2}$$

so we obtain, for all $z \in \mathbf{R}^3$,

$$|K^+(t,\tau,z)| \leq C_{r,R} e^{-\frac{1}{2}tr^2} \int_{\mathbf{R}} \frac{d\xi_2}{1+\tau\xi_2^2},$$

which proves Lemma 2.12. □

Lemma 2.12 yields the following theorem, whose proof is quite classical in the context of Strichartz estimates (it is based on the so-called TT^* argument) and is omitted.

Theorem 2.6 *For any positive constants r and R such that $r < R$, let $\mathcal{C}_{r,R}$ be the frequency domain defined in* (2.5.1)*. Then a constant $C_{r,R}$ exists such that if $v^0 \in L^2(\mathbf{R}^3)$ and $f \in L^1(\mathbf{R}^+; L^2(\mathbf{R}^3))$ are two vector fields whose Fourier transform is supported in $\mathcal{C}_{r,R}$, and if v is the solution of the linear equation (LR_ε) with forcing term f and initial data v^0, then for all $p \in [1, +\infty]$,*

$$\|v\|_{L^p(\mathbf{R}^+;L^\infty(\mathbf{R}^3))} \leq C_{r,R}\, \varepsilon^{\frac{1}{4p}} \left(\|v^0\|_{L^2(\mathbf{R}^3)} + \|f\|_{L^1(\mathbf{R}^+;L^2(\mathbf{R}^3))} \right).$$

We see that the solution of the linearized equations converges strongly to zero as ε goes to zero. Now let us conclude the proof of Theorem 2.3. We define $w_\varepsilon = u_\varepsilon - \overline{u}$, and we are going to prove that w_ε goes to zero as ε goes to zero, in $L^2_{loc}(\mathbf{R}^+; L^q(\mathbf{R}^3))$ for any $2 < q < 6$. One can prove, by an energy estimate on the equation satisfied by w_ε, that

$$\forall t \geq 0, \quad \|w_\varepsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla w_\varepsilon(t')\|_{L^2}^2 \, dt' \leq \|w^0\|_{L^2}^2 \exp\left(C \|\overline{u}^0\|_{L^2(\mathbf{R}^2)}^2 \right).$$

Indeed we have formally

$$\|w_\varepsilon(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w_\varepsilon(t')\|_{L^2}^2 \, dt' = \int_0^t \left| \int_{\mathbf{R}^3} (w_\varepsilon \cdot \nabla)\overline{u} \cdot w_\varepsilon(t', x) \, dx \right| dt',$$

and using the two-dimensional Gagliardo-Nirenberg inequality we can write

$$\begin{aligned} \left| \int_{\mathbf{R}^3} (w_\varepsilon \cdot \nabla)\overline{u} \cdot w_\varepsilon \, dx \right| &\leq \int_{\mathbf{R}} \|\nabla \overline{u}\|_{L^2} \|w_\varepsilon(\cdot, x_3)\|_{L^4(\mathbf{R}^2)}^2 \, dx_3 \\ &\leq \|\nabla \overline{u}\|_{L^2} \int_{\mathbf{R}} \|w_\varepsilon(\cdot, x_3)\|_{L^2(\mathbf{R}^2)} \|\nabla w_\varepsilon(\cdot, x_3)\|_{L^2(\mathbf{R}^2)} \, dx_3 \\ &\leq \|\nabla w_\varepsilon\|_{L^2(\mathbf{R}^3)}^2 + C \|\nabla \overline{u}\|_{L^2}^2 \|w_\varepsilon\|_{L^2(\mathbf{R}^3)}^2, \end{aligned}$$

and the above estimate follows from Gronwall's inequality.

In order to use the Strichartz estimates of Theorem 2.6, we have to get rid of high frequencies and low vertical frequencies. Let us define the following truncation operator

$$\mathcal{P}_R f \stackrel{\text{def}}{=} \chi\left(\frac{D}{R}\right) f, \quad \text{where} \quad \chi \in \mathcal{D}(]-2, 2[), \ \chi(x) = 1 \text{ for } |x| \leq 1.$$

In other words we have

$$\mathcal{F}\mathcal{P}_R f(\xi) = \chi\left(\frac{\xi}{R}\right) f(\xi).$$

Let us observe that, thanks to Sobolev embeddings and the energy estimate, we have, for any $q \in [2, 6[$,

$$\begin{aligned} \|w_\varepsilon - \mathcal{P}_R w_\varepsilon\|_{L^2(\mathbf{R}^+;L^q(\mathbf{R}^3))} &\leq C \|w_\varepsilon - \mathcal{P}_R w_\varepsilon\|_{L^2(\mathbf{R}^+;\dot{H}^{3\left(\frac{1}{2}-\frac{1}{q}\right)})} \\ &\leq C R^{-\alpha_q} \|w_\varepsilon\|_{L^2(\mathbf{R}^+;\dot{H}^1)} \qquad (2.5.2) \\ &\leq C R^{-\alpha_q} \|w^0\|_{L^2} \exp\left(C \|\overline{u}^0\|_{L^2(\mathbf{R}^2)}^2 \right) \end{aligned}$$

with $\alpha_q \stackrel{\text{def}}{=} \frac{3}{q} - \frac{1}{2}\cdot$ Now let us define $\chi\Big(\frac{D_3}{r}\Big) f \stackrel{\text{def}}{=} \mathcal{F}^{-1}\Big(\chi\Big(\frac{\xi_3}{r}\Big)\widehat{f}(\xi)\Big)$. We have

$$\Big\|\chi\Big(\frac{D_3}{r}\Big)\mathcal{P}_R w_\varepsilon\Big\|_{L^2(\mathbf{R}^+;L^\infty)} \leq \Big\|\chi\Big(\frac{\xi_3}{r}\Big)\mathcal{F}(\mathcal{P}_R w_\varepsilon)\Big\|_{L^2(\mathbf{R}^+;L^1)}$$

so using the fact that $|\xi| \leq R$ and a Cauchy-Schwartz inequality one can prove that

$$\Big\|\chi\Big(\frac{D_3}{r}\Big)\mathcal{P}_R w_\varepsilon\Big\|_{L^2(\mathbf{R}^+;L^\infty)} \leq C_R\, r^{\frac{1}{2}} \|w^0\|_{L^2} \exp\Big(C\|\overline{u}^0\|^2_{L^2(\mathbf{R}^2)}\Big)\,. \tag{2.5.3}$$

Let us define $\mathcal{P}_{r,R} \stackrel{\text{def}}{=} \bigg(\mathrm{Id} - \chi\Big(\frac{D_3}{r}\Big)\bigg)\mathcal{P}_R$. The following lemma, whose proof is postponed for a moment, describes the dispersive effects due to fast rotation.

Lemma 2.13 *For any positive real numbers r, R and T, and for any q in $]2,+\infty[$,*

$$\forall \varepsilon > 0\,,\ \Big\|\mathcal{P}_{r,R}\, w_\varepsilon\Big\|_{L^2([0,T];L^q(\mathbf{R}^3))} \leq C\varepsilon^{\frac{1}{8}\left(1-\frac{2}{q}\right)},$$

the constant C above depending on r, q, R, T, $\|\overline{u}^0\|_{L^2}$ and $\|w^0\|_{L^2}$ but not on ε.

Together with Inequalities (2.5.2) and (2.5.3), this lemma implies that, for any positive r, R and T, for $q \in]2,6[$,

$$\forall \varepsilon > 0\,,\ \|w_\varepsilon\|_{L^2([0,T];L^q)} \leq CR^{-\alpha_q} + C_R\, r^{\frac{1}{2}} + C_3 \varepsilon^{\frac{1}{8}\left(1-\frac{2}{q}\right)},$$

the constant C_3 above depending on r, R, T, $\|\overline{u}^0\|_{L^2}$ and $\|w^0\|_{L^2}$ but not on ε. We deduce that, for any positive r, R and T, for $q \in]2,6[$,

$$\limsup_{\varepsilon\to 0} \|w_\varepsilon\|_{L^2([0,T];L^q)} \leq CR^{-\alpha_q} + C_R\, r^{\frac{1}{2}}.$$

Passing to the limit when r tends to 0 and then when R tends to ∞ gives Theorem 2.6, provided of course we prove Lemma 2.13. $\square$

Proof of Lemma 2.13. Thanks to Duhamel's formula we have,

$$\begin{aligned}
\mathcal{P}_{r,R} w_\varepsilon(t) &= \sum_{j=1}^{3} \mathcal{P}^j_{r,R} w_\varepsilon(t) \quad \text{with}\\
\mathcal{P}^1_{r,R} w_\varepsilon(t) &\stackrel{\text{def}}{=} \mathcal{G}^\varepsilon\Big(\frac{t}{\varepsilon}\Big)\mathcal{P}_{r,R} w^0,\\
\mathcal{P}^2_{r,R} w_\varepsilon(t) &\stackrel{\text{def}}{=} \int_0^t \mathcal{G}^\varepsilon\Big(\frac{t-t'}{\varepsilon}\Big)\mathcal{P}_{r,R} Q(w_\varepsilon(t'), w_\varepsilon(t'))\, dt' \quad \text{and}\\
\mathcal{P}^3_{r,R} w_\varepsilon(t) &\stackrel{\text{def}}{=} \int_0^t \mathcal{G}^\varepsilon\Big(\frac{t-t'}{\varepsilon}\Big)\mathcal{P}_{r,R}\big(Q(w_\varepsilon(t'), \overline{u}(t')) + Q(\overline{u}(t'), w_\varepsilon(t'))\big)\, dt'.
\end{aligned}$$

We have defined $Q(a,b) = \mathbf{P}(a\cdot\nabla b)$. Theorem 2.6 implies that

$$\|\mathcal{P}^1_{r,R} w_\varepsilon\|_{L^2(\mathbf{R}^+;L^\infty)} \leq C_{r,R}\, \varepsilon^{\frac{1}{8}} \|w^0\|_{L^2}.$$

By interpolation with the energy bound, we infer that

$$\|\mathcal{P}^1_{r,R} w_\varepsilon\|_{L^2([0,T];L^q)} \leq C_{r,R,T} \varepsilon^{\frac{1}{8}\left(1-\frac{2}{q}\right)} \|w^0\|_{L^2} \exp\left(C\|\overline{u}^0\|^2_{L^2(\mathbf{R}^2)}\right).$$

Using again Theorem 2.6, we have

$$\|\mathcal{P}^2_{r,R} w_\varepsilon\|_{L^2([0,T];L^\infty)} \leq C_{r,R}\, \varepsilon^{\frac{1}{8}} \|\mathcal{P}_R Q(w_\varepsilon, w_\varepsilon)\|_{L^1([0,T];L^2)}.$$

Let us recall Bernstein's lemma : if a function F has its Fourier transform supported in a ball of radius R, then for all $k \in \mathbf{N}$ and all $1 \leq p \leq q \leq \infty$,

$$\sup_{|\alpha|=k} \|\partial^\alpha F\|_{L^q(\mathbf{R}^d)} \leq C R^{k+d(\frac{1}{p}-\frac{1}{q})} \|F\|_{L^p(\mathbf{R}^d)}.$$

That lemma together with the energy estimate implies that

$$\begin{aligned}
\|\mathcal{P}_R Q(w_\varepsilon, w_\varepsilon)\|_{L^1([0,T];L^2)} &\leq CR \|\mathcal{P}_R (w_\varepsilon \otimes w_\varepsilon)\|_{L^1([0,T];L^2)} \\
&\leq CR^{1+\frac{3}{2}} \|w_\varepsilon \otimes w_\varepsilon\|_{L^1([0,T];L^1)} \\
&\leq C_R T \|w^0\|^2_{L^2(\mathbf{R}^3)} \exp\left(C\|\overline{u}^0\|^2_{L^2(\mathbf{R}^2)}\right).
\end{aligned}$$

By interpolation with the energy bound, we infer

$$\|\mathcal{P}^2_{r,R} w_\varepsilon\|_{L^2([0,T];L^q)} \leq C_{r,R,T}\, \varepsilon^{\frac{1}{8}\left(1-\frac{2}{q}\right)} \|w^0\|^{2(1-\frac{1}{q})}_{L^2} \exp\left(C\|\overline{u}^0\|^2_{L^2(\mathbf{R}^2)}\right).$$

Still using Theorem 2.6, we have

$$\|\mathcal{P}^3_{r,R} w_\varepsilon\|_{L^2([0,T];L^\infty)} \leq C_{r,R}\, \varepsilon^{\frac{1}{8}} \|\mathcal{P}_R\big(Q(w_\varepsilon, \overline{u}) + Q(\overline{u}, w_\varepsilon)\big)\|_{L^1([0,T];L^2)}.$$

Similarly, using an anisotropic-type Bernstein inequality, we can prove that

$$\|\mathcal{P}^3_{r,R} w_\varepsilon\|_{L^2([0,T];L^\infty)} \leq C_{r,R,T}\, \varepsilon^{\frac{1}{8}} \|\overline{u}^0\|_{L^2} \|w^0\|_{L^2} \exp\left(C\|\overline{u}^0\|^2_{L^2}\right),$$

and by interpolation with the energy bound, we get

$$\|\mathcal{P}^3_{r,R} w_\varepsilon\|_{L^2([0,T];L^q)} \leq C_{r,R,T}\, \varepsilon^{\frac{1}{8}\left(1-\frac{2}{q}\right)} \|\overline{u}^0\|^{1-\frac{2}{q}}_{L^2} \|w^0\|_{L^2} \exp\left(C\|\overline{u}^0\|^2_{L^2}\right).$$

The lemma is proved. □

Once the behaviour of weak solutions has been investigated, it is natural to consider strong solutions. The question of their convergence is easily settled due to Theorem 2.3 (in particular if the initial data is in $H^{\frac{1}{2}}(\mathbf{R}^3)$ then the strong solutions converge to zero, and if it is in $L^2(\mathbf{R}^2) + H^{\frac{1}{2}}(\mathbf{R}^3)$ then they will converge towards a two-dimensional vector field satisfying the two dimensional Navier-Stokes equations). However since that limit system is known to be globally well posed, one can try to use this information to recover a better bound on the life span of the solutions to the rotating fluid equations in $H^{\frac{1}{2}}(\mathbf{R}^3)$, depending on ε. This is achieved through the following theorem, which is a precised version of Theorem 2.4 stated in Section 2.2.

Theorem 2.7 *Let $\overline{u}^0$ and w^0 be two divergence free vector fields, respectively in $L^2(\mathbf{R}^2)$ and $H^{\frac{1}{2}}(\mathbf{R}^3)$. Then a positive ε_0 exists such that for all $\varepsilon \leq \varepsilon_0$, there is a unique global solution u_ε to the system (RF_ε). More precisely, denoting by $\overline{u}$ the (unique) solution of the two dimensional Navier- Stokes equations associated with $\overline{u}^0$, by v_F the solution of (LR_ε) with initial data w^0 (with $f = 0$), and defining $w_\varepsilon \stackrel{\text{def}}{=} u_\varepsilon - \overline{u}$, then for ε small enough, w_ε is unique in $L^\infty(\mathbf{R}^+, \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)) \cap L^2(\mathbf{R}^+, \dot{H}^{\frac{3}{2}}(\mathbf{R}^3))$ and we have, as ε goes to zero,*

$$
\begin{aligned}
w_\varepsilon \in C_b(\mathbf{R}^+; H^{\frac{1}{2}}(\mathbf{R}^3)) \quad &\text{and} \quad \nabla w_\varepsilon \in L^2(\mathbf{R}^+; H^{\frac{1}{2}}(\mathbf{R}^3)),\\
w_\varepsilon - v_F \to 0 \quad &\text{in} \quad L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}(\mathbf{R}^3))\\
\text{and } \nabla(w_\varepsilon - v_F) \to 0 \quad &\text{in} \quad L^2(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)).
\end{aligned}
$$

Proof of Theorem 2.7. We will not give the details of the proof here but simply some indications. The first step consists in checking that there is indeed a unique solution $u_\varepsilon = \overline{u} + w_\varepsilon$ to (RF_ε) in $H^{\frac{1}{2}}(\mathbf{R}^3)$ for some finite time. This is achieved in a similar way to the case of the Navier-Stokes equations (up to the presence of the perturbation term involving $\overline{u}$, which is not in $L^\infty(\mathbf{R}^+; H^{\frac{1}{2}})$ but only in the energy space; however it only depends on two variables so an anisotropic Gagliardo-Nirenberg inequality gives the desired estimates). To prove that w_ε exists globally in time one needs to use more than an energy estimate, since such an estimate would be similar to the case of the 3D Navier-Stokes system, for which the global existence in time of a unique solution is not known. The idea is to subtract from w_ε the solution v_F of (LR_ε), which we know goes to zero (at least for restricted frequencies) by Strichartz estimates. We are then led to solving the system satisfied by $w_\varepsilon - v_F$, which has small data (involving the extreme frequencies of w^0) and small source terms (due to the Strichartz estimates of Theorem 2.6). The usual methods for the 3D Navier-Stokes equations can then be used. Of course there are a few additional difficulties, the main one being that one needs to cope with the interaction of two dimensional and three dimensional vector fields; for that an anistropic-type Strichartz estimate is needed, but we shall not pursue this question here and refer to [13] for details. □

2.5.2 The periodic case

This paragraph deals with the rotating fluid equations (RF_ε) in a purely periodic setting: we define the periodic box $\mathbf{T}^3 \stackrel{\text{def}}{=} (\mathbf{R}_{/\mathbf{Z}})^3$. All the vector fields considered in this paragraph will be supposed to be mean free. We are interested in the (strong) asymptotic behaviour of u_ε as ε goes to zero, proving Theorem 2.5. The first step of the analysis consists in deriving a limit system for (RF_ε), which will enable us to state and prove a convergence theorem for weak solutions. The main issue will then consist in studying the behaviour of strong solutions. The proof of Theorem 2.5 relies on the construction of families of approximate solutions. Let us state the key lemma, where we have used the following notation:

$$
\|u\|_{\frac{1}{2}}^2 \stackrel{\text{def}}{=} \sup_{t\geq 0}\Big(\|u(t)\|_{H^{\frac{1}{2}}}^2 + 2\int_0^t \|\nabla u(t')\|_{H^{\frac{1}{2}}}^2\,dt'\Big).
$$

Lemma 2.14 *Let u^0 be a divergence free vector field in $H^{\frac{1}{2}}$. For any positive real number η, a family $(u_{app})_{\varepsilon,\eta}$ exists such that $\limsup_{\eta\to 0}\limsup_{\varepsilon\to 0}\|u_{app}\|_{H^{\frac{1}{2}}}<\infty$. Moreover, the families $(u_{app})_{\varepsilon,\eta}$ are approximate solutions of (RF_ε) in the sense that u_{app} satisfies*

$$\begin{cases} \partial_t u_{app} - \Delta u_{app} + \mathbf{P}(u_{app}\cdot\nabla u_{app}) + \dfrac{1}{\varepsilon}\mathbf{P}(u_{app}^{\perp}) = R \quad in \quad \mathbf{T}^3 \\ \lim_{\eta\to 0}\lim_{\varepsilon\to 0}\|u_{app|t=0}-u^0\|_{H^{\frac{1}{2}}}=0, \end{cases} \tag{2.5.4}$$

with

$$\lim_{\eta\to 0}\lim_{\varepsilon\to 0}\|R\|_{L^2(\mathbf{R}^+,H^{-1/2})}=0$$

Remark 2.15 *The stability of strong solutions to the Navier-Stokes equations enables us to prove that as soon as ε and η are small enough, the solution u_ε to (RF_ε) remains arbitrarily close to the solution u_{app} of (2.5.4): indeed both equations are the same up to the initial data and forcing terms, which can be made arbitrarily close. In particular u_ε satisfies*

$$\lim_{\eta\to 0}\limsup_{\varepsilon\to 0}\|u_\varepsilon-u_{app}\|_{H^{\frac{1}{2}}}=0,$$

which implies directly the global existence result of Theorem 2.5. It follows that the construction of the families $(u_{app})_{\varepsilon,\eta}$ is the main step in the analysis of (RF_ε) in a periodic box. Moreover it will also enable us to further describe the asymptotics of u_ε as ε goes to zero, thus achieving the proof of Theorem 2.5.

Let us give the main steps of the construction of the approximate family. Fast time oscillations prevent any result of strong convergence to a fixed function. In order to bypass this difficulty, we are going to introduce a procedure of filtering of the time oscillations. This will lead us to the concept of limit system. So we start by defining the filtering operator, the limit system and to establish that the weak closure of $(u_\varepsilon)_{\varepsilon>0}$ is included, after filtration, in the set of weak solutions of the limit system. Then we prove that the nonlinear terms in the limit system have a special structure, very close to the structure of the nonlinear term in the 2D Navier–Stokes equations, which makes it possible to prove the global wellposedness of the limit system. The families $(u_{app})_{\varepsilon,\eta}$ can then be constructed.

So let us start by finding a limit system. We know that there is a bounded family of solutions $(u_\varepsilon)_{\varepsilon>0}$ associated with the initial data, so one can extract a subsequence and find a weak limit to $(u_\varepsilon)_{\varepsilon>0}$. Then we will need some refined analysis to understand the asymptotic behaviour of $(u_\varepsilon)_{\varepsilon>0}$. Let $\mathcal{L}$ be the evolution group associated with the Coriolis operator L defined in (2.2.1): the vector field $\mathcal{L}(t)v^0$ is the solution at time t of the equation

$$\partial_t v + Lv = 0, \quad v_{|t=0}=v^0.$$

As L is skew–symmetric, the operator $\mathcal{L}(t)$ is unitary for all times t, in all spaces $H^s(\mathbf{T}^3)$. In particular the "filtered solution" associated with u_ε

$$\widetilde{u}_\varepsilon \stackrel{\mathrm{def}}{=} \mathcal{L}\left(-\frac{t}{\varepsilon}\right)u_\varepsilon\,,$$

is uniformly bounded in the space $L^\infty(\mathbf{R}^+; L^2(\mathbf{T}^3)) \cap L^2(\mathbf{R}^+; \dot H^1(\mathbf{T}^3))$. It satisfies the following system:

$$(\widetilde{RF}_\varepsilon) \quad \begin{cases} \partial_t \widetilde{u}_\varepsilon - \mathcal{Q}_\varepsilon(\widetilde{u}_\varepsilon, \widetilde{u}_\varepsilon) - \Delta \widetilde{u}_\varepsilon = 0 \\ \widetilde{u}_{\varepsilon|t=0} = u^0, \end{cases}$$

noticing that $\mathcal{L}(t/\varepsilon)$ is equal to Identity when $t = 0$. We have used the fact that the operator L commutes with all derivation operators, and we have noted

$$\mathcal{Q}_\varepsilon(a,b) \stackrel{\text{def}}{=} -\frac{1}{2}\left(\mathcal{L}(-\frac{t}{\varepsilon})\,\mathbf{P}\Big(\mathcal{L}(\frac{t}{\varepsilon})\,a \cdot \nabla\,\mathcal{L}(\frac{t}{\varepsilon})\,b\Big) + \mathcal{L}(-\frac{t}{\varepsilon})\,\mathbf{P}\Big(\mathcal{L}(\frac{t}{\varepsilon})\,b \cdot \nabla\,\mathcal{L}(\frac{t}{\varepsilon})\,a\Big)\right). \qquad (2.5.5)$$

The point in introducing the filtered vector field $\widetilde{u}_\varepsilon$ is that one can find a limit system to $(\widetilde{RF}_\varepsilon)$ (contrary to the case of (RF_ε)): if u^0 is in $L^2(\mathbf{T}^3)$, it is not difficult to see that contrary to the original system, the family $(\partial_t \widetilde{u}_\varepsilon)_{\varepsilon>0}$ is bounded, for instance in the space $L^{\frac{4}{3}}([0,T]; H^{-1}(\mathbf{T}^3))$ for all $T > 0$. A compactness argument enables us, up to the extraction of a subsequence, to obtain a limit to the sequence $\widetilde{u}_\varepsilon$, called u. The linear terms $\partial_t \widetilde{u}_\varepsilon$ and $\Delta \widetilde{u}_\varepsilon$ converge towards $\partial_t u$ and Δu respectively in $\mathcal{D}'((0,T) \times \mathbf{T}^3)$, so the point is to find the limit of the quadratic form $\mathcal{Q}_\varepsilon(\widetilde{u}_\varepsilon, \widetilde{u}_\varepsilon)$. Let us study that term more precisely. For any vector field $u = (u_1, u_2, u_3)$, $\overline{u}$ is the quantity

$$\overline{u}(x_h) \stackrel{\text{def}}{=} \int_{\mathbf{T}} u(x_h, x_3)\, dx_3.$$

Note that if u is divergence free, then so is $\overline{u}_h = (u_1, u_2)$. Finally we decompose u into

$$u = \overline{u} + u_{osc},$$

where the notation u_{osc} stands for the "oscillating part" of u. Now in order to derive formally the limit of $\mathcal{Q}_\varepsilon$, let us compute more explicitly the operators L and $\mathcal{L}$. As in Paragraph 2.5.1, the eigenvalues of L_n (where $n \in \mathbf{Z}^3$ denotes the Fourier variables) are 0, $in_3/|n|$, and $-in_3/|n|$. We will call $e^\pm(n)$ the associate eigenvectors, as defined in the previous paragraph. Now we are ready to find the limit of the quadratic form $\mathcal{Q}_\varepsilon$. In the following, we denote $\sigma \stackrel{\text{def}}{=} (\sigma_1, \sigma_2, \sigma_3) \in \{+,-\}^3$ any triplet of pluses or minuses, and for any vector field h, its projection (in Fourier variables) along those vector fields is denoted

$$\forall n \in \mathbf{Z}^3, \quad \forall j \in \{1,2,3\}, \quad h^{\sigma_j}(n) \stackrel{\text{def}}{=} (\mathcal{F}h(n) \cdot e^{\sigma_j}(n))\, e^{\sigma_j}(n).$$

Proposition 2.16 *Let $\mathcal{Q}_\varepsilon$ be the quadratic form defined in (2.5.5), and let a and b be two smooth vector fields on $\mathbf{T}^3$. Then one can define*

$$\mathcal{Q}(a,b) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \mathcal{Q}_\varepsilon(a,b) \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^+ \times \mathbf{T}^3),$$

and we have

$$\mathcal{F}\mathcal{Q}(a,b)(n) = -\sum_{\substack{\sigma \in \{+,-\}^3 \\ k \in \mathcal{K}_n^\sigma}} [a^{\sigma_1}(k) \cdot (n-k)]\,[b^{\sigma_2}(n-k) \cdot e^{\sigma_3}(n)] e^{\sigma_3}(n),$$

where $\mathcal{K}_n^\sigma$ is the "resonant set" defined, for any n in $\mathbf{Z}^3$ and any σ in $\{+,-\}^3$, as

$$\mathcal{K}_n^\sigma \stackrel{\text{def}}{=} \left\{ k \in \mathbf{Z}^3 \ / \ \sigma_1 \frac{k_3}{|k|} + \sigma_2 \frac{n_3 - k_3}{|n-k|} - \sigma_3 \frac{n_3}{|n|} = 0 \right\}. \qquad (2.5.6)$$

Proof of Proposition 2.16. We shall write the proof for $a = b$ to simplify. We can write

$$-\mathcal{F}\mathcal{Q}_\varepsilon(a,a)(n)(t) = \sum_{\substack{(k,m)\in\mathbf{Z}^6,\sigma\in\{+,-\}^3\\ k+m=n}} e^{-i\frac{t}{\varepsilon}\left(\sigma_1\frac{k_3}{|k|}+\sigma_2\frac{m_3}{|m|}-\sigma_3\frac{n_3}{|n|}\right)}[a^{\sigma_1}(k)\cdot m]\,[a^{\sigma_2}(m)\cdot e^{\sigma_3}(n)]e^{\sigma_3}(n).$$

To find the limit of that expression in the sense of distributions as ε goes to zero, one integrates it against a smooth function $\varphi(t)$. That can be seen as the Fourier transform of φ at the point $\dfrac{1}{\varepsilon}\left(\sigma_1\dfrac{k_3}{|k|}+\sigma_2\dfrac{m_3}{|m|}-\sigma_3\dfrac{n_3}{|n|}\right)$, which clearly goes to zero as ε goes to zero, if $\sigma_1\dfrac{k_3}{|k|}+\sigma_2\dfrac{m_3}{|m|}-\sigma_3\dfrac{n_3}{|n|}$ is not zero. That is also known as the non stationary phase theorem. In particular defining, for any $(n,\sigma)\in\mathbf{Z}^3\setminus\{0\}\times\{+,-\}^3$,

$$\omega_n^\sigma(k) \stackrel{\rm def}{=} \sigma_1\frac{k_3}{|k|}+\sigma_2\frac{n_3-k_3}{|n-k|}-\sigma_3\frac{n_3}{|n|},$$

we get

$$-\mathcal{F}\mathcal{Q}(a,a)(n) = \sum_{\sigma\in\{+,-\}^3}\sum_{\substack{k\in\mathbf{Z}^3\\ \omega_n^\sigma(k)=0}}[a^{\sigma_1}(k)\cdot(n-k)][a^{\sigma_2}(n-k)\cdot e^{\sigma_3}(n)]e^{\sigma_3}(n),$$

and Proposition 2.16 is proved. □

So the limit system is the following:

$$(RFL) \qquad \begin{cases} \partial_t u - \Delta u - \mathcal{Q}(u,u) &= 0\\ u_{|t=0} &= u^0, \end{cases}$$

and we have proved the following theorem.

Theorem 2.8 *Let u^0 be a divergence free vector field in $L^2(\mathbf{T}^3)$, and let $(u_\varepsilon)_{\varepsilon>0}$ be a family of weak solutions to (RF_ε). Then as ε goes to zero, the weak closure of $(\mathcal{L}(-\frac{t}{\varepsilon})u_\varepsilon)_{\varepsilon>0}$ is included in the set of weak solutions of (RFL).*

Now let us concentrate on the quadratic form $\mathcal{Q}$: we are going to see that it has particular properties which make it very similar to the two dimensional product arising in the 2D incompressible Navier–Stokes equations. We state the following fundamental result without proof — its proof requires a careful analysis of the resonances in the nonlinear term, and is based on the fact that if the frequencies $n\in\mathbf{Z}^3, k_1\in\mathbf{Z}$ and $k_2\in\mathbf{Z}$ are fixed, then there is a finite number of k_3 satisfying the resonance condition (2.5.6), contrary to a classical product with no such condition, where the number of k_3 is infinite.

Proposition 2.17 *The quadratic form $\mathcal{Q}$ given in Proposition 2.16 satisfies the following properties.*

1) *For any smooth divergence free vector field h, we have*

$$-\int_{\mathbf{T}}\mathcal{Q}(h,h)\,dx_3 = \mathbf{P}(\overline{h}\cdot\nabla\overline{h}).$$

2) *If u, v and w are three divergence free vector fields, then*

$$\forall s \geq 0, \quad \big(\mathcal{Q}(\overline{u}, v_{osc}) \big| (-\Delta)^s v_{osc}\big)_{L^2(\mathbf{T}^3)} = 0, \quad \text{and}$$

$$\Big| \big(\mathcal{Q}(u_{osc}, v_{osc}) \big| w_{osc}\big)_{H^{\frac{1}{2}}} \Big| \leq C \Big(\|u_{osc}\|_{H^{\frac{1}{2}}} \|v_{osc}\|_{H^1} + \|v_{osc}\|_{H^{\frac{1}{2}}} \|u_{osc}\|_{H^1} \Big) \|w_{osc}\|_{H^{\frac{3}{2}}}.$$

Remark 2.18 *1) The first result of Proposition 2.17 is no surprise if one recalls Proposition 2.10: we saw indeed in Section 2.4.2 that the vertical average of the non linear term at the limit can only involve interactions between two-dimensional vector fields.*

2) The second result of Proposition 2.17 is a typical two dimensional product rule, although the setting here is three dimensional. The estimate means indeed that one gains half a derivative when one takes into account the special structure of the quadratic form $\mathcal{Q}$ compared with a usual product.

Notice that the limit system (RFL) can be split into two parts: indeed if u solves (RFL) then one can decompose u into $u = \overline{u} + u_{osc}$, where $\overline{u}$ satisfies the two dimensional Navier–Stokes equation

$$(NS2D) \quad \begin{cases} \partial_t \overline{u} - \Delta_h \overline{u} + \mathbf{P}_h(\overline{u}_h \cdot \nabla_h \overline{u}) &= \overline{f} \\ \overline{u}_{|t=0} &= \overline{u}^0, \end{cases}$$

where $\mathbf{P}_h$ denotes the two dimensional Leray projector onto two dimensional divergence free vector fields, and u_{osc} satisfies the coupled system

$$\begin{cases} \partial_t u_{osc} - \Delta u_{osc} - \mathcal{Q}(2\overline{u} + u_{osc}, u_{osc}) &= f_{osc} \\ u_{osc|t=0} &= u^0_{osc}. \end{cases}$$

Of course here $u^0 = \overline{u}^0 + u^0_{osc}$ where $\overline{u}^0$ the vertical mean of u^0, and similarly $f = \overline{f} + f_{osc}$, where $\overline{f}$ is the vertical mean of f. Using the proposition stated above on the limit $\mathcal{Q}$, it is not difficult to prove the following global wellposedness result.

Proposition 2.19 *Let u^0 be a divergence free vector fields in $H^{\frac{1}{2}}(\mathbf{T}^3)$. Let us consider also an external force f in $L^2(\mathbf{R}^+; H^{-\frac{1}{2}}(\mathbf{T}^3))$. Then there exists a unique global solution u to the system (RFL), in $C_b(\mathbf{R}^+; H^{\frac{1}{2}}(\mathbf{T}^3)) \cap L^2(\mathbf{R}^+; H^{\frac{3}{2}}(\mathbf{T}^3))$.*

Let us now give the strategy to describe the asymptotic behaviour of u_ε. It is natural to write an asymptotic expansion of u_ε as $\mathcal{L}(\frac{t}{\varepsilon})u + \varepsilon U^1 + \dots$ and to identify the powers of ε after plugging that expansion into the equation. Unfortunately a few drawbacks appear instantly. First such a method is regularity-consuming, since the equation involves derivatives and nonlinear terms, so one needs to start by smoothing out u. That is possible because of the special properties of $\mathcal{Q}$ pointed out above, which in particular imply the stability of the limit system. More precisely one can prove that if u_N converges towards u (in our case u_N will be spectrally supported in a ball of radius N), then $\mathcal{Q}(u_N, u_N)$ converges towards $\mathcal{Q}(u, u)$, in appropriate function spaces. The next difficulty, more serious than the previous one, is that the quadratic form $\mathcal{Q}$ is only a weak limit of the original quadratic form. So it is not clear that the next term in the expansion, εU^1, does indeed exist (in other words it is not clear that the convergence of $u_\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})u$ to zero is strong, and is even a $O(\varepsilon)$). But the difference between $\mathcal{Q}$ and the original quadratic form is oscillatory in time, and U^1 will roughly correspond to a time integral of that difference (which again can be defined because the frequencies of u have been restricted to a fixed ball; only in the very end will we let N go to infinity).

To make this sketch more precise, we are going to construct the smooth, approximate family $(u_{app})_{\varepsilon,\eta}$, and prove Lemma 2.14 using the previous results. In particular we will then only be dealing with smooth functions. As we proceed in the construction we will in fact also show that $(u_{app})_{\varepsilon,\eta}$ is a (strong) approximation of the limit solution $\mathcal{L}(\frac{t}{\varepsilon})u$, by writing an Ansatz of the type sketched above, and identifying the powers of ε in the equation. In doing so we will prove the following theorem, which gives of course Lemma 2.14, and Theorem 2.5 due to Remark 2.15.

Theorem 2.9 *Let u^0 be a divergence free vector field in $H^{\frac{1}{2}}$ and let u be the unique, global solution of the limit system (RFL) associated with u^0 constructed in Proposition 2.19 (with $f = 0$). For any positive real number η, a family $(u_{app})_{\varepsilon,\eta}$ exists such that*

$$\lim_{\eta\to 0}\limsup_{\varepsilon\to 0}\|u_{app} - \mathcal{L}(\frac{t}{\varepsilon})u\|_{H^{\frac{1}{2}}} = 0.$$

Moreover, the family $(u_{app})_{\varepsilon,\eta}$ satisfies the conclusions of Lemma 2.14.

Proof of Theorem 2.9. Let η be an arbitrary positive number. We define, for any positive integer N,

$$u_N = \mathbf{P}_N u \stackrel{\text{def}}{=} \mathcal{F}^{-1}\left(\mathbf{1}_{|n|\le N}\widehat{u}(n)\right),$$

and obviously there is $N_\eta > 0$ such that

$$\|\mathcal{L}(\frac{t}{\varepsilon})(u_{N_\eta} - u)\|_{H^{\frac{1}{2}}} \le \rho_{\varepsilon,\eta},$$

where $\rho_{\varepsilon,\eta}$ denotes from now on any non negative quantity such that

$$\lim_{\eta\to 0}\limsup_{\varepsilon\to 0}\rho_{\varepsilon,\eta} = 0.$$

We will also denote generically by $R_{\varepsilon,\eta}$ any vector field satisfying

$$\|R_{\varepsilon,\eta}\|_{L^2(\mathbf{R}^+;H^{-\frac{1}{2}})} = \rho_{\varepsilon,\eta}.$$

We can, from now on, concentrate on u_{N_η}, and our goal is to approximate $\mathcal{L}(\frac{t}{\varepsilon})u_{N_\eta}$ in such a way as to satisfy system (2.5.4). So let us write

$$u_{app} = \mathcal{L}(\frac{t}{\varepsilon})u_{N_\eta} + \varepsilon U^1$$

where U^1 is a smooth, divergence free vector field to be determined. To simplify we also define

$$U^0 \stackrel{\text{def}}{=} \mathcal{L}(\frac{t}{\varepsilon})u_{N_\eta},$$

as well as the operator

$$L_\varepsilon w \stackrel{\text{def}}{=} \partial_t w - \Delta w + \frac{1}{\varepsilon} w^\perp.$$

Then we have

$$L_\varepsilon u_{app} + u_{app} \cdot \nabla u_{app} = L_\varepsilon U^0 + \varepsilon L_\varepsilon U^1 + u_{app} \cdot \nabla u_{app}, \tag{2.5.7}$$

and the only point left to prove is that there is a smooth, divergence free vector field U^1 such that the right-hand side of (2.5.7) is a remainder term. We notice that by definition of U^0,

$$\begin{aligned} \mathbf{P} L_\varepsilon U^0 &= \mathbf{P}(\partial_t U^0 - \Delta U^0 + \frac{1}{\varepsilon}(U^0)^\perp) \\ &= \mathcal{L}(\frac{t}{\varepsilon})(\partial_t u_{N_\eta} - \Delta u_{N_\eta}) + \frac{1}{\varepsilon}\partial_\tau \mathcal{L}(\frac{t}{\varepsilon}) u_{N_\eta} + \frac{1}{\varepsilon}\mathbf{P}\left(\mathcal{L}(\frac{t}{\varepsilon}) u_{N_\eta}\right)^\perp \\ &= \mathcal{L}(\frac{t}{\varepsilon}) \mathbf{P}_{N_\eta} \mathcal{Q}(u,u). \end{aligned}$$

But it is easy to prove (using the special form of $\mathcal{Q}$) that

$$\|\mathcal{L}(\frac{t}{\varepsilon}) \mathbf{P}_{N_\eta} \mathcal{Q}(u,u) - \mathcal{L}(\frac{t}{\varepsilon}) \mathcal{Q}(u_{N_\eta}, u_{N_\eta})\|_{L^2(\mathbf{R}^+; H^{-\frac{1}{2}}(\mathbf{T}^3))} \leq \rho_{\varepsilon,\eta}.$$

We infer that

$$\mathbf{P} L_\varepsilon u_{app} + \mathbf{P}(u_{app} \cdot \nabla u_{app}) = R_{\varepsilon,\eta} + \mathcal{L}(\frac{t}{\varepsilon}) \mathcal{Q}(u_{N_\eta}, u_{N_\eta}) + \varepsilon \mathbf{P} L_\varepsilon U^1 + \mathbf{P}(u_{app} \cdot \nabla u_{app}).$$

Now we write, by definition of $\mathcal{Q}_\varepsilon$,

$$\mathbf{P}(u_{app} \cdot \nabla u_{app}) = -\mathcal{L}(\frac{t}{\varepsilon}) \mathcal{Q}_\varepsilon(u_{N_\eta}, u_{N_\eta}) + F_{\varepsilon,\eta},$$

where

$$F_{\varepsilon,\eta} \stackrel{\text{def}}{=} -\varepsilon \mathcal{L}(\frac{t}{\varepsilon}) \mathcal{Q}_\varepsilon \left(u_{N_\eta}, \mathcal{L}(-\frac{t}{\varepsilon}) U^1\right) - \varepsilon^2 \mathcal{Q}^\varepsilon \left(\mathcal{L}(-\frac{t}{\varepsilon}) U^1, \mathcal{L}(-\frac{t}{\varepsilon}) U^1\right).$$

Going back to the equation on u_{app} we find that

$$\mathbf{P} L_\varepsilon u_{app} + \mathbf{P}(u_{app} \cdot \nabla u_{app}) = R_{\varepsilon,\eta} + \mathcal{L}(\frac{t}{\varepsilon}) \mathcal{Q}(u_{N_\eta}, u_{N_\eta}) - \mathcal{L}\left(\frac{t}{\varepsilon}\right) \mathcal{Q}_\varepsilon(u_{N_\eta}, u_{N_\eta}) + F_{\varepsilon,\eta} + \varepsilon \mathbf{P} L_\varepsilon U^1.$$

Let us postpone for a while the proof of the following lemma.

Lemma 2.20 *Let $\eta > 0$ be given. There is a family of divergence free vector fields U^1, bounded in $(L^\infty \cap L^1)(\mathbf{R}^+; H^s(\mathbf{T}^3))$ for all $s \geq 0$, such that*

$$\mathcal{L}(\frac{t}{\varepsilon})(\mathcal{Q}_\varepsilon - \mathcal{Q})(u_{N_\eta}, u_{N_\eta}) = \varepsilon \mathbf{P} L_\varepsilon U^1 + R_{\varepsilon,\eta}.$$

Lemma 2.20 implies that

$$\mathbf{P} L_\varepsilon u_{app} + \mathbf{P}(u_{app} \cdot \nabla u_{app}) = R_{\varepsilon,\eta} + F_{\varepsilon,\eta}$$

and the only point left to check is that $F_{\varepsilon,\eta}$ is a remainder term. But that is obvious due to the smoothness of U^1 and u_{N_η}. So the theorem is proved, up to the proof of Lemma 2.20. □

Proof of Lemma 2.20. We start by noticing that by definition,

$$(\mathcal{Q}_\varepsilon - \mathcal{Q})(u_{N_\eta}, u_{N_\eta}) = -\mathcal{F}^{-1} \sum_{\substack{k \notin \mathcal{K}_n^\sigma \\ \sigma \in \{+,-\}^3}} e^{-i\frac{t}{\varepsilon}\omega_n^\sigma(k)} \mathbf{1}_{|k| \leq N_\eta} \mathbf{1}_{|n-k| \leq N_\eta} \times [u^{\sigma_1}(k) \cdot (n-k)][u^{\sigma_2}(n-k) \cdot e^{\sigma_3}(n)] e^{\sigma_3}(n).$$

The frequency truncation implies that $|\omega_n^\sigma(k)|$ is bounded from below, by a constant depending on η. That enables us to define

$$\widetilde{U}^1 \stackrel{\text{def}}{=} \mathcal{F}^{-1} \sum_{\substack{k \notin \mathcal{K}_n^\sigma \\ \sigma \in \{+,-\}^3}} \frac{e^{-i\frac{t}{\varepsilon}\omega_n^\sigma(k)}}{i\omega_n^\sigma(k)} \mathbf{1}_{|k| \leq N_\eta} \mathbf{1}_{|n-k| \leq N_\eta} [u^{\sigma_1}(k) \cdot (n-k)][u^{\sigma_2}(n-k) \cdot e^{\sigma_3}(n)] e^{\sigma_3}(n),$$

and $U^1 \stackrel{\text{def}}{=} \mathcal{L}\Big(\frac{t}{\varepsilon}\Big)\widetilde{U}^1$. Then

$$\varepsilon \partial_t \widetilde{U}^1 = (\mathcal{Q} - \mathcal{Q}_\varepsilon)(u_{N_\eta}, u_{N_\eta}) + \varepsilon R_t,$$

where R_t is the inverse Fourier transform of

$$\sum_{\substack{k \notin \mathcal{K}_n^\sigma \\ \sigma \in \{+,-\}^3}} \frac{e^{-i\frac{t}{\varepsilon}\omega_n^\sigma(k)}}{i\omega_n^\sigma(k)} \mathbf{1}_{|k| \leq N_\eta} \mathbf{1}_{|n-k| \leq N_\eta} [\partial_t u^{\sigma_1}(k) \cdot (n-k)][u^{\sigma_2}(n-k) \cdot e^{\sigma_3}(n)] e^{\sigma_3}(n).$$

Notice that $\varepsilon \widetilde{U}^1$ is defined as the primitive in time of the oscillating term $\mathcal{Q}_\varepsilon - \mathcal{Q}$, as explained in the sketch of proof above, and it is precisely the time oscillations that imply that $\widetilde{U}^1$ is uniformly bounded in ε. We therefore have

$$\begin{aligned} \varepsilon \partial_t U^1 &= \varepsilon \mathcal{L}(\frac{t}{\varepsilon}) \partial_t \widetilde{U}^1 + \partial_\tau \mathcal{L}(\frac{t}{\varepsilon}) \widetilde{U}^1 \\ &= \mathcal{L}(\frac{t}{\varepsilon})(\mathcal{Q} - \mathcal{Q}_\varepsilon)(u_{N_\eta}, u_{N_\eta}) + \varepsilon \mathcal{L}(\frac{t}{\varepsilon}) R_t - \mathbf{P}\left(\mathcal{L}(\frac{t}{\varepsilon})\widetilde{U}^1\right)^\perp, \end{aligned}$$

so finally

$$\varepsilon \partial_t U^1 + \mathbf{P}\left(U^1\right)^\perp = \mathcal{L}(\frac{t}{\varepsilon})(\mathcal{Q} - \mathcal{Q}_\varepsilon)(u_{N_\eta}, u_{N_\eta}) + \varepsilon \mathcal{L}(\frac{t}{\varepsilon}) R_t.$$

Since U^1 is arbitrarily smooth (for a fixed η) and so is R_t, Lemma 2.20 is proved, and so is Theorem 2.5. □

2.5.3 More general boundary conditions

We have presented above two different strong convergence results in the case of a constant rotation, depending on the boundary conditions (whole space or periodic). Those boundary conditions are of course highly unphysical, so it is natural to try to consider now more physical cases. In this short section we will only list a few cases that have been studied in the literature and give references. We will also discuss a few open questions, still in the case of a constant rotation.

The first more general situation was considered by E. Grenier and N. Masmoudi in [32] where they studied the case of a fluid rotating between two horizontal plates, with vanishing Dirichlet boundary conditions. In the case of initial data independent of x_3 (so-called well prepared initial data), they were able to prove the convergence of weak solutions towards a two dimensional vector field satisfying a damped, two dimensional Navier-Stokes system. The damping term is present when the initial viscosity is anisotropic (the vertical viscosity being of the order of ε, or else everything converges strongly towards zero) and is known as the Ekman pumping term (see the Introduction); it is due to the presence of boundary layers which dissipate energy. The general, ill prepared case was first investigated by N. Masmoudi in [44] in the case of periodic horizontal boundary conditions, while the study of both the periodic and the whole space horizontal boundary conditions can be found in [13]: in the case of horizontal variables in $\mathbf{R}^2$, dispersion occurs which gives at the limit the same system as in the well prepared case, whereas in the periodic case, oscillating boundary conditions have to be considered, and the limit system is more complicated (though still damped). One should mention at this point the study of D. Gérard-Varet [26] who considered non smooth boundaries, meaning that the horizontal plates are replaced by rugous plates with a periodic rugosity of size ε. D. Bresch, B. Desjardins and D. Gérard-Varet [7] considered the case of a cylindrical domain, and under a generic assumption on the domain and a spectral assumption on the spectrum of the Coriolis operator they studied the asymptotics of the rotating fluid equations. Note that more recently, C. Bardos, F. Golse, A. Mahalov and B. Nicolaenko were able to prove in [5] that in the case of a cylindrical domain, the spectrum of the Coriolis operator is discrete.

We leave out here the widely open cases concerning yet more general domains, like rotating spheres for instance, and refer to [13] or [53] for a discussion on those subjects.

2.6 References and Remarks

The rotating fluid equations presented in this chapter have been the object of a number of mathematical studies in the past decade. Let us mention the pionneering works of E. Grenier [31] and of A. Babin, A. Mahalov and B. Nicolaenko [2]-[4], who were interested in the wellposedness and the limiting behaviour as ε goes to zero, in the periodic case, using S. Schochet's method [54] presented in Section 2.5.2. The fact that the limiting system (RFL) is globally wellposed is due to [4] and putting together the works [4] and [21] gives Theorems 2.5 and 2.9. The whole space case was studied a little later, mainly in [12], where the dispersive character of the Coriolis operator was pointed out, along with the strong convergence theorems. The compensated compactness result can be found in [24].

It should be finally noted that the study of the asymptotics of rotating fluid equations in the constant case is part of a general program analyzing the asymptotics of hyperbolic, parabolic, or mixed hyperbolic-parabolic equations, penalized by a skew-symmetric operator. One has to mention here the fundamental works of J.-L. Joly, G. Métivier and J. Rauch (among other references one can refer to [33] or [34]) concerning abstract equations, as well as the study of the incompressible limit ([14],[16],[18],[22],[23],[41],[45]), or the gyrokinetic limit ([29])... Note that we have not considered other models where similar methods can be used, like for example the primitive equations (see [10], [11]).

Chapter 3

Taking into account spatial variations at midlatitudes

3.1 Introduction

As noted in the introduction, one cannot reasonably study the movement of the atmosphere or the ocean if one neglects the spatial variations of the Coriolis force. The preceding chapter enabled us to go quite far in the description of the waves generated by a constant coefficient rotation; in this chapter we shall replace that rotation by a variable one. Of course the price to pay is that the analysis can no longer be so precise, and in particular we will have no way in general of describing precisely the waves generated by a variable-coefficient rotation. We will not be considering the most general penalization operators, but with the application to geophysical flows in mind (or to magneto-hydrodynamics), we will suppose that the Coriolis operator is

$$Lu = \mathbf{P}(u \wedge B), \quad \text{where} \quad B = b(x_h)e_3$$

and b is a smooth function, which does not vanish, and which only depends on the horizontal coordinate $x_h = (x_1, x_2)$. We recall that $\mathbf{P}$ denotes the Leray projector onto divergence free vector fields. More assumptions on b will be made as we go along. We will study the system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \dfrac{1}{\varepsilon} u \wedge B + \nabla p = 0 \quad \text{on } \mathbf{R}^+ \times \Omega, \\ \nabla\cdot u = 0 \quad \text{on } \mathbf{R}^+ \times \Omega, \\ u_{|t=0} = u^0 \quad \text{on } \Omega \end{cases} \tag{3.1.1}$$

where $\Omega = \Omega_h \times \Omega_3$, and Ω_h denotes either the whole space $\mathbf{R}^2$ or any periodic domain of $\mathbf{R}^2$, and similarly Ω_3 denotes $\mathbf{R}$ or $\mathbf{T}$. As in the previous chapter, we will address the questions of the uniform existence of weak or strong solutions, and we will study their asymptotic behaviour as ε goes to zero. Considering the generality of the setting, we will not be able to write as precise computations as in the constant case studied in the previous chapter, and in particular the question of the strong convergence will not be raised except for some remarks in Section 3.3.3.

3.2 Statement of the main results

The first question to be addressed, and which is easily dismissed, concerns the existence of uniformly bounded weak solutions. The Coriolis operator here is no longer skew-symmetric in all Sobolev spaces, since it has variable coefficients, nevertheless it still disappears in L^2 energy estimates, and it is therefore easy to prove the following theorem, which we leave as an exercise to the reader.

Theorem 3.1 *Let u^0 be any divergence free vector field in $L^2(\Omega)$. Then for all $\varepsilon > 0$, Equation (3.1.1) has at least one weak solution $u_\varepsilon \in L^\infty(\mathbf{R}^+, L^2) \cap L^2(\mathbf{R}^+, \dot{H}^1)$. Moreover, for all $t > 0$, the following energy estimate holds:*

$$\|u_\varepsilon(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u_\varepsilon(t')\|_{L^2}^2 \, dt' \leq \|u^0\|_{L^2}^2.$$

The existence of strong solutions is a much more intricate problem, since the Coriolis operator no longer disappears, as soon as one takes derivatives of the equation. We will see that it is nevertheless possible to prove the uniform local existence and uniqueness of a solution in H^s (global for small data), using the fact that B does not depend on the third variable. The precise theorem is the following; it is proved in Section 3.4.

Theorem 3.2 *Let $s > 1/2$ be given, and suppose that $B = b(x_h)e_3$ is a smooth, bounded function. Then there is a constant c such that the following result holds. Suppose that u^0 is a divergence free vector field in $H^s(\Omega)$, such that $\|u^0\|_{H^s} \leq c$. Then for all $\varepsilon > 0$, the system (3.1.1) has a unique, global solution u_ε, which is bounded in the space $C_b(\mathbf{R}^+; H^s) \cap L^2(\mathbf{R}^+; H^{s+1})$.*

Moreover if B only depends on x_2, then for any $s > 1/2$ and u^0 in $H^s(\Omega)$ divergence free, there is a time $T > 0$ such that for all $\varepsilon > 0$, the system (3.1.1) has a unique solution u_ε, bounded in the space $C([0,T]; H^s) \cap L^2([0,T]; H^{s+1})$.

Remark 3.1 *A more general theorem can in fact be proved, where the Laplacian in the equation is replaced by Δ_h only. To simplify the presentation we have chosen here to state the less general result although it should be clear from the proof that the diffusion in the vertical variable plays no role in the analysis.*

The next question concerns the asymptotics of the solutions. We will only state results concerning the weak asymptotics of Leray-type solutions. Before stating the theorem, let us make the following additional assumption. We suppose that b has non degenerate critical points in the following sense: denoting by $\mu(X)$ the Lebesgue measure of any set X we suppose that

$$\lim_{\delta \to 0} \mu\left(\{x \in \Omega_h \,/\, |\nabla b(x)| \leq \delta\}\right) = 0. \tag{3.2.1}$$

The convergence theorem is the following. It is proved in Section 3.3 below.

Theorem 3.3 *Suppose that $B = be_3$ where $b = b(x_h)$ is a smooth function which does not vanish, with non degenerate critical points in sense of (3.2.1). Let u^0 be any divergence free vector field in $L^2(\Omega)$, and let u_ε be any weak solution of (3.1.1) in the sense of Theorem 3.1. Then u_ε converges weakly in $L^2_{loc}(\mathbf{R}^+ \times\Omega)$ to a limit u belonging to Ker L. If $\Omega_3 = \mathbf{R}$ then u is identically zero, and if $\Omega_3 = \mathbf{T}$ it is defined as follows: the third component u_3 belongs to $L^\infty(\mathbf{R}^+; L^2) \cap L^2(\mathbf{R}^+; \dot{H}^1)$ satisfies the transport-diffusion equation*

$$\partial_t u_3 - \Delta_h u_3 + u_h \cdot \nabla_h u_3 = 0, \quad \partial_3 u_3 = 0, \quad u_{3|t=0} = \int_{\mathbf{T}} u_3^0(x_h, x_3)\, dx_3 \quad \text{in} \quad \mathbf{R}^+ \times\Omega,$$

while the horizontal component $u_h \in C(\mathbf{R}^+; H^{-1}(\Omega_h)) \cap L^2_{loc}(\mathbf{R}^+; H^1(\Omega_h))$ satisfies the following property: for any vector field $\Phi \in H^1(\Omega_h) \cap \mathrm{Ker}(L)$ and for any time $t > 0$,

$$(u_h(t)|\Phi_h)_{L^2(\Omega_h)} + \int_0^t (\nabla_h u_h(t')|\nabla_h \Phi_h)_{L^2(\Omega_h)}\, dt' = (u_h^0|\Phi_h)_{L^2(\Omega_h)}. \tag{3.2.2}$$

Remark 3.2 *Formally Equation (3.2.2) can be written as a heat equation on* $\mathrm{Ker}(L)$, *as writing* Π *the orthogonal projector in* L^2 *onto* $\mathrm{Ker}(L)$ *the equation formally reads*

$$\partial_t u_h - (\Pi\Delta_h u)_h = 0.$$

That result is surprising as all non linear terms have disappeared in the limiting process. This can be understood as some sort of turbulent behaviour, where all scales are mixed due to the variation of b. Technically the result is due to the fact that the kernel of L is very small as soon as b is not a constant, which induces a lot of rigidity in the limit equation.

The rest of this chapter is devoted to the proof of those results, starting with the weak convergence result in the next section. The proof of that theorem will follow the same lines as the proof of Theorem 2.2 in Chapter 2, with the additional difficulty of course that the rotation vector is no longer homogeneous. The proof of Theorem 3.2 in Section 3.4 will only be sketched, as it involves techniques which do not have much to do with the fast rotation limit but consists in rather subtle anisotropic estimates, and are beyond the scope of this review article.

3.3 Weak asymptotics

In this section we are concerned with the weak asymptotics of the solutions to the rotating fluid equations with a variable Coriolis force, and we will prove Theorem 3.3. Let us start by noticing that as soon as the initial data is in L^2, it generates a bounded family u_ε of solutions to (3.1.1), so up to the extraction of a subsequence there exists $u \in L^\infty(\mathbf{R}^+, L^2) \cap L^2(\mathbf{R}^+, \dot{H}^1)$, such that

$$u_\varepsilon \rightharpoonup u \text{ in } w\text{-}L^2_{loc}(\mathbf{R}^+ \times\Omega) \text{ as } \varepsilon \to 0.$$

As in the constant case studied in the previous chapter, we will prove that u belongs to the kernel $\mathrm{Ker}(L)$ of L, so in the next section we present the operator L and study its main properties (in particular its kernel). The following section is devoted to the end of the proof of Theorem 3.3, using a compensated-compactness argument to deal with the nonlinear terms.

3.3.1 Study of the Coriolis operator

The kernel Ker(L) of L is characterized in the following proposition.

Proposition 3.3 *If u is a divergence free vector field in $L^2(\Omega)$ belonging to* Ker(L), *then u is in $L^2(\Omega_h)$ and satisfies the following properties:*

$$\begin{cases} \operatorname{div}_h u_h = 0 \\ u_h \cdot \nabla_h b = 0 \\ \displaystyle\int_{\Omega_h} u_h \wedge B \, dx_h = 0. \end{cases}$$

Remark 3.4 *1) In the case when $\Omega_3 = \mathbf{R}$, Proposition 3.3 shows that the kernel of L is reduced to zero since $L^2(\Omega_h) \cap L^2(\Omega) = \{0\}$.*

2) In the case when $\Omega_h = \mathbf{T}^2$, the fact that $\operatorname{div}_h u_h = 0$ does not necessarily mean that u_h can be written as $u_h = \nabla_h^\perp \varphi$ for some function φ because the horizontal mean of u_h is not preserved by the equation.

Proof of Proposition 3.3. If u belongs to Ker(L) then we have $\mathbf{P}(u \wedge B) = 0$, so in particular

$$\int_\Omega u_h \wedge B \, dx = 0.$$

Moreover in the sense of distributions, rot $(u \wedge B) = 0$, which can be rewritten

$$(\nabla \cdot B)u + (B \cdot \nabla)u - (u \cdot \nabla)B - (\nabla \cdot u)B = 0.$$

As $\nabla \cdot B = \nabla \cdot u = 0$ and $B = be_3$, we get

$$b\partial_3 u - (u \cdot \nabla) b e_3 = 0. \tag{3.3.1}$$

In particular, $\partial_3 u_1 = \partial_3 u_2 = 0$ from which we deduce that u_h belongs to $L^2(\Omega_h)$. Note that in the case where $\Omega_3 = \mathbf{R}$, the invariance with respect to x_3 and the fact that u belongs to $L^2(\Omega)$ imply that $u_1 = u_2 = 0$ (and therefore $u_3 = 0$ by the divergence free condition).

We suppose from now on that $\Omega_3 = \mathbf{T}$. Differentiating the incompressibility constraint with respect to x_3 leads then to

$$\partial^2_{33} u_3 = -\partial^2_{13} u_1 - \partial^2_{23} u_2 = 0$$

in the sense of distributions. The function $\partial_3 u_3$ depends only on x_1 and x_2, and satisfies $\displaystyle\int \partial_3 u_3 dx_3 = 0$. So $\partial_3 u_3 = 0$, u_3 belongs to $L^2(\Omega_h)$, and $\partial_1 u_1 + \partial_2 u_2 = 0$. Finally by (3.3.1) we get $u_h \cdot \nabla_h b = 0$ and $\displaystyle\int_{\Omega_h} u_h \wedge b \, dx_h = 0$. The proposition is proved. □

Remark 3.5 *Before applying this result to the characterization of the weak limit u, let us just specify it in two important cases. If $\nabla b = 0$ almost everywhere (for instance if the*

Coriolis operator is constant, which corresponds to the case studied in the previous chapter), then $u \in L^2$ is a divergence free vector field in $\mathrm{Ker}(L)$ *if and only if*

$$u = \nabla_h^\perp \varphi + \alpha e_3,$$

for some $\nabla_h \varphi \in L^2(\Omega_h)$ and $\alpha \in L^2(\Omega_h)$. If $\nabla b \neq 0$ almost everywhere, then the condition arising on u is much more restrictive : if $u \in L^2$ is a divergence free vector field in $\mathrm{Ker}(L)$ *then it can be written*

$$u = \frac{u \cdot \nabla^\perp b}{|\nabla^\perp b|^2} \nabla^\perp b + \alpha e_3$$

for some $\alpha \in L^2(\Omega_h)$, with the additional condition that

$$\mathrm{div}_h \left(\frac{u_h \cdot \nabla^\perp b}{|\nabla^\perp b|^2} \nabla^\perp b \right) = 0 \quad \text{and} \quad \int b \frac{u_h \cdot \nabla^\perp b}{|\nabla^\perp b|^2} \nabla^\perp b \, dx = 0.$$

Using this characterization of $\mathrm{Ker}(L)$, we deduce some constraints on the weak limit u. The proof of the following result is exactly the same as in the constant case (Proposition 2.5 page 43), so we leave it to the reader.

Proposition 3.6 *Let u^0 be any divergence free vector field in $L^2(\Omega)$. Denote by $(u_\varepsilon)_{\varepsilon>0}$ a family of weak solutions of (3.1.1), and by u any of its limit points. Then*

$$u \in L^\infty(\mathbf{R}^+; L^2(\Omega_h)) \cap L^2(\mathbf{R}^+; \dot{H}^1(\Omega_h))$$

and satisfies the following properties:

$$\begin{cases} \mathrm{div}_h \, u_h = 0 \\ u_h \cdot \nabla_h b = 0 \\ \displaystyle\int_{\Omega_h} u_h \wedge B \, dx_h = 0. \end{cases}$$

3.3.2 Proof of the weak convergence theorem

In this section we shall prove Theorem 3.3. If $\Omega_3 = \mathbf{R}$, then $u = 0$ due to Remark 3.4, so from now on we can suppose that $\Omega_3 = \mathbf{T}$. The strategy of the proof is quite similar to the constant case: we have first to give a precise description of the different oscillating modes, and then to prove that these oscillations do not occur in the limiting equation. Finally we need to show that the limiting equation is in fact linear.

As in the constant case, vertical modes generate fast oscillations in the system, meaning that the whole part of the velocity field corresponding to Fourier modes with $k_3 \neq 0$ converges weakly to zero. The corresponding vertical oscillations depend directly on the order of magnitude of b. The main difference comes then from the fact that, in the case of a heterogeneous rotation, the kernel of the penalization is much smaller: restricting our attention to the horizontal modes ($k_3 = 0$), we see that the Coriolis term penalizes all the fields which are parallel to ∇b, which implies in particular that the vertical average of the horizontal velocity is no longer strongly compact. The corresponding two-dimensional oscillations are then governed by ∇b, and possibly become singular if ∇b cancels.

In the following we will therefore only be able to prove that the vertical average of the vertical velocity is strongly compact, and the use of that information alone, coupled with some compensated compactness argument, will enable us to establish the equation satisfied by the weak limit of the velocity field.

Proposition 3.7 *Let u^0 be a divergence free vector field in $L^2(\Omega)$. For all $\varepsilon > 0$, denote by u_ε a weak solution of (3.1.1) and by $\overline{u}_\varepsilon \stackrel{\text{def}}{=} \int u_\varepsilon dx_3$. Then, for all $T > 0$, $(\overline{u}_{\varepsilon,3})_{\varepsilon>0}$ is strongly compact in $L^2([0,T] \times \Omega)$.*

Proof of Proposition 3.7. The computation is similar to the constant case studied in the previous chapter (Proposition 2.8 page 44), only for the fact that one must restrict one's attention to the vertical component only. By the energy estimate, u_ε and consequently $\overline{u}_\varepsilon$ are uniformly bounded in $L^2([0,T];H^1)$. Integrating with respect to x_3 the vertical component of the penalized Navier-Stokes equation leads to

$$\partial_t \overline{u}_{\varepsilon,3} + \int \nabla \cdot (u_\varepsilon u_{\varepsilon,3}) dx_3 - \Delta_h \overline{u}_{\varepsilon,3} = 0,$$

from which we deduce that $\partial_t \overline{u}_{\varepsilon,3}$ is uniformly bounded in $L^2([0,T], H^{-3/2}(\Omega))$, and the result follows by Aubin's lemma. $\square$

Now let us describe the oscillations.

Lemma 3.8 *Let u^0 be a divergence free vector field in $L^2(\Omega)$. For all $\varepsilon > 0$, denote by u_ε a weak solution of (3.1.1), by $\overline{u}_\varepsilon = \int u_\varepsilon dx_3$ and by $\widetilde{u}_\varepsilon = u_\varepsilon - \overline{u}_\varepsilon$. Define*

$$\omega_\varepsilon = \partial_1 u_{\varepsilon,2} - \partial_2 u_{\varepsilon,1}, \quad \overline{\omega}_\varepsilon = \int_{\mathbf{T}} \omega_\varepsilon \, dx_3, \quad \widetilde{\omega}_\varepsilon = \omega_\varepsilon - \overline{\omega}_\varepsilon,$$
$$\text{and} \quad \partial_3 \widetilde{\Omega}_{\varepsilon,h} = \nabla_h^\perp \widetilde{u}_{\varepsilon,3} - \partial_3 \widetilde{u}_{\varepsilon,h}^\perp, \quad \text{with} \quad \int_{\mathbf{T}} \widetilde{\Omega}_{\varepsilon,h} \, dx_3 = 0.$$

Then, regularizing by a kernel κ_δ as in (2.4.2), we get the following description of the oscillations

$$\begin{aligned} \varepsilon \partial_t \overline{\omega}_\varepsilon^\delta - \overline{u}_{\varepsilon,h}^\delta \cdot \nabla b &= -\varepsilon r_\varepsilon^\delta - s_\varepsilon^\delta \\ \nabla_h \cdot \overline{u}_{\varepsilon,h}^\delta &= 0 \\ \varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta + b \widetilde{u}_{\varepsilon,h}^\delta &= -\varepsilon r_\varepsilon^\delta - s_\varepsilon^\delta \\ \varepsilon \partial_t \widetilde{\omega}_\varepsilon^\delta - \nabla \cdot (b \widetilde{u}_{\varepsilon,h}^\delta) &= -\varepsilon r_\varepsilon^\delta - s_\varepsilon^\delta \end{aligned} \tag{3.3.2}$$

denoting generically by r_ε^δ and s_ε^δ some quantities satisfying

$$\forall \delta > 0, \quad \forall T > 0, \quad \sup_\varepsilon \| r_\varepsilon^\delta \|_{L^2([0,T]\times\Omega)} < +\infty$$

$$\text{and} \quad \forall T > 0, \quad \sup_{\varepsilon,\delta} \delta^{-1} \| s_\varepsilon^\delta \|_{L^2([0,T]\times\Omega)} < +\infty.$$

Proof of Lemma 3.8. Denote, as in the previous chapter, by F_ε the flux term

$$F_\varepsilon = -\nabla \cdot (u_\varepsilon \otimes u_\varepsilon) + \Delta u_\varepsilon.$$

The energy inequality and standard bilinear estimates yield that F_ε is uniformly bounded in $L^2([0,T], H^{-3/2}(\Omega))$. Using that notation, (3.1.1) can be simply rewritten

$$\begin{aligned} \varepsilon\partial_t \overline{u}_\varepsilon + \overline{u}_\varepsilon \wedge B + \nabla_h \overline{p}_\varepsilon &= \varepsilon \overline{F}_\varepsilon, \\ \nabla_h \cdot \overline{u}_{\varepsilon,h} &= 0, \\ \varepsilon\partial_t \widetilde{u}_\varepsilon + \widetilde{u}_\varepsilon \wedge B + \nabla \widetilde{p}_\varepsilon &= \varepsilon \widetilde{F}_\varepsilon, \\ \nabla \cdot \widetilde{u}_\varepsilon &= 0, \end{aligned}$$

splitting the purely 2D modes ($k_3 = 0$) and the vertical Fourier modes ($k_3 \neq 0$). Using the vorticity formulation for the horizontal component of $\overline{u}_\varepsilon$, we get

$$\begin{aligned} \varepsilon\partial_t \overline{\omega}_\varepsilon - \overline{u}_{\varepsilon,h} \cdot \nabla b &= -\varepsilon \nabla_h^\perp \cdot \overline{F}_{\varepsilon,h}, \\ \nabla_h \cdot \overline{u}_{\varepsilon,h} &= 0. \end{aligned} \tag{3.3.3}$$

Then taking the rotational of the other part of the equation yields

$$\varepsilon\partial_t \nabla \wedge \widetilde{u}_\varepsilon + \nabla \wedge (\widetilde{u}_\varepsilon \wedge B) = \varepsilon \nabla \wedge \widetilde{F}_\varepsilon$$

and integrating the horizontal component with respect to x_3 leads to

$$\begin{aligned} \varepsilon\partial_t \widetilde{\Omega}_{\varepsilon,h} + b\widetilde{u}_{\varepsilon,h} &= \varepsilon(\nabla \wedge \widetilde{G}_\varepsilon)_h, \\ \varepsilon\partial_t \widetilde{\omega}_\varepsilon - \nabla_h \cdot (\widetilde{u}_{\varepsilon,h} b) &= -\varepsilon \nabla_h^\perp \cdot \widetilde{F}_{\varepsilon,h} \end{aligned} \tag{3.3.4}$$

where $\widetilde{G}_\varepsilon$ is just defined by $\partial_3 \widetilde{G}_\varepsilon = \widetilde{F}_\varepsilon$ and $\displaystyle\int_{\mathbf{T}} \widetilde{G}_\varepsilon dx_3 = 0$, and thus satisfies the same uniform estimates as $\widetilde{F}_\varepsilon$.

The second step of the proof consists then in regularizing the previous wave equations (3.3.3) and (3.3.4). We therefore introduce, as in the previous chapter, a smoothing family κ_δ defined by $\kappa_\delta(x) = \delta^{-3}\kappa(\delta^{-1}x)$ where κ is a function of $C_c^\infty(\mathbf{R}^3, \mathbf{R}^+)$ such that $\kappa(x) = 0$ if $|x| \geq 1$ and $\int \kappa dx = 1$. By convolution, we then obtain

$$\begin{aligned} \varepsilon\partial_t \overline{\omega}_\varepsilon^\delta - \overline{u}_{\varepsilon,h}^\delta \cdot \nabla b = -\varepsilon \nabla_h^\perp \cdot \overline{F}_{\varepsilon,h}^\delta - \overline{u}_{\varepsilon,h}^\delta \cdot \nabla b + (\overline{u}_{\varepsilon,h} \cdot \nabla b)^\delta, \\ \nabla_h \cdot \overline{u}_{\varepsilon,h}^\delta = 0, \end{aligned}$$

and

$$\begin{aligned} \varepsilon\partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta + b\widetilde{u}_{\varepsilon,h}^\delta &= \varepsilon(\nabla \wedge \widetilde{G}_\varepsilon^\delta)_h + b\widetilde{u}_{\varepsilon,h}^\delta - (b\widetilde{u}_{\varepsilon,h})^\delta, \\ \varepsilon\partial_t \widetilde{\omega}_\varepsilon^\delta + \nabla_h \cdot (\widetilde{u}_{\varepsilon,h}^\delta b) &= -\varepsilon \nabla_h^\perp \cdot \tilde{F}_{\varepsilon,h}^\delta + \nabla_h \cdot (\widetilde{u}_{\varepsilon,h}^\delta b) - \nabla_h \cdot (\widetilde{u}_{\varepsilon,h} b)^\delta. \end{aligned}$$

It remains only to check that the source terms satisfy the convenient a priori estimates. It is easy to see that

$$\|\kappa_\delta\|_{W^{5/2,1}(\mathbf{R}^3)} \leq \delta^{-5/2} \|\kappa\|_{L^1(\mathbf{R}^3)},$$

so the terms generically called r_ε^δ satisfy a uniform bound for any fixed δ :

$\nabla_h^\perp \cdot \overline{F}_{\varepsilon,h}^\delta$, $(\nabla \wedge \widetilde{G}_\varepsilon^\delta)_h$ and $\nabla_h^\perp \cdot \widetilde{F}_{\varepsilon,h}^\delta$ are uniformly bounded in $L^2([0,T] \times \Omega)$ (of order $\delta^{-5/2}$),

since $\nabla_h^\perp \cdot \overline{F}_{\varepsilon,h}, (\nabla \wedge \widetilde{G}_\varepsilon)_h$ and $\nabla_h^\perp \cdot \widetilde{F}_{\varepsilon,h}$ are bounded in $L^2([0,T]; H^{-5/2}(\Omega))$. We then have to estimate quantities of the form $u_\varepsilon^\delta \psi - (u_\varepsilon \psi)^\delta$ for smooth functions ψ. We have

$$\begin{aligned} |u_\varepsilon^\delta \psi(x) - (u_\varepsilon \psi)^\delta(x)| &= \left| \int \kappa_\delta(y) u_\varepsilon(x-y)(\psi(x) - \psi(x-y)) dy \right| \\ &\leq \delta \|\nabla \psi\|_{L^\infty(\Omega)} (\kappa^\delta * |u_\varepsilon|)(x), \end{aligned}$$

so in particular,

$$\begin{aligned} \|\overline{u}_{\varepsilon,h}^\delta \cdot \nabla b - (\overline{u}_{\varepsilon,h} \cdot \nabla b)^\delta\|_{L^2([0,T]\times\Omega)} &\leq \delta \|D^2 b\|_{L^\infty(\Omega)} \|u_\varepsilon\|_{L^2([0,T]\times\Omega)} \\ \|b\widetilde{u}_{\varepsilon,h}^\delta - (b\widetilde{u}_{\varepsilon,h})^\delta\|_{L^2([0,T]\times\Omega)} &\leq \delta \|Db\|_{L^\infty(\Omega)} \|u_\varepsilon\|_{L^2([0,T]\times\Omega)} \\ \|\nabla_h \cdot (\widetilde{u}_{\varepsilon,h}^\delta b) - \nabla_h \cdot (\widetilde{u}_{\varepsilon,h} b)^\delta\|_{L^2([0,T]\times\Omega)} &\leq \delta (\|Db\|_{L^\infty(\Omega)} + \|D^2 b\|_{L^\infty(\Omega)}) \|u_\varepsilon\|_{L^2([0,T],H^1(\Omega))} \end{aligned}$$

meaning that the terms generically called s_ε^δ converge to 0 as $\delta \to 0$ uniformly in ε, according to the bound

$$\forall T > 0, \quad \sup_{\varepsilon,\delta} \delta^{-1} \|s_\varepsilon^\delta\|_{L^2([0,T]\times\Omega)} < +\infty.$$

Lemma 3.8 is proved. □

Now let us compute the coupling term. As remarked in the introduction of this paragraph, the fact that ∇b can get very small could lead to a defect of compactness of vertical averages. The non degeneracy assumption (3.2.1) will enable us to deal with regions of space where ∇b is small, simply using a cut-off function. Let us state the result.

Proposition 3.9 *Let u^0 be any divergence free vector field in $L^2(\Omega)$. For all $\varepsilon > 0$, denote by u_ε a weak solution of (3.1.1). Define the truncation χ_δ by*

$$\chi_\delta(x) = \chi(\delta^{-1/4} \nabla b(x))$$

where χ is a function of $C_c^\infty(\mathbf{R}^3, \mathbf{R}^+)$ such that $\chi(x) = 1$ if $|x| \leq 1$. Then, with the same notation as in Lemma 3.8, the averaged nonlinear term in (3.1.1) can be rewritten

$$\begin{aligned} &\int \left(\nabla \cdot (u_\varepsilon^\delta \otimes u_\varepsilon^\delta) - \nabla \frac{|u_\varepsilon^\delta|^2}{2} \right) dx_3 \\ =& \nabla_h \cdot (\overline{u}_{\varepsilon,h}^\delta \overline{u}_{\varepsilon,3}^\delta) e_3 - \nabla \frac{|\overline{u}_{\varepsilon,3}^\delta|^2}{2} + \varepsilon \rho_\varepsilon^\delta + \sigma_\varepsilon^\delta \\ &- \frac{\varepsilon}{2} \partial_t |\overline{\omega}_\varepsilon^\delta|^2 (1-\chi_\delta) \frac{\nabla^\perp b}{|\nabla b|^2} - (1-\chi_\delta)(\overline{u}_{\varepsilon,h}^\delta \cdot \nabla^\perp b) \frac{\nabla b}{|\nabla b|^2} \overline{\omega}_\varepsilon^\delta \\ &+ \frac{\varepsilon}{b} \partial_t \int \widetilde{\omega}_\varepsilon^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3 + \frac{\varepsilon}{2b^2} \partial_t \int (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla b)^2 dx_3 \frac{\nabla^\perp b}{|\nabla b|^2} \\ &- \frac{1}{b^2} \int (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla^\perp b)(\varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla b) dx_3 \frac{\nabla b}{|\nabla b|^2} + \frac{\varepsilon}{2b} \partial_t \int (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp) dx_3 e_3 \end{aligned}$$

where ρ_ε^δ and $\sigma_\varepsilon^\delta$ are quantities satisfying the following estimates

$$\forall\delta>0,\quad \forall T>0,\quad \sup_{\varepsilon\to 0}\|\rho_\varepsilon^\delta\|_{L^1([0,T];L^{6/5}\Omega))}<+\infty,$$

$$\textit{and}\quad \forall T>0,\quad \lim_{\delta\to 0}\sup_{\varepsilon}\|\sigma_\varepsilon^\delta\|_{L^1([0,T];L^{6/5}\Omega))}=0.$$

Proof of Proposition 3.9. Let us first remark that

$$\int \nabla\cdot(u_\varepsilon^\delta\otimes u_\varepsilon^\delta)dx_3=\nabla\cdot(\overline{u}_\varepsilon^\delta\otimes\overline{u}_\varepsilon^\delta)+\int\nabla\cdot(\widetilde{u}_\varepsilon^\delta\otimes\widetilde{u}_\varepsilon^\delta)dx_3$$

which allows us to consider separately purely 2D modes and vertical modes. As in the constant case, due to (2.4.3) we can in fact further restrict our attention to the quantities $-\overline{u}_\varepsilon^\delta\wedge(\nabla\wedge\overline{u}_\varepsilon^\delta)$ and $-\int\widetilde{u}_\varepsilon^\delta\wedge(\nabla\wedge\widetilde{u}_\varepsilon^\delta)dx_3$. We will finally simplify the computations by neglecting all the remainder terms in Lemma 3.8, and leave the precise computations to the reader, as in the constant case in Chapter 2.

(i) We start with the study of the purely 2D modes. A simple computation leads to

$$\begin{aligned}-\overline{u}_\varepsilon^\delta\wedge(\nabla\wedge\overline{u}_\varepsilon^\delta)&=-\overline{u}_\varepsilon^\delta\wedge(\nabla_h^\perp\overline{u}_{\varepsilon,3}^\delta+\overline{\omega}_\varepsilon^\delta e_3)\\&=-\overline{\omega}_\varepsilon^\delta(\overline{u}_{\varepsilon,h}^\delta)^\perp-\nabla_h\frac{|\overline{u}_{\varepsilon,3}^\delta|^2}{2}+\nabla_h\cdot(\overline{u}_{\varepsilon,h}^\delta\overline{u}_{\varepsilon,3}^\delta)e_3.\end{aligned}\tag{3.3.5}$$

We can decompose $\overline{u}_{\varepsilon,h}^\delta$ as follows

$$\overline{u}_{\varepsilon,h}^\delta=(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla b)\frac{\nabla b}{|\nabla b|^2}+(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)\frac{\nabla^\perp b}{|\nabla b|^2}$$

as soon as $\nabla b\neq 0$, and we will actually do so, using the truncation χ, only if $|\nabla b|\geq\delta^{1/4}$.

Using the first identity in (3.3.2), and neglecting remainder terms, we obtain

$$\overline{u}_{\varepsilon,h}^\delta=\varepsilon\partial_t\overline{\omega}_\varepsilon^\delta\frac{\nabla b}{|\nabla b|^2}+(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)\frac{\nabla^\perp b}{|\nabla b|^2}$$

and replacing in (3.3.5) provides finally

$$\begin{aligned}-\overline{u}_\varepsilon^\delta\wedge(\nabla\wedge\overline{u}_\varepsilon^\delta)=&-(1-\chi_\delta)\varepsilon\partial_t\frac{|\overline{\omega}_\varepsilon^\delta|^2}{2}\frac{\nabla^\perp b}{|\nabla b|^2}-(1-\chi_\delta)(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)\frac{\nabla b}{|\nabla b|^2}\overline{\omega}_\varepsilon^\delta\\&-\chi_\delta\overline{\omega}_\varepsilon^\delta(\overline{u}_{\varepsilon,h}^\delta)^\perp-\nabla_h\frac{|\overline{u}_{\varepsilon,3}^\delta|^2}{2}+\nabla_h\cdot(\overline{u}_{\varepsilon,h}^\delta\overline{u}_{\varepsilon,3}^\delta)e_3.\end{aligned}\tag{3.3.6}$$

That concludes the first step of the proof since

$$\begin{aligned}\left\|\chi_\delta\overline{\omega}_\varepsilon^\delta\overline{u}_{\varepsilon,h}^\delta\right\|_{L^1([0,T],L^{6/5}(\Omega))}&\leq\|\chi_\delta\|_{L^6(\Omega)}\|\overline{\omega}_\varepsilon^\delta\|_{L^2([0,T]\times\Omega)}\|\overline{u}_{\varepsilon,h}^\delta\|_{L^2([0,T],L^6(\Omega))}\\&\leq C\left(\mu\{x\in\Omega_h\ /\ |\nabla b(x)|\leq\delta^{\frac14}\}\right)^{\frac16},\end{aligned}$$

which goes to zero with δ according to Assumption (3.2.1), hence can be incorporated in the term $\sigma_\varepsilon^\delta$.

(ii) We have now to deal with the vertical modes. A simple computation leads to

$$\begin{aligned}-\widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta) &= -\widetilde{u}_\varepsilon^\delta \wedge (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta + \widetilde{\omega}_\varepsilon^\delta e_3)\\ &= -\widetilde{\omega}_\varepsilon^\delta (\widetilde{u}_{\varepsilon,h}^\delta)^\perp + \widetilde{u}_{\varepsilon,3}^\delta \partial_3 (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp - (\widetilde{u}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp) e_3\end{aligned}$$

so that using the divergence free condition,

$$-\int \widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta) dx_3 = \int \left(-\widetilde{\omega}_\varepsilon^\delta (\widetilde{u}_{\varepsilon,h}^\delta)^\perp + (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp (\nabla_h \cdot \widetilde{u}_{\varepsilon,h}^\delta) \right) dx_3 - \int (\widetilde{u}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp) dx_3 e_3.$$

In order to determine the horizontal component, we then use the last two identities in (3.3.2)

$$\begin{aligned}-\int (\widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta))_h dx_3 &= \int \widetilde{\omega}_\varepsilon^\delta \frac{1}{b} (\varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3 + \int (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \frac{1}{b} (\varepsilon \partial_t \widetilde{\omega}_\varepsilon^\delta - \widetilde{u}_{\varepsilon,h}^\delta \cdot \nabla b) dx_3\\ &= \frac{\varepsilon}{b} \partial_t \int \widetilde{\omega}_\varepsilon^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3 + \int (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \frac{1}{b^2} \varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla b dx_3.\end{aligned}$$

We can decompose $\widetilde{\Omega}_{\varepsilon,h}^\delta$ as follows

$$\widetilde{\Omega}_{\varepsilon,h}^\delta = (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla b) \frac{\nabla b}{|\nabla b|^2} + (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla^\perp b) \frac{\nabla^\perp b}{|\nabla b|^2}$$

as soon as $\nabla b \neq 0$, that is almost everywhere by assumption. Finally we get

$$\begin{aligned}-\int (\widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta))_h dx_3 =& \frac{\varepsilon}{b} \partial_t \int \widetilde{\omega}_\varepsilon^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3 + \frac{\varepsilon}{2b^2} \partial_t \int (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla b)^2 dx_3 \frac{\nabla^\perp b}{|\nabla b|^2}\\ &- \frac{1}{b^2} \int (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla^\perp b)(\varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla b) dx_3 \frac{\nabla b}{|\nabla b|^2},\end{aligned} \tag{3.3.7}$$

which is the expected formula. In order to determine the vertical component, we use the third identity in (3.3.2) and an integration by parts with respect to x_3, to find

$$\begin{aligned}\int \widetilde{u}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3 =& -\int \frac{1}{b} \varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3\\ =& -\frac{1}{2b} \int \left((\varepsilon \partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta) \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp - (\varepsilon \partial_t \partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta) \cdot (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \right) dx_3,\end{aligned}$$

from which we deduce

$$\int \widetilde{u}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3 = -\frac{\varepsilon}{2b} \partial_t \int (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp) dx_3. \tag{3.3.8}$$

Combining (3.3.6), (3.3.7) and (3.3.8) gives finally the proper decomposition of the averaged nonlinear term. Proposition 3.9 is proved (up to the computation of the remainder terms). □

Now we are ready to take the limit.

Proposition 3.10 *Let u^0 be any divergence free vector field in $L^2(\Omega)$. For all $\varepsilon > 0$, denote by u_ε a weak solution of (3.1.1) and by $\overline{u}_\varepsilon = \int u_\varepsilon \, dx_3$. Then, for all $\phi \in H^1(\Omega) \cap \mathrm{Ker}(L)$, we have the following limit in $W^{-1,1}([0,T])$, for all $T > 0$:*

$$\int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h} \overline{u}_{\varepsilon,3}) \phi_3 dx \to 0 \text{ as } \varepsilon \to 0.$$

Proof of Proposition 3.10. We first introduce the same regularization as in the previous paragraphs, and split the integral as follows

$$\begin{aligned}
&\int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h} \overline{u}_{\varepsilon,3}) \phi_3 dx \\
&\quad = \int_\Omega \nabla \cdot (u_\varepsilon^\delta \otimes u_\varepsilon^\delta) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h}^\delta \overline{u}_{\varepsilon,3}^\delta) \phi_3 dx \\
&\quad + \int_\Omega \nabla \cdot ((u_\varepsilon - u_\varepsilon^\delta) \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot ((\overline{u}_{\varepsilon,h} - \overline{u}_{\varepsilon,h}^\delta) \overline{u}_{\varepsilon,3}) \phi_3 dx \\
&\quad + \int_\Omega \nabla \cdot (u_\varepsilon^\delta \otimes (u_\varepsilon - u_\varepsilon^\delta)) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h}^\delta (\overline{u}_{\varepsilon,3} - \overline{u}_{\varepsilon,3}^\delta)) \phi_3 dx.
\end{aligned} \tag{3.3.9}$$

By the energy estimate, we deduce that the four last terms converge to 0 as $\delta \to 0$ uniformly in ε : indeed,

$$\begin{aligned}
\left\| \int_\Omega \nabla \cdot ((u_\varepsilon - u_\varepsilon^\delta) \otimes u_\varepsilon) \cdot \phi dx \right\|_{L^1([0,T])} &\le \|\nabla \phi\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2([0,T],L^6(\Omega))} \|u_\varepsilon^\delta - u_\varepsilon\|_{L^2([0,T],L^3(\Omega))} \\
&\le \omega(\delta) \|\nabla \phi\|_{L^2(\Omega)} \|u_\varepsilon\|^2_{L^2([0,T],H^1(\Omega))},
\end{aligned}$$

where the function $\omega(\delta)$ goes to zero as δ goes to zero.

We are then interested in the difference between the first two terms. By Proposition 3.9, and using the fact that

$$\partial_3 \phi = 0, \quad \nabla \cdot \phi = 0 \text{ and } \phi \cdot \nabla b = 0,$$

it can be rewritten

$$\begin{aligned}
&\int_\Omega \nabla \cdot (u_\varepsilon^\delta \otimes u_\varepsilon^\delta) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h}^\delta \overline{u}_{\varepsilon,3}^\delta) \phi_3 dx \\
&\quad = \int_\Omega \phi \cdot (\varepsilon \rho_{\varepsilon,\delta} + \sigma_{\varepsilon,\delta}) dx - \frac{\varepsilon}{2} \partial_t \int_\Omega |\overline{\omega}_\varepsilon^\delta|^2 (1 - \chi_\delta) \frac{\nabla^\perp b}{|\nabla b|^2} \cdot \phi dx \\
&\quad + \varepsilon \partial_t \int_\Omega \frac{1}{b} \widetilde{\omega}_\varepsilon^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \cdot \phi dx + \varepsilon \partial_t \int_\Omega \frac{1}{2b^2} (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \nabla b)^2 \frac{\nabla^\perp b}{|\nabla b|^2} \cdot \phi dx \\
&\quad + \varepsilon \partial_t \int_\Omega \frac{1}{2b} (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot (\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp) \phi_3 dx.
\end{aligned} \tag{3.3.10}$$

We just need to check that all the terms in the right-hand side of (3.3.10) can be made arbitrarily small. For instance the second term in the right-hand side converges to 0 as $\delta \to 0$ uniformly in ε, since

$$\left\| \int_\Omega \sigma_{\varepsilon,\delta} \cdot \phi dx \right\|_{L^1([0,T])} \le \|\phi\|_{L^6(\Omega)} \|\sigma_{\varepsilon,\delta}\|_{L^1([0,T],L^{6/5}(\Omega))}.$$

The other terms are dealt with as easily, and are left to the reader. Taking limits as $\varepsilon \to 0$ and then as $\delta \to 0$ in (3.3.9)-(3.3.10) shows that, for all $\phi \in H^1(\Omega) \cap \mathrm{Ker}(L)$,

$$\int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h} \overline{u}_{\varepsilon,3}) \phi_3 dx \to 0 \text{ as } \varepsilon \to 0$$

in $W^{-1,1}([0,T])$, which proves Proposition 3.10. □

Theorem 3.3 follows easily. □

3.3.3 Some general remarks

Remarks on the strong convergence. Now that the weak asymptotic behaviour of weak solutions has been understood, one can try to study the strong asymptotics, as in the constant case. However this seems quite a difficult task, as we have no information in general on the nature of the spectrum of the variable-coefficient Coriolis operator. As the next chapter will show (in the case of a model for the tropics), it is possible to write explicit computations if one relaxes the generality of the setting, even if the Fourier transform is not available.

A few general remarks are in order however. Due to the RAGE theorem [52] one can expect the continuous spectrum to have no influence on the convergence. That was clear in the constant case: in the case of the whole space, the spectrum is indeed continuous and a Strichartz theorem (which one can understand as a precise version of the RAGE theorem) enabled us to get rid of all oscillations and to find that the weak convergence was in fact strong. In the periodic case however there is no continuous spectrum and the weak convergence only becomes strong once the oscillations are filtered out. In the variable case one expects therefore to find a strong convergence once the discrete spectrum has been "filtered out" in some way, but the precise way to do so is not so clear. Furthermore even if one does manage to understand precisely the oscillations and to introduce the corresponding filtering operator, the question of the existence of solutions to the limit system is unclear: in the periodic case the terms that could cause some trouble to solve the system miraculously satisfy 2D-type energy estimates. If that were not the case then the limit system would only have a short time life span (supposing the spectral projectors are continuous in $H^{\frac{1}{2}}$, which is also far from clear). To understand the limit system better one would probably have to introduce some non resonant conditions. That program will be carried out in the coming chapter, in the particular case of the tropics.

Remarks on the role of b and ∇b. As suggested before, the parameter b is responsible for the vertical waves, whereas ∇b rules horizontal waves. If ∇b vanishes on sets of nonzero measure, one therefore expects to recover the constant b situation, that is, a limit satisfying the 2D Navier-Stokes system inside such sets. Of course that generates transmission problems on the boundary of those sets, with a possible degeneracy of the horizontal waves.

On the contrary if b vanishes on sets of nonzero measure, then the penalization itself disappears from the equation in such regions. The equation being nonlocal this will have an incidence everywhere in the system and can create coupling problems. If b vanishes at a point only, with a non degenerate singularity, then one can make the weak compactness argument work, although one must be careful with the functional setting; a special case is considered in the next chapter, with a model for the tropics.

The case when the direction of B is not fixed. It seems reasonable, from a physical point of view, to retain only the vertical component of the rotation vector in the Coriolis force. This should be mathematically justified by considering more general models where the direction of the rotation vector is allowed to vary. The algebraic compensated- compactness argument in that case seems to still hold (under the same type of non degeneracy condition as that required in this chapter). However serious geometrical problems appear to understand precisely the structure of the kernel of the rotation operator: the constraint established here on the vertical averages should be replaced by a constraint on the averages over level lines of B, which implies some geometrical understanding on those level lines (are they closed or not, have they a finite length or not...).

3.4 Strong solutions

In this section we want to investigate the question of strong solutions to (3.1.1), and to prove Theorem 3.2. The usual methods to prove the local existence and uniqueness of solutions for the 3D Navier-Stokes equations yield the existence and uniqueness of a solution to (3.1.1) as soon as the initial data is in $H^{\frac{1}{2}}(\Omega)$, but unfortunately one realizes quickly that with such methods, the life span of the solution decays to zero as ε goes to zero, while all norms (other than the energy norm) blow up. On the contrary to ensure large time existence of a unique solution one would need to require the norm of the initial data to go to zero with ε. That is due to the fact that contrary to the case of a constant rotation studied in Chapter 2, the Coriolis operator does not commute with derivatives, and creates large, unbounded terms in the estimates. Our aim in this section is nevertheless to prove the existence and uniqueness of a solution on a uniform time interval, or the global existence and uniqueness for small initial data, independently of ε. For technical reasons, the local in time theorem only holds if the rotation vector only depends on one variable, say x_2 (which as noted in the introductory chapter is consistent with some models of geophysical flows, like the tropics).

Let us explain the structure of the proof of Theorem 3.2. The idea is that since B does not depend on x_3, one is allowed as many vertical derivatives as one likes in the energy estimates. Only horizontal derivatives create an unbounded commutator term. So the first step of the analysis consists in proving the global existence and uniqueness of a solution for small data in an anisotropic-type Sobolev space, where derivatives are only placed on x_3. The local existence and uniqueness for arbitrary data in such an anistropic Sobolev space can also be proved, as long as B only depends on x_1 — the proof is rather technical however, compared to the global existence result. Once that step is accomplished, one proves a propagation of regularity result, enabling the replacement of the anisotropic Sobolev space by H^s. Those steps are explained in more detail in the next sections. To simplify the analysis we will place ourselves in the case where $\Omega = \mathbf{R}^3$; the periodic case can be proved by slight modifications of that case. Moreover we will not be giving any details of the anisotropic-type estimates involved in the proof, as they are quite technical and beyond the scope of this review article;

we merely want to point out here the main ideas and estimates giving the result, and we refer to [43] for all the details.

3.4.1 Global solutions for small data

Let us give the definition of the anisotropic Sobolev spaces we will be using. Calling as usual $\widehat{u}$ the Fourier transform of u, we define the Hilbert space $H^{s,s'}$ by the norm

$$\|u\|_{H^{s,s'}} \stackrel{\text{def}}{=} \left(\int_{\mathbf{R}^3} (1+|\xi_h|^2)^s (1+|\xi_3|^2)^{s'} |\widehat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

We will need to write an energy estimate in such spaces. The following inequality is the main ingredient to prove the next proposition. We refer to [12] and [48] for a proof. For any vector fields u and v, with u divergence free,

$$|(u\cdot\nabla v|v)_{H^{0,s}}| \leq C\left(\|u\|_{H^{0,s}}^{\frac{1}{2}} \|\nabla_h u\|_{H^{0,s}}^{\frac{1}{2}} \|v\|_{H^{0,s}}^{\frac{1}{2}} \|\nabla_h v\|_{H^{0,s}}^{\frac{3}{2}} + \|\nabla_h u\|_{H^{0,s}} \|v\|_{H^{0,s}} \|\nabla_h v\|_{H^{0,s}} \right).$$

Using that inequality and noticing that the Coriolis operator is skew- symmetric in $H^{0,s}$, it is not too difficult to prove the following proposition, stating the global wellposedness of (3.1.1) in $H^{0,s}(\Omega)$ for small enough data.

Proposition 3.11 *Let $s > 1/2$ be given. There is a constant c such that the following result holds. Suppose that u^0 is a divergence free vector field in $H^{0,s}(\Omega)$, such that $\|u^0\|_{H^{0,s}} \leq c$. Then for all $\varepsilon > 0$, the system (3.1.1) has a unique, global solution u_ε, which is bounded in the space $C_b(\mathbf{R}^+; H^{0,s}) \cap L^2(\mathbf{R}^+; H^{1,s})$ and satisfies*

$$\forall t \geq 0, \quad \|u_\varepsilon(t)\|_{H^{0,s}}^2 + \int_0^t \|\nabla_h u_\varepsilon(t')\|_{H^{0,s}}^2 \, dt' \leq \|u^0\|_{H^{0,s}}^2.$$

Once that result is obtained, on can infer the first part of Theorem 3.2, by writing an energy estimate in H^s. Of course this time the Coriolis operator does not disappear, but since B is smooth and bounded one has

$$\frac{1}{\varepsilon}\Big|(u\wedge B|u)_{H^s}\Big| \leq \frac{C}{\varepsilon}\|u\|_{H^s}^2.$$

The main point is then that one can prove, using an anisotropic Littlewood-Paley decomposition and an anisotropic-type paraproduct algorithm (this is quite technical and omitted), that

$$|(u\cdot\nabla u|u)_{H^s}| \leq \frac{1}{2}\|\nabla_h u\|_{H^s}^2 + C\|\nabla_h u\|_{H^{0,s}}^2 \|u\|_{H^s}^2 (1+\|u\|_{H^{0,s}}^2).$$

That estimate is better than a standard H^s estimate, as it involves the $H^{0,s}$ norm of u and ∇u. An H^s energy estimate therefore yields, using the energy estimate of Proposition 3.11 and a Gronwall lemma,

$$\|u(t)\|_{H^s}^2 \leq \|u^0\|_{H^s}^2 \exp\left(\frac{Ct}{\varepsilon} + C\|u^0\|_{H^s}^2 + C\|u^0\|_{H^s}^4\right),$$

which allows to prove the first part of the theorem.

Remark 3.12 *It should be noted that the H^s norm of the solution is unbounded with ε. The global existence result is therefore not as satisfactory as in the constant case, since one does not have a bounded family of solutions in H^s, as ε goes to zero.*

3.4.2 Local solutions for large data

In this section we suppose that the rotation vector B only depends on x_2. This appears like a technical assumption but it is not clear how to deal with the more general case. As in the previous section, we start by proving a result in an anisotropic space.

Proposition 3.13 *Suppose that B only depends on x_2, and let $s > 1/2$ be given. Suppose that u^0 is a divergence free vector field in $H^{0,s}(\Omega)$. Then there is a time $T > 0$ such that for all $\varepsilon > 0$, the system (3.1.1) has a unique solution, bounded in $C([0,T]; H^{0,s}) \cap L^2([0,T]; H^{1,s})$.*

Proof of Proposition 3.13. The first step consists in solving the linearized equation

$$\partial_t v_\varepsilon - \Delta v_\varepsilon + \frac{1}{\varepsilon}\mathbf{P}(v_\varepsilon \wedge B) = 0$$

for smooth initial data, say $v_{\varepsilon|t=0} = \chi(|D|/N)u^0$, where χ is a smooth cut-off function in a ball centered at zero and N is a large enough integer. Then clearly v_ε is globally defined and bounded in $C_b(\mathbf{R}^+; H^{0,s}) \cap L^2(\mathbf{R}^+; H^{1,s})$, and since B depends neither on x_1 nor on x_3, its frequencies in the ξ_1 and ξ_3 direction are in a ball of size N. Then one needs to solve the perturbed equation satisfied by $w_\varepsilon = u_\varepsilon - v_\varepsilon$, and prove it has a solution on a uniform time interval. The equation is the following:

$$\begin{cases} \partial_t w_\varepsilon + w_\varepsilon \cdot \nabla w_\varepsilon + w_\varepsilon \cdot \nabla v_\varepsilon + v_\varepsilon \cdot \nabla w_\varepsilon - \Delta w_\varepsilon + \dfrac{1}{\varepsilon} w_\varepsilon \wedge B + \nabla p = -v_\varepsilon \cdot \nabla v_\varepsilon \\ \nabla \cdot w_\varepsilon = 0 \\ w_{\varepsilon|t=0} = \Big(1 - \chi(|D|/N)\Big)u^0. \end{cases}$$

This is a 3D Navier-Stokes type equation, with a non constant rotating term which is harmless since we will write an energy estimate in $H^{0,s}$. The initial data can be made arbitrarily small as soon as N is large enough. It moreover has a forcing term due to the presence of v_ε, and transport-reaction terms. Those latter terms classically do not cause much trouble as they contribute in an exponential in the final estimate (through a Gronwall lemma), which is independent of ε and N. More troublesome is the forcing term $-v_\varepsilon \cdot \nabla v_\varepsilon$, but using the fact that two frequency directions of v_ε are bounded, it is possible to write an estimate of the type

$$\left| \int_0^t (v_\varepsilon \cdot \nabla v_\varepsilon | w_\varepsilon)_{H^{0,s}}(t')\, dt' \right| \leq C(N, \|u^0\|_{L^2}) t^{\frac{1}{2}} + \frac{1}{2}\|\nabla_h w_\varepsilon\|^2_{H^{0,s}}.$$

Proving such an estimate is of course the main difficulty of the analysis and is left out. It is here that the fact that B does not depend on x_2 is crucial: without that assumption, the constant $C(N, \|u^0\|_{L^2})$ above would depend on ε, which would prevent the life span from being independent of ε. Once that estimate is proved, one finds that for a small enough time, depending on N but not on ε, one can solve the system on w_ε, hence going back to the original equation, there is a solution u_ε on a time interval independent of ε. □

To infer the second part of Theorem 3.2 one uses again a propagation of regularity type result. We omit the details. □

3.5 References and remarks

The analysis of the weak convergence of weak solutions presented in this chapter is probably the first attempt in understanding mathematically the behaviour of a variable coefficient Coriolis operator, and the original analysis can be found in [24]. Note that the study is not unrelated to works on the incompressible limit. As recalled in Section 2.6, the idea of using compensated compactness methods originates in the article [41] for the incompressible limit. The uniform existence of strong solutions presented in Section 3.4 is due to M. Majdoub and M. Paicu [43], and concerning the difficulty of studying strong solutions one can also refer, among other studies to the paper by G. M étivier and S. Schochet [47], concerning nonisentropic, compressible Navier-Stokes equations (see also T. Alazard [1]), or to the recent works [8] and [9].

Chapter 4

The tropics

4.1 Introduction

In this chapter we will be concerned with a shallow water system governing the movement of the ocean at the tropics, presented in the introduction (see (1.2.13). Using the cartesian approximation (1.2.10) of the latitude and the longitude, and the shallow water approximation of the Navier-Stokes system with free surface, we obtain the following system for the depth fluctuation η and the horizontal velocity u:

$$\begin{aligned}
&\partial_t\eta + \frac{1}{\varepsilon}\nabla\cdot\Big((1+\varepsilon\eta)u\Big) = 0,\\
&\partial_t\Big((1+\varepsilon\eta)u\Big) + \nabla\cdot\Big((1+\varepsilon\eta)u\otimes u\Big) + \frac{\beta x_2}{\varepsilon}(1+\varepsilon\eta)u^\perp + \frac{1}{\varepsilon}(1+\varepsilon\eta)\nabla\eta\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -A(1+\varepsilon\eta,u) = 0,\\
&\eta_{|t=0} = \eta^0,\ u_{|t=0} = u^0.
\end{aligned} \tag{4.1.1}$$

We will suppose that the space variable $x = (x_1, x_2)$ belongs to $\mathbf{T}\times\mathbf{R}$. As in the previous chapters, we have denoted $u^\perp = (u_2, -u_1)$. The operator A represents the viscous effects, and from a physical point of view, it would be relevant to model such effects by the following operator

$$A(1+\varepsilon\eta, u) = \nu\nabla\cdot((1+\varepsilon\eta)\nabla u),$$

meaning in particular that the viscosity cancels when $1+\varepsilon\eta$ vanishes. Then, in order for the Cauchy problem to be globally well-posed, it is necessary to get some control on the cavitation. Results by D. Bresch and B. Desjardins [6] show that capillary or friction effects can prevent the formation of singularities in the Saint-Venant system (without Coriolis force). On the other hand, in the absence of such dissipative effects, A. Mellet and A. Vasseur [46] have proved the weak stability of this same system under a suitable integrability assumption on the initial velocity field. All these results are based on a new entropy inequality [6] which controls in particular the first derivative of $\sqrt{1+\varepsilon\eta}$. In particular, they cannot be easily extended to (4.1.1) since the betaplane approximation of the Coriolis force prevents from deriving such an entropy inequality.

For the sake of simplicity, since we are interested in an asymptotic regime where the depth $h = 1+\varepsilon\eta$ is just a fluctuation around a mean value, we will consider the following viscosity operator

$$A(h,u) = \nu\Delta u,$$

so that the usual theory of the isentropic Navier-Stokes equations can be applied.

Note also that we do not consider realistic boundary conditions in the x_1 variable, but that enables us to give a complete description of the asymptotics. In the presence of boundaries one would have to take into account boundary layers (namely Munk-type boundary layers; see [17] for instance).

As in the previous chapters, the questions we shall address are first to solve this system uniformly in ε, and then to understand the asymptotic behaviour of the solutions as ε goes to zero. The mathematical setting is not quite the one studied in the previous chapter, since the rotation vector vanishes for $x_2 = 0$. However the advantage of our situation is that it is an explicit function, so it will be possible to carry out computations further than in the abstract case studied in the previous chapter.

4.2 Statement of the main results

We obtain the following result as a consequence of the global existence of weak solutions to the isentropic Navier-Stokes equations, remarking that the penalization (which is a skew-symmetric operator) does not modify the energy inequality.

Theorem 4.1 *Let $(\eta^0,u^0) \in L^2(\mathbf{T}\times\mathbf{R})$ and consider a sequence $((\eta^0_\varepsilon,u^0_\varepsilon))_{\varepsilon>0}$ such that*

$$\begin{aligned}\sup_{\varepsilon>0}\left(\frac{1}{2}\int\left(|\eta^0_\varepsilon|^2+(1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon|^2\right)dx\right) \leq \mathcal{E}^0 \quad \textit{and}\\ (\eta^0_\varepsilon,u^0_\varepsilon)\to(\eta^0,u^0) \textit{ in } L^2(\mathbf{T}\times\mathbf{R}).\end{aligned} \tag{4.2.1}$$

Then, for all $\varepsilon > 0$, System (4.1.1) has at least one weak solution $(\eta_\varepsilon,u_\varepsilon)$ with initial data $(\eta^0_\varepsilon,u^0_\varepsilon)$, satisfying the uniform bound

$$\sup_{\varepsilon>0}\left(\frac{1}{2}\int\left(\eta^2_\varepsilon+(1+\varepsilon\eta_\varepsilon)|u_\varepsilon|^2\right)(t,x)dx+\nu\int_0^t\int|\nabla u_\varepsilon|^2(t',x)dxdt'\right)\leq\mathcal{E}^0. \tag{4.2.2}$$

In particular, there exist η and u belonging respectively to the spaces $L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))$ and $L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))\cap L^2(\mathbf{R}^+;\dot H^1(\mathbf{T}\times\mathbf{R}))$ such that, up to extraction of a subsequence,

$$(\eta_\varepsilon,u_\varepsilon)\rightharpoonup(\eta,u) \textit{ in } w\text{-}L^2_{loc}(\mathbf{R}^+\times\mathbf{T}\times\mathbf{R}). \tag{4.2.3}$$

It therefore makes sense to inquire on the limit behaviour of the solution as ε goes to zero. We will start by studying the weak asymptotics, and establishing that as the rotation increases, the geostrophic flow is governed by a linear equation. The proof of the following result can be found in Section 4.3 below.

Theorem 4.2 *Let $(\eta^0, u^0) \in L^2(\mathbf{T}\times\mathbf{R})$ and $(\eta^0_\varepsilon, u^0_\varepsilon)$ satisfy (4.2.1). For all $\varepsilon > 0$, denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (4.1.1) with initial data $(\eta^0_\varepsilon, u^0_\varepsilon)$. Then up to the extraction of a subsequence, $(\eta_\varepsilon, u_\varepsilon)$ converges weakly in $L^2_{loc}(\mathbf{R}^+\times\mathbf{T}\times\mathbf{R})$ to the solution (η, u) in $L^\infty(\mathbf{R}^+; L^2(\mathbf{R}))$, with u also belonging to $L^2(\mathbf{R}^+; \dot H^1(\mathbf{R}))$, of the following linear equation (given in weak formulation)*

$$u_2 = 0, \quad -\beta x_2 u_1 + \partial_2 \eta = 0, \tag{4.2.4}$$

and for all $(\eta^*, u^*) \in L^2 \times H^1(\mathbf{R})$ satisfying (4.2.4)

$$\int (\eta\eta^* + u_1 u_1^*)(t,x)\,dx + \nu \int_0^t \int \nabla u_1 \cdot \nabla u_1^*(t',x)\,dx\,dt' = \int (\eta^0\eta^* + u_1^0 u_1^*)(x)\,dx. \tag{4.2.5}$$

Once the mean flow has been described, it is natural to address the question of the strong convergence of solutions. As in the case of midlatitudes (when the Coriolis penalization is assumed to be constant), for periodic boundary conditions we need to filter out the oscillatory modes before taking the strong limit. Indeed equatorial waves are known to be trapped (see Chapter 1), thus we cannot expect to establish any dispersion.

In the next theorem we have defined the operator $\mathcal{L}(t) = e^{-tL}$ where L is the Coriolis operator

$$L : (\eta, u) \in L^2(\mathbf{T}\times\mathbf{R}) \mapsto (\nabla\cdot u, \beta x_2 u^\perp + \nabla\eta). \tag{4.2.6}$$

We moreover denote by Π_0 the L^2 projection onto the kernel of L, and by $\Pi_\perp$ the projection onto $(\mathrm{Ker} L)^\perp$. Finally for any three-component vector field Φ, we denote by Φ' its two last components. In the following statement, a limit system is referred to, which is obtained as in Chapter 2, by a filtering method. It will be studied in Section 4.4. Special function spaces are also used, they are defined by the following norm:

$$\forall s \geq 0, \quad \|\Phi\|_{H^s_L} \stackrel{\text{def}}{=} \|(\mathrm{Id} - \Delta + \beta^2 x_2^2)^{s/2}\Phi\|_{L^2(\mathbf{T}\times\mathbf{R})}. \tag{4.2.7}$$

The limit system is presented in Paragraph 4.4.2, and the main steps of the result are described in Paragraph 4.4.5.

Theorem 4.3 *Let $\Phi^0 = (\eta^0, u^0) \in L^2(\mathbf{T}\times\mathbf{R})$, and consider a family $((\eta^0_\varepsilon, u^0_\varepsilon))_{\varepsilon>0}$ such that*

$$\begin{aligned} \frac{1}{2}\int \left(|\eta^0_\varepsilon|^2 + (1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon|^2\right) dx \leq \mathcal{E}^0 \quad \text{and} \\ \frac{1}{2}\int \left(|\eta^0_\varepsilon - \eta^0|^2 + (1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon - u^0|^2\right) dx \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.2.8}$$

For all $\varepsilon > 0$ denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (4.1.1) with initial data $(\eta^0_\varepsilon, u^0_\varepsilon)$. Then

- *there exists a weak solution in $L^\infty(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R}))$ to the limit filtered system (given to simplify notation in compact formulation rather than in weak formulation as in (4.2.5) above)*

$$\begin{aligned} \partial_t \Phi + Q_L(\Phi,\Phi) - \nu\Delta'_L \Phi = 0 \\ \Phi_{|t=0} = \Phi^0, \end{aligned} \tag{4.2.9}$$

where Δ'_L and Q_L are defined in (4.4.11). Moreover $\Pi_\perp\Phi$ belongs to the space $L^2(\mathbf{R}^+; H^1_L)$. If $\Pi_\perp\Phi^0$ belongs to H^α_L for some $\alpha > 1/2$, then for all but a countable number of β, the weak solution satisfies for all $t \in \mathbf{R}^+$

$$\int_0^t \|\nabla\cdot\Phi'(t')\|_{L^\infty(\mathbf{T}\times\mathbf{R})}dt' < +\infty. \tag{4.2.10}$$

• *If we further assume that $\Pi_\perp\Phi^0$ belongs to $H^{1/2}_L$, then there exists a maximal time interval $[0,T^*[$, with $T^* = +\infty$ under the smallness assumption*

$$\|\Pi_0\Phi^0\|_{L^2(\mathbf{T}\times\mathbf{R})} + \|\Pi_\perp\Phi^0\|_{H^{1/2}_L} \leq C\nu,$$

such that Φ is the unique (strong) solution to (4.2.9), and $\Pi_\perp\Phi$ belongs to $L^\infty_{loc}([0,T^*[, H^{1/2}_L)\cap L^2_{loc}([0,T^*[, H^{3/2}_L)$.

• *Finally if $\Pi_\perp\Phi^0$ belongs to H^α_L for some $\alpha > 1/2$, then for all but a countable number of β, the sequence of filtered solutions (Φ_ε) to (4.1.1) defined by*

$$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon),$$

converges strongly towards Φ in $L^2_{loc}([0,T^*[; L^2(\mathbf{T}\times\mathbf{R}))$.

Remark 4.1 *The limit equation (4.2.9) is obtained as usual (see Chapter 2) by studying resonances in the nonlinear term. It so happens that the limit quadratic form is shown to satisfy three-dimensional type estimates in H^s_L spaces, although the setting here is purely two-dimensional. That is due to the particular structure of the eigenvalues and eigenvectors of the penalization operator L and will be discussed in Paragraph 4.4 below. That is the reason why we are only able to prove the local in time wellposedness of (4.2.9).*

The following, final result, is an intermediate statement between the two convergence theorems stated above. The proof is presented in the final section of this survey, Section 4.5 below. We have denoted by $\mathfrak{S}$ the set of all the eigenvalues of L (which turns out to be exactly the spectrum of L).

Theorem 4.4 *Let $(\eta^0, u^0) \in L^2(\mathbf{T}\times\mathbf{R})$ and $(\eta^0_\varepsilon, u^0_\varepsilon)$ satisfy (4.2.1). For all $\varepsilon > 0$, denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (4.1.1) with initial data $(\eta^0_\varepsilon, u^0_\varepsilon)$, and by*

$$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon).$$

Then up to the extraction of a subsequence, Φ_ε converges strongly in $L^2_{loc}(\mathbf{R}^+; H^s_{loc}(\mathbf{T}\times\mathbf{R}))$ (for all $s < 0$) to some weak solution Φ of the following limiting filtered system: for all $i\lambda \in \mathfrak{S}$, there is a bounded measure $\upsilon_\lambda \in \mathcal{M}(\mathbf{R}^+\times\mathbf{T}\times\mathbf{R})$ (which vanishes if $\lambda = 0$), such that for all smooth $\Phi^*_\lambda \in \mathrm{Ker}(L - i\lambda Id)$,

$$\int \Phi\cdot\bar{\Phi}^*_\lambda(x)\,dx - \nu\int_0^t\int \Delta'_L\Phi\cdot\bar{\Phi}^*_\lambda(t',x)\,dxdt'$$
$$+\int_0^t\int Q_L(\Phi,\Phi)\cdot\bar{\Phi}^*_\lambda(t',x)\,dxdt' + \int_0^t\int \nabla\cdot(\bar{\Phi}^*_\lambda)'\upsilon_\lambda(dt'dx) = \int \Phi^0\cdot\bar{\Phi}^*_\lambda(x)\,dx,$$

where Q_L and Δ'_L are defined by (4.4.11), and where $\Phi^0 = (\eta^0, u^0)$.

Remark 4.2 • *Note that, by interpolation with the uniform $L^2_{loc}(\mathbf{R}^+, H^1(\mathbf{T}\times\mathbf{R}))$ bound on u_ε, we get the strong convergence of u_ε in $L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{T}\times\mathbf{R}))$: up to extraction of a subsequence,*

$$\left\| u_\varepsilon - \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right)' \right\|_{L^2(\mathbf{T}\times\mathbf{R})} \to 0 \text{ in } L^2_{loc}([0,T]).$$

• *The presence of the defect measure υ_λ at the limit is due to a possible defect of compactness in space of the sequence $(\eta_\varepsilon)_{\varepsilon>0}$. As the proof of the theorem shows, that measure is zero if one is able to prove some equicontinuity in space on η_ε, or even on $\varepsilon\eta_\varepsilon$. Since we have been unable to prove such a result, we study in the final paragraph of this chapter a slightly different model, where capillarity effects are added in order to gain that compactness. Note that the model introduced in Paragraph 4.5.4 is unfortunately not very physical due to the particular form of the capillarity operator (see its definition in (4.5.8) below).*

4.3 Weak asymptotics

In this paragraph we intend to prove Theorem 4.2 stated above. The structure of the proof is similar to the previous chapters: we study the kernel of the penalization operator and show that the limit is necessarily in that kernel, and a compensated compactness argument allows to take limits in the nonlinear terms.

4.3.1 The geostrophic constraint

The first step consists in proving that the weak limit defined by (4.2.3) satisfies the geostrophic constraint (4.2.4), or in other words belongs to the kernel of L. We skip the proof of the following proposition: as in the previous chapters one proves that the limit is in Ker$L$ by multiplying the system (4.1.1) by ε and taking limits in the sense of distributions thanks to the uniform bounds coming from the energy estimate. The constraint (4.3.1) is easily shown to characterize elements of Ker$L$.

Proposition 4.3 *Let $(\eta^0, u^0) \in L^2(\mathbf{T}\times\mathbf{R})$ and $(\eta^0_\varepsilon, u^0_\varepsilon)$ be initial data satisfying (4.2.1). Denote by $((\eta_\varepsilon, u_\varepsilon))_{\varepsilon>0}$ a family of solutions of (4.1.1) with respective initial data $(\eta^0_\varepsilon, u^0_\varepsilon)$, and by (η, u) any of its limit points. Then, $(\eta, u) \in L^\infty(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R}))$ satisfies the constraints*

$$u_2 = 0, \quad -\beta x_2 u_1 + \partial_2\eta = 0. \tag{4.3.1}$$

To go further in the description of the weak limit (η, u), we have to isolate the fast oscillations generated by the singular perturbation L, which produce "big" terms in (4.1.1), but converge weakly to 0. The idea to get the mean motion is to consider the weak form of the evolution equations, testing (4.1.1) against smooth functions of Ker$L$.

Note that contrary to the previous chapters, we are missing regularity in the unknown η_ε (which does not satisfy a uniform $L^2_{loc}(\mathbf{R}^+, H^1(\mathbf{T}\times\mathbf{R}))$ bound), so we will need to use smoother functions in the kernel of L than merely H^1 functions as in the previous chapter. In fact a careful study of the constraint (4.3.1) indicates that the Hermite functions are naturally associated with Ker$L$. Let us therefore introduce the Hermitian basis of $L^2(\mathbf{R})$ constituted of Hermite functions $(\psi_n)_{n\in\mathbf{N}}$ where

$$-\psi_n'' + \beta^2 x_2^2 \psi_n = \beta(2n+1)\psi_n.$$

We recall that

$$\psi_n(x_2) = \exp\left(-\frac{\beta x_2^2}{2}\right) P_n(x_2\sqrt{\beta})$$

where P_n is the n-th Hermite polynomial, as well as the identities

$$\begin{aligned}\psi_n'(x_2) + \beta x_2 \psi_n(x_2) &= \sqrt{2\beta n}\,\psi_{n-1}(x_2),\\ \psi_n'(x_2) - \beta x_2 \psi_n(x_2) &= -\sqrt{2\beta(n+1)}\,\psi_{n+1}(x_2).\end{aligned} \tag{4.3.2}$$

Then decomposing any element of Ker$L$ on the Hermite basis one can show that it is a linear combination of the following

$$(\eta_0, u_0) = \begin{pmatrix} -\psi_0(x_2) \\ \psi_0(x_2) \\ 0 \end{pmatrix} \quad \text{and}$$

$$(\eta_n, u_n) = \begin{pmatrix} \sqrt{\frac{\beta(n+1)}{2}}\psi_{n-1}(x_2) + \sqrt{\frac{\beta n}{2}}\psi_{n+1}(x_2) \\ \sqrt{\frac{\beta(n+1)}{2}}\psi_{n-1}(x_2) - \sqrt{\frac{\beta n}{2}}\psi_{n+1}(x_2) \\ 0 \end{pmatrix} \quad \text{for } n \geq 1.$$

We will therefore restrict our attention to those particular vector fields, which are smooth and integrable against any polynomial in x_2, and then conclude by a density argument.

Using the conservations of mass and momentum (4.1.1) it is easy to see that, defining $m_\varepsilon = (1+\varepsilon\eta_\varepsilon)u_\varepsilon$,

$$\begin{aligned} &\int (\eta_\varepsilon \eta_n + m_{\varepsilon,1} u_{n,1})(t,x)\,dx + \nu\int_0^t\int \nabla u_{\varepsilon,1}\cdot\nabla u_{n,1}(t',x)\,dx\,dt' \\ = &\int (\eta_\varepsilon^0 \eta_n + m_{\varepsilon,1}^0 u_{n,1})(x)\,dx + \int_0^t\int (m_\varepsilon\cdot(u_\varepsilon\cdot\nabla u_n))(t',x)\,dxdt'. \end{aligned}$$

The difficulty is then to take limits the nonlinear terms, which can be simply written

$$\int_0^t\int m_{\varepsilon,1}u_{\varepsilon,2}\partial_2 u_{n,1}(t',x)\,dxdt'.$$

This is achieved by a compensated compactness technique presented in the next section.

4.3.2 The compensated compactness argument

The analysis of the nonlinear terms lies essentially on the structure of the oscillations. A rough description of those fast oscillations will be enough to prove that they do not produce any constructive interference, and therefore do not occur in the equation governing the mean (geostrophic) motion. As in the previous chapters, κ denotes a regularizing kernel.

Lemma 4.4 *Let us define*

$$\eta_\varepsilon^\delta = \kappa_\delta \star \eta_\varepsilon \text{ and } m_\varepsilon^\delta = \kappa_\delta \star ((1+\varepsilon\eta_\varepsilon)u_\varepsilon) = u_\varepsilon^\delta + \varepsilon(\eta_\varepsilon u_\varepsilon)^\delta$$

which converge uniformly in ε *as* $\delta \to 0$ *to* η_ε *and* m_ε *in* $L^\infty(\mathbf{R}^+, H^s_{loc}(\mathbf{T}\times\mathbf{R}))$ *for any* $s<0$. *We also introduce the approximate vorticity*

$$\omega_\varepsilon^\delta = \nabla^\perp \cdot m_\varepsilon^\delta.$$

Then the following approximate wave equations hold

$$\begin{aligned} \varepsilon\partial_t\eta_\varepsilon^\delta + \nabla\cdot m_\varepsilon^\delta &= 0,\\ \varepsilon\partial_t m_\varepsilon^\delta + \beta x_2 (m_\varepsilon^\delta)^\perp + \nabla\eta_\varepsilon^\delta &= \varepsilon s_\varepsilon^\delta + \delta\sigma_\varepsilon^\delta,\\ \varepsilon\partial_t(\omega_\varepsilon^\delta - \beta x_2\eta_\varepsilon^\delta) + \beta m_{\varepsilon,2}^\delta &= \varepsilon q_\varepsilon^\delta + \delta p_\varepsilon^\delta, \end{aligned} \tag{4.3.3}$$

denoting by $s_\varepsilon^\delta, q_\varepsilon^\delta$ *and* $\sigma_\varepsilon^\delta, p_\varepsilon^\delta$ *some quantities satisfying, for all* $T>0$,

$$\begin{aligned} &\sup_{\delta>0}\sup_{\varepsilon>0}\Big(\|\sigma_\varepsilon^\delta\|_{L^2([0,T];H^1(\mathbf{T}\times\mathbf{R}))} + \|p_\varepsilon^\delta\|_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))}\Big) < +\infty,\\ &\forall\delta>0,\quad \sup_{\varepsilon>0}\Big(\|s_\varepsilon^\delta\|_{L^1([0,T];H^1(\mathbf{T}\times\mathbf{R}))} + \|q_\varepsilon^\delta\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))}\Big) < \infty. \end{aligned} \tag{4.3.4}$$

In order to prove this lemma, we proceed in two steps as in the previous chapters, first stating the wave equations for $(\eta_\varepsilon, m_\varepsilon)$, then introducing the regularization $(\eta_\varepsilon^\delta, m_\varepsilon^\delta)$. We omit the details.

Equipped with this preliminary result, we are now able to establish the compensated compactness result, which implies that the nonlinear term actually converges to zero.

Proposition 4.5 *With the previous notations, we have locally uniformly in* t

$$\lim_{\varepsilon\to 0}\int_0^t\int m_{\varepsilon,1}u_{\varepsilon,2}\partial_2 u_{n,1}(t',x)\,dxdt' = 0.$$

Proof. Let us define, as in Lemma 4.4,

$$\eta_\varepsilon^\delta = \eta_\varepsilon \star \kappa_\delta, \quad u_\varepsilon^\delta = u_\varepsilon \star \kappa_\delta \quad \text{and} \quad m_\varepsilon^\delta = m_\varepsilon \star \kappa_\delta.$$

Then

$$\begin{aligned}\int_0^t \int m_{\varepsilon,1} u_{\varepsilon,2} \partial_2 u_{n,1}(t',x)\, dxdt' = &\int_0^t \int m^\delta_{\varepsilon,1} m^\delta_{\varepsilon,2} \partial_2 u_{n,1}(t',x)\, dxdt' \\ &+ \int_0^t \int m^\delta_{\varepsilon,1} (u^\delta_{\varepsilon,2} - m^\delta_{\varepsilon,2}) \partial_2 u_{n,1}(t',x)\, dxdt' \\ &+ \int_0^t \int m^\delta_{\varepsilon,1} (u_{\varepsilon,2} - u^\delta_{\varepsilon,2}) \partial_2 u_{n,1}(t',x)\, dxdt' \\ &+ \int_0^t \int (m_{\varepsilon,1} - m^\delta_{\varepsilon,1}) u_{\varepsilon,2} \partial_2 u_{n,1}(t',x)\, dxdt'.\end{aligned} \tag{4.3.5}$$

• From the energy estimates we can prove that the two last integrals converge towards zero as δ goes to zero uniformly in ε. Indeed for all $\alpha > 0$ there exists some bounded subset $\mathbf{T} \times \Omega_\alpha$ of $\mathbf{T} \times \mathbf{R}$ such that

$$\|\partial_2 u_{n,1}\|_{W^{1,\infty}(\mathbf{R} \setminus \Omega_\alpha)} \leq \alpha.$$

Then, for $0 < s < 1$,

$$\begin{aligned}&\left| \int_0^t \int (m_{\varepsilon,1} - m^\delta_{\varepsilon,1}) u_{\varepsilon,2} \partial_2 u_{n,1}(t',x)\, dxdt' \right| \\ &\leq \|m_{\varepsilon,1} - m^\delta_{\varepsilon,1}\|_{L^2([0,T];H^{-s}(\mathbf{T} \times \Omega_\alpha))} \|u_{\varepsilon,2}\|_{L^2([0,T];H^1(\mathbf{T} \times \mathbf{R}))} \|\partial_2 u_{n,1}\|_{W^{1,\infty}(\mathbf{R})} \\ &+ 2\alpha \|m_{\varepsilon,1}\|_{L^2([0,T];H^{-s}(\mathbf{T} \times \mathbf{R}))} \|u_{\varepsilon,2}\|_{L^2([0,T];H^1(\mathbf{T} \times \mathbf{R}))},\end{aligned}$$

which goes to zero as α then δ go to zero, uniformly in ε by (4.2.2) and Lemma 4.4.

Similarly, we get, for $0 < s < 1$,

$$\begin{aligned}&\left| \int_0^t \int m^\delta_{\varepsilon,1} (u_{\varepsilon,2} - u^\delta_{\varepsilon,2}) \partial_2 u_{n,1}(t',x)\, dxdt' \right| \\ &\leq \|m^\delta_{\varepsilon,1}\|_{L^2([0,T];H^{-s}(\mathbf{T} \times \mathbf{R}))} \|u_{\varepsilon,2} - u^\delta_{\varepsilon,2}\|_{L^2([0,T];H^s(\mathbf{T} \times \Omega_\alpha))} \|\partial_2 u_{n,1}\|_{W^{1,\infty}(\mathbf{T} \times \mathbf{R})} \\ &+ 2\alpha \|m_{\varepsilon,2}\|_{L^2([0,T];H^{-s}(\mathbf{T} \times \mathbf{R}))} \|u_{\varepsilon,1}\|_{L^2([0,T];H^1(\mathbf{T} \times \mathbf{R}))}\end{aligned}$$

which goes to zero as α then δ go to zero, uniformly in ε by (4.2.2) and Lemma 4.4.

Next we prove that for all $\delta > 0$, the second integral in the right-hand side of (4.3.5) goes to zero as $\varepsilon \to 0$. But $\eta_\varepsilon u_\varepsilon$ and consequently m_ε are uniformly bounded in $L^2([0,T]; H^s(\mathbf{T} \times \mathbf{R}))$ for $s < 0$. Therefore, for fixed $\delta > 0$, $(\eta_\varepsilon u_\varepsilon)^\delta$ and m^δ_ε are uniformly bounded in $L^2([0,T] \times \mathbf{T} \times \mathbf{R})$. Then

$$\begin{aligned}&\left| \int_0^t \int m^\delta_{\varepsilon,1} (u^\delta_{\varepsilon,2} - m^\delta_{\varepsilon,2}) \partial_2 u_{n,1}(t',x)\, dxdt' \right| \\ &\qquad \leq \varepsilon \|m^\delta_{\varepsilon,1}\|_{L^2([0,T] \times \mathbf{T} \times \mathbf{R})} \|(\eta_\varepsilon u_{\varepsilon,2})^\delta\|_{L^2([0,T] \times \mathbf{T} \times \mathbf{R})} \|\partial_2 u_{n,1}\|_{L^\infty(\mathbf{R})}\end{aligned}$$

which goes to zero as $\varepsilon \to 0$ for all fixed $\delta > 0$.

• So finally we need to consider the first term in the right-hand side of (4.3.5). We are going to prove that the limit of that term is zero using Lemma 4.4. Integrating by parts, we have

$$
\begin{aligned}
&\int_0^t \int m^\delta_{\varepsilon,1} m^\delta_{\varepsilon,2} \partial_2 u_{n,1}(t',x)\, dxdt' \\
&\quad = -\int_0^t \int \Big((\partial_2 m^\delta_{\varepsilon,1}) m^\delta_{\varepsilon,2} + m^\delta_{\varepsilon,1} (\partial_2 m^\delta_{\varepsilon,2}) \Big) u_{n,1}(t',x)\, dxdt' \\
&\quad = -\int_0^t \int \Big((\omega^\delta_\varepsilon + \partial_1 m^\delta_{\varepsilon,2}) m^\delta_{\varepsilon,2} + m^\delta_{\varepsilon,1} (\nabla \cdot m^\delta_\varepsilon - \partial_1 m^\delta_{\varepsilon,1}) \Big) u_{n,1}(t',x)\, dxdt' \\
&\quad = -\int_0^t \int \Big((\omega^\delta_\varepsilon - \beta x_2 \eta^\delta_\varepsilon) m^\delta_{\varepsilon,2} + \eta^\delta_\varepsilon (\beta x_2 m^\delta_{\varepsilon,2} + \partial_1 \eta^\delta_\varepsilon) + m^\delta_{\varepsilon,1} \nabla \cdot m^\delta_\varepsilon \Big) u_{n,1}(t',x)\, dxdt' \\
&\qquad - \frac{1}{2} \int_0^t \int \partial_1 \Big((m^\delta_{\varepsilon,2})^2 - (m^\delta_{\varepsilon,1})^2 - (\eta^\delta_\varepsilon)^2 \Big) u_{n,1}(t',x) dxdt'
\end{aligned}
$$

and the last term is zero because $\partial_1 u_{n,1} = 0$.

Lemma 4.4 now implies that

$$
\begin{aligned}
&\int_0^t \int m^\delta_{\varepsilon,1} m^\delta_{\varepsilon,2} \partial_2 u_{n,1}(t',x)\, dxdt' \\
&\quad = \int_0^t \int \left(\frac{\varepsilon}{2\beta} \partial_t (\beta x_2 \eta^\delta_\varepsilon - \omega^\delta_\varepsilon)^2 + \frac{\varepsilon}{\beta} (\beta x_2 \eta^\delta_\varepsilon - \omega^\delta_\varepsilon) q^\delta_\varepsilon + \frac{\delta}{\beta} (\beta x_2 \eta^\delta_\varepsilon - \omega^\delta_\varepsilon) p^\delta_\varepsilon \right) u_{n,1}(t',x)\, dxdt' \\
&\quad + \int_0^t \int \Big(\varepsilon \partial_t (\eta^\delta_\varepsilon m^\delta_{\varepsilon,1}) - \varepsilon \eta^\delta_\varepsilon s^\delta_{\varepsilon,1} - \delta \eta^\delta_\varepsilon \sigma^\delta_{\varepsilon,1} \Big) u_{n,1}(t',x) dxdt'.
\end{aligned}
$$

Now we notice that

$$
\begin{aligned}
\left| \int_0^t \int (\beta x_2 \eta^\delta_\varepsilon - \omega^\delta_\varepsilon) p^\delta_\varepsilon u_{n,1}(t',x)\, dxdt' \right| &\leq C \|(1+x_2^2)^{1/2} u_{n,1}\|_{L^\infty(\mathbf{R})} \|p^\delta_\varepsilon\|_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))} \\
&\quad \times \Big(T^{1/2} \|\eta^\delta_\varepsilon\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))} + \|\omega^\delta_\varepsilon\|_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))} \Big),
\end{aligned}
$$

and similarly

$$
\left| \int_0^t \int \eta^\delta_\varepsilon \sigma^\delta_{\varepsilon,1} u_{n,1}(t',x)\, dxdt' \right| \leq C T^{1/2} \|\eta^\delta_\varepsilon\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))} \|u_{n,1}\|_{L^\infty(\mathbf{R})} \|\sigma^\delta_\varepsilon\|_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))}.
$$

So writing

$$
\|\omega^\delta_\varepsilon\|_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R})))} \leq \|\nabla^\perp \cdot u^\delta_\varepsilon\|_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))} + \varepsilon \|\nabla^\perp \cdot (\eta^\delta_\varepsilon u^\delta_\varepsilon)\|_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))},
$$

we infer that

$$
\begin{aligned}
&\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \left(\frac{\delta}{\beta} \left| \int_0^t \int (\beta x_2 \eta^\delta_\varepsilon - \omega^\delta_\varepsilon) p^\delta_\varepsilon u_{n,2}(t',x)\, dxdt' \right| \right) = 0, \quad \text{and} \\
&\lim_{\delta\to 0} \left(\delta \left| \int_0^t \int \eta^\delta_\varepsilon \sigma^\delta_{\varepsilon,1} u_{n,1}(t',x)\, dxdt' \right| \right) = 0, \quad \text{uniformly in } \varepsilon.
\end{aligned}
$$

On the other hand,

$$\left|\int_0^t \int (\beta x_2 \eta_\varepsilon^\delta - \omega_\varepsilon^\delta) q_\varepsilon^\delta u_{n,2}(t',x)\, dxdt'\right| \leq C\left(\|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))} + \|\omega_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))}\right)$$
$$\times \|(1+x_2^2)^{1/2} u_{n,2}\|_{L^\infty(\mathbf{T}\times\mathbf{R}))}\|q_\varepsilon^\delta\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))}$$
$$\leq C\left(\|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))} + \frac{1}{\delta}\|u_\varepsilon\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))} + \varepsilon\|\nabla^\perp\cdot(\eta_\varepsilon^\delta u_\varepsilon^\delta)\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))}\right)$$
$$\times \|(1+x_2^2) u_{n,2}\|_{L^\infty(\mathbf{T}\times\mathbf{R})}\|q_\varepsilon^\delta\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))},$$

and

$$\left|\int_0^t \int \eta_\varepsilon^\delta s_{\varepsilon,1}^\delta u_{n,1}(t',x)\, dxdt'\right| \leq C\|\eta_\varepsilon^\delta\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))}\|u_{n,1}\|_{L^\infty(\mathbf{T}\times\mathbf{R})}\|s_\varepsilon^\delta\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))}$$

so

$$\lim_{\varepsilon\to 0}\left(\frac{\varepsilon}{\beta}\left|\int_0^t \int (\beta x_2 \eta_\varepsilon^\delta - \omega_\varepsilon^\delta) q_\varepsilon^\delta u_{n,1}(t',x)\, dxdt'\right|\right) = 0, \quad \text{for all } \delta > 0,$$
$$\lim_{\varepsilon\to 0}\left(\varepsilon\left|\int_0^t \int \eta_\varepsilon^\delta s_{\varepsilon,1}^\delta u_{n,1}(t',x)\, dxdt'\right|\right) = 0, \quad \text{for all } \delta > 0.$$

So we simply need to let ε go to zero, then δ, and the result follows. □

4.4 Strong asymptotics

In this section we aim at getting a complete description of the asymptotic behaviour of the ocean in the fast rotation limit, including the various equatorial waves – thus proving the strong convergence result stated in Theorem 4.3. In the first Paragraph of this section (Paragraph 4.4.1) we present the various waves involved, which are eigenvectors of the singular perturbation and constitute a Hilbertian basis of $L^2(\mathbf{T}\times\mathbf{R})$. That basis enables us to introduce the filtering operator and to formally derive the limit filtered system. Then, proving that the limit filtered system has strong solutions (see Paragraphs 4.4.2, 4.4.3 and 4.4.4) and using the strong-weak stability of (4.1.1) (see Paragraph 4.4.5) leads to the strong convergence result.

4.4.1 The equatorial waves

In view of the structure of the rotating shallow-water equations (4.1.1), we expect the oscillations of $(\eta_\varepsilon, u_\varepsilon)$ to be mainly governed by the singular perturbation L. The crucial point is that the description of the eigenmodes of L can be achieved using the Fourier transform with respect to x_1 and the decomposition on the Hermite functions $(\psi_n)_{n\in\mathbf{N}}$ with respect to x_2.

In order to investigate the spectrum of L (which is an unbounded skew-symmetric operator), we are interested in the non trivial solutions to

$$L(\eta, u) = i\tau(\eta, u).$$

Rewriting that equation as on equation on u_2 only, one checks easily that necessarily

$$\tau^3 - (k^2 + \beta(2n+1))\tau + \beta k = 0, \tag{4.4.1}$$

for some $n \in \mathbf{N}$.

• If $k \neq 0$ and $n \neq 0$, (4.4.1) admits three solutions

$$\tau(n,k,-1) < \tau(n,k,0) < \tau(n,k,1),$$

and one can check that these solutions are eigenvalues of L associated to the following unitary eigenvectors (the coefficient $C_{n,k,j}$ ensures they are unitary)

$$\Psi_{n,k,j} = C_{n,k,j} e^{ikx_1} \begin{pmatrix} \dfrac{i}{k-\tau(n,k,j)}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_2) + \dfrac{i}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_2) \\ \dfrac{i}{k-\tau(n,k,j)}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_2) - \dfrac{i}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_2) \\ \psi_n(x_2) \end{pmatrix}. \tag{4.4.2}$$

The modes corresponding to $\tau(n,k,-1)$ and $\tau(n,k,1)$ are called *Poincaré modes* because

$$\tau(n,k,\pm 1) \sim \pm\sqrt{k^2 + \beta(2n+1)} \text{ as } |k|, n \to \infty,$$

which are the frequencies of the gravity waves.

The modes corresponding to $\tau(n,k,0)$ are called *Rossby modes* because

$$\tau(n,k,0) \sim \frac{\beta k}{k^2 + \beta(2n+1)} \text{ as } |k|, n \to \infty,$$

meaning that the oscillation frequency is very small : the planetary waves $\Psi_{n,k,0}$ satisfy indeed the quasigeostrophic approximation.

• If $k = 0$ and $n \neq 0$, the three solutions to (4.4.1) are the two Poincaré modes $\tau(n,0,\pm 1) = \pm\sqrt{\beta(2n+1)}$ and the *non-oscillating mode* $\tau(n,0,0) = 0$. The corresponding eigenvectors of L are given by (4.4.2) if $j \neq 0$ and by

$$\Psi_{n,0,0} = C_{n,0,0} \begin{pmatrix} \sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n-1}(x_2) + \sqrt{\dfrac{\beta n}{2}}\psi_{n+1}(x_2) \\ \sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n-1}(x_2) - \sqrt{\dfrac{\beta n}{2}}\psi_{n+1}(x_2) \\ 0 \end{pmatrix}. \tag{4.4.3}$$

• If $n = 0$, the three solutions to (4.4.1) are the two Poincaré and *mixed Poincaré-Rossby modes*

$$\tau(0,k,\pm 1) = -\frac{k}{2} \pm \frac{1}{2}\sqrt{k^2 + 4\beta}$$

with asymptotic behaviours given by

$$\tau(0,k,-\ \mathrm{sgn}(k)) \sim -k \text{ as } |k| \to \infty,$$
$$\tau(0,k,\ \mathrm{sgn}(k)) \sim \frac{\beta}{k} \text{ as } |k| \to \infty,$$

and the *Kelvin mode* $\tau(0,k,0) = k$. The corresponding eigenvectors of L are given by (4.4.2) if $j \neq 0$ and by

$$\Psi_{0,k,0} = \frac{1}{\sqrt{4\pi}} e^{ikx_1} \begin{pmatrix} -\psi_0(x_2) \\ \psi_0(x_2) \\ 0 \end{pmatrix}. \tag{4.4.4}$$

One can then prove the following diagonalization result (whose technical proof is omitted here).

Proposition 4.6 *For all $(n,k,j) \in \mathbf{N} \times \mathbf{Z} \times \{-1,0,1\}$, denote by $\tau(n,k,j)$ the three roots of (4.4.1) and by $\Psi_{n,k,j}$ the unitary vector defined above. Then $(\Psi_{n,k,j})_{(n,k,j)\in \mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}}$ is a Hilbertian basis of $L^2(\mathbf{T}\times\mathbf{R})$ constituted of eigenvectors of L :*

$$L\Psi_{n,k,j} = i\tau(n,k,j)\Psi_{n,k,j}. \tag{4.4.5}$$

Furthermore we have the following estimate : for all $s > 0$, there exists a nonnegative constant C_s such that, for all $(n,k,j) \in \mathbf{N} \times \mathbf{Z} \times \{-1,0,1\}$,

$$\|\Psi_{n,k,j}\|_{L^\infty(\mathbf{T}\times\mathbf{R})} \leq C_0 \quad \text{and} \quad \|\Psi_{n,k,j}\|_{H^s\cap W^{s,\infty}(\mathbf{T}\times\mathbf{R})} \leq C_s(1+|k|^2+n)^{s/2}. \tag{4.4.6}$$

Moreover the following property holds : if $\tau(n,k,j) = \tau(n^*,k,j^*)$, then $n = n^*$ and $j = j^*$. Finally the eigenspace associated with any $i\lambda \neq 0$ is of finite dimension.

As mentioned in the introduction, the adjustment processes are therefore somewhat special in the vicinity of the equator (when the Coriolis acceleration vanishes). A very important property of the equatorial zone is that it acts as a *waveguide*, i.e., disturbances are trapped in the vicinity of the equator. The waveguide effect is due entirely to the variation of Coriolis parameter with latitude. Note that another important effect of the waveguide is the separation into a discrete set of modes $n = 0, 1, 2, ...$ as occurs in a channel.

The next definition will be useful in the following.

Definition 4.7 *With the previous notation, let us define*

$$P = Vect\{\Psi_{n,k,j} \,/\, (n,k,j) \in \mathbf{N}^* \times \mathbf{Z} \times \{-1,1\} \setminus \{0\} \times \{(k, -sgn(k) \,/\, k \in \mathbf{Z}^*\}\},$$
$$R = Vect\{\Psi_{n,k,0} \,/\, (n,k) \in \mathbf{N}^* \times \mathbf{Z}^*\},$$
$$M = Vect\{\Psi_{0,k,j} \,/\, k \in \mathbf{Z}^*, j = -sgn(k)\},$$
$$K = Vect\{\Psi_{0,k,0} \,/\, k \in \mathbf{Z}^*\},$$

so that

$$L^2(\mathbf{T}\times\mathbf{R}) = P \oplus R \oplus M \oplus K \oplus \mathrm{Ker}L.$$

Then we denote by Π_P (resp. Π_R, Π_M, Π_K and Π_0) the L^2 orthogonal projection on P (resp. on R, M, K and $\mathrm{Ker}L$).

We are now able to define the "filtering operator" associated with the system. Let $\mathcal{L}$ be the semi-group generated by L : we write $\mathcal{L}(t) = \exp(-tL)$. Then, for any three-component vector field $\Phi \in L^2(\mathbf{T} \times \mathbf{R})$, we have

$$\mathcal{L}(t)\Phi = \sum_{i\lambda \in \mathfrak{S}} e^{-it\lambda} \Pi_\lambda \Phi, \tag{4.4.7}$$

where Π_λ denotes the L^2 orthogonal projection on the eigenspace of L corresponding to the eigenvalue $i\lambda$, and $\mathfrak{S}$ is the set of all eigenvalues of L.

Now let us consider $(\eta_\varepsilon, u_\varepsilon)$ a weak solution to (4.1.1), and let us define

$$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon). \tag{4.4.8}$$

Conjugating formally equation (4.1.1) by the semi-group leads to

$$\partial_t \Phi_\varepsilon + \mathcal{L}\left(-\frac{t}{\varepsilon}\right) Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon\right) - \nu \mathcal{L}\left(-\frac{t}{\varepsilon}\right)\Delta' \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon = R_\varepsilon, \tag{4.4.9}$$

where Δ' and Q are the linear and symmetric bilinear operator defined by

$$\Delta'\Phi = (0, \Delta\Phi') \text{ and } Q(\Phi, \Phi) = (\nabla \cdot (\Phi_0 \Phi'), (\Phi' \cdot \nabla)\Phi') \tag{4.4.10}$$

denoting by Φ_0 the first coordinate and by Φ' the two other coordinates of Φ, and where

$$R_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(0, -\nu \frac{\varepsilon\eta_\varepsilon}{1 + \varepsilon\eta_\varepsilon}\Delta u_\varepsilon).$$

We therefore expect to get a bound on the time derivative of Φ_ε in some space of distributions. A formal passage to the limit in (4.4.9) as ε goes to zero (based on formula (4.4.7) and on a nonstationary phase argument) leads then to

$$\partial_t \Phi + Q_L(\Phi, \Phi) - \nu \Delta'_L \Phi = 0,$$

where Δ'_L and Q_L denote the linear and symmetric bilinear operator defined by

$$\Delta'_L \Phi = \sum_{i\lambda \in \mathfrak{S}} \Pi_\lambda \Delta' \Pi_\lambda \Phi \quad \text{and} \quad Q_L(\Phi, \Phi) = \sum_{\substack{i\lambda, i\mu, i\tilde{\mu} \in \mathfrak{S} \\ \lambda = \mu + \tilde{\mu}}} \Pi_\lambda Q(\Pi_\mu \Phi, \Pi_{\tilde{\mu}} \Phi). \tag{4.4.11}$$

Note that this formulation makes a priori no sense, but should be understood in weak form.

The definition of the quadratic form naturally addresses the question of the resonances induced by L, which will be studied in Section 4.4.3.

4.4.2 The quasigeostrophic motion

In this section we shall investigate the wellposedness of the limit system derived formally in the previous section. The aim of this section is therefore to prove all the results of Theorem 4.3 except for the final, convergence result. Those existence results are based on a precise study of the structure of (4.2.9), and in particular of the ageostrophic part of that equation, meaning its projection onto $(\mathrm{Ker}L)^\perp$. One can prove in particular that the ageostrophic part of (4.2.9) is in fact fully parabolic. That should be compared to the case of the incompressible limit of the compressible Navier-Stokes equations, where again the limit system is parabolic, contrary to the original compressible system (see [15], [22], [44]). Note however that (4.2.9) actually satisfies the same type of trilinear estimates as the three-dimensional incompressible Navier-Stokes system, which accounts for the fact that unique solutions are only obtained for a short life span (despite the fact that the space variable x runs in the two dimensional domain $\mathbf{T}\times\mathbf{R}$).

We will not give all the details of the proof here but indicate the main steps, which enable us to rely on the theory of the three dimensional Navier-Stokes equations.

Let us start by considering the existence of weak solutions. The main argument, as pointed out above, is that the ageostrophic part of the limit system is actually fully parabolic, in the following sense: recalling the definition of the spaces H^s_L given in (4.2.7) above, one can prove that for any $\Phi\in(\mathrm{Ker}L)^\perp\cap H^s_L$,

$$\|\Phi\|^2_{H^s_L}\sim\sum_{i\lambda\in\mathfrak{S}\setminus\{0\}}\|\Pi_\lambda\Phi\|^2_{H^s(\mathbf{T}\times\mathbf{R})}<+\infty. \tag{4.4.12}$$

Then it can be proved that for all $s\geq 0$,

$$\forall\Phi\in(\mathrm{Ker}L)^\perp,\quad \|\Phi\|^2_{H^{s+1}(\mathbf{T}\times\mathbf{R})}\leq C\|\Phi\|^2_{H^{s+1}_L}\leq C(\Phi|-\Delta'_L\Phi)_{H^s(\mathbf{T}\times\mathbf{R})},$$

which implies in particular that once projected onto $(\mathrm{Ker}L)^\perp$, the system (4.2.9) is fully parabolic. The proof of those inequalities relies on three main arguments. First, the structure of the eigenmodes shows that the diffusion, acting a priori only on the velocity field, also has a smoothing effect on the depth flucutation. Then one proves the orthogonality in $H^s(\mathbf{T}\times\mathbf{R})$ of the eigenmodes corresponding to the same eigenvalue $i\lambda\neq 0$, and finally a "quasi-orthogonality" property on the eigenmodes: one can prove that

$$\forall\Phi\in(\mathrm{Ker}L)^\perp,\quad \|\nabla\Phi\|^2_{H^s}\leq C\sum_{i\lambda\in\mathfrak{S}\setminus\{0\}}\|\nabla(\Pi_\lambda\Phi)\|^2_{H^s}.$$

In the following we will also use the fact that

$$\begin{aligned}\forall\Phi\in(\mathrm{Ker}L)^\perp,\quad \|\Phi_0\|^2_{H^s}&\leq C\sum_{i\lambda\in\mathfrak{S}\setminus\{0\}}\|(\Pi_\lambda\Phi)_0\|^2_{H^s}\\ \text{and}\quad \|\Phi'\|^2_{H^s}&\leq C\sum_{i\lambda\in\mathfrak{S}\setminus\{0\}}\|(\Pi_\lambda\Phi)'\|^2_{H^s}.\end{aligned} \tag{4.4.13}$$

We recall that we have denoted $\Phi=(\Phi_0,\Phi')$. Once those results are obtained, the existence of weak solutions satisfying the usual energy estimate is obtained with a classical approximation method (the approximate sequence being a truncation to a finite number of $\Psi_{n,k,j}$'s).

The proof of the uniqueness of strong solutions is more delicate. Indeed bilinear estimates have to be established on the quadratic form Q_L in function spaces compatible with the diffusion operator Δ'_L, typically the spaces H^s_L. One proves the continuity of the quadratic form, and in particular an estimate of the following type: for any Φ_*, Φ and Φ^* in H^1_L, the following estimate is satisfied

$$\begin{aligned}\Big|(\Phi_*|Q_L(\Phi,\Phi^*))_{L^2(\mathbf{T}\times\mathbf{R})}\Big| \leq C\|\Pi_\perp\Phi_*\|^{3/4}_{H^1_L}\|\Pi_\perp\Phi_*\|^{1/4}_{L^2(\mathbf{T}\times\mathbf{R})}\|\Pi_\perp\Phi\|^{3/4}_{H^1_L}\|\Pi_\perp\Phi^*\|^{3/4}_{H^1_L}\\ \times\Big(\|\Pi_\perp\Phi^*\|^{1/4}_{H^1_L}\|\Pi_\perp\Phi\|^{1/4}_{L^2(\mathbf{T}\times\mathbf{R})}+\|\Pi_\perp\Phi\|^{1/4}_{H^1_L}\|\Pi_\perp\Phi^*\|^{1/4}_{L^2(\mathbf{T}\times\mathbf{R})}\Big)\\ +C\|\Pi_\perp\Phi_*\|_{L^2(\mathbf{T}\times\mathbf{R})}\Big(\|\Pi_0\Phi\|_{L^2(\mathbf{T}\times\mathbf{R})}\|\Pi_\perp\Phi^*\|_{H^1_L}+\|\Pi_0\Phi^*\|_{L^2(\mathbf{T}\times\mathbf{R})}\|\Pi_\perp\Phi\|_{H^1_L}\Big)\,.\end{aligned}$$

This is exactly the analogue of the usual trilinear estimate for the three-dimensional Navier-Stokes equations:

$$\begin{aligned}\Big|(\Phi_*|\mathrm{div}\,(\Phi\otimes\Phi^*))_{L^2(\mathbf{R}^3)}\Big| \leq C\|\Phi_*\|_{\dot H^{\frac34}(\mathbf{R}^3)}\Big(\|\Phi\|_{\dot H^{\frac34}(\mathbf{R}^3)}\|\nabla\Phi^*\|_{L^2(\mathbf{R}^3)}\\ +\|\Phi^*\|_{\dot H^{\frac34}(\mathbf{R}^3)}\|\nabla\Phi\|_{L^2(\mathbf{R}^3)}\Big)\end{aligned}$$

whereas in two space dimensions one would expect

$$\begin{aligned}\Big|(\Phi_*|\mathrm{div}\,(\Phi\otimes\Phi^*))_{L^2(\mathbf{R}^2)}\Big| \leq C\|\Phi_*\|_{\dot H^{\frac12}(\mathbf{R}^2)}\Big(\|\Phi\|_{\dot H^{\frac12}(\mathbf{R}^2)}\|\nabla\Phi^*\|_{L^2(\mathbf{R}^2)}\\ +\|\Phi^*\|_{\dot H^{\frac12}(\mathbf{R}^2)}\|\nabla\Phi\|_{L^2(\mathbf{R}^2)}\Big).\end{aligned}$$

Similarly, three dimensional-type estimates can be derived for $(\Phi_*|Q_L(\Phi,\Phi^*))_{H^{1/2}_L}$, and the usual theory of the three dimensional Navier-Stokes equations enables us to prove the expected existence and uniqueness result. We refer to [25] for all the technicalities; let us simply mention that the reason for the loss of one half derivative compared to the usual dwo dimensional case is linked to the fact that differentiation with respect to x_2 correponds to a multiplication by $\sqrt{n}$ instead of n. □

4.4.3 Interactions between equatorial waves

Unfortunately in order to prove the strong convergence result of Theorem 4.3, more regularity is required on the limit system. We postpone to Section 4.4.5 the end of the proof of the theorem, and will pursue in this section and the next the study of the limit system, in order to gather more useful information. In particular we need to study more precisely the resonance condition $\lambda=\mu+\mu^*$, which can be written

$$\tau(n,k,j)+\tau(n^*,k^*,j^*)=\tau(m,k+k^*,\ell).$$

Recalling that the eigenvalues of the penalization operator L are defined as the roots of a countable number of polynomials whose coefficients depend (linearly) on the ratio β, we deduce that for fixed $n,n^*,m\in\mathbf{N}$ and $k,k^*\in\mathbf{Z}$, the occurence of such a resonant triad is controlled by the cancellation of some polynomial in β. Therefore, either this polynomial has a finite number of zeros, or it is identically zero. The difficulty here is that we are not able to eliminate the second possibility using only the asymptotics $\beta \to \infty$. Because of the possible resonance with $j = j^* = \ell = 0$ which can occur even for large β, we have to refine the previous argument introducing an auxiliary polynomial. We refer to [25] for the details, enabling one to conclude that the limit filtered system can be rewritten in the following manner for all but a countable number of β:

$$\begin{aligned} \partial_t \Pi_0 \Phi - \nu \Pi_0 \Delta'_L \Pi_0 \Phi = 0, \\ \partial_t \Pi_R \Phi + 2Q'_L(\Pi_0\Phi, \Pi_R\Phi) + Q'_L(\Pi_R\Phi, \Pi_R\Phi) - \nu \Pi_R \Delta'_L \Phi = 0, \\ \partial_t \Pi_M \Phi + 2Q'_L(\Pi_0\Phi, \Pi_M\Phi) - \nu \Pi_M \Delta'_L \Phi = 0, \\ \partial_t \Pi_P \Phi + 2Q'_L(\Pi_0\Phi, \Pi_P\Phi) - \nu \Pi_P \Delta'_L \Phi = 0, \\ \partial_t \Pi_K \Phi + 2Q'_L(\Pi_0\Phi, \Pi_K\Phi) + Q'_L(\Pi_K\Phi, \Pi_K\Phi) - \nu \Pi_K \Delta'_L \Phi = 0, \end{aligned} \tag{4.4.14}$$

It can be noticed that the only nonlinear interactions are due to Kelvin or to Rossby-type waves, which will be crucial in the proof of the propagation of regularity.

Let us prove estimate (4.2.10). We can decompose Φ in the basis of eigenvectors of L, and will estimate each projection separately. Clearly we have div $(\Pi_0\Phi)' = 0$, so let us consider now the projection onto Rossby modes $\Pi_R\Phi$. By definition of the Rossby modes we deduce the following relation

$$\forall \Phi \in R, \quad \nabla \cdot \Phi' = \sum_{i\lambda \in \mathfrak{S}_R} \nabla \cdot \Phi'_\lambda = \sum_{i\lambda \in \mathfrak{S}_R} i\lambda (\Phi_\lambda)_0$$

with the notation $\Phi_\lambda = \Pi_\lambda \Phi$, and where $\mathfrak{S}_R$ denotes the set of Roosby modes. It follows that, using (4.4.13),

$$\begin{aligned} \|\nabla \cdot \Phi'\|^2_{H^2(\mathbf{T}\times\mathbf{R})} &= \Big\| \sum_{i\lambda \in \mathfrak{S}_R} i\lambda (\Phi_\lambda)_0 \Big\|^2_{H^2(\mathbf{T}\times\mathbf{R})} \\ &\le C \sum_{i\lambda \in \mathfrak{S}_R} \|\lambda(\Phi_\lambda)_0\|^2_{H^2(\mathbf{T}\times\mathbf{R})} \,. \end{aligned}$$

But, as Rossby waves correspond to $j = 0$, we have (denoting by $\Pi_{n,k,j}$ the orthogonal projection onto $\Psi_{n,k,j}$)

$$\|\lambda(\Phi_\lambda)_0\|^2_{H^2(\mathbf{T}\times\mathbf{R})} \le C|\lambda|^2 \sum_{\tau(n,k,0)=\lambda} \|(\Pi_{n,k,0}\Phi)_0\|^2_{H^2(\mathbf{T}\times\mathbf{R})}.$$

Recalling the explicit form of $(\Psi_{n,k,j})_0$, we see that

$$\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^2(\mathbf{T}\times\mathbf{R})} \le C(1+n+k^2)\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^1(\mathbf{T}\times\mathbf{R})}.$$

But for Rossby modes, the following asymptotics hold as $|k|$ or n goes to infinity:

$$\lambda = \tau(n,k,0) \sim \frac{\beta k}{k^2 + \beta(2n+1)}.$$

So we infer that as $|k|$ or n goes to infinity,

$$|\lambda|^2 \|(\Pi_{n,k,0}\Phi)_0\|^2_{H^2(\mathbf{T}\times\mathbf{R})} \leq C\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^1(\mathbf{T}\times\mathbf{R})}.$$

Finally we infer that

$$\begin{aligned}\|\nabla\cdot\Phi'\|^2_{H^2} &\leq C\sum_{i\lambda\in\mathfrak{S}_R}\|\lambda(\Phi_\lambda)_0\|^2_{H^2(\mathbf{T}\times\mathbf{R})}\\ &\leq C\sum_{(n,k,0)\in\mathfrak{S}_R}\|(\Pi_{n,k,0}\Phi)_0\|^2_{H^1(\mathbf{R}\times\mathbf{T})}\\ &\leq C\|\Phi\|^2_{H^1_L}.\end{aligned}$$

By the embedding of $H^2(\mathbf{T}\times\mathbf{R})$ into $L^\infty(\mathbf{T}\times\mathbf{R})$ we conclude that $\nabla\cdot(\Pi_R\Phi)'$ belongs to the space $L^2([0,T];L^\infty(\mathbf{T}\times\mathbf{R}))$. The same result can easily be extended to the mixed Poincaré-Rossby modes (it is in fact easier since $n=0$ in that case) and we obtain

$$\|\Pi_M\Phi\|_{L^2([0,T],H^1_L)} \leq C_T, \quad \|\nabla\cdot(\Pi_M\Phi)'\|_{L^2([0,T],L^\infty(\mathbf{T}\times\mathbf{R}))} \leq C_T\,.$$

Finally we deduce that

$$\|\nabla\cdot((\Pi_0+\Pi_R+\Pi_M)\Phi)'\|_{L^2([0,T],L^\infty(\mathbf{T}\times\mathbf{R}))} \leq C_T\,.$$

Let us now consider the equation governing the Poincaré modes which can be seen as a linear parabolic equation whose coefficients depend on $\Pi_0\Phi$. We can write

$$\Pi_P\Phi = \sum_{(n,k,j)\in S_P}\varphi_{n,k,j}\Psi_{n,k,j},$$

where

$$S_P = \mathbf{N}^*\times\mathbf{Z}\times\{-1,1\}\,\cup\,\{0\}\times\mathbf{Z}^+_*\times\{1\}\,\cup\,\{0\}\times\mathbf{Z}^-_*\times\{-1\}.$$

We can use Proposition 4.6 to deduce that for each (n,k,j) in S_P the equation governing $\varphi_{n,k,j}$ can be decoupled (recall that $\Pi_0\Phi$ only depends on x_2):

$$\partial_t\varphi_{n,k,j} - \nu\varphi_{n,k,j}(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{T}\times\mathbf{R})} = -2\varphi_{n,k,j}(\Psi_{n,k,j}|Q(\Psi_{n,k,j},\Pi_0\Phi))_{L^2(\mathbf{T}\times\mathbf{R})}$$

which can be rewritten

$$\begin{aligned}&\partial_t\left(\varphi_{n,k,j}\exp\left(-\nu t(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{T}\times\mathbf{R})}\right)\right)\\ &\quad= -2\varphi_{n,k,j}(\Psi_{n,k,j}|Q(\Psi_{n,k,j},\Pi_0\Phi))_{L^2(\mathbf{T}\times\mathbf{R})}\exp\left(-\nu t(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{T}\times\mathbf{R})}\right).\end{aligned}$$

From Gronwall's lemma and the following estimates (due to (4.4.6) and to the bound of $\Pi_0\Phi$ in $L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))$),

$$\begin{aligned}\left|(\Psi_{n,k,j}|Q(\Psi_{n,k,j},\Pi_0\Phi))_{L^2(\mathbf{T}\times\mathbf{R})}\right| &\leq C_1(n+k^2)^{1/2},\\ -(\Psi_{n,k,j}|\Delta'\Psi_{n,k,j})_{L^2(\mathbf{T}\times\mathbf{R})} &\geq C_2(n+k^2),\end{aligned}$$

we then deduce that there exists a nonnegative constant C_ν (depending only on ν) such that,

$$\forall (n,k,j) \in S_P, \quad |\varphi_{n,k,j}(t)| \le |\varphi_{n,k,j}(0)| \exp(-C_\nu(n+k^2)t). \tag{4.4.15}$$

Now we write

$$\begin{aligned}\left\|\nabla\cdot(\Pi_P\Phi)'(t)\right\|_{L^\infty(\mathbf{T}\times\mathbf{R})} &\le \sum_{(n,k,j)\in S_P} |\varphi_{n,k,j}(t)| \left\|\nabla\cdot(\Psi_{n,k,j})'(t)\right\|_{L^\infty(\mathbf{T}\times\mathbf{R})}\\ &\le C\sum_{(n,k,j)\in S_P} |\varphi_{n,k,j}(t)|(n+k^2)^{1/2}\end{aligned}$$

since $(\Psi_{n,k,j})$ is uniformly bounded in $L^\infty(\mathbf{T}\times\mathbf{R})$. Thus, by (4.4.15), we infer that

$$\left\|\nabla\cdot(\Pi_P\Phi)'(t)\right\|_{L^\infty(\mathbf{T}\times\mathbf{R})} \le C\sum_{(n,k,j)\in S_P} |\varphi_{n,k,j}(0)|(n+k^2)^{1/2}\exp(-C_\nu(n+k^2)t).$$

Integrating with respect to time leads then to

$$\begin{aligned}\left\|\nabla\cdot(\Pi_P\Phi)'\right\|_{L^1([0,T];L^\infty(\mathbf{T}\times\mathbf{R}))} &\le C'_\nu \sum_{(n,k,j)\in S_P} |\varphi_{n,k,j}(0)|(n+k^2)^{-1/2}\\ &\le C'_\nu\left(\sum_{(n,k,j)\in S_P}|\varphi_{n,k,j}(0)|^2(n+k^2)^{\alpha}\right)^{1/2}\left(\sum_{(n,k,j)\in S_P}(n+k^2)^{-1-\alpha}\right)^{1/2},\end{aligned}$$

from which we deduce that for $\alpha > 1/2$,

$$\left\|\nabla\cdot(\Pi_P\Phi)'\right\|_{L^1([0,T];L^\infty(\mathbf{T}\times\mathbf{R}))} \le C\|\Pi_P\Phi^0\|_{\tilde H^\alpha_L(\mathbf{T}\times\mathbf{R})}$$

where C depends only on ν and α.

Finally we are left with the Kelvin modes. The difficulty here is that the equation is nonlinear, and the argument of the Rossby part does not work (there is no natural smoothing of the divergence). However $\Pi_K\Phi$ satisfies an equation which is actually one-dimensional (modulo a smooth function with respect to x_2), and thus the energy estimate is supercritical in the sense that the H^1 norm allows to control the stability. We first note that for the Kelvin modes, since the decomposition of the eigenmodes of L corresponds to the Fourier decomposition, we have

$$(\Pi_K\Phi|Q_L(\Pi_K\Phi,\Pi_K\Phi))_{H^\alpha(\mathbf{T}\times\mathbf{R})} = (\Pi_K\Phi|Q(\Pi_K\Phi,\Pi_K\Phi))_{H^\alpha(\mathbf{T}\times\mathbf{R})}\,.$$

Therefore, using the fact that $H^\alpha(\mathbf{T})$ is an algebra for all $\alpha > 1/2$, we get

$$\begin{aligned}\left|(\Pi_K\Phi|Q(\Pi_K\Phi,\Pi_K\Phi))_{H^\alpha(\mathbf{T}\times\mathbf{R})}\right| &\le C\|\Pi_K\Phi\|_{H^{\alpha+1}(\mathbf{T}\times\mathbf{R})}\|\Pi_K\Phi\|_{H^\alpha(\mathbf{R}\times\mathbf{T})}\|\Pi_K\Phi\|_{H^1(\mathbf{T}\times\mathbf{R})}\\ &\le \nu\|\Pi_K\Phi\|^2_{H^{\alpha+1}(\mathbf{T}\times\mathbf{R})}+\frac{C}{\nu}\|\Pi_K\Phi\|^2_{H^\alpha(\mathbf{T}\times\mathbf{R})}\|\Pi_K\Phi\|^2_{H^1(\mathbf{T}\times\mathbf{R})}\end{aligned}$$

by the Cauchy-Schwarz inequality.

Estimating the linear term as before, we get by Gronwall's lemma

$$\|\Pi_K\Phi(t)\|^2_{H^\alpha(\mathbf{T}\times\mathbf{R})} + \nu\int_0^t \|\Pi_K\Phi(t')\|^2_{H^{\alpha+1}(\mathbf{T}\times\mathbf{R})}dt' \leq \|\Pi_K\Phi^0\|^2_{H^\alpha(\mathbf{T}\times\mathbf{R})} \times \exp\left(\frac{C}{\nu}\int \|\Phi(\tau)\|^2_{H^1_L(\mathbf{T}\times\mathbf{R})}d\tau\right).$$

Then,

$$\|\Pi_K\Phi\|_{L^\infty([0,T],\tilde{H}^\alpha_L(\mathbf{T}\times\mathbf{R}))} \leq C_T \quad \text{and} \quad \|\Pi_K\Phi\|_{L^2([0,T],\tilde{H}^{\alpha+1}_L(\mathbf{T}\times\mathbf{R}))} \leq C_T\,, \tag{4.4.16}$$

under the suitable initial assumption. From the orthogonality properties mentioned earlier, along with the Sobolev embeddings $H^\alpha(\mathbf{T}) \subset L^\infty(\mathbf{T})$ we infer that

$$\|\nabla\cdot(\Pi_K\Phi)'\|_{L^2([0,T],L^\infty(\mathbf{T}\times\mathbf{R}))} \leq C_T\,, \tag{4.4.17}$$

provided that $\alpha > 1/2$.

The estimate (4.2.10) is proved. □

4.4.4 Propagation of regularity

In this paragraph we shall state without proof some useful results concerning the propagation of regularity for the limit system.

Propagation of regularity for $\Pi_0\Phi$

Let us notice that the weak formulation of the limit equation given in the statement of Theorem 4.2 could be written in the more compact way

$$\partial_t\Phi - \nu\Pi_0\Delta'\Pi_0\Phi = 0, \tag{4.4.18}$$

were it not for the fact that the operator $\Pi_0\Delta'\Pi_0$ is a priori not defined on $L^2(\mathbf{R})$. The projection Π_0 is a pseudo-differential operator, whose symbol is

$$\begin{pmatrix} \frac{(\beta x_2)^2}{(\beta x_2)^2+\xi_2^2} & \frac{-i\beta x_2\xi_2}{(\beta x_2)^2+\xi_2^2} & 0 \\ \frac{i\beta x_2\xi_2}{(\beta x_2)^2+\xi_2^2} & \frac{\xi_2^2}{(\beta x_2)^2+\xi_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular extending Π_0 to Sobolev spaces requires some techniques of microlocal analysis like the Weyl-Hörmander calculus. The singularity at $x_2 = 0$ unfortunately prevents one of using this theory blindfolded, but inspired by the results given by that theory, in particular its commutator estimates, one can work "by hand" (see [25] for details) to prove the following proposition.

Proposition 4.8 *Denote by $\underline{\Phi}$ the (unique) weak solution to the geostrophic equation*

$$\partial_t \underline{\Phi} - \nu \Pi_0 \Delta' \Pi_0 \underline{\Phi} = 0, \quad \underline{\Phi}(t=0) = \Pi_0 \Phi^0.$$

Then, if the initial data satisfies the regularity assumption

$$\|\Pi_0 \Phi^0\|_{H^s_L} \leq C_0$$

for some $s \geq 1$, the solution satisfies for all $T > 0$ the regularity estimate

$$\|\underline{\Phi}\|_{L^\infty([0,T],H^s_L)} \leq C_T.$$

Propagation of regularity for $\Pi_\perp \Phi$

One can prove bilinear estimates in H^s_L for Q_L for $s \in [1/2, 1[$, which allow to deduce easily the following result.

Proposition 4.9 *Denote by Φ the (unique) strong solution on $[0, T^*[$ to the envelope equations*

$$\partial_t \Phi + Q_L(\Phi, \Phi) - \nu \Delta'_L \Phi = 0$$
$$\Pi_\perp \Phi(t=0) = \Pi_\perp \Phi^0 \in H^{1/2}_L$$
$$\Pi_0 \Phi(t=0) = \Pi_0 \Phi^0 \in L^2(\mathbf{T} \times \mathbf{R}).$$

If the initial data satisfies the regularity assumption

$$\|\Pi_\perp \Phi^0\|_{H^s_L} \leq C_0$$

for some $s \in [1/2, 1[$, then the solution satisfies for all $T < T^*$ the regularity estimate

$$\|\Pi_\perp \Phi\|_{L^\infty([0,T],H^s_L) \cap L^2([0,T],H^{s+1}_L)} \leq C_T.$$

4.4.5 Stability and strong convergence

In this final section we shall gather the previous results in order to prove the strong convergence of the filtered solutions. The idea is, as usual in filtering methods, to start by approximating the solution of the limit system, and then to use a weak-strong stability method to conclude.

So let us consider the solution Φ constructed in the previous paragraph, which we truncate in the following way:

$$\Phi_N = J_N \Pi_\perp \Phi + \Pi_0 \Phi_N, \tag{4.4.19}$$

where J_N is the spectral truncation defined by

$$J_N = \sum_{i\lambda \in \mathfrak{S}_N} \Pi_\lambda \tag{4.4.20}$$

with

$$\mathfrak{S}_N = \left\{ i\tau(n,k,j) \in \mathfrak{S} \Big/ n \leq N, \ |k| \leq N \right\},$$

and $\Pi_\perp$ denotes as previously the projection onto $(\mathrm{Ker}L)^\perp$. Finally $\Pi_0\Phi_N$ solves

$$\partial_t \Pi_0 \Phi_N - \nu \Pi_0 \Delta' \Pi_0 \Phi_N = 0$$
$$\Pi_0 \Phi_{N|t=0} = \sum_{0\leq n\leq N} \Pi_{n,0,0}\Phi^0,$$

where $\Pi_{n,0,0}$ denotes the projection onto the eigenvector $\Psi_{n,0,0}$ of Ker$L$. According to Proposition 4.8, for all fixed $N \in \mathbf{N}$ we have

$$\Pi_0\Phi_N \text{ belongs to } L^\infty(\mathbf{R}^+; H^\sigma_L),\ \forall \sigma \geq 0. \tag{4.4.21}$$

Recall that such a result means that $\Pi_0\Phi_N$ is as smooth as needed, and decays as fast as needed when x_2 goes to infinity. Moreover by the stability of the limit system (which is linear) we have of course

$$\lim_{N\to\infty} \|\Pi_0\Phi_N - \Pi_0\Phi\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))} = 0,\ \forall T > 0.$$

Note also that for all fixed $N \in \mathbf{N}$, using the smoothness and the decay of the eigenvectors of L, we get for any polynomial $Q \in \mathbf{R}[X]$

$$Q(x_2)\Phi_N \in L^\infty([0,T]; C^\infty(\mathbf{T}\times\mathbf{R}))$$

We have moreover of course

$$\forall T < T^*, \quad \|\Pi_\perp(\Phi - \Phi_N)\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))} \to 0 \text{ as } N \to \infty,$$

and

$$\forall T < T^*, \quad \|\Pi_\perp(\Phi - \Phi_N)\|_{L^2([0,T];H^{\alpha+1}_L)} \to 0 \text{ as } N \to \infty.$$

Finally since J_N commutes with Δ'_L, the vector field Φ_N satisfies the approximate limit filtered system

$$\begin{aligned} \partial_t\Phi_N + J_N Q_L(\Phi,\Phi) - \nu\Delta'_L\Phi_N &= 0, \\ \Phi_{N|t=0} &= J_N\Phi^0. \end{aligned} \tag{4.4.22}$$

Conjugating this equation by the semi-group $\mathcal{L}$ leads then to

$$\partial_t\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + \frac{1}{\varepsilon}L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + J_N Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right) - \nu\Delta'_L\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N = 0,$$

using the definitions (4.4.11) of Q_L and Δ'_L. We are going now to follow the same method as that used in Chapter 2, in the periodic case: we start by rewriting this last equation in a convenient way

$$\begin{aligned} &\partial_t\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + \frac{1}{\varepsilon}L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) - \nu\Delta'\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N \\ &= (Q - Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) - \nu(\Delta' - \Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N \\ &+ (Id - J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right) + Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N - \Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N + \Phi)\right). \end{aligned}$$

The two last terms in the right-hand side are expected to be small when N is large, uniformly in ε, using the stability of the limit system. So we are left with the first two terms, which as usual cannot be dealt with so easily since they do not converge strongly towards zero. However they are fast oscillating terms, and will be treated by introducing a small quantity $\varepsilon\phi_N$ (which will be small when ε goes to zero, for each fixed N), so that

$$(\partial_t + \frac{1}{\varepsilon}L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\varepsilon\phi_N\right) \sim -(Q - Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + \nu(\Delta' - \Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N.$$

Let us now define

$$\phi_N = -\sum_{\substack{\lambda\neq\mu+\tilde{\mu} \\ i\lambda\in\mathfrak{S}, i\mu, i\tilde{\mu}\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu-\tilde{\mu})}}{i(\lambda-\mu-\tilde{\mu})}\Pi_\lambda Q(\Pi_\mu\Phi_N, \Pi_{\tilde{\mu}}\Phi_N) + \nu \sum_{\substack{\lambda\neq\mu, \\ i\lambda\in\mathfrak{S}, i\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu)}}{i(\lambda-\mu)}\Pi_\lambda\Delta'\Pi_\mu\Phi_N, \tag{4.4.23}$$

and consider

$$\Phi_{\varepsilon,N} = \Phi_N + \varepsilon\phi_N.$$

The proof of the following result follows essentially the same lines as in the constant, periodic case of Chapter 2 (up to the fact that λ is not truncated here) and we refer to [25] for details.

Proposition 4.10 *For all but a countable number of β, the following result holds. Consider a vector field $\Phi^0 = (\eta_0, u_0) \in L^2(\mathbf{T}\times\mathbf{R})$, with $\Pi_\perp\Phi^0$ in H^α_L for some $\alpha > 1/2$. Denote by Φ the associate solution of the limit system on $[0, T^*[$. Then there exists a family $(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_{\varepsilon,N}$ such that $\Pi_\perp(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is uniformly bounded in the space $L^\infty_{loc}([0, T^*[, H^\alpha_L) \cap L^2_{loc}([0, T^*[, H^{\alpha+1}_L)$, satisfying the following properties:*

- *$\Phi_{\varepsilon,N}$ behaves asymptotically as Φ as $\varepsilon \to 0$ and $N \to \infty$:*

$$\forall T < T^*, \quad \lim_{N\to\infty}\lim_{\varepsilon\to 0}\|\Phi_{\varepsilon,N} - \Phi\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))} = 0; \tag{4.4.24}$$

- *for all $N \in \mathbf{N}$, $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is smooth: for all $T \in [0, T^*[$ and all $Q \in \mathbf{R}[X]$,*

$$Q(x_2)(\eta_{\varepsilon,N}, u_{\varepsilon,N}) \text{ is bounded in } L^\infty([0,T]; C^\infty(\mathbf{T}\times\mathbf{R})), \text{ uniformly in } \varepsilon; \tag{4.4.25}$$

- *$(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ satisfies the uniform regularity estimate*

$$\forall T \in [0, T^*[, \quad \sup_{N\in\mathbf{N}}\lim_{\varepsilon\to 0}\|\nabla\cdot u_{\varepsilon,N}\|_{L^1([0,T];L^\infty(\mathbf{T}\times\mathbf{R}))} \leq C_T; \tag{4.4.26}$$

- *$(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ satisfies approximatively the viscous Saint-Venant system (SW_ε) :*

$$\partial_t(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + \frac{1}{\varepsilon}L(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + Q\left((\eta_{\varepsilon,N}, u_{\varepsilon,N}), (\eta_{\varepsilon,N}, u_{\varepsilon,N})\right) - \nu\Delta'(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = R_{\varepsilon,N} \tag{4.4.27}$$

where $R_{\varepsilon,N}$ goes to 0 as $\varepsilon \to 0$ then $N \to \infty$:

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\left(\|R_{\varepsilon,N}\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))} + \varepsilon\|R_{\varepsilon,N}\|_{L^\infty([0,T]\times\mathbf{T}\times\mathbf{R})}\right) = 0. \tag{4.4.28}$$

Equipped with that result, we are now ready to prove the strong convergence theorem. The method relies on a weak-strong stability method which we shall now detail. We are going to prove that

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\|(\eta_\varepsilon,u_\varepsilon)-(\eta_{\varepsilon,N},u_{\varepsilon,N})\|_{L^2([0,T]\times\mathbf{T}\times\mathbf{R})}=0, \tag{4.4.29}$$

where $(\eta_{\varepsilon,N},u_{\varepsilon,N})$ is the approximate solution to (4.1.1) defined in Proposition 4.10. Note that combining this estimate with the fact that $(\eta_{\varepsilon,N},u_{\varepsilon,N})$ is close to $\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi$ provides the expected convergence, namely the fact that

$$\forall T\in[0,T^*[,\quad \lim_{\varepsilon\to 0}\left\|(\eta_\varepsilon,u_\varepsilon)-\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right\|_{L^2([0,T]\times\mathbf{T}\times\mathbf{R})}=0.$$

The key to the proof of (4.4.29) lies in the following proposition.

Proposition 4.11 *There is a constant C such that the following property holds. Let (η^0,u^0) and $(\eta^0_\varepsilon,u^0_\varepsilon)$ satisfy assumption (4.2.8), and let $T>0$ be given. For all $\varepsilon>0$, denote by $(\eta_\varepsilon,u_\varepsilon)$ a solution of (4.1.1) with initial data (η^0,u^0). For any vector field $(\bar\eta,\bar u)$ belonging to $L^\infty([0,T];C^\infty(\mathbf{T}\times\mathbf{R}))$ and rapidly decaying with respect to x_2, define*

$$\mathcal{E}_\varepsilon(t)=\frac12\int\left((\eta_\varepsilon-\bar\eta)^2+(1+\varepsilon\eta_\varepsilon)|u_\varepsilon-\bar u|^2\right)(t,x)dx+\nu\int_0^t\int|\nabla(u_\varepsilon-\bar u)|^2(t',x)dxdt'.$$

Then the following stability inequality holds for all $t\in[0,T]$:

$$\begin{aligned}&\mathcal{E}_\varepsilon(t)\le C\mathcal{E}_\varepsilon(0)\exp\left(\chi(t)\right)+\omega_\varepsilon(t)\\ &+C\int_0^t e^{\chi(t)-\chi(t')}\int\left(\partial_t\bar\eta+\frac1\varepsilon\nabla\cdot\bar u+\nabla\cdot(\bar\eta\bar u)\right)(\bar\eta-\eta_\varepsilon)(t',x)dxdt'\\ &+C\int_0^t e^{\chi(t)-\chi(t')}\int(1+\varepsilon\eta_\varepsilon)\left(\partial_t\bar u+\frac1\varepsilon(\beta x_2\bar u^\perp+\nabla\bar\eta)+(\bar u\cdot\nabla)\bar u-\nu\Delta\bar u\right)\cdot(\bar u-u_\varepsilon)(t',x)dxdt',\end{aligned}$$

where $\omega_\varepsilon(t)$ depends on $\bar u$ and goes to zero with ε, uniformly in time, and where

$$\chi(t)=C\int_0^t\left(\|\nabla\cdot\bar u\|_{L^\infty(\mathbf{T}\times\mathbf{R})}+\|\nabla\bar u\|^2_{L^2(\mathbf{T}\times\mathbf{R})}\right)(t')dt'.$$

Let us postpone the proof of that result, and end the proof of the strong convergence. We apply that proposition to $(\bar\eta,\bar u)=(\eta_{\varepsilon,N},u_{\varepsilon,N})$, where $(\eta_{\varepsilon,N},u_{\varepsilon,N})$ is the approximate solution on $[0,T^*[$ given by Proposition 4.10. We will denote by $\chi_{\varepsilon,N}$ and $\mathcal{E}_{\varepsilon,N}$ the quantities defined in Proposition 4.11, where $(\bar\eta,\bar u)$ has been replaced by $(\eta_{\varepsilon,N},u_{\varepsilon,N})$.

Because of the uniform regularity estimates on $(\eta_{\varepsilon,N},u_{\varepsilon,N})$, we have

$$\forall T\in[0,T^*[,\quad \sup_N\lim_{\varepsilon\to 0}\left(\|\nabla u_{\varepsilon,N}\|^2_{L^2([0,T],L^2(\mathbf{T}\times\mathbf{R}))}+\|\nabla\cdot u_{\varepsilon,N}\|_{L^1([0,T];L^\infty(\mathbf{T}\times\mathbf{R}))}\right)\le C_T,$$

so we get a uniform bound on $\chi_{\varepsilon,N}$:

$$\sup_N\lim_{\varepsilon\to 0}\|\chi_{\varepsilon,N}\|_{L^\infty([0,T])}\le C_T.$$

Then, from the initial convergence (4.2.8) we obtain that

$$\forall N \in \mathbf{N}, \quad \mathcal{E}_{\varepsilon,N}(0)\exp\left(\chi_{\varepsilon,N}(t)\right) \to 0 \text{ as } \varepsilon \to 0 \text{ in } L^\infty([0,T]).$$

Moreover by Proposition 4.11 we have

$$\partial_t(\eta_{\varepsilon,N},u_{\varepsilon,N}) + \frac{1}{\varepsilon}L(\eta_{\varepsilon,N},u_{\varepsilon,N}) + Q((\eta_{\varepsilon,N},u_{\varepsilon,N}),(\eta_{\varepsilon,N},u_{\varepsilon,N})) - \nu\Delta'(\eta_{\varepsilon,N},u_{\varepsilon,N}) = R_{\varepsilon,N}. \tag{4.4.30}$$

Let us estimate the contribution of the remainder term. We can write

$$\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot\left((\eta_{\varepsilon,N}-\eta_\varepsilon),(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)\right)(t',x)dxdt' \\ = I^{(1)}_{\varepsilon,N}(t) + I^{(2)}_{\varepsilon,N}(t),$$

with

$$\begin{aligned} I^{(1)}_{\varepsilon,N}(t) &\stackrel{\text{def}}{=} \int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N,0}(\eta_{\varepsilon,N}-\eta_\varepsilon)(t',x)dxdt', \quad \text{and} \\ I^{(2)}_{\varepsilon,N}(t) &\stackrel{\text{def}}{=} \int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R'_{\varepsilon,N}(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)(t',x)dxdt'. \end{aligned}$$

The first term can be estimated in the following way:

$$|I^{(1)}_{\varepsilon,N}(t)| \le C_T\|R_{\varepsilon,N}\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))}\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))}.$$

For the second term we can write

$$|I^{(2)}_{\varepsilon,N}(t)| \le C_T\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N}-u_\varepsilon)\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))}\|\sqrt{1+\varepsilon\eta_\varepsilon}R_{\varepsilon,N}\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))}.$$

Now we can write

$$\|\sqrt{1+\varepsilon\eta_\varepsilon}R_{\varepsilon,N}\|^2_{L^2(\mathbf{T}\times\mathbf{R})} \le C(\|R_{\varepsilon,N}\|^2_{L^2(\mathbf{T}\times\mathbf{R})} + \varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})}\|R_{\varepsilon,N}\|^2_{L^4(\mathbf{T}\times\mathbf{R})}).$$

Since

$$\varepsilon\|R_{\varepsilon,N}\|^2_{L^4(\mathbf{T}\times\mathbf{R})} \le \varepsilon\|R_{\varepsilon,N}\|_{L^\infty(\mathbf{T}\times\mathbf{R})}\|R_{\varepsilon,N}\|_{L^2(\mathbf{T}\times\mathbf{R})},$$

we infer that the quantity $\varepsilon^{\frac12}R_{\varepsilon,N}$ goes to zero as ε goes to zero and N goes to infinity, in the space $L^2([0,T];L^4(\mathbf{T}\times\mathbf{R}))$, so in particular

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{\frac12}\|R_{\varepsilon,N}\|_{L^1([0,T];L^4(\mathbf{T}\times\mathbf{R}))} = 0.$$

Finally by the uniform bound on η_ε in $L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))$ and by the smallness assumptions on $R_{\varepsilon,N}$, we deduce that

$$\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot\left((\eta_{\varepsilon,N}-\eta_\varepsilon),(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)\right)(t',x)dxdt' \\ \le \frac{1}{2}(\|\eta_{\varepsilon,N}-\eta_\varepsilon\|^2_{L^\infty([0,T];L^2)} + \|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N}-u_\varepsilon)\|^2_{L^\infty([0,T];L^2)}) + \omega_{\varepsilon,N}(t),$$

where

$$\lim_{N\to\infty}\lim_{\varepsilon\to0}\|\omega_{\varepsilon,N}(t)\|_{L^\infty([0,T])}=0.$$

We now recall that by Proposition 4.11, using (4.4.30), we have

$$\begin{aligned}\mathcal{E}_{\varepsilon,N}(t)&\leq C\mathcal{E}_{\varepsilon,N}(0)\exp\left(\chi_{\varepsilon,N}(t)\right)+\omega_{\varepsilon,N}(t)\\&\quad+C\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot\left((\eta_{\varepsilon,N}-\eta_\varepsilon),(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)\right)(t',x)dxdt'\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}_{\varepsilon,N}(t)=\frac12\Big(\|(\eta_\varepsilon-\eta_{\varepsilon,N})(t)\|_{L^2}^2+\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-u_{\varepsilon,N})(t)\|_{L^2}^2\Big)\\+\nu\int_0^t\|\nabla(u_\varepsilon-u_{\varepsilon,N})(t')\|_{L^2}^2(t')dt'.\end{aligned}$$

Putting together the previous results we get that $\lim\limits_{N\to\infty}\lim\limits_{\varepsilon\to0}\mathcal{E}_{\varepsilon,N}(t)=0$ uniformly on $[0,T]$, hence that

$$\begin{aligned}\lim_{N\to\infty}\lim_{\varepsilon\to0}\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))}&=0,\\\lim_{N\to\infty}\lim_{\varepsilon\to0}\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N}-u_\varepsilon)\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))}&=0,\\\lim_{N\to\infty}\lim_{\varepsilon\to0}\|u_{\varepsilon,N}-u_\varepsilon\|_{L^2([0,T],\dot H^1(\mathbf{T}\times\mathbf{R}))}&=0.\end{aligned}$$

By interpolation we therefore find that

$$\lim_{N\to\infty}\lim_{\varepsilon\to0}\left(\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))}+\|u_{\varepsilon,N}-u_\varepsilon\|_{L^2([0,T],H^1(\mathbf{T}\times\mathbf{R}))}\right)=0,$$

hence (4.4.29) is proved. □

To conclude the proof of Theorem 4.3 it remains to give an idea of the proof of Proposition 4.11. As the energy is a Lyapunov functional for (4.1.1), we have

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)&\leq\int_0^t\frac{d}{dt}\int\Big((\frac12\bar\eta^2-\bar\eta\eta_\varepsilon)+(1+\varepsilon\eta_\varepsilon)(\frac12|\bar u|^2-\bar u\cdot u_\varepsilon)\Big)(t',x)dxdt'\\&\qquad+\int_0^t\int\nu(\nabla\bar u-2\nabla u_\varepsilon)\cdot\nabla\bar u(t',x)dxdt'\\&\leq\int_0^t\int\left(\partial_t\bar\eta(\bar\eta-\eta_\varepsilon)+(1+\varepsilon\eta_\varepsilon)\partial_t\bar u\cdot(\bar u-u_\varepsilon)\right)(t',x)dxdt'\\&\qquad-\int_0^t\int\Big(\partial_t\eta_\varepsilon\bar\eta+\partial_t((1+\varepsilon\eta_\varepsilon)u_\varepsilon)\cdot\bar u-\frac{\varepsilon}{2}\partial_t\eta_\varepsilon|\bar u|^2)\Big)(t',x)dxdt'\\&\qquad-\int_0^t\int\nu\left(\Delta\bar u\cdot(\bar u-u_\varepsilon)-\Delta u_\varepsilon\cdot\bar u\right)(t',x)dxdt'.\end{aligned}$$

Using the conservation of mass and of momentum we get

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)\le&\int_0^t\int\left(\partial_t\bar\eta(\bar\eta-\eta_\varepsilon)+(1+\varepsilon\eta_\varepsilon)(\partial_t\bar u-\nu\Delta\bar u)\cdot(\bar u-u_\varepsilon)\right)(t',x)dxdt'\\&+\int_0^t\int\frac{1}{\varepsilon}\nabla\cdot((1+\varepsilon\eta_\varepsilon)u_\varepsilon)\left(\bar\eta-\frac{\varepsilon}{2}|\bar u|^2\right)(t',x)dxdt'\\&+\int_0^t\int\left(\frac{(1+\varepsilon\eta_\varepsilon)}{\varepsilon}(\beta x_2u_\varepsilon^\perp+\nabla\eta_\varepsilon)+\nabla\cdot((1+\varepsilon\eta_\varepsilon)u_\varepsilon\otimes u_\varepsilon)\right)\cdot\bar u(t',x)dxdt'\\&+\int_0^t\int\varepsilon\nu\eta_\varepsilon\Delta\bar u\cdot(\bar u-u_\varepsilon)(t',x)dxdt'.\end{aligned}$$

Integrating by parts leads then to

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)\le&\int_0^t\int\left(\partial_t\bar\eta+\frac{1}{\varepsilon}\nabla\cdot\bar u+\nabla\cdot(\bar\eta\bar u)\right)(\bar\eta-\eta_\varepsilon)(t',x)dxdt'\\&+\int_0^t\int(1+\varepsilon\eta_\varepsilon)\left(\partial_t\bar u+\frac{1}{\varepsilon}(\beta x_2\bar u^\perp+\nabla\bar\eta)+(\bar u\cdot\nabla)\bar u-\nu\Delta\bar u\right)\cdot(\bar u-u_\varepsilon)(t',x)dxdt'\\&-\int_0^t\int(1+\varepsilon\eta_\varepsilon)D\bar u:(\bar u-u_\varepsilon)^{\otimes2}(t',x)dxdt'\\&-\int_0^t\int\left(\frac12\eta_\varepsilon^2\nabla\cdot\bar u+(\bar\eta-\eta_\varepsilon)\nabla\cdot(\bar\eta\bar u)+\eta_\varepsilon\bar u\cdot\nabla\bar\eta\right)(t',x)dxdt'+R_\varepsilon,\end{aligned}\tag{4.4.31}$$

where

$$R_\varepsilon(t)=\int_0^t\int\varepsilon\nu\eta_\varepsilon\Delta\bar u\cdot(\bar u-u_\varepsilon)(t',x)dxdt'.$$

The last term is rewritten in a convenient form by integrating by parts

$$\begin{aligned}-\int_0^t\int&\left(\frac12\eta_\varepsilon^2\nabla\cdot\bar u+(\bar\eta-\eta_\varepsilon)\nabla\cdot(\bar\eta\bar u)+\eta_\varepsilon\bar u\cdot\nabla\bar\eta\right)(t',x)dxdt'\\&=-\int_0^t\int\left(\frac12\eta_\varepsilon^2\nabla\cdot\bar u+(\bar\eta-\eta_\varepsilon)(\bar u\cdot\nabla\bar\eta+\bar\eta\nabla\cdot\bar u)+\eta_\varepsilon\bar u\cdot\nabla\bar\eta\right)(t',x)dxdt'\\&=-\int_0^t\int\left(\frac12\eta_\varepsilon^2\nabla\cdot\bar u+(\bar\eta-\eta_\varepsilon)\bar\eta\nabla\cdot\bar u+\frac12\bar u\cdot\nabla\bar\eta^2\right)(t',x)dxdt'\\&=-\int_0^t\int\frac12(\eta_\varepsilon-\bar\eta)^2\nabla\cdot\bar u(t',x)dxdt'.\end{aligned}\tag{4.4.32}$$

Plugging (4.4.32) into (4.4.31) leads to

$$\begin{aligned}\mathcal{E}_\varepsilon(t)-\mathcal{E}_\varepsilon(0)\le&\int_0^t\int\left(\partial_t\bar\eta+\frac{1}{\varepsilon}\nabla\cdot\bar u+\nabla\cdot(\bar\eta\bar u)\right)(\bar\eta-\eta_\varepsilon)(t',x)dxdt'\\&+\int_0^t\int(1+\varepsilon\eta_\varepsilon)\left(\partial_t\bar u+\frac{1}{\varepsilon}(\beta x_2\bar u^\perp+\nabla\bar\eta)+(\bar u\cdot\nabla)\bar u-\nu\Delta\bar u\right)\cdot(\bar u-u_\varepsilon)(t',x)dxdt'\\&-\int_0^t\int(1+\varepsilon\eta_\varepsilon)D\bar u:(\bar u-u_\varepsilon)^{\otimes2}(t',x)dxdt'-\int_0^t\int\frac12(\eta_\varepsilon-\bar\eta)^2\nabla\cdot\bar u(t',x)dxdt'+R_\varepsilon(t).\end{aligned}\tag{4.4.33}$$

In order to get an inequality of Gronwall type, one has to control the right hand side in terms of $\mathcal{E}_\varepsilon$. We start by estimating the flux term. We have

$$
\begin{aligned}
-\int_0^t \int (1+\varepsilon\eta_\varepsilon)\nabla\bar{u} : (\bar{u}-u_\varepsilon)^{\otimes 2}(t',x)\,dx\,dt' \\
&\le \int_0^t \left(\|\nabla\bar{u}\|_{L^2(\mathbf{T}\times\mathbf{R})} + \varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})}\|\nabla\bar{u}\|_{L^\infty(\mathbf{T}\times\mathbf{R})}\right)\|\bar{u}-u_\varepsilon\|^2_{L^4(\mathbf{T}\times\mathbf{R})}(t')\,dt' \\
&\le C\int_0^t \left(\|\nabla\bar{u}\|_{L^2(\mathbf{T}\times\mathbf{R})} + \varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})}\|\nabla\bar{u}\|_{L^\infty(\mathbf{T}\times\mathbf{R})}\right)\|\bar{u}-u_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})} \\
&\qquad\qquad \times \|\bar{u}-u_\varepsilon\|_{\dot{H}^1(\mathbf{T}\times\mathbf{R})}(t')\,dt'
\end{aligned}
$$

and

$$
\begin{aligned}
\|\bar{u}-u_\varepsilon\|^2_{L^2(\mathbf{T}\times\mathbf{R})} \le \|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\bar{u})\|^2_{L^2(\mathbf{T}\times\mathbf{R})} \\
+\varepsilon\|\eta_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})}\|\bar{u}-u_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})}\|\bar{u}-u_\varepsilon\|_{\dot{H}^1(\mathbf{T}\times\mathbf{R})}
\end{aligned}
$$

which implies

$$
\|\bar{u}-u_\varepsilon\|^2_{L^2(\mathbf{T}\times\mathbf{R})} \le 2\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\bar{u})\|^2_{L^2(\mathbf{T}\times\mathbf{R})} + 16\varepsilon^2\|\eta_\varepsilon\|^2_{L^2(\mathbf{T}\times\mathbf{R})}\|\bar{u}-u_\varepsilon\|^2_{\dot{H}^1(\mathbf{T}\times\mathbf{R})}.
$$

Therefore, using the uniform bounds on η_ε, $\sqrt{1+\varepsilon\eta_\varepsilon}u_\varepsilon$ and on u_ε given by the energy estimate, we gather that

$$
\begin{aligned}
-\int_0^t \int (1+\varepsilon\eta_\varepsilon)\nabla\bar{u} : (\bar{u}-u_\varepsilon)^{\otimes 2}(t',x)\,dx\,dt' \\
&\le C\int_0^t (\|\nabla\bar{u}\|_{L^2} + \varepsilon\|\eta_\varepsilon\|_{L^2}\|\nabla\bar{u}\|_{L^\infty})\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\bar{u})\|_{L^2}\|\bar{u}-u_\varepsilon\|_{\dot{H}^1}(t')\,dt' \\
&\qquad + C\varepsilon\int_0^t (\|\nabla\bar{u}\|_{L^2} + \varepsilon\|\eta_\varepsilon\|_{L^2}\|\nabla\bar{u}\|_{L^\infty})\|\bar{u}-u_\varepsilon\|^2_{\dot{H}^1}(t')\,dt' \\
&\le \frac{\nu}{4}\int \|\bar{u}-u_\varepsilon\|^2_{\dot{H}^1}(t')\,dt' + \frac{C}{\nu}\int \|\nabla\bar{u}\|^2_{L^2}\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-\bar{u})\|^2_{L^2}(t')\,dt' + \omega_\varepsilon(t).
\end{aligned}
\tag{4.4.34}
$$

We also have

$$
-\int_0^t \int \frac{1}{2}(\eta_\varepsilon-\bar{\eta})^2\nabla\cdot\bar{u}(t',x)\,dx\,dt' \le \frac{1}{2}\int_0^t \|\nabla\cdot\bar{u}\|_{L^\infty(\mathbf{T}\times\mathbf{R})}\|\bar{\eta}-\eta_\varepsilon\|^2_{L^2(\mathbf{T}\times\mathbf{R})}(t')\,dt', \tag{4.4.35}
$$

so we are left with the study of the remainder R_ε. We have

$$
R_\varepsilon(t) \le \varepsilon\nu\|\eta_\varepsilon\|_{L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))}\int_0^t \|\Delta\bar{u}\|_{L^4(\mathbf{T}\times\mathbf{R})}\|\bar{u}-u_\varepsilon\|_{L^4(\mathbf{T}\times\mathbf{R})}(t')\,dt'.
$$

The above estimate on $\|\bar{u}-u_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})}$ implies in particular that $\|\bar{u}-u_\varepsilon\|_{L^2(\mathbf{T}\times\mathbf{R})}$ is bounded in $L^2([0,T])$, hence we get that $\|\bar{u}-u_\varepsilon\|_{L^4(\mathbf{T}\times\mathbf{R})}$ is also bounded in $L^2([0,T])$. So we infer directly that $R_\varepsilon(t)$ goes to zero in $L^\infty([0,T])$ as ε goes to zero. That result, joined to (4.4.34) and (4.4.35) allows to deduce from (4.4.33) the following estimate:

$$\frac{1}{2}\mathcal{E}_\varepsilon(t) - \mathcal{E}_\varepsilon(0) \leq \int_0^t \int \left(\partial_t\bar\eta + \frac{1}{\varepsilon}\nabla\cdot\bar u + \nabla\cdot(\bar\eta\bar u)\right)(\bar\eta - \eta_\varepsilon)(t',x)dxdt'$$
$$+ \int_0^t \int (1+\varepsilon\eta_\varepsilon)\left(\partial_t\bar u + \frac{1}{\varepsilon}(\beta x_2\bar u^\perp + \nabla\bar\eta) + (\bar u\cdot\nabla)\bar u - \nu\Delta\bar u\right)\cdot(\bar u - u_\varepsilon)(t',x)dxdt'$$
$$+ \frac{C}{\nu}\int \|\nabla\bar u\|_{L^2}^2 \|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon - \bar u)\|^2_{L^2(\mathbf{T}\times\mathbf{R})}(t')dt' + \frac{1}{2}\int_0^t \|\nabla\cdot\bar u\|_{L^\infty}\|\bar\eta - \eta_\varepsilon\|^2_{L^2}(t')dt' + \omega_\varepsilon(t)$$

thus applying Gronwall's lemma provides the expected stability inequality. □

4.5 A hybrid result

In this final paragraph we are going to put together some results obtained in the previous sections, to prove the strong convergence theorem presented in the introduction of this chapter, namely Theorem 4.4. Due to the unfortunate presence of a defect measure in the limit system, we propose in Pararaph 4.5.4 an alternate model with capillarity, whose virtue is that it gives the lacking compactness on $\varepsilon\eta_\varepsilon$. Its disadvantage is its unphysical character, along with the fact that weak solutions are only known to exist for small data.

4.5.1 Strong compactness of $\Pi_\lambda\Phi_\varepsilon$

In order to characterize completely the asymptotic behaviour of $(\eta_\varepsilon, u_\varepsilon)$ we know from the previous section that it is necessary to introduce the filtering operator

$$\mathcal{L}\left(\frac{t}{\varepsilon}\right) \stackrel{\text{def}}{=} \sum_{i\lambda\in\mathfrak{S}} \exp\left(-\frac{it\lambda}{\varepsilon}\right)\Pi_\lambda,$$

where Π_λ is the projection on the eigenspace of L associated with the eigenvalue $i\lambda$.

Lemma 4.12 *With the notation of Theorem 4.4, the following results hold.*

- *For all $i\lambda \in \mathfrak{S}\setminus\{0\}$, $\Pi_\lambda\Phi_\varepsilon$ is strongly compact in $L^2([0,T],H^s(\mathbf{T}\times\mathbf{R}))$ for all $T>0$ and all $s\in\mathbf{R}$;*
- *$\Pi_0\Phi_\varepsilon$ is strongly compact in $L^2([0,T],H^s_{loc}(\mathbf{T}\times\mathbf{R}))$ for all $T>0$ and all $s<0$.*

Proof.

- For all $\lambda \neq 0$, we recall that by Proposition 4.6, the eigenspace of L associated with the eigenvalue $i\lambda$ is a finite dimensional subspace of $H^\infty(\mathbf{T}\times\mathbf{R})$. Therefore the only point to be checked is the compactness with respect to time, which is obtained as follows.

Let $(n,k,j) \in \mathbf{N}\times\mathbf{Z}\times\{-1,0,1\}$ be given, such that $\lambda = \tau(n,k,j) \neq 0$. Multiplying (4.1.1) by $\Psi_{n,k,j} = (\eta_{n,k,j}, u_{n,k,j})$ (which is smooth and rapidly decaying as $|x_2|$ goes to infinity) and integrating with respect to x leads to

$$\partial_t \int (\eta_\varepsilon \bar{\eta}_{n,k,j} + m_\varepsilon \cdot \bar{u}_{n,k,j})(t,x)dx + \frac{i\tau(n,k,j)}{\varepsilon} \int (\eta_\varepsilon \bar{\eta}_{n,k,j} + m_\varepsilon \cdot \bar{u}_{n,k,j})(t,x)dx$$
$$+\nu \int \nabla u_\varepsilon : \nabla \bar{u}_{n,k,j}(t,x)dx - \int m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)\bar{u}_{n,k,j}(t,x)dx$$
$$-\frac{1}{2} \int \eta_\varepsilon^2 \nabla \cdot \bar{u}_{n,k,j}(t,x)dx = 0$$

where $\bar{u}$ denotes the complex conjugute of u, or equivalently

$$\partial_t \left(\exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right) \int (\eta_\varepsilon \bar{\eta}_{n,k,j} + m_\varepsilon \cdot \bar{u}_{n,k,j})(t,x)dx \right)$$
$$+\nu \int \nabla \left(\exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right) u_\varepsilon \right) : \nabla \bar{u}_{n,k,j}(t,x)dx \tag{4.5.1}$$
$$- \int \exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right) \left(m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)\bar{u}_{n,k,j} + \frac{1}{2}\eta_\varepsilon^2 \nabla \cdot \bar{u}_{n,k,j} \right)(t,x)dx = 0.$$

From the uniform estimates coming from the energy inequality we then deduce that

$$\partial_t \left(\exp\left(\frac{it\tau(n,k,j)}{\varepsilon}\right) \int (\eta_\varepsilon \bar{\eta}_{n,k,j} + m_\varepsilon \cdot \bar{u}_{n,k,j})(t,x)dx \right) \text{ is uniformly bounded in } \varepsilon.$$

Therefore the family

$$\left(\exp\left(\frac{it\lambda}{\varepsilon}\right) \Pi_\lambda(\eta_\varepsilon, m_\varepsilon) \right)_{\varepsilon>0} \text{ is compact in } L^2([0,T]; H^s(\mathbf{T}\times\mathbf{R})) \text{ for any } s \in \mathbf{R},$$

and since $\varepsilon\eta_\varepsilon u_\varepsilon$ converges to 0 in $L^2(\mathbf{R}^+; H^s(\mathbf{T}\times\mathbf{R}))$ for all $s < 0$, we deduce that

$$\exp\left(\frac{it\lambda}{\varepsilon}\right) \Pi_\lambda(\eta_\varepsilon, u_\varepsilon) = \Pi_\lambda \Phi_\varepsilon \text{ is compact in } L^2([0,T]; H^s(\mathbf{T}\times\mathbf{R})) \text{ for any } s \in \mathbf{R}\,.$$

• For $\Pi_0\Phi_\varepsilon = \Pi_0(\eta_\varepsilon, u_\varepsilon)$ the study is a little more difficult since the compactness with respect to spatial variables has to be taken into account. From the energy estimate we have the uniform bound

$$\Phi_\varepsilon \text{ is uniformly bounded in } L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{T}\times\mathbf{R})).$$

Recall that we have defined

$$H^s_L = \left\{ \psi \in \mathcal{S}'(\mathbf{T}\times\mathbf{R}) \,/\, (\mathrm{Id} - \Delta + \beta^2 x_2^2)^{\frac{s}{2}} \psi \in L^2(\mathbf{T}\times\mathbf{R}) \right\}.$$

Equivalently we have

$$H^s_L = \left\{ \psi \in \mathcal{S}'(\mathbf{T}\times\mathbf{R}) \,/\, \sum_{n,k,j\in S} (1+n+k^2)^s \, (\Psi_{n,k,j}|\psi)^2_{L^2(\mathbf{T}\times\mathbf{R})} < +\infty \right\},$$

where $S = \mathbf{N} \times \mathbf{Z} \times \{-1, 0, 1\}$.

As $(\Psi_{n,0,0})_{n\in\mathbf{N}}$ is a Hilbertian basis of Ker$L$, we have for all $T > 0$ and all $s < 0$

$$\left\| \sum_{n\leq N} (\Psi_{n,0,0}|\Phi_\varepsilon)_{L^2(\mathbf{T}\times\mathbf{R})} \Psi_{n,0,0} - \Pi_0\Phi_\varepsilon \right\|_{L^2([0,T],H^s_L)} \to 0 \text{ as } N \to \infty \text{ uniformly in } \varepsilon.$$

Let Ω be any relatively compact open subset of $\mathbf{T}\times\mathbf{R}$. It is easy to see that, for all $s > 0$

$$H^s_0(\Omega) \subset H^s_L \subset H^s(\mathbf{T}\times\mathbf{R}),$$

and conversely for $s < 0$,

$$H^s(\mathbf{T}\times\mathbf{R}) \subset H^s_L \subset H^s(\Omega).$$

Here $H^s_0(\Omega)$ denotes, for $s \geq 0$, the closure of $\mathcal{D}(\Omega)$ for the H^s norm, and $H^{-s}(\Omega)$ is its dual space.

Thus for all $s < 0$ and all $T > 0$, we have

$$\left\| \sum_{n\leq N} (\Psi_{n,0,0}|\Phi_\varepsilon)_{L^2(\mathbf{T}\times\mathbf{R})} \Psi_{n,0,0} - \Pi_0\Phi_\varepsilon \right\|_{L^2([0,T];H^s(\Omega))} \to 0 \text{ as } N \to \infty \text{ uniformly in } \varepsilon.$$

Moreover the same computation as previously shows that for any $n \in \mathbf{N}$,

$$\begin{aligned} \partial_t \left(\int (\eta_\varepsilon \bar\eta_{n,0,0} + m_\varepsilon \cdot \bar u_{n,0,0})(t,x)dx \right) + \nu \int \nabla u_\varepsilon : \nabla \bar u_{n,0,0}(t,x)dx \\ - \int m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)\bar u_{n,0,0}(t,x)dx = 0, \end{aligned} \tag{4.5.2}$$

and, since $\varepsilon\eta_\varepsilon u_\varepsilon$ converges to 0 in $L^2(\mathbf{R}^+; H^s(\mathbf{T}\times\mathbf{R}))$ for any $s < 0$ we get

$$\sum_{n\leq N} \Pi_{n,0,0}(\eta_\varepsilon, u_\varepsilon) \text{ is compact in } L^2([0,T]\times\mathbf{T}\times\mathbf{R}).$$

Combining both results shows finally that

$$\Pi_0\Phi_\varepsilon \text{ is compact in } L^2([0,T]; H^s_{loc}(\mathbf{T}\times\mathbf{R}))$$

for all $T > 0$ and all $s < 0$. □

As $\mathfrak{S}$ is countable, we are therefore able to construct (by diagonal extraction) a subsquence of Φ_ε, and some $\Phi_\lambda \in \mathrm{Ker}(L - i\lambda Id)$ such that for all $s < 0$ and all $T > 0$

$$\forall i\lambda \in \mathfrak{S}, \quad \Pi_\lambda\Phi_\varepsilon \to \Phi_\lambda \text{ in } L^2([0,T]; H^s_{loc}(\mathbf{T}\times\mathbf{R})).$$

Note that the Φ_λ defined as the strong limit of $\Pi_\lambda\Phi_\varepsilon$ can also be obtained as the weak limit of $\exp\left(\frac{it\lambda}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon)$.The following lemma is easily proved.

Lemma 4.13 *With the notation of Theorem 4.4, consider a subsequence of* $(\Phi_\varepsilon)_{\varepsilon>0}$*, and some* $\Phi_\lambda \in \mathrm{Ker}(L - i\lambda Id)$ *such that for all* $s < 0$ *and all* $T > 0$

$$\forall i\lambda \in \mathfrak{S}, \quad \Pi_\lambda \Phi_\varepsilon \to \Phi_\lambda \text{ in } L^2([0,T]; H^s_{loc}(\mathbf{T}\times\mathbf{R})).$$

Then, for all $i\lambda \in \mathfrak{S}$, $e^{\frac{it\lambda}{\varepsilon}}(\eta_\varepsilon, u_\varepsilon)$ *converges to* Φ_λ *weakly in* $L^2([0,T]\times\mathbf{T}\times\mathbf{R})$*. In particular, for all* $i\lambda \in \mathfrak{S}$, Φ'_λ *is bounded in* $L^2([0,T]; H^1(\mathbf{T}\times\mathbf{R}))$ *uniformly in* λ*.*

4.5.2 Strong convergence of Φ_ε

As a corollary of the previous mode by mode convergence results, we get the following convergence for Φ_ε.

Lemma 4.14 *With the notation of Theorem 4.4, the following results hold. Consider a subsequence of* (Φ_ε)*, and some* $\Phi_\lambda \in \mathrm{Ker}(L - i\lambda Id)$ *such that for all* $s < 0$ *and all* $T > 0$

$$\forall i\lambda \in \mathfrak{S}, \quad \Pi_\lambda \Phi_\varepsilon \to \Phi_\lambda \text{ in } L^2([0,T]; H^s_{loc}(\mathbf{T}\times\mathbf{R})).$$

Then,

$$\Phi_\varepsilon \rightharpoonup \Phi = \sum_{i\lambda\in\mathfrak{S}} \Phi_\lambda \text{ weakly in } L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R})),$$

$$\text{and} \quad \Phi_\varepsilon \to \Phi \text{ strongly in } L^2_{loc}(\mathbf{R}^+; H^s_{loc}(\mathbf{T}\times\mathbf{R})) \text{ for all } s<0.$$

Moreover, defining $K_N = \displaystyle\sum_{\substack{(n,k,j)\in S \\ (n+|k|^2)^{1/2}\le N}} \Pi_{n,k,j}$*, we have for any relatively compact subset* Ω *of* $\mathbf{T}\times\mathbf{R}$*, for all* $T>0$ *and for all* $s<0$*,*

$$\|(Id - K_N)\Phi_\varepsilon\|_{L^2([0,T];H^s(\Omega))} + \|(Id-K_N)\mathcal{L}(\frac{t}{\varepsilon})\Phi_\varepsilon\|_{L^2([0,T];H^s(\Omega))} \to 0 \text{ as } N\to\infty, \quad (4.5.3)$$

uniformly in ε*.*

Proof. The first convergence statement comes directly from the uniform bound on Φ_ε in $L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R}))$ and the L^2 continuity of Π_λ. In order to establish the strong convergence result, the crucial argument is to approximate (uniformly) Φ_ε by a finite number of modes, i.e. to prove (4.5.3). The main idea is the same as for the approximation of $\Pi_0\Phi_\varepsilon$ in Lemma 4.12. We have for all $T>0$ and all $s<0$

$$\left\| \sum_{n\le N, |k|\le N} (\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{T}\times\mathbf{R})} \Psi_{n,k,j} - \Phi_\varepsilon \right\|_{L^2([0,T];H^s_L(\mathbf{T}\times\mathbf{R}))} \to 0 \text{ as } N\to\infty \text{ uniformly in } \varepsilon,$$

and similarly

$$\left\| \sum_{n\le N, |k|\le N} e^{-i\tau(n,k,j)\frac{t}{\varepsilon}} (\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{T}\times\mathbf{R})} \Psi_{n,k,j} - \mathcal{L}(\frac{t}{\varepsilon})\Phi_\varepsilon \right\|_{L^2([0,T];H^s_L(\mathbf{T}\times\mathbf{R}))} \to 0 \text{ as } N\to\infty,$$

uniformly in ε. Therefore for all relatively compact subsets Ω of $\mathbf{T}\times\mathbf{R}$, the embedding of H^s_L into $H^s(\Omega)$ implies that both quantities

$$\sum_{n\le N,|k|\le N} (\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{T}\times\mathbf{R})}\,\Psi_{n,k,j}-\Phi_\varepsilon$$

and

$$\sum_{n\le N,|k|\le N} e^{-i\tau(n,k,j)\frac{t}{\varepsilon}}\,(\Psi_{n,k,j}|\Phi_\varepsilon)_{L^2(\mathbf{T}\times\mathbf{R})}\,\Psi_{n,k,j}-\mathcal{L}(\frac{t}{\varepsilon})\Phi_\varepsilon$$

converge strongly towards zero in $L^2([0,T];H^s(\Omega))$ as N goes to infinity, uniformly in ε. So (4.5.3) is proved.

The strong convergence is then directly obtained from the following decomposition:

$$\Phi_\varepsilon-\Phi=(Id-K_N)\Phi_\varepsilon+K_N(\Phi_\varepsilon-\Phi)-(Id-K_N)\Phi.$$

The result is proved. □

4.5.3 Taking limits in the equation on $\Pi_\lambda\Phi_\varepsilon$

The final step is now to obtain the evolution equation for each mode Φ_λ, taking limits in (4.5.1) and (4.5.2). In the following proposition, we recall that the first result (concerning the geostrophic motion) relies on a compensated compactness argument, i.e. on both the algebraic structure of the coupling term and the particular form of the oscillating modes, which implies that there is no contribution of the equatorial waves to the geostrophic flow. That result was proved in Section 4.3. Here we will prove the second part of the statement, concerning the limit ageostrophic motion.

Proposition 4.15 *With the notation of Theorem 4.4, consider a subsequence of (Φ_ε), and some family $(\Phi_\lambda)_{i\lambda\in\mathfrak{S}}$ such that $\Phi_\lambda\in\mathrm{Ker}(L-i\lambda Id)$ and such that for all $s<0$ and all $T>0$*

$$\forall i\lambda\in\mathfrak{S},\quad \Pi_\lambda\Phi_\varepsilon\to\Phi_\lambda \text{ in } L^2([0,T];H^s_{loc}(\mathbf{T}\times\mathbf{R})).$$

Then, $\Phi_0=(\eta_0,u_0)$ satisfies the geostrophic equation : for all (η^*,u^*) belonging to $\mathrm{Ker}L$ and satisfying $u^*\in H^1(\mathbf{T}\times\mathbf{R})$,

$$\int(\eta_0\eta^*+u_{0,1}u_1^*)(t,x)\,dx+\nu\int_0^t\int\nabla u_{0,1}\cdot\nabla u_1^*(t',x)\,dx\,dt'=\int(\eta^0\eta^*+u_1^0u_1^*)(x)\,dx. \quad (4.5.4)$$

Moreover for $\lambda\neq 0$, $\Phi_\lambda=(\Phi^0_\lambda,\Phi'_\lambda)$ satisfies the following envelope equation : there is a measure υ_λ in $\mathcal{M}(\mathbf{R}^+\times\mathbf{T}\times\mathbf{R})$, such that for all smooth $\Phi^*_\lambda=(\Phi^*_{\lambda,0},(\Phi^*_\lambda)')\in\mathrm{Ker}(L-i\lambda Id)$,

$$\int\Phi_\lambda\cdot\bar\Phi^*_\lambda(t,x)\,dx+\nu\int_0^t\int\nabla\Phi'_\lambda:\nabla(\bar\Phi^*_\lambda)'(t',x)\,dx\,dt'+\int_0^t\int\nabla\cdot(\bar\Phi^*_\lambda)'\upsilon_\lambda(dt',dx)$$
$$+\sum_{\substack{i\mu,i\tilde\mu\in\mathfrak{S}\\ \lambda=\mu+\tilde\mu}}\int_0^t Q(\Phi_\mu,\Phi_{\tilde\mu})\cdot\bar\Phi^*_\lambda(t',x)\,dx\,dt'=\int\Phi^0\cdot\bar\Phi^*_\lambda(x)\,dx,$$

where Q is defined by (4.4.10).

Proof. Let us first recall that for $\lambda \neq 0$, $\mathrm{Ker}(L - i\lambda Id)$ is constituted of smooth, rapidly decaying vector fields, so that it makes sense to apply Π_λ to any distribution.

Starting from (4.5.1) we get that for all smooth $\Phi^*_\lambda = (\Phi^*_{\lambda,0}, (\Phi^*_\lambda)') \in \mathrm{Ker}(L - i\lambda Id)$

$$\begin{aligned}\int \exp\left(\frac{it\lambda}{\varepsilon}\right)(\eta_\varepsilon \bar\Phi^*_{\lambda,0} + m_\varepsilon \cdot (\bar\Phi^*_\lambda)')(t,x)dx - \int (\eta^0_\varepsilon \bar\Phi^*_{\lambda,0} + m^0_\varepsilon \cdot (\bar\Phi^*_\lambda)')(x)dx \\ +\nu \int_0^t \int \nabla\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right) : \nabla(\bar\Phi^*_\lambda)'(t',x)dxdt' \qquad (4.5.5)\\ -\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right)\left(m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar\Phi^*_\lambda)' + \frac{1}{2}\eta^2_\varepsilon \nabla \cdot (\bar\Phi^*_\lambda)'\right)(t',x)dxdt' = 0.\end{aligned}$$

Taking limits as $\varepsilon \to 0$ in the three first terms is immediate using Lemma 4.13 and the assumption on the initial data. The limit as $\varepsilon \to 0$ in the two nonlinear terms is given in the following proposition.

Proposition 4.16 *With the previous notation, we have*

$$\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt' \to \int_0^t \int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Phi'_\mu \cdot (\Phi'_{\tilde\mu} \cdot \nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt',$$

and

$$\frac{1}{2}\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta^2_\varepsilon \nabla \cdot (\bar\Phi^*_\lambda)'(t',x)dxdt' \to \frac{1}{2}\int_0^t \int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}} \Phi_{\mu,0}\Phi_{\tilde\mu,0} \nabla \cdot (\bar\Phi^*_\lambda)'(t',x)dxdt' - \int_0^t \nabla \cdot (\bar\Phi^*_\lambda)' \upsilon_\lambda(dt'dx).$$

The fact that this result gives Proposition 4.15 is an algebraic computation left to the reader. Let us prove Proposition 4.16. The idea is to decompose Φ_ε on the eigenmodes of L, by writing

$$(\eta_\varepsilon, u_\varepsilon)(t,x) = \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon(t,x) = \sum_{i\lambda\in\mathfrak{S}} e^{-\frac{it\lambda}{\varepsilon}} \Pi_\lambda \Phi_\varepsilon(t,x).$$

Note in particular that by (4.5.3), for any $s < 0$,

$$(\eta_\varepsilon, u_\varepsilon)(t) - \mathcal{L}\left(\frac{t}{\varepsilon}\right) K_N \Phi_\varepsilon(t) \to 0 \text{ in } L^2_{loc}(\mathbf{R}^+; H^s_{loc}(\mathbf{T}\times\mathbf{R}))$$

as N goes to infinity, uniformly in ε. Let us also introduce the notation

$$\begin{aligned}\Phi_{\varepsilon,N} &= \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = K_N\Phi_\varepsilon, \quad \text{and}\\ \Phi_{\varepsilon,\lambda,N} &= \Pi_\lambda \Phi_{\varepsilon,N}.\end{aligned}$$

We will start by considering the first nonlinear term in Proposition 4.16, namely

$$\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'.$$

We can notice that

$$\begin{aligned}&\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'\\ =&\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \varepsilon\eta_\varepsilon u_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'\\ &+\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'.\end{aligned}$$

The uniform bounds coming from the energy estimate imply clearly that the first term converges to 0 as $\varepsilon \to 0$. Then we can decompose the second contribution in the following way:

$$\begin{aligned}&\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'\\ =&\int_0^t \int_{\mathbf{T}\times(\mathbf{R}\setminus[-R,R])} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'\\ &+\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (u_\varepsilon - u_{\varepsilon,N}) \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt' \qquad (4.5.6)\\ &+\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_{\varepsilon,N} \cdot ((u_\varepsilon - u_{\varepsilon,M}) \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'\\ &+\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_{\varepsilon,N} \cdot (u_{\varepsilon,M} \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'.\end{aligned}$$

Let us consider now all the terms in the right-hand side of (4.5.6). The uniform bound on u_ε and the decay of Φ_λ^* imply that the first term on the right-hand side converges to 0 as $R \to \infty$ uniformly in ε.

By the inequality

$$\begin{aligned}&\left|\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) (u_\varepsilon - u_{\varepsilon,N}) \cdot (u_\varepsilon \cdot \nabla)(\bar{\Phi}_\lambda^*)'(t',x)dxdt'\right|\\ &\qquad \le C\|u_\varepsilon - u_{\varepsilon,N}\|_{L^2([0,T];H^s(\mathbf{T}\times[-R,R]))}\|u_\varepsilon\|_{L^2([0,T];H^1(\mathbf{T}\times\mathbf{R}))}\|\Phi_\lambda^*\|_{W^{2,\infty}(\mathbf{T}\times\mathbf{R})},\end{aligned}$$

with $-1 < s < 0$, we deduce that the third term converges to 0 as $N \to \infty$ uniformly in ε.

Now let us consider the third term on the right-hand side. Since $u_{\varepsilon,N}$ corresponds to the projection of Φ_ε onto a finite number of eigenvectors of L, we deduce that

$$\forall N \in \mathbf{N}, \exists C_N, \forall \varepsilon > 0, \quad \|u_{\varepsilon,N}\|_{L^\infty(\mathbf{R}^+;H^1(\mathbf{T}\times\mathbf{R}))} \le C_N.$$

Thus

$$\left|\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_{\varepsilon,N}\cdot((u_\varepsilon - u_{\varepsilon,M})\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'\right|$$
$$\le C_N \|u_\varepsilon - u_{\varepsilon,M}\|_{L^2([0,T];H^s(\mathbf{T}\times[-R,R]))}\|\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{T}\times\mathbf{R})}$$

and, for all fixed N and R, the fourth term converges to 0 as $M\to\infty$ uniformly in ε.

It remains then to take limits in the last term of (4.5.6). It can be rewritten

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) u_{\varepsilon,N}\cdot(u_{\varepsilon,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'$$
$$= \int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right) (\Phi_{\varepsilon,\mu,N})'\cdot(\Phi'_{\varepsilon,\tilde\mu,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

This in turn can be written in the following way:

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right) \Phi'_{\varepsilon,\mu,N}\cdot(\Phi'_{\varepsilon,\tilde\mu,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'$$
$$= \int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right) (\Phi'_{\varepsilon,\mu,N}-\Phi'_{\mu,N})\cdot(\Phi'_{\varepsilon,\tilde\mu,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'$$
$$+ \int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right) \Phi'_{\mu,N}\cdot((\Phi'_{\varepsilon,\tilde\mu,M}-\Phi'_{\tilde\mu,M})\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'$$
$$+ \int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right) \Phi'_{\mu,N}\cdot(\Phi'_{\tilde\mu,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

We have denoted

$$\Phi_{\mu,N} = \Pi_\mu\Phi_N, \quad \text{where} \quad \Phi_N = K_N\Phi.$$

The first two terms on the right-hand side go to zero as ε goes to zero, for all given N, M and R, due to the following estimates: for $-1 < s < 0$,

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \left|(\Phi'_{\varepsilon,\mu,N}-\Phi'_{\mu,N})\cdot(\Phi'_{\varepsilon,\tilde\mu,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)\right| dxdt'$$
$$\le C_{N,M}\|\Phi'_{\varepsilon,N}-\Phi'_N\|_{L^2([0,T];H^s(\mathbf{T}\times[-R,R]))}\|\Phi_{\varepsilon,M}\|_{L^\infty([0,T];H^1(\mathbf{T}\times\mathbf{R}))}\|\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{T}\times\mathbf{R})},$$

and similarly

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \left|\Phi'_{\mu,N}\cdot((\Phi'_{\varepsilon,\tilde\mu,M}-\Phi'_{\tilde\mu,M})\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)\right| dxdt'$$
$$\le C_{N,M}\|\Phi'_{\varepsilon,M}-\Phi'_M\|_{L^2([0,T];H^s(\mathbf{T}\times[-R,R]))}\|\Phi_{\varepsilon,N}\|_{L^\infty([0,T];H^1(\mathbf{T}\times\mathbf{R}))}\|\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{T}\times\mathbf{R})}.$$

Finally let us consider the last term, which can be decomposed in the following way:

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{i\mu,i\tilde\mu\in\mathfrak{S}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right) \Phi'_{\mu,N}\cdot(\Phi'_{\tilde\mu,N}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'$$
$$= \int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{\substack{i\mu,i\tilde\mu\in\mathfrak{S}\\ \lambda=\mu+\tilde\mu}} \Phi'_{\mu,N}\cdot(\Phi'_{\tilde\mu,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'$$
$$+ \int_0^t \int_{\mathbf{T}\times[-R,R]} \sum_{\substack{i\mu,i\tilde\mu\in\mathfrak{S}\\ \lambda\neq\mu+\tilde\mu}} \exp\left(\frac{it'(\lambda-\mu-\tilde\mu)}{\varepsilon}\right) \Phi'_{\mu,N}\cdot(\Phi'_{\tilde\mu,M}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

For fixed N and M, the nonstationary phase theorem (which corresponds here to a simple integration by parts in the t' variable) shows that the second term is a finite sum of terms converging to 0 as $\varepsilon\to 0$. And the first term (which does not depend on ε) converges to

$$\int_0^t \int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\lambda,i\mu,i\tilde\mu\in\mathfrak{S}}} \Phi'_\mu\cdot(\Phi'_{\tilde\mu}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'$$

as $N,M,R\to\infty$, because Φ'_N converges towards Φ' strongly in $L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))$ when N goes to infinity, and then by Lebesgue's theorem when R goes to infinity.

Therefore, taking limits as $\varepsilon\to 0$, then $M\to\infty$, then $N\to\infty$, then $R\to\infty$ in (4.5.6) leads to

$$\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) m_\varepsilon\cdot(u_\varepsilon\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt' \to \int_0^t \int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\lambda,i\mu,i\tilde\mu\in\mathfrak{S}}} \Phi'_\mu\cdot(\Phi'_{\tilde\mu}\cdot\nabla)(\bar\Phi^*_\lambda)'(t',x)dxdt'.$$

Finally let us consider the second term of the proposition, namely

$$\frac12\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2\nabla\cdot(\bar\Phi^*_\lambda)'(t',x)\,dxdt'.$$

The first step of the above study remains valid, in the sense that one can write

$$\frac12\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2\nabla\cdot(\bar\Phi^*_\lambda)'(t',x)\,dxdt'$$
$$= \frac12\int_0^t \int_{\mathbf{R}\setminus\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2\nabla\cdot(\bar\Phi^*_\lambda)'(t',x)\,dxdt'$$
$$+\frac12\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_\varepsilon^2\nabla\cdot(\bar\Phi^*_\lambda)'(t',x)\,dxdt',$$

and the first term converges to zero uniformly in ε as R goes to infinity, due to the spatial decay of the eigenvectors of L. For such a result, a uniform bound of η_ε in $L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))$ is sufficient. However the next steps of the above study do not work here, as we have no smoothness on η_ε other than that energy bound. In order to conclude, let us nevertheless decompose the remaining term as above, for any integers N and M to be chosen large enough below:

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon^2\nabla\cdot(\bar\Phi_\lambda^*)'(t',x)\,dxdt' \\
&= \frac{1}{2}\int_0^t\int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon-\eta_{\varepsilon,N})\eta_\varepsilon\nabla\cdot(\bar\Phi_\lambda^*)'(t',x)\,dxdt' \\
&+ \frac{1}{2}\int_0^t\int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_{\varepsilon,N}(\eta_\varepsilon-\eta_{\varepsilon,M})\nabla\cdot(\bar\Phi_\lambda^*)'(t',x)\,dxdt' \\
&+ \frac{1}{2}\int_0^t\int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_{\varepsilon,N}\eta_{\varepsilon,M}\nabla\cdot(\bar\Phi_\lambda^*)'(t',x)\,dxdt'.
\end{aligned} \tag{4.5.7}$$

The sequence $-\frac{1}{2}\exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon-\eta_{\varepsilon,N})\eta_\varepsilon$ is uniformly bounded in $N\in\mathbf{N}$ and $\varepsilon>0$ in the space $L^1_{loc}(\mathbf{R}^+\times\mathbf{T}\times\mathbf{R})$, so up to the extraction of a subsequence it converges weakly, as ε goes to zero, towards a measure $\upsilon_{\lambda,N}$, which in turn is uniformly bounded in $\mathcal{M}(\mathbf{R}^+\times\mathbf{T}\times\mathbf{R})$. Denoting by υ_λ its limit in $\mathcal{M}(\mathbf{R}^+\times\mathbf{T}\times\mathbf{R})$ as N goes to infinity, we find that

$$\frac{1}{2}\int_0^t\int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon-\eta_{\varepsilon,N})\eta_\varepsilon\nabla\cdot(\bar\Phi_\lambda^*)'(t',x)\,dxdt' \\ \to -\int_0^t\int_{\mathbf{T}\times[-R,R]}\nabla\cdot(\bar\Phi_\lambda^*)'\upsilon_\lambda(dt'dx)$$

as ε goes to zero and N goes to infinity, which in turn converges to

$$-\int_0^t\int\nabla\cdot(\bar\Phi_\lambda^*)'\upsilon_\lambda(dt'dx)$$

as R goes to infinity, due to the smoothness of $\nabla\cdot(\bar\Phi_\lambda^*)'$. Note that as $\mathfrak{S}$ is countable, one can choose a subsequence such that for all $i\lambda\in\mathfrak{S}$, the sequence $-\frac{1}{2}\exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon-\eta_{\varepsilon,N})\eta_\varepsilon$ converges towards υ_λ as ε goes to zero and N goes to infinity.

Finally the two last terms in (4.5.7) are dealt with as in the previous case, and we leave the details to the reader.

Proposition 4.16 is proved. □

4.5.4 The case when capillarity is added

In this final paragraph we propose an adaptation to the Saint-Venant model which provides some additional smoothness on $\varepsilon\eta_\varepsilon$, and which enables one to get rid of the defect measure present in the above study. The model is presented in the next part, and the convergence result stated and proved below.

The model

Let us define the capillarity operator

$$K(h) = \kappa(-\Delta)^{2\alpha} h, \tag{4.5.8}$$

where $\kappa > 0$ and $\alpha > 1/2$ are given constants. The system we shall study is the following:

$$\begin{aligned} \partial_t \eta + \frac{1}{\varepsilon} \nabla \cdot \Big((1+\varepsilon\eta) u \Big) = 0, \\ \partial_t u + u \cdot \nabla u + \frac{\beta x_2}{\varepsilon} u^\perp + \frac{1}{\varepsilon} \nabla \eta - \frac{\nu}{1+\varepsilon\eta} \Delta u + \varepsilon\kappa \nabla(-\Delta)^{2\alpha} \eta = 0, \\ \eta_{|t=0} = \eta^0, \quad u_{|t=0} = u^0. \end{aligned} \tag{4.5.9}$$

In the next part we discuss the existence of bounded energy solutions to that system of equations (under a smallness assumption), and the following part consists in the proof of the analogue of Theorem 4.4 in that setting. One should emphasize here that the additional capillarity term that is added in the system will not appear in the limit, since it comes as a $O(\varepsilon)$ term. Moreover it is a linear term, so it should not change the other asymptotics proved in this chapter. However its unphysical character (as well as the smallness condition on the initial data) made us prefer to study the original Saint-Venant system for all the convergence results of this chapter.

Existence of solutions

The following theorem is an easy adaptation of the result by D. Bresch and B. Desjardins in [6] (see also [39] for the compressible Navier-Stokes system).

Theorem 4.5 *There is a constant $C > 0$ such that the following result holds. Let $(\eta_\varepsilon^0, u_\varepsilon^0)$ be a family of $H^{2\alpha} \times L^2(\mathbf{T} \times \mathbf{R})$ such that for all $\varepsilon > 0$,*

$$\frac{1}{2} \int \Big((\eta_\varepsilon^0)^2 + \kappa\varepsilon^2 |(-\Delta)^\alpha \eta_\varepsilon^0|^2 + (1+\varepsilon\eta_\varepsilon^0)|u_\varepsilon^0|^2 \Big)(x)\, dx \leq \mathcal{E}^0.$$

If $\mathcal{E}^0 \leq C$, then there is a family $(\eta_\varepsilon, u_\varepsilon)$ of weak solutions to (4.5.9), satisfying the energy estimate

$$\frac{1}{2} \int \Big(\eta_\varepsilon^2 + \kappa\varepsilon^2 |(-\Delta)^\alpha \eta_\varepsilon|^2 + (1+\varepsilon\eta_\varepsilon)|u_\varepsilon|^2 \Big)(t,x)\, dx + \nu \int_0^t \int |\nabla u_\varepsilon|^2(t',x)\, dx dt' \leq \mathcal{E}^0.$$

Convergence

In this section our aim is to show that the capillarity term enables us to get rid of the defect measure present in the conclusion of Theorem 4.4. As the proof is very similar to that theorem, up to the compactness of η_ε, we will not give the full details. The result is the following.

Theorem 4.6 *Under the assumptions of Theorem 4.5, denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (4.5.9) with initial data $(\eta_\varepsilon^0, u_\varepsilon^0)$, and define*

$$\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon).$$

Up to the extraction of a subsequence, Φ_ε converges weakly in $L^2_{loc}(\mathbf{R}^+; H^s_{loc}(\mathbf{T}\times\mathbf{R}))$ (for all $s<0$) toward some solution Φ of the following limiting filtered system: for all $i\lambda$ in $\mathfrak{S}$ and for all smooth Φ^*_λ in $\mathrm{Ker}(L-i\lambda Id)$,

$$\int \Phi\cdot\bar\Phi^*_\lambda(x)dx-\nu\int_0^t\int \Delta'_L\Phi\cdot\bar\Phi^*_\lambda(t',x)dxdt'+\int_0^t\int Q_L(\Phi,\Phi)\cdot\bar\Phi^*_\lambda(t',x)dxdt' = \int\Phi^0\cdot\bar\Phi^*_\lambda(x)dx,$$

where $\Phi^0=(\eta^0,u^0)$.

Let us prove that result. We will follow the lines of the proof of Theorem 4.4; the only difference consists in taking the limit as ε goes to zero, of the equation on $\Pi_\lambda\Phi_\varepsilon$.

Equation (4.5.5) page 113 can be written here as follows: for all smooth $\Phi^*_\lambda=(\Phi^*_{\lambda,0},(\Phi^*_\lambda)')$ belonging to $\mathrm{Ker}(L-i\lambda Id)$,

$$\begin{aligned}\int \exp\left(\frac{it\lambda}{\varepsilon}\right)(\eta_\varepsilon\bar\Phi^*_{\lambda,0}+u_\varepsilon\cdot(\bar\Phi^*_\lambda)')(t,x)dx-\int(\eta^0_\varepsilon\bar\Phi^*_{\lambda,0}+u^0_\varepsilon\cdot(\bar\Phi^*_\lambda)')(x)dx\\ -\varepsilon\kappa\int_0^t\int(-\Delta)^\alpha\exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon\nabla\cdot(-\Delta)^\alpha(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ -\int_0^t\int\frac{\nu}{1+\varepsilon\eta_\varepsilon}\Delta\left(\exp\left(\frac{it'\lambda}{\varepsilon}\right)u_\varepsilon\right)\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ +\int_0^t\int\exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt'\\ -\int_0^t\int\exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon u_\varepsilon\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt'=0.\end{aligned}\tag{4.5.10}$$

Remark 4.17 *We have chosen to keep the unknowns $(\eta_\varepsilon,u_\varepsilon)$ and not write the analysis in terms of $(\eta_\varepsilon,m_\varepsilon)$ as previously (recall that $m_\varepsilon=(1+\varepsilon\eta_\varepsilon)u_\varepsilon$): the study of m_ε rather than u_ε is indeed unnecessary here as the factor $\dfrac{1}{1+\varepsilon\eta_\varepsilon}$ which appears in the diffusion term in the equation on u_ε can be controled in this situation, contrary to the previous case. The advantage of writing the equations on $(\eta_\varepsilon,u_\varepsilon)$ is that there is no nonlinear term in η_ε, contrary to the previous study, but of course the difficulty is transfered to the study of the diffusion operator; the gain of regularity in η_ε will appear here.*

Taking limits as $\varepsilon\to 0$ in the two first terms is immediate. For the third term, we simply recall that η_ε is bounded in $L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))$ and $\varepsilon\eta_\varepsilon$ is bounded in $L^\infty(\mathbf{R}^+;H^{2\alpha}(\mathbf{T}\times\mathbf{R}))$, so $\varepsilon\eta_\varepsilon$ goes strongly to zero in $L^\infty(\mathbf{R}^+;H^s(\mathbf{T}\times\mathbf{R}))$ for every $s<2\alpha$. Since Φ^*_λ is smooth, we infer that

$$\varepsilon\kappa\int_0^t\int(-\Delta)^\alpha\exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon\nabla\cdot(-\Delta)^\alpha(\bar\Phi^*_\lambda)'(t',x)dxdt'\to 0,\quad\text{as }\varepsilon\to 0.$$

Let us now consider the fourth term,

$$-\int_0^t \int \frac{\nu}{1+\varepsilon\eta_\varepsilon} \Delta \left(\exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \right) \cdot (\bar{\Phi}_\lambda^*)'(t',x) dx dt'.$$

It is here that the presence of capillarity enables us to get a better control. Let us write

$$\begin{aligned} &-\int_0^t \int \frac{\nu}{1+\varepsilon\eta_\varepsilon} \Delta \left(\exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \right) \cdot (\bar{\Phi}_\lambda^*)'(t',x) dx dt' \\ &= \nu \int_0^t \int \nabla \left(\exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \right) : \nabla (\bar{\Phi}_\lambda^*)'(t',x) dx dt' \\ &- \nu \int_0^t \int \nabla \left(\exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \right) : \nabla \left(\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon} (\bar{\Phi}_\lambda^*)' \right) (t',x) dx dt'. \end{aligned}$$

Clearly the first term on the right-hand side converges towards the expected limit: we have

$$\nu \int_0^t \int \nabla \left(\exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \right) : \nabla (\bar{\Phi}_\lambda^*)'(t',x) dx dt' \to \nu \int_0^t \int \nabla \Phi_\lambda' : \nabla (\bar{\Phi}_\lambda^*)'(t',x) dx dt',$$

as ε goes to 0. To study the second one, we can notice that

$$\nabla \left(\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon} (\bar{\Phi}_\lambda^*)' \right) = \nabla \left(\frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon} \right) (\bar{\Phi}_\lambda^*)' + \frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon} \nabla (\bar{\Phi}_\lambda^*)',$$

and since the second term on the right-hand side is obviously easier to study than the first one, let us concentrate on the first term. We have

$$\nabla \frac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon} = \frac{\varepsilon \nabla \eta_\varepsilon}{1+\varepsilon\eta_\varepsilon} - \frac{\varepsilon^2 \eta_\varepsilon \nabla \eta_\varepsilon}{(1+\varepsilon\eta_\varepsilon)^2}.$$

Since $\varepsilon\eta_\varepsilon$ is bounded in $L^\infty(\mathbf{R}^+; H^{2\alpha}(\mathbf{T}\times\mathbf{R}))$, we infer easily, by product laws in Sobolev spaces, that

$$\varepsilon^2 \eta_\varepsilon \nabla \eta_\varepsilon \quad \text{is bounded in} \quad L^\infty(\mathbf{R}^+; H^\sigma(\mathbf{T}\times\mathbf{R})), \text{ for some } \sigma > 0.$$

But on the other hand η_ε is bounded in $L^\infty(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R}))$, so we have also

$$\varepsilon^2 \eta_\varepsilon \nabla \eta_\varepsilon \to 0 \quad \text{in} \quad L^\infty(\mathbf{R}^+; H^{2\alpha-2}(\mathbf{T}\times\mathbf{R})).$$

By interpolation we gather that

$$\varepsilon^2 \eta_\varepsilon \nabla \eta_\varepsilon \to 0 \quad \text{in} \quad L^\infty(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R})),$$

and the lower bound on $1+\varepsilon\eta_\varepsilon$ ensures that

$$\frac{\varepsilon^2 \eta_\varepsilon \nabla \eta_\varepsilon}{(1+\varepsilon\eta_\varepsilon)^2} \to 0 \quad \text{in} \quad L^\infty(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R})).$$

The argument is similar (and easier) for the term $\dfrac{\varepsilon\nabla\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}$, so we can conclude that

$$-\int_0^t \int \frac{\nu}{1+\varepsilon\eta_\varepsilon} \Delta \left(\exp\left(\frac{it'\lambda}{\varepsilon}\right) u_\varepsilon \right) \cdot (\bar{\Phi}_\lambda^*)'(t',x) dx dt' \to \nu \int_0^t \int \nabla \Phi_\lambda : \nabla (\bar{\Phi}_\lambda^*)'(t',x) dx dt'.$$

Finally we are left with the nonlinear terms: let us study the limit of

$$\int_0^t \int \exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt' - \int_0^t\int \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon u_\varepsilon \nabla\cdot\bar\Phi^*_{\lambda,0}(t',x)dxdt'.$$

The study is very similar to the case studied above (see Proposition 4.16), so we will not give all the details but merely point out the differences. First, one can truncate the integral in $x_2\in\mathbf{R}$ to $x_2\in[-R,R]$, where R is a parameter to be chosen large enough in the end. As previously that is simply due to the decay of the eigenvectors of L at infinity. So we are reduced to the study of

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt' \quad \text{and}$$

$$-\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon u_\varepsilon\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt'.$$

The limit of the first term is obtained in an identical way to above, since u_ε satisfies the same bounds, so we have

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt' \\ \to \int_0^t\int \sum_{\substack{\mu+\tilde\mu=\lambda\\ i\mu,i\tilde\mu\in\mathfrak{S}}}(\Phi'_\mu\cdot\nabla)\Phi'_{\tilde\mu}\cdot(\bar\Phi^*_\lambda)'(t',x)dxdt',$$

as ε goes to 0 and R goes to infinity.

Now let us concentrate on the last nonlinear term. With the notation defined in the previous section, we can write

$$\begin{aligned}
&\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_\varepsilon u_\varepsilon\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt' \\
&= \int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon-\eta_{\varepsilon,N})u_\varepsilon\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt' \\
&+ \int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_{\varepsilon,N}(u_\varepsilon-u_{\varepsilon,M})\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt' \\
&+ \int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)\eta_{\varepsilon,N}u_{\varepsilon,M}\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt'.
\end{aligned}$$

The first two terms on the right-hand side converge to zero, due to the following estimates: for some $-1<s<0$ and for all $t\in[0,T]$,

$$\left|\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right)(\eta_\varepsilon-\eta_{\varepsilon,N})u_\varepsilon\cdot\nabla\bar\Phi^*_{\lambda,0}(t',x)dxdt'\right| \\ \leq C_T\|\eta_\varepsilon-\eta_{\varepsilon,N}\|_{L^\infty([0,T];H^s(\mathbf{T}\times[-R,R]))}\|u_\varepsilon\|_{L^2([0,T];H^1(\mathbf{T}\times[-R,R]))}\|\bar\Phi^*_\lambda\|_{W^{2,\infty}(\mathbf{T}\times\mathbf{R})},$$

and similarly

$$\left| \int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N}(u_\varepsilon - u_{\varepsilon,M}) \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \right| \\ \leq C_{T,N} \|u_\varepsilon - u_{\varepsilon,M}\|_{L^2([0,T];H^s(\mathbf{T}\times[-R,R]))} \|\bar{\Phi}^*_\lambda\|_{W^{2,\infty}(\mathbf{T}\times\mathbf{R})}.$$

Finally the limit of the third term is obtained by the (by now) classical nonstationary phase theorem, namely we find, exactly as in the proof of Proposition 4.16, that

$$\int_0^t \int_{\mathbf{T}\times[-R,R]} \exp\left(\frac{it'\lambda}{\varepsilon}\right) \eta_{\varepsilon,N} u_{\varepsilon,M} \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt' \\ \to \int_0^t \int \sum_{\substack{\mu+\tilde{\mu}=\lambda \\ i\mu,i\tilde{\mu}\in\mathfrak{S}}} \Phi_{\mu,0}\Phi'_{\tilde{\mu}} \cdot \nabla\bar{\Phi}^*_{\lambda,0}(t',x)dxdt',$$

as ε goes to 0 and M, N and R go to infinity.

That concludes the proof of the theorem. □

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Index