On the global wellposedness of the 3-D Navier-Stokes equations with large initial data

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Abstract

We give a condition for the periodic, three dimensional, incompressible Navier-Stokes equations to be globally wellposed. This condition is not a smallness condition on the initial data, as the data is allowed to be arbitrarily large in the scale invariant space $B^{-1}_{\infty,\infty}$, which contains all the known spaces in which there is a global solution for small data. The smallness condition is rather a nonlinear type condition on the initial data; an explicit example of such initial data is constructed, which is arbitrarily large and yet gives rise to a global, smooth solution.

Keywords Navier-Stokes equations, global wellposedness.

1 Introduction

The purpose of this text is to establish a condition of global wellposedness for regular initial data for the incompressible Navier-Stokes system on the three dimensional torus $\mathbf{T}^3 = (\mathbf{R}/2\pi\mathbf{Z})^3$. Let us recall the system:

$$(NS) \begin{cases} \partial_t u - \Delta u + u\cdot\nabla u = -\nabla p \\ \operatorname{div} u = 0 \\ u_{t=0} = u_0. \end{cases}$$

Here u is a mean free three-component vector field $u = (u^1, u^2, u^3) = (u^h, u^3)$ representing the velocity of the fluid, and p is a scalar denoting the pressure, both are unknown functions of the space variable $x \in \mathbf{T}^3$, and the time variable $t \in \mathbf{R}^+$. We have chosen the kinematic viscosity of the fluid to be equal to one for simplicity. We recall that the pressure can be eliminated by projecting (NS) onto the space of divergence free vector fields, using the Leray projector

$$\mathbf{P} = \mathrm{Id} - \nabla\Delta^{-1}\mathrm{div}.$$

Thus we shall be using in the following the equivalent system

$$\partial_t u - \Delta u + \mathbf{P}(u \cdot \nabla u) = 0.$$

Our motivation is the study of the size of the initial data yielding global existence of solutions to that system, rather than the minimal regularity one can assume on the initial data. Thus, in all this work, we shall assume that u_0 is a mean free vector field with components in the Sobolev space $H^{\frac{1}{2}}(\mathbf{T}^3)$: we recall that $H^{\frac{1}{2}}(\mathbf{T}^3)$ is a scale invariant space for (NS), and that smooth solutions exist for a short time if the initial data belongs to $H^{\frac{1}{2}}(\mathbf{T}^3)$, globally in time if the data is small enough. The problem of global wellposedness for general data in $H^{\frac{1}{2}}(\mathbf{T}^3)$ is known to be open. The search of smallness conditions on u_0 the least restrictive as possible is a long story, essentially initiated by J. Leray (in the whole space $\mathbf{R}^3$ but the phenomenon is similar in the torus) in the seminal paper [9], continued in particular by H. Fujita and T. Kato in [5], M. Cannone, Y. Meyer et F. Planchon in [2] and H. Koch and D. Tataru in [8]. The theorem proved in [8] claims that if $\|u_0\|_{\partial BMO}$ is small, which means that the components of u_0 are derivatives of BMO functions and are small enough, then (NS) is globally wellposed in the sense that a global (and of course unique) solution exists in $C(\mathbf{R}^+; H^{\frac{1}{2}})$. Our aim is to prove a theorem of global wellposedness which allows for very large data in ∂BMO, under a nonlinear smallness condition on the initial data. In fact the initial data will even be large in $B^{-1}_{\infty,\infty}$, which contains strictly ∂BMO and which is the largest scale invariant Banach space in which one can hope to prove a wellposedness result. Before stating the result, let us recall that the question is only meaningful in three or more space dimensions. We recall indeed that according to J. Leray [10], there is a unique, global solution to the two dimensional Navier-Stokes system as soon as the initial data is in $L^2(\mathbf{T}^2)$, and if there is a forcing term it should belong for instance to $L^1(\mathbf{R}^+; L^2(\mathbf{T}^2))$.

In order to state our result, we shall need the following notation: one can decompose any function f defined on $\mathbf{T}^3$ as

$$f = \overline{f} + \widetilde{f}, \quad \text{where} \quad \overline{f}(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} f(x_1, x_2, x_3)\, dx_3.$$

Similarly we shall define the horizontal mean $\overline{u}$ of any vector field as $\overline{u} = (\overline{u}^1, \overline{u}^2, \overline{u}^3)$. It will also be convenient to use the following alternative notation: we denote by M the projector onto vector fields defined on $\mathbf{T}^2$,

$$\mathrm{M} f = \overline{f} \quad \text{and} \quad (\mathrm{Id} - \mathrm{M}) f = \widetilde{f}.$$

we shall denote the heat semiflow by $\mathrm{S}(t) = e^{t\Delta}$. Finally let us define negative index Besov spaces.

Definition 1.1 *Let s be a positive real number, and let p and q be two real numbers in $[1, +\infty]$. The Besov space $B^{-s}_{p,r}(\mathbf{T}^3)$ is the space of distributions in $\mathbf{T}^3$ such that*

$$\|u\|_{B^{-s}_{p,q}} \stackrel{\text{def}}{=} \Big\| t^{\frac{s}{2}} \|\mathrm{S}(t) u\|_{L^p} \Big\|_{L^q(\mathbf{R}^+, \frac{dt}{t})} < +\infty.$$

Remark we shall see an equivalent definition in terms of Littlewood-Paley theory in Section 2 (see Definition 2.2).

Now let us consider the following subspace of $H^{\frac12}(\mathbf{T}^3)$, where we have noted, for all vector fields a and b,

$$Q(a,b) \stackrel{\text{def}}{=} \mathbf{P}\,\mathrm{div}(a\otimes b + b\otimes a).$$

Definition 1.2 *Let A and B be two positive real numbers and let p in $]3,+\infty]$. We define the set*

$$\mathcal{I}_p(A,B) = \Big\{u_0 \in H^{\frac12}(\mathbf{T}^3) \Big/ \ \mathrm{div}\, u_0 = 0 \ \textit{and} \ \text{(H1)},\ \text{(H2)},\ \text{(H3)} \textit{ are satisfied}\Big\}, \quad \textit{where}$$

$$\begin{array}{ll} \text{(H1)} & \|\overline{u}_0\|_{L^2(\mathbf{T}^2)} + \|\mathrm{M}\mathbf{P}(u_F\cdot\nabla u_F)\|_{L^1(\mathbf{R}^+;L^2(\mathbf{T}^2))} \leq A \\ \text{(H2)} & \|\widetilde{u_0}\|_{B^{-1}_{\infty,2}} \leq A \\ \text{(H3)} & \|(\mathrm{Id}-\mathrm{M})\mathbf{P}(u_F\cdot\nabla u_F) + Q(u_{2D},u_F)\|_{L^1(\mathbf{R}^+;B^{-1+\frac3p}_{p,2})} \leq B, \end{array}$$

where we have noted $u_F(t) = \mathrm{S}(t)\widetilde{u}_0$ and where u_{2D} is a three component vector field defined on $\mathbf{T}^2$, satisfying the following two dimensional Navier-Stokes equation, in the case when the initial data is $v_0 = \overline{u}_0$ and the force is $f = -\mathrm{M}\mathbf{P}(u_F\cdot\nabla u_F)$:

$$(NS2D) \quad \left\{ \begin{array}{c} \partial_t v + \mathbf{P}(v^h\cdot\nabla^h v) - \Delta_h v = f \\ v_{|t=0} = v_0, \end{array}\right.$$

where Δ_h denotes the horizontal Laplacian $\Delta_h = \partial_1^2 + \partial_2^2$ and where $\nabla^h = (\partial_1,\partial_2)$.

Now let us state the main result of this paper.

Theorem 1 *Let $p \in]6,+\infty[$ be given. There is a constant $C_0 > 0$ such that the following holds. Consider two positive real numbers A and B satisfying*

$$B\exp\Big(C_0A^2\left(1 + A\log(e+A)\right)^2\Big) \leq C_0^{-1}. \tag{1.1}$$

Then for any vector field $u_0 \in \mathcal{I}_p(A,B)$, there is a unique, global solution u to (NS) associated with u_0, satisfying

$$u \in C_b(\mathbf{R}^+; H^{\frac12}(\mathbf{T}^3)) \cap L^2(\mathbf{R}^+; H^{\frac32}(\mathbf{T}^3)).$$

Remarks

1) Condition (1.1) appearing in the statement of Theorem 1 should be understood as a nonlinear smallness condition on the initial data: the parameter A, measuring through $(H1)$ and $(H2)$ the norm of the initial data in a scale-invariant space, may be as large as wanted, as long as the parameter B, which measures a nonlinear quantity in a scale-invariant space, is small enough. We give below an example of such initial data, which is a smooth vector field with arbitrarily large $B^{-1}_{\infty,\infty}$ norm, and which generates a unique, global solution to (NS): see the statement of Theorem 2.

2) Some results of global existence for large data can be found in the literature. To our knowledge they all involve either an initial vector field which is close enough to a two dimensional vector field (see for instance [11], [6] or [7]), or initial data such that after a change of

coordinates, the equation is transformed into the three dimensional rotating fluid equations (for which global existence is known), see [1]. Here we are in neither of those situations.

Let us now give an example where condition (1.1) holds. As mentioned in the remarks above, in that example the initial data can be arbitrarily large in $B^{-1}_{\infty,\infty}$, and nevertheless generates a global solution. We have noted by $\widehat{u}$ the Fourier transform of any vector field u.

Theorem 2 *Let N_0 be a given positive integer. A positive integer N_1 exists such, if N is an integer larger than N_1, it satisfies the following properties. If v_0^h is any two component, divergence free vector fields defined on $\mathbf{T}^2$ such that*

$$\operatorname{Supp}\, \widehat{v}_0^h \subset [-N_0, N_0]^2 \quad \textit{and} \quad \|v_0^h\|_{L^2(\mathbf{T}^2)} \leq (\log N)^{\frac{1}{9}},$$

then a unique, global smooth solution to (NS) exists, associated with the initial data

$$u_0(x) = \Big(N v_0^h(x_h)\cos(Nx_3), -\operatorname{div}_h v_0^h(x_h)\sin(Nx_3)\Big).$$

Moreover the vector field u_0 satisfies

$$\|u_0^h\|_{B^{-1}_{\infty,\infty}} \geq \frac{1}{4\pi\sqrt{e}}\|v_0^h\|_{L^2(\mathbf{T}^2)}. \tag{1.2}$$

Remarks

1) Since the L^2 norm of v_0^h can be chosen arbitarily large, the lower bound given in (1.2) implies that the $B^{-1}_{\infty,\infty}$ norm of the initial data may be chosen arbitrarily large.

2) One can rewrite this example in terms of the Reynolds number of the fluid: let $\mathrm{re} \in \mathbf{N}$ be its Reynolds number, and define the rescaled velocity field $v(t,x) = \dfrac{1}{\mathrm{re}} u(\dfrac{t}{\mathrm{re}}, x)$. Then v satisfies the Navier-Stokes equation

$$\partial_t v + \mathbf{P}(v\cdot\nabla v) - \nu\Delta v = 0$$

where $\nu = 1/\mathrm{re}$, and Theorem 2 states the following: if $v_{|t=0}$ is equal to

$$v_{0,\nu} = \left(v_0^h(x_h)\cos\left(\frac{x_3}{\nu}\right), -\nu\operatorname{div}_h v_0^h(x_h)\sin\left(\frac{x_3}{\nu}\right)\right)$$

where $\widehat{v}_0^h$ is supported in $[-N_0, N_0]^2$ and satisfies

$$\|v_0^h\|_{L^2(\mathbf{T}^2)} \leq \left(\log\frac{1}{\nu}\right)^{\frac{1}{9}},$$

then for ν small enough there is a unique, global, smooth solution.

The rest of the paper is devoted to the proof of Theorems 1 and 2. The proof of Theorem 1 relies on the following idea: if u denotes the solution of (NS) associated with u_0, which exists at least for a short time since u_0 belongs to $H^{\frac{1}{2}}(\mathbf{T}^3)$, then it can be decomposed as follows, with the notation of Definition 1.2:

$$u = u^{(0)} + R, \quad \text{where} \quad u^{(0)} = u_F + u_{2D}.$$

Note that the Leray theorem in dimension two mentioned above, namely the existence and uniqueness of a smooth solution for an initial data in $L^2(\mathbf{T}^2)$ and a forcing term in $L^1(\mathbf{R}^+; L^2(\mathbf{T}^2))$, holds even if the vector fields have three components rather than two (as is the case for the equation satisfied by u_{2D}). One notices that the vector field R satisfies the perturbed Navier-Stokes system

$$(PNS) \quad \begin{cases} \partial_t R + \mathbf{P}(R \cdot \nabla R) + Q(u^{(0)}, R) - \Delta R = F \\ R_{|t=0} = R_0, \end{cases}$$

where

$$R_{|t=0} = 0 \quad \text{and} \quad F = -(\mathrm{Id} - \mathrm{M})\mathbf{P}(u_F \cdot \nabla u_F) - Q(u_F, u_{2D}).$$

The proof of Theorem 1 consists in studying both systems, the two dimensional Navier-Stokes system and the perturbed three dimensional Navier-Stokes system. In particular a result on the two dimensional Navier-Stokes system will be proved in Section 3, which, as far as we know is new, and may have its own interest. It is stated below.

Theorem 3 *There is a constant $C > 0$ such that the following result holds. Let v be the solution of $(NS2D)$ with initial data $v_0 \in L^2$ and external force f in $L^1(\mathbf{R}^+; L^2)$. Then we have*

$$\|v\|^2_{L^2(\mathbf{R}^+; L^\infty)} \leq C E_0 \Big(1 + E_0 \log^2(e + E_0^{\frac{1}{2}})\Big) \quad \text{with} \quad E_0 \stackrel{\text{def}}{=} \|v_0\|^2_{L^2} + \Big(\int_0^\infty \|f(t)\|_{L^2} dt\Big)^2.$$

The key to the proof of Theorem 1 is the proof of the global wellposedness of the perturbed three dimensional system (PNS). That is achieved in Section 4 below, where a general statement is proved, concerning the global wellposedness of (PNS) for general R_0 and F satisfying a smallness condition. That result is joint to Theorem 3 to prove Theorem 1 in Section 5. Finally Theorem 2 is proved in Section 6. The coming section is devoted to some notation and the recollection of well known results on Besov spaces and the Littlewood-Paley theory which will be used in the course of the proofs.

2 Notation and useful results on Littlewood-Paley theory

In this short section we shall present some well known facts on the Littlewood-Paley theory. Let us start by giving the definition of Littlewood-Paley operators on $\mathbf{T}^d$.

Definition 2.1 *Let χ be a nonnegative function in $C^\infty(\mathbf{T}^d)$ such that $\widehat{\chi} = 1$ for $|\xi| \leq 1$ and $\widehat{\chi} = 0$ for $|\xi| > 2$, and define $\chi_j(x) = 2^{jd}\chi(2^j|x|)$. Then the Littlewood-Paley frequency localization operators are defined by*

$$S_j = \chi_j * \cdot, \quad \Delta_j = S_j - S_{j-1}.$$

As is well known, one of the interests of this decomposition is that the Δ_j operators allow to count derivatives easily. More precisely we recall the Bernstein inequality. A constant C exists such that

$$\forall\, k \in \mathbf{N}, \, \forall\, 1 \leq p \leq q \leq \infty, \quad \sup_{|\alpha| = k} \|\partial^\alpha \Delta_j u\|_{L^q(\mathbf{T}^d)} \leq C^{k+1} 2^{jk} 2^{jd\left(\frac{1}{p} - \frac{1}{q}\right)} \|\Delta_j u\|_{L^p(\mathbf{T}^d)}. \quad (2.1)$$

Using those operators we can give a definition of Besov spaces for all indexes, and we recall the classical fact that the definition in the case of a negative index coincides with the definition given in the introduction using the heat kernel (Definition 1.1 above).

Definition 2.2 *Let f be a mean free function in $\mathcal{D}'(\mathbf{T}^d)$, and let $s \in \mathbf{R}$ and $(p,q) \in [1,+\infty]^2$ be given real numbers. Then f belongs to the Besov space $B^s_{p,q}(\mathbf{T}^d)$ if and only if*

$$\|f\|_{B^s_{p,q}} \stackrel{\text{def}}{=} \big\|2^{js}\|\Delta_j f\|_{L^p}\big\|_{\ell^q(\mathbf{Z})} < +\infty.$$

Using the Bernstein inequality (2.1), it is easy to see that the following continuous embedding holds:

$$B^{s+\frac{d}{p_1}}_{p_1,r_1}(\mathbf{T}^d) \hookrightarrow B^{s+\frac{d}{p_2}}_{p_2,r_2}(\mathbf{T}^d), \tag{2.2}$$

for all real numbers s, p_1, p_2, r_1, r_2 such that p_i and r_i belong to the interval $[1,\infty]$ and such that $p_1 \leq p_2$ and $r_1 \leq r_2$.

We recall that Sobolev spaces are special cases of Besov spaces, since $H^s = B^s_{2,2}$.

Throughout this article we shall denote by the letters C or c all universal constants. we shall sometimes replace an inequality of the type $f \leq Cg$ by $f \lesssim g$. we shall also denote by $(c_j)_{j\in\mathbf{Z}}$ any sequence of norm 1 in $\ell^2(\mathbf{Z})$.

3 An L^∞ estimate for Leray solutions in dimension two

The purpose of this section is the proof of Theorem 3. Let us write the solution v of $(NS2D)$ as the sum of v_1 and v_2 with

$$\left\{\begin{array}{c} \partial_t v_1 - \Delta_h v_1 = \mathbf{P}f \\ v_{1|t=0} = v_0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{c} \partial_t v_2 - \Delta_h v_2 = -\mathbf{P}\,\mathrm{div}(v\otimes v) \\ v_{2|t=0} = 0. \end{array}\right. \tag{3.1}$$

Duhamel's formula gives

$$v_1(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-t')\Delta}\mathbf{P}f(t')dt',$$

thus we get that

$$\|v_1\|_{L^2(\mathbf{R}^+;L^\infty)} \leq \|e^{t\Delta}v_0\|_{L^2(\mathbf{R}^+;L^\infty)} + \int_0^\infty \big\|e^{\tau\Delta}f(t)\big\|_{L^2(\mathbf{R}^+_\tau;L^\infty)}\,dt.$$

Due to (2.2), we have $L^2 \hookrightarrow B^{-1}_{\infty,2}$ so by Definition 1.1 we get that

$$\|v_1\|_{L^2(\mathbf{R}^+;L^\infty)} \lesssim \|v_0\|_{L^2} + \int_0^\infty \|f(t)\|_{L^2}\,dt. \tag{3.2}$$

Now let us estimate $\|v_2\|_{L^2(\mathbf{R}^+;L^\infty)}$. It relies on the following technical proposition.

Proposition 3.1 *Let v be the solution of $(NS2D)$ with initial data v_0 in L^2 and external force f in $L^1(\mathbf{R}^+;L^2)$. Then we have*

$$\sum_j \|\Delta_j v\|^2_{L^\infty(\mathbf{R}^+;L^2)} \lesssim E_0(e + E_0^{\frac{1}{2}}) \quad \textit{with} \quad E_0 = \|v_0\|^2_{L^2} + \Big(\int_0^\infty \|f(t)\|_{L^2}dt\Big)^2.$$

Proof of Proposition 3.1 Applying Δ_j to the $(NS2D)$ system and doing an L^2 energy estimate gives, neglecting (only here) the smoothing effect of the heat flow,

$$\|\Delta_j v(t)\|_{L^2}^2 \leq \|\Delta_j v_0\|_{L^2}^2 + \int_0^t \big|\big(\Delta_j(v(t')\cdot\nabla v(t'))|\Delta_j v(t')\big)_{L^2}\big|\,dt' + \int_0^t \big|\langle \Delta_j f(t'),\Delta_j v(t')\rangle\big|\,dt'.$$

Lemma 1.1 of [3] and the conservation of energy tell us that

$$\begin{aligned}\big|\big(\Delta_j(v(t)\cdot\nabla v(t))|\Delta_j v(t)\big)_{L^2}\big| &\lesssim c_j(t)\|\nabla v(t)\|_{L^2}\|v(t)\|_{L^2}2^j\|\Delta_j v(t)\|_{L^2}\\ &\lesssim c_j^2(t)\|\nabla v(t)\|_{L^2}^2\|v(t)\|_{L^2}\\ &\lesssim E_0^{\frac12}c_j^2(t)\|\nabla v(t)\|_{L^2}^2.\end{aligned}$$

Since

$$\begin{aligned}|\langle\Delta_j f(t),\Delta_j v(t)\rangle| &\leq \|\Delta_j f(t)\|_{L^2}\|\Delta_j v(t)\|_{L^2}\\ &\lesssim E_0^{\frac12}c_j^2(t)\|f(t)\|_{L^2},\end{aligned}$$

we infer that

$$\|\Delta_j v\|_{L^\infty(\mathbf{R}^+;L^2)}^2 \lesssim \|\Delta_j v_0\|_{L^2}^2 + E_0^{\frac12}\int_0^\infty c_j^2(t)\left(\|\nabla v(t)\|_{L^2}^2+\|f(t)\|_{L^2}\right)dt.$$

Taking the sum over j concludes the proof of the proposition. ■

Conclusion of the proof of Theorem 3 Let us first observe that interpolating the result of Proposition 3.1 with the energy estimate, we find that a constant C exists such that, for any p in $[2,\infty]$, we have

$$\sum_j 2^{j\frac4p}\|\Delta_j v\|_{L^p(\mathbf{R}^+;L^2)}^2 \leq CE_0(e+E_0^{\frac12})^{1-\frac2p}. \tag{3.3}$$

Then by Bernstein's inequality (2.1), we have

$$2^{-j\left(1-\frac2p\right)}\|S_j v\|_{L^p(\mathbf{R}^+;L^\infty)} \leq C\sum_{j'\leq j-1}2^{(j'-j)\left(1-\frac2p\right)}2^{j'\frac2p}\|\Delta_{j'}v\|_{L^p(\mathbf{R}^+;L^2)}.$$

Using Young's inequality on series and (3.3), we infer that a constant C exists such that, for any p in $]2,\infty]$,

$$2^{-j\left(1-\frac2p\right)}\|S_j v\|_{L^p(\mathbf{R}^+;L^\infty)} \leq Cc_j\frac{p}{p-2}E_0^{\frac12}(e+E_0^{\frac12})^{\frac12-\frac1p}. \tag{3.4}$$

Now using Bernstein's inequality and Fourier-Plancherel, we get by (3.1)

$$\begin{aligned}\|\Delta_j v_2(t)\|_{L^\infty} &\lesssim 2^j\|\Delta_j v_2(t)\|_{L^2}\\ &\lesssim 2^{2j}\int_0^t e^{-c2^{2j}(t-t')}\left\|\Delta_j\mathbf{P}\left(v(t')\otimes v(t')\right)\right\|_{L^2}dt'.\end{aligned} \tag{3.5}$$

Using Bony's decomposition, let us write that for any a and b,

$$\Delta_j(a(t)b(t)) = \sum_{j'\geq j-N_0} \Delta_j(S_{j'}a(t)\Delta_{j'}b(t)) + \sum_{j'\geq j-N_0} \Delta_j(\Delta_{j'}a(t)S_{j'+1}b(t)).$$

We have

$$\|S_{j'}a\Delta_{j'}b\|_{L^{\frac{2p}{p+2}}(\mathbf{R}^+;L^2)} \leq \|S_{j'}a\|_{L^p(\mathbf{R}^+;L^\infty)}\|\Delta_{j'}b\|_{L^2(\mathbf{R}^+;L^2)}.$$

Using (3.4), we deduce that a constant C exists such that, for any p in $]2,\infty]$,

$$\begin{aligned}\|\Delta_j\mathbf{P}\,(v\otimes v)\|_{L^{\frac{2p}{p+2}}(\mathbf{R}^+;L^2)} &\leq C\sum_{j'\geq j-N_0}\|S_{j'}v\|_{L^p(\mathbf{R}^+;L^\infty)}\|\Delta_{j'}v\|_{L^2(\mathbf{R}^+;L^2)}\\ &\leq C\frac{p}{p-2}E_0^{\frac12}(e+E_0^{\frac12})^{\frac12-\frac1p}\sum_{j'\geq j-N_0}c_{j'}\|\Delta_{j'}v\|_{L^2(\mathbf{R}^+;L^2)}2^{-j'\left(\frac2p-1\right)}.\end{aligned}$$

Using Young's inequality in time in (3.5) gives

$$\begin{aligned}\|\Delta_jv_2\|_{L^2(\mathbf{R}^+;L^\infty)} &\leq C2^{2j}\|e^{-c2^{2j}\cdot}\|_{L^{\frac{p}{p-1}}}\|\Delta_j\mathbf{P}\,(v\otimes v)\|_{L^{\frac{2p}{p+2}}(\mathbf{R}^+;L^2)}\\ &\leq C\frac{p}{p-2}E_0^{\frac12}(e+E_0^{\frac12})^{\frac12-\frac1p}\sum_{j'\geq j-N_0}c_{j'}\|\Delta_{j'}v\|_{L^2(\mathbf{R}^+;L^2)}2^{j'}2^{(j-j')\frac2p}.\end{aligned}$$

By Young's inequality on series we find that a constant C exists such that, for any p in $]2,\infty[$,

$$\|\Delta_jv_2\|_{L^2(\mathbf{R}^+;L^\infty)} \leq Cc_j^2\frac{p^2}{p-2}E_0^{\frac12}(e+E_0^{\frac12})^{\frac12-\frac1p}$$

and thus

$$\|v_2\|_{L^2(\mathbf{R}^+;L^\infty)} \leq C\frac{p^2}{p-2}E_0(e+E_0^{\frac12})^{\frac12-\frac1p}.$$

Then let us choose p such that

$$\frac2p = 1-\frac{2}{\log(e+E_0^{\frac12})}\cdot$$

Then we have that

$$\|v_2\|_{L^2(\mathbf{R}^+;L^\infty)} \leq CE_0\log(e+E_0^{\frac12}), \tag{3.6}$$

and putting (3.2) and (3.6) together proves Theorem 3. ∎

This theorem will enable us to infer the following useful corollary.

Corollary 3.1 *Let $p\in]2,+\infty[$ and let u_0 be a vector field in $\mathcal{I}_p(A,B)$. Then $u^{(0)}=u_F+u_{2D}$ satisfies*

$$\|u^{(0)}\|^2_{L^2(\mathbf{R}^+;L^\infty)} \lesssim A^2(1+A\log(e+A))^2.$$

Proof of Corollary 3.1 As in the proof of (3.2) above, we have clearly by Definition 1.1 and (H2),

$$\begin{aligned}\|u_F\|_{L^2(\mathbf{R}^+;L^\infty)} &\leq \|\widetilde{u}_0\|_{B^{-1}_{\infty,2}}\\ &\leq A.\end{aligned}$$

Then by Theorem 3 we have

$$\|u_{2D}\|^2_{L^2(\mathbf{R}^+;L^\infty)} \lesssim E_0(1+E_0\log^2(e+E_0^{\frac{1}{2}})),$$

where by definition of E_0 and by (H1),

$$\begin{aligned}E_0 &= \|\overline{u}_0\|^2_{L^2}+\|\mathrm{M}\mathbf{P}(u_F\cdot\nabla u_F)\|^2_{L^1(\mathbf{R}^+;L^2)}\\ &\leq A^2.\end{aligned}$$

As a result we get

$$\|u^{(0)}\|^2_{L^2(\mathbf{R}^+;L^\infty)} \lesssim A^2+A^2(1+A\log(e+A))^2$$

and the corollary is proved. ■

4 Global wellposedness of the perturbed system

In this section we shall study the global wellposedness of the system (PNS). The result is the following.

Theorem 4 *Let $p\in\,]3,+\infty[$ be given. There is a constant $C_0>0$ such that for any R_0 in $B^{-1+\frac{3}{p}}_{p,2}$, F in $L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})$ and $u^{(0)}$ in $L^2(\mathbf{R}^+;L^\infty)$ satisfying*

$$\|R_0\|_{B^{-1+\frac{3}{p}}_{p,2}}+\|F\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} \leq C_0^{-1}e^{-C_0\|u^{(0)}\|^2_{L^2(\mathbf{R}^+;L^\infty)}}, \tag{4.1}$$

there is a unique, global solution R to (PNS) associated with R_0 and F, such that

$$R\in C_b(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})\cap L^2(\mathbf{R}^+;B^{\frac{3}{p}}_{p,2}).$$

Proof of Theorem 4 Using Duhamel's formula, the system (PNS) turns out be

$$\begin{aligned}R &= \mathcal{R}_0+L_0R+B_{NS}(R,R) \quad \text{with}\\ \mathcal{R}_0(t) &\stackrel{\text{def}}{=} e^{t\Delta}R_0+\int_0^t e^{(t-t')\Delta}F(t')dt'\,,\\ L_0R(t) &\stackrel{\text{def}}{=} -\int_0^t e^{(t-t')\Delta}Q(u^{(0)}(t'),R(t'))dt' \quad \text{and}\\ B_{NS}(R,R)(t) &\stackrel{\text{def}}{=} -\int_0^t e^{(t-t')\Delta}\mathbf{P}\,\mathrm{div}\,\big(R(t')\otimes R(t')\big)\,dt'.\end{aligned}$$

The proof of the global wellposedness of (PNS) relies on the following classical fixed point lemma in a Banach space, the proof of which is omitted.

Lemma 4.1 *Let X be a Banach space, let L be a continuous linear map from X to X, and let B be a bilinear map from $X \times X$ to X. Let us define*

$$\|L\|_{\mathcal{L}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=1} \|Lx\| \quad \textit{and} \quad \|B\|_{\mathcal{B}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=\|y\|=1} \|B(x,y)\|.$$

If $\|L\|_{\mathcal{L}(X)} < 1$, then for any x_0 in X such that

$$\|x_0\|_X < \frac{(1 - \|L\|_{\mathcal{L}(X)})^2}{4\|B\|_{\mathcal{B}(X)}},$$

the equation

$$x = x_0 + Lx + B(x,x)$$

has a unique solution in the ball of center 0 and radius $\dfrac{1 - \|L\|_{\mathcal{L}(X)}}{2\|B\|_{\mathcal{B}(X)}}\cdot$

Solving system (PNS) consists therefore in finding a space X in which we shall be able to apply Lemma 4.1. Let us define, for any positive real number λ and for any p in $]3,\infty]$, the following space.

Definition 4.1 *The space X_λ is the space of distributions a on $\mathbf{R}^+ \times \mathbf{T}^3$ such that*

$$\|a\|^2_{X_\lambda} \stackrel{\text{def}}{=} \sum_j 2^{-2j\left(1-\frac{3}{p}\right)} \left(\|\Delta_j a_\lambda\|^2_{L^\infty(\mathbf{R}^+;L^p)} + 2^{2j}\|\Delta_j a_\lambda\|^2_{L^2(\mathbf{R}^+;L^p)}\right) < \infty \quad \textit{with}$$

$$a_\lambda(t) \stackrel{\text{def}}{=} \exp\Big(-\lambda \int_0^t \|u^{(0)}(t')\|^2_{L^\infty} dt'\Big) a(t).$$

Remark If a belongs to X_λ, then a_λ belongs to $L^\infty(\mathbf{R}^+; B^{-1+\frac{3}{p}}_{p,2}) \cap L^2(\mathbf{R}^+; B^{\frac{3}{p}}_{p,2})$ and, as $u^{(0)}$ is in $L^2(\mathbf{R}^+; L^\infty)$, we have

$$\|a\|_{L^\infty(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} + \|a\|_{L^2(\mathbf{R}^+;B^{\frac{3}{p}}_{p,2})} \leq \|a\|_{X_\lambda} \exp\left(\lambda \|u^{(0)}\|^2_{L^2(\mathbf{R}^+;L^\infty)}\right).$$

The fact that X_λ equipped with this norm is a Banach space is a routine exercise left to the reader. The introduction of this space is justified by the following proposition which we shall prove at the end of this section.

Proposition 4.1 *For any p in $]3,\infty[$, a constant C exists such that, for any positive λ,*

$$\|L_0\|_{\mathcal{L}(X_\lambda)} \leq \frac{C}{\lambda^{\frac{1}{2}}} \quad \textit{and} \quad \|B_{NS}\|_{\mathcal{B}(X_\lambda)} \leq C e^{\lambda \|u^{(0)}\|^2_{L^2(\mathbf{R}^+;L^\infty)}}.$$

Conclusion of the proof of Theorem 1 In order to apply Lemma 4.1, let us choose λ such that $\|L_0\|_{\mathcal{L}(X_\lambda)} \leq 1/2$. Then, the condition required to apply Lemma 4.1 is

$$\|\mathcal{R}_0\|_{X_\lambda} \leq \frac{1}{16C} e^{-4C^2\|u^{(0)}\|^2_{L^2(\mathbf{R}^+;L^\infty)}}. \tag{4.2}$$

In order to ensure this condition, let us recall Lemma 2.1 of [4].

Lemma 4.2 *A constant c exists such that, for any integer j, any positive real number t and any p in $[1,\infty]$,*

$$\|\Delta_j e^{t\Delta} a\|_{L^p} \leq \frac{1}{c} e^{-c2^{2j}t} \|\Delta_j a\|_{L^p}.$$

This lemma and the Cauchy-Schwarz inequality for the measure $\|F(t')\|_{B^{-1+\frac{3}{p}}_{p,2}} dt'$ give

$$\begin{aligned}
\|\Delta_j \mathcal{R}_{0,\lambda}(t)\|_{L^p} &\leq Ce^{-c2^{2j}t}\|\Delta_j R_0\|_{L^p} + C\int_0^t e^{-c2^{2j}(t-t')}\|\Delta_j F(t')\|_{L^p} dt' \\
&\leq C2^{j\left(1-\frac{3}{p}\right)}\left(e^{-c2^{2j}t} c_j \|R_0\|_{B^{-1+\frac{3}{p}}_{p,2}} + \int_0^t e^{-c2^{2j}(t-t')} c_j(t')\|F(t')\|_{B^{-1+\frac{3}{p}}_{p,2}} dt'\right) \\
&\leq C2^{j\left(1-\frac{3}{p}\right)}\Bigg(e^{-c2^{2j}t} c_j \|R_0\|_{B^{-1+\frac{3}{p}}_{p,2}} + \left(\int_0^t e^{-c2^{2j}(t-t')}\|F(t')\|_{B^{-1+\frac{3}{p}}_{p,2}} dt'\right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\int_0^t e^{-c2^{2j}(t-t')} c_j^2(t')\|F(t')\|_{B^{-1+\frac{3}{p}}_{p,2}} dt'\right)^{\frac{1}{2}}\Bigg).
\end{aligned}$$

Then we infer immediately that

$$\begin{aligned}
\|\Delta_j \mathcal{R}_{0,\lambda}\|_{L^\infty(\mathbf{R}^+;L^p)} &\lesssim 2^{j\left(1-\frac{3}{p}\right)} c_j \|R_0\|_{B^{-1+\frac{3}{p}}_{p,2}} \\
&\quad + 2^{j\left(1-\frac{3}{p}\right)}\left(\int_0^\infty c_j^2(t)\|F(t)\|_{B^{-1+\frac{3}{p}}_{p,2}} dt\right)^{\frac{1}{2}} \|F\|^{\frac{1}{2}}_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} \quad \text{and} \\
\|\Delta_j \mathcal{R}_{0,\lambda}\|_{L^2(\mathbf{R}^+;L^p)} &\lesssim 2^{-j\frac{3}{p}} c_j \|R_0\|_{B^{-1+\frac{3}{p}}_{p,2}} \\
&\quad + 2^{-j\frac{3}{p}}\left(\int_0^\infty c_j^2(t)\|F(t)\|_{B^{-1+\frac{3}{p}}_{p,2}} dt\right)^{\frac{1}{2}} \|F\|^{\frac{1}{2}}_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})}.
\end{aligned}$$

This gives

$$\|\mathcal{R}_0\|_{X_\lambda} \lesssim \|R_0\|_{B^{-1+\frac{3}{p}}_{p,2}} + \|F\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})}.$$

It follows that the smallness condition (4.1) implies precisely condition (4.2). So we can apply Lemma 4.1 which gives a global, unique solution R to (PNS) such that

$$R \in L^\infty(\mathbf{R}^+; B^{-1+\frac{3}{p}}_{p,2}) \cap L^2(\mathbf{R}^+; B^{\frac{3}{p}}_{p,2}).$$

We leave the classical proof of the continuity in time to the reader. Theorem 4 is proved, provided we prove Proposition 4.1. ∎

Proof of Proposition 4.1 It relies mainly on Lemma 4.2 and in a Bony type decomposition. In order to prove the estimate on B_{NS}, let us observe that Lemma 4.2 implies that

$$\|\Delta_j (B_{NS}(R,R'))_\lambda(t)\|_{L^p} \leq e^{\lambda\int_0^\infty \|u^{(0)}(t)\|^2_{L^\infty} dt} \int_0^t e^{-c2^{2j}(t-t')}\|\Delta_j \mathbf{P}\,\mathrm{div}(R_\lambda(t') \otimes R'_\lambda(t'))\|_{L^p} dt'.$$

Proposition 3.1 of [4] implies that

$$\|\Delta_j \mathbf{P}\,\mathrm{div}(R_\lambda \otimes R'_\lambda)\|_{L^2(\mathbf{R}^+;L^p)} \leq C c_j 2^{j\left(-1+\frac{3}{p}\right)} \|R\|_{X_\lambda} \|R'\|_{X_\lambda}.$$

Young's inequality in time ensures the estimate on B_{NS}.

The study of L_0 follows the ideas of [4]. Let us decompose $(L_0 a)_\lambda \stackrel{\text{def}}{=} L_0 a_\lambda$ as a sum of two operators $L_{1,\lambda}$ and $L_{2,\lambda}$ defined by

$$\begin{aligned}
(L_{n,\lambda}R)(t) &\stackrel{\text{def}}{=} \int_0^t e^{(t-t')\Delta - \lambda \int_{t'}^t \|u^{(0)}(t'')\|^2_{L^\infty} dt''} \mathbf{P}\,\mathrm{div}\,\mathcal{T}_n(u^{(0)}(t'), R_\lambda(t'))dt' \quad \text{with} \\
\mathcal{T}_1(a,b) &\stackrel{\text{def}}{=} \sum_j \left(\Delta_j a \otimes S_{j-1} b + S_{j-1} b \otimes \Delta_j a\right) \quad \text{and} \\
\mathcal{T}_2(a,b) &\stackrel{\text{def}}{=} \sum_j \left(S_{j+2} a \otimes \Delta_j b + \Delta_j b \otimes S_{j+2} a\right).
\end{aligned}$$

As

$$\Delta_j \mathcal{T}_1(a,b) = \sum_{|j'-j|\leq 5} \Delta_j \left(\Delta_{j'} a \otimes S_{j'-1} b + S_{j'-1} b \otimes \Delta_{j'} a\right),$$

we have

$$\|\Delta_j \mathcal{T}_1(a,b)\|_{L^p} \leq C \sum_{|j'-j|\leq 5} \|\Delta_{j'} a\|_{L^\infty} \|S_{j'-1} b\|_{L^p}.$$

Noticing that

$$\|S_{j-1} R_\lambda\|_{L^\infty(\mathbf{R}^+;L^p)} \leq c_j 2^{j\left(1-\frac{3}{p}\right)} \|R\|_{X_\lambda},$$

we obtain

$$\|\Delta_j \mathcal{T}_1(u^{(0)}(t), R_\lambda(t))\|_{L^p} \leq C c_j 2^{j\left(1-\frac{3}{p}\right)} \|R\|_{X_\lambda} \|u^{(0)}(t)\|_{L^\infty}.$$

Using Bernstein's inequality and Lemma 4.2, we have therefore

$$\begin{aligned}
\|\Delta_j (L_{1,\lambda}R)(t)\|_{L^p} &\leq C 2^j \int_0^t e^{-c2^{2j}(t-t') - \lambda \int_{t'}^t \|u^{(0)}(t'')\|^2_{L^\infty} dt''} \|\mathcal{T}_1(u^{(0)}(t'), R_\lambda(t'))\|_{L^p} dt' \\
&\leq C c_j 2^{j\left(2-\frac{3}{p}\right)} \|R\|_{X_\lambda} \int_0^t e^{-c2^{2j}(t-t') - \lambda \int_{t'}^t \|u^{(0)}(t'')\|^2_{L^\infty} dt''} \|u^{(0)}(t')\|_{L^\infty} dt'.
\end{aligned}$$

Thus we get, by Young's inequality,

$$\|\Delta_j (L_{1,\lambda}R)\|_{L^\infty(\mathbf{R}^+;L^p)} + 2^j \|\Delta_j (L_{1,\lambda}R)\|_{L^2(\mathbf{R}^+;L^p)} \leq \frac{C}{\lambda^{\frac{1}{2}}} c_j 2^{j\left(1-\frac{3}{p}\right)} \|R\|_{X_\lambda}. \tag{4.3}$$

Let us now estimate $L_{2,\lambda}R$. As

$$\Delta_j \mathcal{T}_2(a,b) = \sum_{j'-j\geq N_0} \Delta_j \left(\Delta_{j'} a \otimes S_{j'-1} b + S_{j'-1} b \otimes \Delta_{j'} a\right),$$

we have

$$\begin{aligned}
\|\Delta_j \mathcal{T}_2(a,b)\|_{L^p} &\leq C \sum_{j'-j\geq N_0} \|S_{j'+2} a\|_{L^\infty} \|\Delta_{j'} b\|_{L^p} \\
&\leq C \|a\|_{L^\infty} \sum_{j'-j\geq N_0} \|\Delta_{j'} b\|_{L^p}.
\end{aligned}$$

As for the estimate of $L_{1,\lambda}$ we get that

$$\|\Delta_j(L_{2,\lambda}R)(t)\|_{L^p} \leq C2^j \sum_{j'\geq j-N_0} \int_0^t e^{-c2^{2j}(t-t')-\lambda\int_{t'}^t \|u^{(0)}(t'')\|_{L^\infty}^2 dt''} \times \|u^{(0)}(t')\|_{L^\infty}\|\Delta_{j'}R_\lambda(t')\|_{L^p}dt'.$$

The Cauchy-Schwarz inequality implies that

$$\|\Delta_j(L_{2,\lambda}R)(t)\|_{L^p} \leq C2^j \sum_{j'\geq j-N_0} \Big(\int_0^t e^{-c2^{2j}(t-t')}\|\Delta_{j'}R_\lambda(t')\|_{L^p}^2 dt'\Big)^{\frac12} \times \Big(\int_0^t e^{-2\lambda\int_{t'}^t \|u^{(0)}(t'')\|_{L^\infty}^2 dt''}\|u^{(0)}(t')\|_{L^\infty}^2 dt'\Big)^{\frac12}.$$

Then we infer that

$$2^{j\frac3p}\Big(\|\Delta_j(L_{2,\lambda}R)\|_{L^\infty(\mathbf{R}^+;L^p)} + 2^j\|\Delta_j(L_{2,\lambda}R)\|_{L^2(\mathbf{R}^+;L^p)}\Big) \leq \frac{C}{\lambda^{\frac12}} \sum_{j'\geq j-N_0} 2^{(j-j')\frac3p}2^{j'\frac3p}\|\Delta_{j'}R_\lambda\|_{L^2(\mathbf{R}^+;L^p)}.$$

Young's inequality on series and (4.3) allow to conclude the proof of Proposition 4.1. ■

5 End of the proof of Theorem 1

Now we are ready to prove Theorem 1. The idea, as presented in the introduction, is to write

$$u = u^{(0)} + R,$$

where R satisfies (PNS) with $R_0 = 0$ and $F = -(\mathrm{Id}-\mathrm{M})\mathbf{P}(u_F\cdot\nabla u_F) - Q(u_F, u_{2D})$, and where $u^{(0)} = u_F + u_{2D}$. According to the assumptions of Theorem 1, we know that u_0 belongs to $\mathcal{I}_p(A,B)$, so in particular by (H3) we have

$$\|F\|_{L^1(\mathbf{R}^+;B_{p,2}^{-1+\frac3p})} \leq B.$$

Moreover by Corollary 3.1 we have

$$\|u^{(0)}\|_{L^2(\mathbf{R}^+;L^\infty)}^2 \lesssim A^2(1+A\log(e+A))^2.$$

Due to Theorem 4, the global wellposedness of (PNS) is guaranteed if

$$\|F\|_{L^1(\mathbf{R}^+;B_{p,2}^{-1+\frac3p})} \leq C_0^{-1}e^{-C_0\|u^{(0)}\|_{L^2(\mathbf{R}^+;L^\infty)}^2}.$$

Clearly the smallness assumption (1.1) implies directly that inequality, so under the assumptions of Theorem 1, we have

$$R \in C_b(\mathbf{R}^+;B_{p,2}^{-1+\frac3p}) \cap L^2(\mathbf{R}^+;B_{p,2}^{\frac3p}).$$

To end the proof of Theorem 1 we still need to prove that u is in $C_b(\mathbf{R}^+;H^{\frac12})\cap L^2(\mathbf{R}^+;H^{\frac32})$. It is well known (see for instance [3]) that the blow up condition for $H^{\frac12}(\mathbf{T}^3)$ data is the blow up of the norm L^2 in time with values in $H^{\frac32}$. As u_0 is in $H^{\frac12}$, so are $\overline{u}_0$ and $\widetilde{u}_0$. Then thanks to the propagation of regularity in $(NS2D)$ (see for instance [3]) and the properties of the heat flow, u_F and u_{2D} belong to

$$L^\infty(\mathbf{R}^+;H^{\frac12})\cap L^2(\mathbf{R}^+;H^{\frac32}) \quad \text{and thus to} \quad L^\infty(\mathbf{R}^+;B_{p,2}^{-1+\frac3p})\cap L^2(\mathbf{R}^+;B_{p,2}^{\frac3p})$$

by the embedding recalled in (2.2). Thus as R belongs also to this space, it is enough to prove the following blow up result, which we prove for the reader's convenience.

Proposition 5.1 *If the maximal time $T^\star$ of existence in $L^\infty_{loc}(\mathbf{R}^+;H^{\frac12})\cap L^2_{loc}(\mathbf{R}^+;H^{\frac32})$ of a solution u of (NS) is finite, then for any p,*

$$\int_0^{T^\star}\|u(t)\|^4_{B_{p,\infty}^{-\frac12+\frac3p}}dt=+\infty.$$

Proof of Proposition 5.1 An energy estimate in $H^{\frac12}$ gives, for some positive c,

$$\|u(t)\|^2_{H^{\frac12}}+c\int_0^t\|u(t')\|^2_{H^{\frac32}}dt'\le\|u_0\|^2_{H^{\frac12}}+2\int_0^t\big(\operatorname{div}(u(t')\otimes u(t'))|u(t')\big)_{H^{\frac12}}dt'.$$

We can assume that $p>6$. Laws of product in Besov spaces imply that

$$\|u(t')\otimes u(t')\|_{H^{\frac12}}\le C\|u(t')\|_{B_{p,\infty}^{-\frac12+\frac3p}}\|u(t')\|_{H^1}.$$

Thus by interpolation we infer that

$$\big(\operatorname{div}(u(t')\otimes u(t'))|u(t')\big)_{H^{\frac12}}\le C\|u(t')\|_{B_{p,\infty}^{-\frac12+\frac3p}}\|u(t')\|^{\frac12}_{H^{\frac12}}\|u(t')\|^{\frac32}_{H^{\frac32}}.$$

Using the convexity inequality $ab\le 3/4a^{\frac43}+1/4b^4$ gives

$$\|u(t)\|^2_{H^{\frac12}}+\frac c2\int_0^t\|u(t')\|^2_{H^{\frac32}}dt'\le\|u_0\|^2_{H^{\frac12}}+C\int_0^t\|u(t')\|^4_{B_{p,\infty}^{-\frac12+\frac3p}}\|u(t')\|^2_{H^{\frac12}}dt'.$$

A Gronwall lemma concludes the proof of Proposition 5.1, and therefore of Theorem 1. ■

6 Proof of Theorem 2

In this final section we shall prove Theorem 2. In order to do so, two points must be checked: first, that the initial data defined in the statement of the theorem satisfies the assumptions of Theorem 1, namely the nonlinear smallness assumption (1.1), in which case the global wellposedness will follow as a consequence of that theorem. Second, that the initial data satisfies the lower bound (1.2). Those two points are dealt with in Sections 6.1 and 6.2 respectively.

6.1 The nonlinear smallness assumption

Let us check that the initial data defined in the statement of Theorem 2 belongs to the space $\mathcal{I}_p(A,B)$ with the smallness condition (1.1). Recall that A and B are chosen so that

$$\begin{array}{ll}
\text{(H1)} & \|\overline{u}_0\|_{L^2(\mathbf{T}^2)} + \|\mathrm{M}\mathbf{P}(u_F\cdot\nabla u_F)\|_{L^1(\mathbf{R}^+;L^2(\mathbf{T}^2))} \leq A \\
\text{(H2)} & \|\widetilde{u_0}\|_{B^{-1}_{\infty,2}} \leq A \\
\text{(H3)} & \|(\mathrm{Id}-\mathrm{M})\mathbf{P}(u_F\cdot\nabla u_F) + Q(u_{2D},u_F)\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} \leq B.
\end{array}$$

Let us start with Assumption (H1). We first notice directly that $\overline{u}_0=0$, so we just have to check that $\mathrm{M}\mathbf{P}(u_F\cdot\nabla u_F)$ belongs to $L^1(\mathbf{R}^+;L^2(\mathbf{T}^2))$, and to compute its bound.
We have

$$u_F\cdot\nabla u_F = \mathrm{div}(u_F\otimes u_F) \quad \text{hence} \quad \mathrm{M}(u_F\cdot\nabla u_F^j) = \mathrm{M}\,\mathrm{div}_h(u_F^j u_F^h) \quad \text{for} \quad j\in\{1,2,3\}.$$

On the one hand, we have

$$\begin{aligned}
\mathrm{div}_h(u_F^3 u_F^h)(x) &= N\,\mathrm{div}_h\left(-\mathrm{div}_h(e^{t\Delta_h}v_0^h)e^{t\Delta_h}v_0^h\right)(x_h)\left(e^{t\partial_3^2}\sin(Nx_3)\right)\left(e^{t\partial_3^2}\cos(Nx_3)\right) \\
&= \frac{N}{2}e^{-2tN^2}\,\mathrm{div}_h\left(-\mathrm{div}_h(e^{t\Delta_h}v_0^h)e^{t\Delta_h}v_0^h\right)(x_h)\sin(2Nx_3),
\end{aligned}$$

which implies that $\mathrm{M}(u_F\cdot\nabla u_F^3)=0$. Notice that in particular, since $\overline{u}_0=0$, we infer that

$$\forall t\geq 0, \quad u_{2D}^3(t)=0. \tag{6.1}$$

On the other hand, we have

$$\begin{aligned}
\mathrm{div}_h(u_F^h\otimes u_F^h)(x) &= N^2\,\mathrm{div}_h\left(e^{t\Delta_h}v_0^h\otimes e^{t\Delta_h}v_0^h\right)(x_h)\left(e^{t\partial_3^2}\cos(Nx_3)\right)^2 \\
&= \frac{N^2}{2}e^{-2tN^2}\,\mathrm{div}_h\left(e^{t\Delta_h}v_0^h\otimes e^{t\Delta_h}v_0^h\right)(x_h)(1+\cos(2Nx_3)).
\end{aligned}$$

Using the frequency localization of v_0^h and Bernstein's inequality (2.1), we get

$$\begin{aligned}
\left\|\mathrm{M}(u_F\cdot\nabla u_F^h)\right\|_{L^2(\mathbf{T}^2)} &\leq \frac{N^2}{2}e^{-2tN^2}N_0\|e^{t\Delta_h}v_0^h\|^2_{L^4(\mathbf{T}^2)} \\
&\leq CN_0^2N^2e^{-2tN^2}\|v_0^h\|^2_{L^2(\mathbf{T}^2)}.
\end{aligned}$$

Finally we infer that

$$\begin{aligned}
\left\|\mathrm{M}(u_F\cdot\nabla u_F^h)\right\|_{L^1(\mathbf{R}^+;L^2(\mathbf{T}^2))} &\leq CN_0^2\|v_0^h\|^2_{L^2(\mathbf{T}^2)} \\
&\leq C_{N_0}(\log N)^{\frac{2}{9}}.
\end{aligned} \tag{6.2}$$

Let us now consider Assumption (H2). Since $\overline{u}_0=0$, it simply consists in computing the $B^{-1}_{\infty,2}$ norm of u_0. We have

$$u_0^h(x) = Nv_0^h(x_h)\cos(Nx_3),$$

and by definition of Besov norms,

$$\|u_0^h\|_{B^{-1}_{\infty,2}} = \Big\|\tau^{\frac12}\|e^{\tau\Delta}u_0^h\|_{L^\infty}\Big\|_{L^2(\mathbf{R}^+,\frac{d\tau}{\tau})} = \|e^{\tau\Delta}u_0^h\|_{L^2(\mathbf{R}^+;L^\infty)}.$$

It is easy to see that

$$\begin{aligned}\|e^{\tau\Delta}u_0^h\|_{L^\infty} &= N\|e^{\tau\Delta_h}v_0^h(x_h)e^{\tau\partial_3^2}\cos(Nx_3)\|_{L^\infty}\\ &\le CN_0Ne^{-\tau N^2}\|v_0^h\|_{L^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\|u_0^h\|_{B^{-1}_{\infty,2}} &\le CN_0N\|v_0^h\|_{L^2}\|e^{-\tau N^2}\|_{L^2(\mathbf{R}^+)}\\ &\le CN_0\|v_0^h\|_{L^2}.\end{aligned}$$

The computation is similar for u_0^3, so we get, for N large enough,

$$\|u_0\|_{B^{-1}_{\infty,2}} \le C_{N_0}(\log N)^{\frac19}. \tag{6.3}$$

Thus one can choose for the parameter A in (H1) and (H2)

$$A = C_{N_0}(\log N)^{\frac29}. \tag{6.4}$$

Finally let us consider Assumption (H3). We shall start with $(\mathrm{Id}-\mathrm{M})\mathbf{P}(u_F\cdot\nabla u_F)$. We have

$$(\mathrm{Id}-\mathrm{M})(u_F\cdot\nabla u_F) = (\mathrm{Id}-\mathrm{M})(u_F^h\cdot\nabla^h u_F) + (\mathrm{Id}-\mathrm{M})(u_F^3\partial_3 u_F),$$

and we shall concentrate on the first term, as both are treated in the same way. We compute

$$\begin{aligned}(\mathrm{Id}-\mathrm{M})(u_F^h\cdot\nabla^h u_F^h)(x) &= \frac{N^2}{2}\Big(e^{t\Delta_h}v_0^h\cdot\nabla^h e^{t\Delta_h}v_0^h\Big)(x_h)e^{-2tN^2}\cos(2Nx_3) \quad \text{and}\\ (\mathrm{Id}-\mathrm{M})(u_F^h\cdot\nabla^h u_F^3) &= -\frac{N}{2}\Big(e^{t\Delta_h}v_0^h\cdot\nabla^h e^{t\Delta_h}\,\mathrm{div}_h\, v_0^h\Big)(x_h)e^{-2tN^2}\sin(2Nx_3).\end{aligned}$$

So

$$\begin{aligned}\|(\mathrm{Id}-\mathrm{M})(u_F^h\cdot\nabla^h u_F^h)\|_{B_{p,2}^{-1+\frac3p}} &= \Big\|\tau^{\frac12-\frac{3}{2p}}\|e^{\tau\Delta}(\mathrm{Id}-\mathrm{M})(u_F^h\cdot\nabla^h u_F^h)\|_{L^p}\Big\|_{L^2(\mathbf{R}^+,\frac{d\tau}{\tau})}\\ &\le C_{N_0}\frac{N^2}{2}e^{-2tN^2}\Big\|\tau^{\frac12-\frac{3}{2p}}e^{-2\tau N^2}\Big\|_{L^2(\mathbf{R}^+,\frac{d\tau}{\tau})}\|v_0^h\|_{L^2}^2\\ &\le C_{N_0}N^2e^{-2tN^2}N^{\frac3p-1}\|v_0^h\|_{L^2}^2.\end{aligned}$$

It follows that

$$\|(\mathrm{Id}-\mathrm{M})(u_F^h\cdot\nabla^h u_F^h)\|_{L^1(\mathbf{R}^+;B_{p,2}^{-1+\frac3p})} \le C_{N_0}N^{\frac3p-1}\|v_0^h\|_{L^2}^2, \tag{6.5}$$

and similarly

$$\|(\mathrm{Id}-\mathrm{M})(u_F^h\cdot\nabla^h u_F^3)\|_{L^1(\mathbf{R}^+;B_{p,2}^{-1+\frac3p})} \le C_{N_0}N^{\frac3p-2}\|v_0^h\|_{L^2}^2. \tag{6.6}$$

Finally let us estimate the term $Q(u_{2D}, u_F)$. Since by (6.1), u^3_{2D} is identically equal to zero, we have

$$Q(u_{2D}, u_F) = \mathbf{P}\,\mathrm{div}_h(u_{2D} \otimes u_F + u_F \otimes u_{2D})$$

so

$$\|Q(u_{2D}, u^h_F)\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} \leq N\|e^{-tN^2}\,\mathrm{div}_h(e^{t\Delta_h} v^h_0 \otimes u^h_{2D})(x_h)\cos(Nx_3)\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})}\cdot$$

we shall only compute that term, as $Q(u_{2D}, u^3_F)$ is estimated similarly (and contributes in fact one power less in N). Sobolev embeddings imply that $H^s(\mathbf{T}^2) \hookrightarrow L^p(\mathbf{T}^2)$ for $s \stackrel{\text{def}}{=} 1 - \frac{2}{p}\cdot$ So

$$\begin{aligned}
\|\,\mathrm{div}_h(e^{t\Delta_h} v^h_0 \otimes u^h_{2D})\|_{L^p} &\leq \|e^{t\Delta_h} v^h_0 \cdot \nabla^h u^h_{2D}\|_{L^p} + \|u^h_{2D} \cdot \nabla^h e^{t\Delta_h} v^h_0\|_{L^p}\\
&\leq \|e^{t\Delta_h} v^h_0\|_{L^\infty}\|\nabla^h u^h_{2D}\|_{L^p} + \|u^h_{2D}\|_{L^p}\|e^{t\Delta_h} v^h_0\|_{L^\infty}\\
&\leq CN_0\|v^h_0\|_{L^2}\|u^h_{2D}\|_{H^{s+1}} + CN_0^2\|v^h_0\|_{L^2}\|u^h_{2D}\|_{H^s}.
\end{aligned}$$

Propagation of regularity for the two dimensional Navier-Stokes equations is expressed by

$$\|u_{2D}\|_{L^\infty(\mathbf{R}^+;H^2)} \leq \|\mathrm{M}(u_F \cdot \nabla u_F)\|_{L^1(\mathbf{R}^+;H^2)} e^{C\|u_{2D}\|^2_{L^2(\mathbf{R}^+;H^1)}}.$$

Using (6.3) and that the Fourier transform of $\mathrm{M}(u_F \cdot \nabla u_F)$ is supported in $[-2N_0, 2N_0]^2$, we get

$$\begin{aligned}
\|u_{2D}\|_{L^\infty(\mathbf{R}^+;H^2)} &\leq CN_0^2\|\mathrm{M}(u_F \cdot \nabla u_F)\|_{L^1(\mathbf{R}^+;L^2)} e^{C_{N_0}\|v^h_0\|^4_{L^2}}\\
&\leq C_{N_0}\|v^h_0\|^2_{L^2} e^{C_{N_0}\|v^h_0\|^4_{L^2}}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\|Q(u_{2D}, u_F)(t)\|_{B^{-1+\frac{3}{p}}_{p,2}} &\leq C_{N_0} N e^{-tN^2}\left\|\tau^{\frac{1}{2}-\frac{3}{2p}} e^{-\tau N^2}\right\|_{L^2(\mathbf{R}^+;\frac{d\tau}{\tau})}\|v^h_0\|^3_{L^2} e^{C_{N_0}\|v^h_0\|^4_{L^2}}\\
&\leq C_{N_0} N^{\frac{3}{p}} e^{-tN^2}\|v^h_0\|^3_{L^2} e^{C_{N_0}\|v^h_0\|^4_{L^2}}.
\end{aligned}$$

Finally

$$\|Q(u_{2D}, u_F)\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} \leq C_{N_0} N^{\frac{3}{p}-2}\|v^h_0\|^3_{L^2} e^{C_{N_0}\|v^h_0\|^4_{L^2}}.$$

Together with (6.5) and (6.6), this gives

$$\begin{aligned}
&\|(\mathrm{Id} - \mathrm{M})\mathbf{P}(u_F \cdot \nabla u_F) + Q(u_{2D}, u_F)\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})}\\
&\qquad\qquad \leq C_{N_0} N^{\frac{3}{p}-1}\|v^h_0\|^2_{L^2}\Big(1 + N^{-1}\|v^h_0\|_{L^2} e^{C_{N_0}\|v^h_0\|^4_{L^2}}\Big).
\end{aligned}$$

Using that $\|v^h_0\|_{L^2(\mathbf{T}^2)} \leq (\log N)^{\frac{1}{9}}$, we infer that, for N large enough,

$$\|(\mathrm{Id} - \mathrm{M})\mathbf{P}(u_F \cdot \nabla u_F) + Q(u_{2D}, u_F)\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} \leq C_{N_0} N^{\frac{3}{p}-1}(\log N)^{\frac{2}{9}}.$$

Choosing $p \geq 6$ gives, still for N large enough,

$$\|(\mathrm{Id} - \mathrm{M})\mathbf{P}(u_F \cdot \nabla u_F) + Q(u_{2D}, u_F)\|_{L^1(\mathbf{R}^+;B^{-1+\frac{3}{p}}_{p,2})} \leq N^{-\frac{1}{4}}.$$

We can therefore choose for the parameter B in (H3) the value $B = N^{-\frac{1}{4}}$. Let us check that with such choices of A and B, the smallness assumption (1.1) holds. With the choice of $A = C_{N_0}(\log N)^{\frac{2}{9}}$ made in (6.4), we have, for N large enough,

$$
\begin{aligned}
\exp\left(C_0 A^2(1+A\log A)^2\right) &\leq \exp\Big(C_{N_0}(\log^{\frac{8}{9}} N)(\log\log N)\Big)\\
&\leq \exp\Big(\frac{1}{8}\log N\Big)\\
&\leq N^{\frac{1}{8}}.
\end{aligned}
$$

Since $B = N^{-\frac{1}{4}}$, the smallness assumption (1.1) is guaranteed for large enough N, and Theorem 1 yields the global wellposedness of the system with that initial data.

6.2 The lower bound

Let us now check that the initial data u_0^h satisfies the lower bound (1.2). We recall that the $B^{-1}_{\infty,\infty}$ norm is defined by

$$
\|u_0^h\|_{B^{-1}_{\infty,\infty}} = \sup_{t\geq 0} t^{\frac{1}{2}}\|e^{t\Delta}u_0^h\|_{L^\infty(\mathbf{T}^3)}.
$$

An easy computation, using the explicit formulation of u_0^h, enables us to write that

$$
\begin{aligned}
e^{t\Delta}u_0^h(x) &= Ne^{t\Delta_h}v_0^h(x_h)e^{t\partial_3^2}\cos(Nx_3)\\
&= Ne^{t\Delta_h}v_0^h(x_h)e^{-tN^2}\cos(Nx_3).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|e^{t\Delta}u_0^h\|_{L^\infty(\mathbf{T}^3)} &= Ne^{-tN^2}\|e^{t\Delta_h}v_0^h\|_{L^\infty(\mathbf{T}^2)}\\
&\geq \frac{N}{2\pi}e^{-tN^2}\|e^{t\Delta_h}v_0^h\|_{L^2(\mathbf{T}^2)}\\
&\geq \frac{N}{2\pi}e^{-2tN^2}\|v_0^h\|_{L^2(\mathbf{T}^2)},
\end{aligned}
$$

for $N \geq N_0$, using the fact that the frequencies of v_0^h are smaller than N_0. Finally we have

$$
\begin{aligned}
\|u_0^h\|_{B^{-1}_{\infty,\infty}} &\geq \frac{N}{2\pi}\|v_0^h\|_{L^2(\mathbf{T}^2)}\sup_{t\geq 0}\left(t^{\frac{1}{2}}e^{-2tN^2}\right)\\
&\geq \frac{1}{4\pi\sqrt{e}}\|v_0^h\|_{L^2(\mathbf{T}^2)},
\end{aligned}
$$

and Theorem 2 follows. ∎

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