INTERPOLATION BETWEEN ENERGY AND SCALING FOR SOME NONLINEAR CAUCHY PROBLEMS

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Abstract. In this series of lectures we focus on the following question: how can one improve usual results on the Cauchy problem for some nonlinear PDEs by making use both of energy estimates and of scaling arguments? Using one only of those special features usually provides with a certain type of result, but incorporating the other one can sometimes improve those results. After some heuristical explanations we will see three different examples showing how to implement the heuristics. Those examples are taken from the papers [8] and [9].

1. Introduction

1.1. **Some heuristics.** Consider a time dependent nonlinear partial differential equation of the type

$$\partial_t u = F(t, x, u, \partial u...) \quad \text{in} \quad \mathbb{R}^d$$

with prescribed initial data u_0 at time $t = 0$. A natural problem is to try and solve this PDE, that is to answer the following question: given an initial data in some functional space, is it possible to find a solution associated with that data ? If so is it unique, for how long does it exist, does it remain in the same functional space as the data, is it stable...? Of course the answer to those questions depends strongly on the equation, and for a given equation it depends on the initial data.

Very often such an equation is given by Physics (for instance it can originate from a model in elasticity, in electromagnetism, in fluid mechanics...), and it translates some fundamental law(s) of Physics. Once the equation is given, it is therefore natural to suppose that the initial data belongs to a functional space linked to the conservation laws in question. Before giving any precise example, let us make the following heuristical observations.

One can associate to the equation some conserved quantity, call it the energy. It is also common for the model to have some "scale", which means that one can build another model having exactly the same properties, simply by rescaling space and time in some way. Let us denote by an abstract symbol X_e, e for energy, a Banach space in the space variable, corresponding to a conservation law of the equation (that means that if the initial data is in X_e, then one expects the solution to remain in X_e for all times since that quantity is conserved), and by X_s, s for scaling, a Banach space in the space variable, invariant under the scaling of the model (which means that the norm of the initial data in X_s does not change if one rescales the space variable according to the scale of the model).

Let us analyze what type of existence and uniqueness theorem one can hope to prove for an initial data in such spaces.

- Suppose that the initial data is in X_e. Then one can try to approximate the equation in such a way that its structure is not destroyed, i.e. so that the approximate equation enjoys the same conservation properties as the original system. On the smoothened equation one can apply the Cauchy-Lipschitz theory in X_e and come up with a smooth, approximate solution on some life span. The conservation of the X_e norm will imply that the life span is infinite, hence the sequence of approximate solutions is defined globally in time and is bounded in X_e. Then (up to the extraction of a subsequence) one can find a weak limit to that sequence, which one can hope to prove it solves the original system — that is far from being obvious since the equation is nonlinear. The question of uniqueness is usually very difficult to address on the other hand (uniqueness may indeed not hold) since the equation satisfied by the difference of two solutions loses the structure of the original system so an energy estimate in X_e is not so clear.

  For an initial data in an energy type space, one can therefore hope to prove the global existence of weak solutions, bounded in the energy space for all time, but not their uniqueness in general.
- Suppose that the initial data is in X_s. A scale invariant space (in the space and time variables) should be understood as a space in which every term of the equation plays the same role. There are no structural properties to use anymore so the idea is rather to apply a fixed-point type procedure. The difficulty is to show that the nonlinear term can be controled in a scale invariant space, by a (nonlinear) function of the solution in that same space. If that is the case then one needs to use some bootstrap argument to absorb that nonlinear term. This will work for small data, or perhaps for arbitrary data on a short time interval. No a priori bound is available in X_s to extend the local solution globally in time.

  For an initial data in a scale invariant space, one can therefore hope to prove the local existence and uniqueness of a solution, with no control of the solution globally in time unless the data is small.

Considering the heuristical arguments above, it seems that the results one can hope to prove depend very much on whether the initial data belongs to an X_e or an X_s-type space. In particular one can expect very different results if X_s is embedded in X_e, or if X_e is embedded in X_s, or if both spaces have the same scaling.

For instance if X_e is "above the scaling" of X_s (for instance a Sobolev space of higher regularity, or a Lebesgue space of larger index), then the fixed point procedure will probably be possible to implement (and in fact will be easier) in X_e, which will guarantee the uniqueness of the solution in X_e and will imply the *global wellposedness* (i.e. the global existence and uniqueness) of the equation in X_e — note however that nothing allows to state conversely that the unique solution in X_s is global in time for all data, for lack of a priori estimates in X_s.

On the other hand if X_e has the same scaling as X_s, then again the fixed point procedure should (just) work in the energy space so *global wellposedness* should hold in X_e. Conversely since X_s has the same scaling as X_e then one can hope to prove the global wellposedness in X_s — note that such a result again is not obvious, for lack of a priori estimates.

Finally the worst case of all is the case when X_e is "below the scaling" of X_s (for instance a Sobolev space of lower regularity, or a Lebesgue space of smaller index). One result one can try to prove is a "weak-strong uniqueness result" of the following type: suppose the initial data is both in the energy space and in a scale invariant space. Then on the one hand there is a weak solution (possibly many) defined for all times in X_e, and on the other hand there is a unique solution in X_s for some short time interval $[0, T]$. The question is then to know whether all those solutions coincide on that time interval. Also, the use of the energy conservation is liable to give some new qualitative information on scale invariant solutions.

In this text we shall present three different examples, dealing with all three situtations: the 3D cubic wave equation, and the 2D and 3D Navier-Stokes equations. In each case we will see that the heuristics hold (and correspond to well-known facts on those equations), and then we will implement the ideas above (which correspond to more recent results) to improve on those well-known facts.

1.2. **Examples.**

1.2.1. *The semilinear wave equation.* Consider the following semilinear wave equation, in three space dimensions

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta u + |u|^{p-1} u &= 0 \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}^3_x \\ (u, \partial_t u)_{|t=0} &= (u_0, u_1). \end{cases}$$

It is easy to find a conserved quantity for this equation, simply by multiplying it formally by $\partial_t u$ and integrating by parts. One finds

$$E_0 = \frac{1}{2} \|\nabla_{t,x} u(t)\|^2_{L^2} + \frac{1}{p+1} \|u(t)\|^{p+1}_{L^{p+1}}.$$

So one can write $X_e = \dot{H}^1 \cap L^{p+1}$.

The invariants of the equation are

- space translations;
- scaling: if u solves (W), then so does $u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda^2 u(\lambda^{p-1} t, \lambda^{p-1} x)$ for all $\lambda > 0$.

So for instance $X_s = \dot{H}^{\frac{3}{2} - \frac{2}{p-1}}, L^{\frac{3}{2}(p-1)}$. Let us recall the Sobolev embeddings

$$\dot{H}^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3), \quad s = 3(\frac{1}{2} - \frac{1}{q}).$$

One can notice that the case $p = 5$ is critical since $\dot{H}^1$ is embedded in L^{p+1} (and X_e has the scaling of X_s) whereas the case $p < 5$ is subcritical (since $\frac{3}{2} - \frac{2}{p-1} < 1$).

Let us recall some classical results in both cases. Before doing so, we shall state the well-known Strichartz estimates, in three space dimensions.

Theorem 1 ([13, 17])**.** *Let (p, q) and $(\tilde{p}, \tilde{q})$ be admissible pairs, i.e. such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $p > 2$, and similarly for $(\tilde{p}, \tilde{q})$. Let $f(x)$, $F(t,x)$ be two functions localized at frequency $|\xi| \sim 2^j$, and denote $\omega = \sqrt{-\Delta}$. Then*

$$\|e^{\pm i\omega t} f(x)\|_{L^p_t(L^q_x)} \lesssim 2^{\frac{2}{p} j} \|f(x)\|_{L^2}, \tag{1.1}$$

and, if $u(x,t) = \Box^{-1}F(x,t)$ *is the solution to the inhomogeneous equation with zero Cauchy data, then*

$$\|u(x,t)\|_{L^\infty_t(L^2_x)} \lesssim 2^{j(\frac{2}{\tilde p}-1)}\|F(x,t)\|_{L^{\tilde p'}_t(L^{\tilde q'}_x)} \tag{1.2}$$

$$\|u(x,t)\|_{L^p_t(L^q_x)} \lesssim 2^{j(\frac{2}{p}+\frac{2}{\tilde p}-1)}\|F(x,t)\|_{L^{\tilde p'}_t(L^{\tilde q'}_x)} \tag{1.3}$$

where p' *denotes the dual exponent of* p*.*

The first result on the Cauchy problem deals with the subcritical case, with an initial data in the energy space. One expects the existence of a global solution, since the energy is conserved. Furthermore, since the setting is subcritical one can expect to prove the uniqueness of the solution.

Theorem 2 (J. Ginibre and G. Velo [12])**.** *Suppose that* $(u_0,u_1) \in \dot H^1 \cap L^4(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$*. Then there is a unique solution* u *to* (W) *for* $1 \le p < 5$*, in* $C(\mathbb{R}, \dot H^1 \cap L^{p+1}(\mathbb{R}^3))$*, such that* u *belongs to the Strichartz spaces* $L^p(\mathbb{R}, L^q(\mathbb{R}^3))$ *with* $2/p + 3/q = 1$*.*

On the other hand, if the initial data belongs to a scale invariant space alone, one loses the fact that the solution is global, as shown by the following theorem.

Theorem 3 (H. Lindblad and C. Sogge [17])**.** *Suppose that* $(u_0,u_1) \in \dot H^{\frac12} \times \dot H^{-\frac12}(\mathbb{R}^3)$*. Then there is a unique maximal time* T^* *and a unique solution* u *to* (W) *for* $p = 3$*, in the space* $C^0([0,T], \dot H^{\frac12}(\mathbb{R}^3))$ *for* $t < T^*$*, such that moreover* u *belongs to the Strichartz spaces.*

It should be noted that more recently F. Planchon [21] proved a similar result in the Besov space $\dot B^{\frac12}_{2,\infty}$ (see Definition 2.2 below).

In the critical case, the energy is at the same level as scaling, and one is able to prove the existence and uniqueness of global solutions.

Theorem 4 (J. Shatah and M. Struwe [22])**.** *Suppose that* (u_0,u_1) *belongs to* $\dot H^1 \times L^2(\mathbb{R}^3)$*. Then there is a unique solution* u *to* (W) *for* $p = 5$*, in the space* $C([0,T], \dot H^1(\mathbb{R}^3))$ *such that furthermore* u *belongs to* $L^5_{loc}(\mathbb{R}_t, L^{10}(\mathbb{R}^3))$*.*

In [2], H. Bahouri and J. Shatah proved that in fact $u \in L^5(\mathbb{R}_t, L^{10}(\mathbb{R}^3))$.

1.2.2. *The Navier-Stokes equation.* The movement of an incompressible, viscous and homogeneous fluid is governed by the following Navier-Stokes equations, where $v(t,x)$ is the velocity field (with d components, and where $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$) and $p(t,x)$ (a scalar field) is the pressure — both are unknown:

$$(NS) \qquad \left\{\begin{array}{rcll} \partial_t v + v\cdot\nabla v - \Delta v &=& -\nabla p & \text{in} \quad \mathbb{R}^+_t \times \mathbb{R}^d_x \\ \operatorname{div} v &=& 0 & \\ v_{|t=0} &=& v_0. & \end{array}\right.$$

The first equation translates conservation of momentum, while the second one expresses conservation of mass. One can note that taking the divergence of the first equation allows to eliminate the pressure, through the following relation:

$$-\Delta p = \operatorname{div}(v\cdot\nabla v).$$

The pressure can therefore be considered as a known quantity (once the velocity field is kown of course).

An easy formal computation (simply take the scalar product of the equation by v and integrate by parts using the divergence free condition) shows that

$$E_0 = \frac{1}{2}\|v_0\|_{L^2}^2 = \frac{1}{2}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(s)\|_{L^2}^2\, ds,$$

that is the energy of the system. With the notation of the previous section, one therefore has $X_e = L^2(\mathbb{R}^d)$.

The invariants naturally associated to the Navier–Stokes equations are

- space translations;
- scaling: if v solves (NS), then so does $v_\lambda(t,x) \stackrel{\text{def}}{=} \lambda v(\lambda^2 t, \lambda x)$ for all $\lambda > 0$.

For example one has $X_s = \dot H^{\frac{d}{2}-1}$, or L^d, or $\dot B^{\frac{d}{p}-1}_{p,q}$. Those spaces are defined in Definition 2.2 below.

One immediately notices that in two space dimensions, the energy space L^2 is scale invariant. One can therefore expect global existence and uniqueness of solutions (see Theorem 5 below). In higher space dimensions however, the energy space is "below" scaling, in the sense that $0 < \frac{d}{2} - 1$, where 0 is the Sobolev index of the energy space, while $\frac{d}{2} - 1$ is the Sobolev index of the scaling space. In dimension higher than two it therefore seems very difficult to prove such a result without any additional assumption.

Let us now recall the two main results on the Navier-Stokes equations: the first one deals with "energy" type solutions, the second one with "scaling" solutions.

Theorem 5 (J. Leray [16]). *Let $v_0 \in L^2(\mathbb{R}^d)$ be a divergence free vector field. There is a global solution v to (NS) associated with v_0, satisfying the energy inequality*

$$\forall t \geq 0, \quad \frac{1}{2}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(s)\|_{L^2}^2\, ds \leq \frac{1}{2}\|v_0\|_{L^2}^2.$$

If $d = 2$ the solution is unique and satisfies the energy equality.

Let us discuss the uniqueness of those solutions (in any space dimension). If we form the difference w of two solutions v_1 and v_2, the equation satisfied by w is of the Navier–Stokes type, in which appears also the linear term $w \cdot \nabla v_1 + v_2 \cdot \nabla w$. The energy of w will be controlled if one is able to estimate a term of the type

$$\int_0^T \int_{\mathbb{R}^d} w \cdot \nabla v_1 \cdot w \, dxdt.$$

But ∇v_1 belongs to $L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))$ so one needs to control w^2 in the space $L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))$, which means to control w in $L^4(\mathbb{R}^+, L^4(\mathbb{R}^d))$. The Gagliardo-Nirenberg inequality in two space dimensions

$$\|w\|_{L^4(\mathbb{R}^+, L^4(\mathbb{R}^2))} \leq \|w\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))} \|\nabla w\|^{\frac{1}{2}}_{L^2(\mathbb{R}^+, L^2(\mathbb{R}^2))}$$

provides the required estimate. In higher dimensions however one needs more smoothness assumptions on the data to get a chance to conclude to uniqueness in that way.

In the article by J. Leray [16] one can find uniqueness results, but we choose here to state H. Fujita and T. Kato's theorem [7].

Theorem 6 (H. Fujita and T. Kato [7])**.** *Let v_0 be a divergence free vector field in the space $\dot H^{\frac{d}{2}-1}(\mathbb{R}^d)$. Then there is a unique maximal time T^* and a unique solution v associated with v_0 such that for any $T < T^*$, v belongs to $C([0,T],\dot H^{\frac{d}{2}-1}(\mathbb{R}^d)) \cap L^2([0,T],\dot H^{\frac{d}{2}}(\mathbb{R}^d))$. Moreover there is a constant $c > 0$ such that if $\|v_0\|_{\dot H^{\frac{d}{2}-1}(\mathbb{R}^d)}$ is smaller than c, then v belongs to $C(\mathbb{R}^+,\dot H^{\frac{d}{2}-1}(\mathbb{R}^d)) \cap L^2(\mathbb{R}^+,\dot H^{\frac{d}{2}}(\mathbb{R}^d))$.*

The largest function space in which such a result is known to be true is the space BMO^{-1} of vector fields, the components of which are linear combinations of derivatives of BMO functions (that is a result by H. Koch and D. Tataru [15]). That type or result (see also the works of M. Cannone, Y. Meyer and F. Planchon [5], [18], [20] for results in Besov type spaces $\dot B^{\frac{d}{p}-1}_{p,\infty}$, $p < +\infty$) does not use in any way the conservation of energy and therefore holds for a much larger class of equations than the Navier-Stokes equations. Those theorems are therefore very general, but in a way lose sight of the Navier-Stokes equations — for instance there are equations (S. Montgomery–Smith [19]) very like the Navier-Stokes equations, and in particular that satisfy all the theorems in scale invariant spaces mentioned above, whose solutions blow up in finite time (such a question is of course open for Navier-Stokes). Note that the uniqueness theorem by J.-Y. Chemin [6] (for data in $C^{-1} \cap L^2(\mathbb{R}^3)$) uses the energy estimate.

1.3. **Structure of the paper.** As recalled above, the aim of this text is to provide examples where one can improve well-known results on the Cauchy problem by making use of particularities of the system. The next section consists in a presentation of notation and definitions which will be of constant use later. Section 3 studies a subcritical case (when the energy spaec is above scaling) and improves the usual global wellposedness results for the cubic wave equation in three space dimensions. The following Section 4 deals with a critical case (when the energy space is a scale invariant space) and again improves usual global wellposedness results, in the case of the two dimensional Navier-Stokes equations. Finally Section 5 considers the three dimensional Navier-Stokes equations (a supercritical equation in the sense that energy is below scaling) and consists in the proof of a weak strong uniqueness result.

2. Some notation

Let us recall here some definitions and notation which will be of use constantly in this text. First we define the Littlewood-Paley operators S_j and Δ_j.

Definition 2.1. *Let $\Phi \in \mathcal{S}(\mathbb{R}^3)$ be such that $\widehat\Phi(\xi) = 1$ for $|\xi| \leq 1$ and $\widehat\Phi(\xi) = 0$ for $|\xi| > 2$. Define, for $j \in \mathbb{Z}$, the function $\Phi_j(x) \stackrel{def}{=} 2^{3j}\Phi(2^j x)$; then the Littlewood–Paley operators are*

$$S_j \stackrel{def}{=} \Phi_j * \cdot \quad \text{and} \quad \Delta_j \stackrel{def}{=} S_{j+1} - S_j.$$

Then we recall the definition of Besov spaces (the Sobolev space $\dot H^s$ being simply the space $\dot B^s_{2,2}$).

Definition 2.2. *If $s < \frac{3}{p}$, then f belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ if and only if the partial sum $\sum_{-m}^{m} \Delta_j f$ converges towards f as a tempered distribution, and the sequence $\epsilon_j \stackrel{\text{def}}{=} 2^{js}\|\Delta_j f\|_{L^p}$ belongs to $\ell^q(\mathbb{Z})$.*

3. Energy above scaling: the 3D cubic wave equation

This section is a summary of the article [9]. The goal is to study the following equation

$$\left\{ \begin{array}{rcl} \partial_t^2\Phi - \Delta\Phi + \Phi^3 & = & 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^3 \\ (\Phi, \partial_t\Phi)_{|t=0} & = & (\Phi_0, \Phi_1), \end{array} \right. \tag{3.1}$$

where Φ is real valued. Recalling the wellposedness results stated in Section 1.2.1, one can naturally wonder whether the local solutions can be extended globally in time, at least for some range $\frac{1}{2} \le s < 1$. In [3] Bourgain introduced a general framework for obtaining results of this type, and applied it to the 2D cubic Schrödinger equation. For equation (3.1), Bourgain's method yields global wellposedness in $\dot{H}^s$ for $s > \frac{3}{4}$, as proved in [14]. We intend to give a different proof of this result, following a strategy introduced in the context of the Navier-Stokes equations in [4] (see also [8] for a more recent approach, developped also in Section 4). When compared, the two methods appear to be somewhat dual of each other. Let us now state the theorem.

Theorem 7. *Let $(\Phi_0, \Phi_1) \in (\dot{H}^s \cap L^4, \dot{H}^{s-1})$ with $s > \frac{3}{4}\cdot$ Then there exists a unique global in time solution to (3.1). Moreover, we have*

$$\|\Phi\|_{\dot{H}^s}(t) \le C(\|u_0\|_{\dot{H}^s\cap L^4})t^{\frac{(1-s)(6s-3)}{4s-3}}.$$

Let us now give the main steps of the proof.

3.1. Step 1: decomposition of the data.

Since the equation is globally wellposed for large data in $\dot{H}^1$ and small data in $\dot{H}^{\frac{1}{2}}$, the idea is to use both pieces of information, and a natural way to do so is to split the data $\Phi_0 \in \dot{H}^s$ into two pieces (and similarly for Φ_1): $\Phi_0 = u_0 + v_0$ where $u_0 \in \dot{H}^1$ with a large norm and $v_0 \in \dot{H}^{\frac{1}{2}}$ with a small norm. One may achieve this by defining $u_0 = S_J\Phi_0$ with large J. Then we have

$$\|u_0\|_{\dot{H}^1} \approx 2^{J(1-s)}\|\Phi_0\|_{\dot{H}^s} \quad \text{and} \quad \|v_0\|_{\dot{H}^\gamma} \approx 2^{J(\gamma-s)}\|\Phi_0\|_{\dot{H}^s}, \text{ for all } \gamma \le s.$$

Note that a scaling argument (see [9]) shows easily that the quantity $\|u_0\|^4_{L^4}$ can be controlled by $\|u_0\|^2_{\dot{H}^1}$, and we assume this to be the case for the rest of the proof.

3.2. Step 2: global existence for the high frequency part.

Let us consider the following equation:

$$\left\{ \begin{array}{rcl} \partial_t^2 v - \Delta v + v^3 & = & 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^3 \\ (v, \partial_t v)_{|t=0} & = & (v_0, v_1), \end{array} \right. \tag{3.2}$$

where $(v_0, v_1) \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^\gamma \times \dot{H}^{-\frac{1}{2}} \cap \dot{H}^{\gamma-1}$, with $\gamma \le s$ to be set later. Then usual results for small data theory imply that there is a constant ε_0 such that if $2^{J(\frac{1}{2}-s)} \lesssim \varepsilon_0$, then there is a unique, global solution v to (3.2), such that

$$v \in C(\mathbb{R}, \dot{H}^{\frac{1}{2}} \cap \dot{H}^\gamma), \quad \partial_t v \in C(\mathbb{R}, \dot{H}^{-\frac{1}{2}} \cap \dot{H}^{\gamma-1}).$$

Moreover if $\frac{2}{p} + \frac{2}{q} = 1$, with $p > 2$ and $q < +\infty$, then $v \in L^p(\mathbb{R}, \dot{B}^{\gamma-\frac{2}{p}}_{q,2})$, with

$$\|v\|_{L^p(\mathbb{R},\dot{B}^{\gamma-\frac{2}{p}}_{q,2})} \lesssim 2^{J(\gamma-s)}. \tag{3.3}$$

For the next step, we remark that any additional regularity is preserved, so that in particular we have (as $v_0 \in \dot{H}^{\frac{1}{2}+\frac{1}{6}}$)

$$\|v\|_{L^\infty(\mathbb{R},\dot{H}^{\frac{1}{2}+\frac{1}{6}})\cap L^3(\mathbb{R},L^6)} \approx 2^{(\frac{1}{2}+\frac{1}{6}-s)J}.$$

3.3. **Step 3: local existence for the low frequency part.** The more difficult part of the analysis consists in the study of the following perturbed equation:

$$\begin{cases} \partial_t^2 u - \Delta u + u^3 + 3uv^2 + 3u^2 v &= 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^3 \\ (u, \partial_t u)_{|t=0} &= (u_0, u_1), \end{cases} \tag{3.4}$$

where v was constructed in the previous part, and with $(u_0, u_1) \in \dot{H}^1 \times L^2$. We denote

$$E^{\frac{1}{2}}(u_0, u_1) \stackrel{\text{def}}{=} \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} \approx 2^{J(1-s)}.$$

A fixed point argument allows one to construct a solution u satisfying

$$u \in C([0,T], \dot{H}^1),\ \partial_t u \in C([0,T], L^2) \text{ and } u \in L^p([0,T], \dot{B}^{1-\frac{2}{p}}_{q,2}),$$

with $\frac{2}{p} + \frac{2}{q} = 1$, $p > 2$, as long as

$$T \lesssim \inf\left(2^{-6j(\frac{1}{2}+\frac{1}{6}-s)}, \frac{1}{\|u_0\|^2_{\dot{H}^1}}\right).$$

Indeed it is enough to prove the nonlinear terms to be $L^1_t(L^2_x)$. Using Sobolev embedding, the term u^3 is in $C_t(L^2_x)$ and hence is locally L^1_t. On the other hand, $v \in L^3_t(L^6_x)$ for v_0 in $\dot{H}^{\frac{1}{2}+\frac{1}{6}}$ (recall Step 2), which yields $v^2 u \in L^1_t(L^2_x)$ using a Hölder estimate. The remaining term $u^2 v$ is controlled by the other two, and denoting by $|||\cdot|||$ the norm in the contraction space, we obtain

$$|||u|||_T \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + 2^{2j(\frac{1}{2}+\frac{1}{6}-s)} T^{\frac{1}{3}} |||u|||_T + T|||u|||^3_T. \tag{3.5}$$

Thus, the linear term on the right can be absorbed on the left, as soon as $T \lesssim 2^{-6j(\frac{1}{2}+\frac{1}{6}-s)}$ and we obtain the desired result. Note that in [9] the local result is proved under the only assumption that $v_0 \in \dot{H}^{\frac{1}{2}}$, which shows that (3.4) is wellposed whenever the perturbation v has at least $\dot{H}^{\frac{1}{2}}$ regularity. The proof of that fact is more technical than the easy argument above (which is enough for our purposes) so we refer to [9] for details.

To extend local solutions to global ones, we then need to obtain an a priori bound on the energy of a solution u. This will be accomplished through the energy inequality, provided one can control the perturbative terms by the energy of u.

3.4. **Step 4: Control of the energy of the low frequency part.** The control comes from the following claim: for any $r < +\infty$,

$$\begin{aligned} E_T(u) &\lesssim E(u_0) + T^{\frac{1}{3}} \|v_0\|^2_{\dot H^{\frac{1}{2}+\frac{1}{6}}} E_T(u) + T^{\frac{1}{2}+\frac{1}{r}} \|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{r}}} E_T(u)^{\frac{3}{2}} \\ &\quad + T^{\frac{2}{3}} \|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{6}}} \|v_0\|^2_{\dot H^{\frac{1}{2}}} E_T(u)^{\frac{3}{2}}, \end{aligned} \tag{3.6}$$

where we have defined $E_T(u) \stackrel{\text{def}}{=} \sup_{t\leq T} E(u)(t)$, with

$$E(u)(t) \stackrel{\text{def}}{=} \left(\frac{1}{2}\|u(t)\|^2_{\dot H^1} + \frac{1}{2}\|\partial_t u(t)\|^2_{L^2}\right).$$

Let us explain how to prove that claim. Multiplying (3.1) by $\partial_t u$, integrating over x and t, we get

$$\begin{aligned} \frac{1}{2}\left(\|u(t)\|^2_{\dot H^1} + \|\partial_t u(t)\|^2_{L^2}\right) + \frac{1}{4}\|u(t)\|^4_{L^4} &\leq \frac{1}{2}(\|u_0\|^2_{\dot H^1} + \|u_1\|^2_{L^2}) + \frac{1}{4}\|u_0\|^4_{L^4} \\ + 3\left|\int_0^t \int_{\mathbb{R}^3} u(s,x)v^2(s,x)\partial_s u \, dxds\right| &+ 3\left|\int_0^t \int_{\mathbb{R}^3} u^2(s,x)v(s,x)\partial_s u \, dxds\right|. \end{aligned}$$

As remarked earlier the quantity $\|u_0\|^4_{L^4}$ is negligible compared to the energy of the initial data, so taking the supremum over $t < T$ we get finally

$$E_T(u) \lesssim E(u_0,u_1) + 3\int_0^T \left|\int_{\mathbb{R}^3} u(t,x)v^2(t,x)\partial_t u \, dxdt\right| + 3\left|\int_0^T \int_{\mathbb{R}^3} u^2(t,x)v(t,x)\partial_t u \, dxdt\right|. \tag{3.7}$$

Let us call I and II the two space–time integrals appearing on the right–hand side of the inequality, and let us start by estimating I, which is the easiest. We have

$$\begin{aligned} I &\leq \int_0^T \|v(t)\|^2_{L^6}\|u(t)\|_{L^6}\|\partial_t u(t)\|_{L^2}\, dt \\ &\leq E_T(u)\int_0^T \|v(t)\|^2_{L^6}\, dt, \end{aligned}$$

and by Strichartz' estimates, we can write

$$\begin{aligned} \|v\|^2_{L^2_T L^6} &\lesssim T^{\frac{1}{3}}\|v\|^2_{L^3_T L^6} \\ &\lesssim T^{\frac{1}{3}}\|v_0\|^2_{\dot H^{\frac{1}{2}+\frac{1}{6}}}. \end{aligned}$$

Finally we get

$$I \lesssim T^{\frac{1}{3}}\|v_0\|^2_{\dot H^{\frac{1}{2}+\frac{1}{6}}} E_T(u). \tag{3.8}$$

Now let us estimate the term II. Using the same type of estimate as above for the term I yields

$$II \lesssim E_T(u)^{\frac{3}{2}} T^{\frac{2}{3}}\|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{6}}}.$$

Recalling that $\|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{6}}} \lesssim 2^{J(\frac{2}{3}-s)}$, a superlinear bootstrap argument (see for instance [1], Lemma 2.2) shows that one needs

$$T^{\frac{4}{3}} 2^{2J(1-s+\frac{2}{3}-s)} \lesssim 1.$$

hence necessarily $s > \dfrac{3}{4} + \dfrac{1}{12}$, which is not the index given by the theorem.

To improve the lower bound on s, one needs to improve the estimate on II. We will not write the details here as they get a little technical, but it is proved in [9] that for all $r < \infty$,

$$|II| \lesssim T^{2/3} \|v_0\|_{\dot{H}^{1/2+1/6}} \|v_0\|^2_{\dot{H}^{1/2}} E_T(u)^{\frac{3}{2}} + T^{\frac{1}{2}+\frac{1}{r}} \|v_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{r}}} E_T(u). \tag{3.9}$$

Note that the first term on the right-hand side comes from the nonlinear part of v whereas the second one (the more tricky one) comes from the linear part.

Putting together (3.8) and (3.9) we find that (3.7) becomes

$$E_T(u) \lesssim 2^{2J(1-s)} + 2^{2J(\frac{1}{2}+\frac{1}{6}-s)} T^{\frac{1}{3}} E_T(u) + 2^{J(\frac{1}{2}+\frac{1}{r}-s)} E_T(u)^{\frac{3}{2}} \left(T^{\frac{1}{2}+\frac{1}{r}} + T^{\frac{2}{3}} 2^{2J(\frac{1}{2}-s)} \right).$$

Now the conclusion is rather straightforward: we start by noticing that since $s > \dfrac{1}{2} + \dfrac{1}{6}$, we can choose J such that, say, $2^{2J(\frac{1}{2}+\frac{1}{6}-s)} T^{\frac{1}{3}} \leq \frac{1}{2}$. Then similarly one can also choose J so that $T^{\frac{2}{3}} 2^{2J(\frac{1}{2}-s)} \leq T^{\frac{1}{2}+\frac{1}{r}}$. So we are left with

$$E_T(u) \lesssim 2^{2J(1-s)} + 2^{J(\frac{1}{2}+\frac{1}{r}-s)} E_T(u)^{\frac{3}{2}} T^{\frac{1}{2}+\frac{1}{r}},$$

and by superlinear bootstrap we find the condition:

$$1 - s + \frac{1}{2} + \frac{1}{r} - s < 0,$$

which implies the desired result (uniqueness follows from local existence, and for the proof of the bound in $\dot{H}^s$ we refer to [9]).

4. Energy at scaling: the 2D Navier-Stokes equation

In this section we shall give an idea of the proof of a global existence and uniqueness theorem for the 2D Navier-Stokes equations, for an initial data which is in a scale invariant space (and not in L^2). The method of proof is similar to the one used for the wave equation above: it consists in decomposing the initial data into two parts, one small part in the scale invariant space, and one large part in the energy space. One first solves globally in time the equation for the small part (by using the classical small data theory), and then one shows that the energy of the large part (satisfying a perturbed equation) is controled for all times (which in turn shows that the scale invariant norm is controlled globally in time, since it is smaller than the energy norm in this situation).

The result is the following (we refer to Section 1.2.2 for a recollection of well-known results on that equation).

Theorem 8 (2D global existence)**.** *Let r and q be two real numbers with $2 \leq r < +\infty$ and with $2 < q < +\infty$. Let $u_0 \in \dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)$ be a divergence free vector field. Then there exists a unique global solution to the two dimensional Navier-Stokes equations in the space $C([0,\infty), \dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2))$. Moreover, if $\dfrac{2}{r} + \dfrac{2}{q} \geq 1$, then there exists a constant $C_{r,q}$ such that*

$$\forall t \geq 0, \quad \|u(t)\|_{\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)} \leq C_{r,q} \|u_0\|^{1+\frac{r+1}{2}}_{\dot{B}^{\frac{2}{r}-1}_{r,q}(\mathbb{R}^2)}. \tag{4.1}$$

Note that recently P. Germain [10] extended that result to the limit space BMO^{-1}.

Let us give a sketch of the proof of that result. We will not prove the a priori estimate here, as it involves some nonlinear interpolation theory and is slightly technical (see [8]).

Step 1: decomposition of the initial data. We consider $\dot{B}^{\frac{2}{r}-1}_{r,q}$ as a real interpolation space between L^2 and $\dot{B}^{\frac{2}{\tilde r}-1}_{\tilde r,\tilde q}$ where $\tilde q > q$ and $\tilde r > r$. Real interpolation theory enables us to write $u_0 = v_0 + w_0$ where $v_0 \in L^2(\mathbb{R}^2)$ and $w_0 \in \dot{B}^{\frac{2}{\tilde r}-1}_{\tilde r,\tilde q}(\mathbb{R}^2)$ has an arbitrarily small norm.

Step 2: Global existence for the small part. By the small data theory we can construct a unique solution w to the Navier-Stokes system with initial data w_0, provided w_0 is small enough in $\dot{B}^{\frac{2}{\tilde r}-1}_{\tilde r,\tilde q}$. In particular we know that there exists a unique global solution w associated to w_0 which is such that $w \in C_b(\mathbb{R}^+, \dot{B}^{\frac{2}{r}-1}_{r,q})$. Moreover we have

$$\sup_t t^{\frac12-\frac1\eta+\frac\alpha2}\|\nabla^\alpha w\|_{L^\eta} \lesssim \|w_0\|_{\dot{B}^{\frac{2}{r}-1}_{r,q}} \text{ for } \alpha = 0, 1 \text{ and } r \le \eta \le \infty. \tag{4.2}$$

In what follows we will assume that $\|w_0\|_{\dot{B}^{\frac{2}{r}-1}_{r,q}} \le \varepsilon_0$ is very small.

Step 3: Local existence for the energy part. Now we need to solve the perturbed equation

$$\left\{\begin{array}{rcl} \dfrac{\partial v}{\partial t} &=& \Delta v - \nabla\cdot(v\otimes w) - \nabla\cdot(w\otimes v) - \nabla\cdot(v\otimes v) - \nabla p, \\ \nabla\cdot v &=& 0, \\ v(x,0) &=& v_0(x),\, x\in\mathbb{R}^2,\, t\ge 0, \end{array}\right. \tag{4.3}$$

where recall that $v_0 \in L^2$.

It is not difficult to implement a fixed point procedure to find a solution satisfying

$$\sup_{t<T}\|u(t)\|_{L^2_x} + \|\nabla u\|_{L^2([0,T],L^2_x)} < \infty$$

for some small enough time T, we leave the details to the reader (see [8]). The difficulty is to prove an energy estimate which will make the solution global in time.

Step 4: Control of the energy. Defining

$$\|f(t)\|^2_{\mathcal{L}} \stackrel{\text{def}}{=} \sup_{0\le s\le t}\|f(s)\|^2_{L^2} + \|\nabla f\|^2_{L^2([0,t],L^2)},$$

we claim that there exists a time $t_0 > 0$, arbitrarily small, such that the function v satisfies for all $t > t_0$

$$\|v(t)\|^2_{\mathcal{L}} \le 2\|v(t_0)\|^2_{L^2}\left(\frac{t}{t_0}\right)^{\varepsilon_0}. \tag{4.4}$$

Let us prove that claim. Formally, we may multiply (4.3) by v and integrate over x and t to get, using the fact that v is divergence free,

$$\|v(t)\|^2_{L^2} + 2\int_0^t\|\nabla v(s)\|^2_{L^2}\,ds + \int_0^t\int_{\mathbb{R}^2}(v\cdot\nabla)vw\,dxds \le \|v_0\|^2_{L^2}. \tag{4.5}$$

To obtain a global bound we shall use the energy inequality on a time interval (t_0, T) with a small $t_0 > 0$. Indeed we know that one has a local solution up to, say, $2t_0$, and at time t_0 the small rough solution w has been smoothed out: it verifies

$$\sup_t \sqrt{t}\|w\|_{L^\infty} < \varepsilon_0,$$

which allows to write by Hölder's inequality,

$$\left|\int_{t_0}^t \int_{\mathbb{R}^2} (v\cdot\nabla)vw\, dxds\right| \lesssim \varepsilon_0 \left(\int_{t_0}^t \|\nabla v(s)\|_{L^2}^2\, ds + \int_{t_0}^t \frac{\|v(s)\|_{L^2}^2}{s}\, ds\right),$$

and that yields the expected bound after applying Gronwall Lemma, so the claim is proved. Note that the formal computation (4.5) is justified since we apply the energy inequality from a time $t > t_0 > 0$, all terms are smooth and there is no difficulty in defining the various quantities.

The proof presented here will not enable us to obtain the a priori estimate given in Theorem 8, since we have no control over the time t_0. In order to find the a priori estimate we need to write the global energy estimate starting from time 0. That requires significantly more work, in particular some paradifferential calculus — which forces us to restrict the range of indexes of the Besov norms (in fact we use Lemma 5.1 below), and we refer the interested reader to [8]. Once that global in time energy estimate is found, the a priori bound follows from some real interpolation arguments.

5. Energy below scaling: the 3D Navier-Stokes equation

In this final section we shall consider the less favourable case when the energy has a lower scaling than scale invariant spaces: in a first study we shall give some weak-strong uniqueness result for the 3D Navier-Stokes system, and then we will try to see how the use of the energy conservation can give some interesting qualitative information on global, scale invariant solutions (namely their stability and their large time asymptotics).

5.1. **Weak-strong uniqueness.** The definition of weak-strong uniqueness was given in the introduction so we will not discuss it here. We shall simply remark that it is an important problem, at least from two points of view. The original point of view is due to J. Leray, who in his famous paper [16] raised the following question: if the initial data is smooth, then does the unique smooth solution coincide (on its life span) with the energy solutions? For J. Leray a positive answer to that question (which he proved) meant that the "turbulent" solutions he had come up with had some meaning. Today the existence of such "turbulent" solutions does not worry the mathematician, who is used to working with low regularity functions (even as solutions of physical PDEs). On the other hand the second point of view of weak-strong uniqueness consists in checking that the unique solution in some "exotic" space such as a Besov space, coincides (on its life span) with the physical, energy solutions. Such a result for the L^3 case (in three space dimensions) was proved by W. von Wahl in [23]. The case of Besov spaces was studied in [8], but for the moment weak-strong uniqueness is only known for a restricted range of indexes. That is no doubt due to the method followed but it is an open question, at the time of the writing, to know whether the weak-strong uniqueness holds, say in BMO^{-1} (see [11] for some recent developments on the subject). Here we shall give an idea of the proof of the following result (see [8]) — note that Theorem 9 obviously implies weak-strong uniqueness in $\dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)$ for $d/r + 2/q > 1$.

We recall that the space $\mathcal{L}$ was defined in Section 4 above.

Theorem 9 (Stability)**.** *Consider $d \geq 2$, and let r and q be two real numbers such that $2 \leq r < +\infty, 2 < q < +\infty$. Suppose additionally that $d/r + 2/q > 1$. Let v_0 and u_0 be two divergence free vector fields in $L^2(\mathbb{R}^d)$, and suppose that u_0 is also an element of $\dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)$. Let $v \in \mathcal{L}$ be any Leray solution associated with v_0, and let u be the unique solution associated with u_0, with $u \in L^q([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)) \cap \mathcal{L}$ for some time $T > 0$. Then $w \stackrel{\text{def}}{=} v - u$ satisfies, for all times $t \leq T$,*

$$\begin{aligned}\|w(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\,ds &\leq \|v_0 - u_0\|^2_{L^2(\mathbb{R}^d)}\\ &\times \exp\left(C\int_0^t \|u(s)\|^q_{\dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}\,ds\right).\end{aligned}$$

We shall follow here the classical method of proof of weak-strong type estimates (see [16], [23]). Let us start by associating with v_0 and u_0 two Leray solutions v and u, in the space $\mathcal{L}$. It follows from the theory of scale-invariant solutions (see [8]) that

$$\forall p \geq q, \quad u \in C([0,T], \dot{B}^{\frac{d}{r}-1}_{r,q}(\mathbb{R}^d)) \cap L^q([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)).$$

Let us now define $w \stackrel{\text{def}}{=} v - u$, which belongs to $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^d))$ by assumption. We can write

$$\begin{aligned}\|w(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\,ds &= \|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^d)}\,ds\\ + \|v(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla v(s)\|^2_{L^2(\mathbb{R}^d)}\,ds &- 2\,(v(t)|u(t)) - 4\int_0^t (\nabla v(s)|\nabla u(s))\;ds,\end{aligned}$$

where $(\cdot\,|\,\cdot)$ denotes the scalar product in $L^2(\mathbb{R}^d)$. The energy estimate on Leray solutions implies that

$$\begin{aligned}\|w(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla w(s)\|^2_{L^2(\mathbb{R}^d)}\,ds &\leq \|u_0\|^2_{L^2(\mathbb{R}^d)} + \|v_0\|^2_{L^2(\mathbb{R}^d)}\\ &\quad -2\,(v(t)|u(t)) - 4\int_0^t (\nabla v(s)|\nabla u(s))\;ds.\end{aligned}$$

Wed claim that for all times $t \leq T$,

$$(v(t)|u(t)) + 2\int_0^t (\nabla v(s)|\nabla u(s))\;ds = (v_0|u_0) + \int_0^t (w \cdot \nabla w(s)|u(s))\;ds.$$

A formal computation yields that result with no difficulty; in order to prove it, the idea is to smoothen out u and v to make the formal computation correct, and then to take the limit — there we will need the extra assumption on u. So we define two sequences of smooth, divergence free vector fields (v_n) and (u_n) such that $\lim_{n\to\infty} v_n = v$ in $L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^d))$, and $\lim_{n\to\infty} u_n = u$ in $L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^d)) \cap L^q([0,T], \dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d))$. Taking the scalar product with v_n and u_n of the Navier–Stokes equations on u and v respectively yields, after integration in time and integration by parts in the space variables,

$$\int_0^t \Big((\partial_s u \mid v_n) + (\nabla u \mid \nabla v_n) + (u\cdot\nabla u \mid v_n) \Big)(s)\, ds = 0,$$
$$\int_0^t \Big((\partial_s v \mid u_n) + (\nabla v \mid \nabla u_n) + (v\cdot\nabla v \mid u_n) \Big)(s)\, ds = 0.$$

It is now a matter of taking the limit in n, and of summing the limits found. It is clear that

$$\lim_{n\to\infty} \Big(\int_0^t (\nabla u \mid \nabla v_n)\,(s)\, ds + \int_0^t (\nabla v \mid \nabla u_n)\,(s)\, ds \Big) = 2\int_0^t (\nabla u \mid \nabla v)\,(s)\, ds.$$

Let us state a fundamental tricontinuity lemma:

Lemma 5.1. *Let $d \geq 2$ be fixed, and let r and q be two real numbers such that $2 \leq r < +\infty, 2 < q < +\infty$ and $d/r + 2/q > 1$. Then for every $T \geq 0$, the trilinear form*

$$(a,b,c) \in \mathcal{L}\times\mathcal{L}\times L^q([0,T],\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)) \mapsto \int_0^T \int_{\mathbb{R}^d} (a\cdot\nabla b)\cdot c(t)\, dxdt$$

is continuous. In particular the following estimate holds:

$$\Big| \int_0^t \int_{\mathbb{R}^d} (a\cdot\nabla a)\cdot c \, dxds \Big| \leq \|\nabla a\|^2_{L^2(\mathbb{R}^+,L^2(\mathbb{R}^d))} + C\int_0^t \|a(s)\|^2_{L^2(\mathbb{R}^d))} \|c(s)\|^q_{\dot B^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}\, ds. \tag{5.1}$$

We refer to [8] for the proof of that result, which relies on some paradifferential calculus. It is here that the restriction on the indexes r and q appear. Lemma 5.1 implies that

$$\lim_{n\to\infty} \int_0^t (v\cdot\nabla v \mid u_n)\,(s)\, ds = \int_0^t (v\cdot\nabla v \mid u)\,(s)\, ds,$$
$$\lim_{n\to\infty} \int_0^t (u\cdot\nabla u \mid v_n)\,(s)\, ds = \int_0^t (u\cdot\nabla u \mid v)\,(s)\, ds.$$

But $\partial_s v = \Delta v - \mathbb{P}(v\cdot\nabla v)$ in $\mathcal{D}'(\mathbb{R}^d)$, where $\mathbb{P}$ stands for the Leray projector onto divergence–free vector fields, so those limits imply in particular that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^t (\partial_s v \mid u_n)\,(s)\, ds &= -\lim_{n\to\infty} \int_0^t \Big((\nabla v \mid \nabla u_n) + (v\cdot\nabla v \mid u_n) \Big)(s)\, ds \\
&= -\int_0^t \Big((\nabla v \mid \nabla u) + (v\cdot\nabla v \mid u) \Big)(s)\, ds \\
&= \int_0^t (\partial_s v \mid u)\,(s),
\end{aligned}$$

and similarly

$$\lim_{n\to\infty} \int_0^t (\partial_s u \mid v_n)\,(s)\, ds = \int_0^t (\partial_s u \mid v)\,(s)\, ds.$$

The claim follows from putting everything together, and noticing that

$$\int_0^t \Big((u\cdot\nabla u \mid v)\,(s) + (v\cdot\nabla v \mid u) \Big)(s)\, ds = \int_0^t (w\cdot\nabla w \mid u)\,(s)\, ds.$$

Now let us go back to the proof of the theorem. Recall that we have obtained

$$\|w(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\int_0^t \|\nabla w(s)\|_{L^2(\mathbb{R}^d)}^2\, ds \leq \|u_0\|_{L^2(\mathbb{R}^d)}^2 + \|v_0\|_{L^2(\mathbb{R}^d)}^2 - 2\left(v(t)|u(t)\right) - 4\int_0^t \left(\nabla v(s)|\nabla u(s)\right)\, ds,$$

so that means that

$$\|w(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\int_0^t \|\nabla w(s)\|_{L^2(\mathbb{R}^d)}^2\, ds \leq \|w_0\|_{L^2(\mathbb{R}^d)}^2 + \left|\int_0^t \left(w\cdot\nabla w|u\right)(s)\, ds\right|.$$

But Lemma 5.1, and in particular estimate (5.1), then yields

$$\|w(t)\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \|\nabla w(s)\|_{L^2(\mathbb{R}^d)}^2\, ds \leq \|w_0\|_{L^2(\mathbb{R}^d)}^2 + C\int_0^t \|w\|_{L^2(\mathbb{R}^d))}^2 \|u\|^q_{\dot{B}^{\frac{d}{r}+\frac{2}{q}-1}_{r,q}(\mathbb{R}^d)}(s)\, ds,$$

and the result follows from a Gronwall inequality.

5.2. **Qualitative behaviour of global scale invariant solutions.** In this final section we shall address the following question: consider a global solution to the 3D Navie-Stokes equations in a scale invariant space (but not necessarily associated with small data). Is there a way to describe its large time behaviour ? Is such a solution stable (i.e. is there a small ball around an initial data generating a global solution, such that all data in that ball generate global solutions) ? It turns out that one can answer all those questions by using the same strategy as that introduced in Sections 3 and 4, namely by a decomposition method.

Let us start by stating the result, then we will give an idea of the proof (see [**?**] for details, and also [**?**] for a self-contained proof in the easier $\dot{H}^{1/2}$ case).

6. Conclusion and open questions

We have seen here three different examples (accounting for the three different situations that can occur) showing how the use of special features of an equation can improve classical theorems on the Cauchy problem. Roughly speaking what the first part of the introduction above (Section 1) shows is that if one's intention is to prove wellposedness results which are specific to the equation one is is considering, then one must go further than the basic procedures consisting in finding global weak solutions in the energy space, and local unique solutions in scale invariant spaces. Such theorems, as difficult to prove as they may be, will never emcompass the full specificity of the equation. If one wishes to go further, then the first step one should follow is to try and bring together energy and scaling. That will probably not be enough, but will certainly be some progress in the understanding of the equation.

That is what we have tried to do in the three examples presented here. There is of course no unique way to bridge the gap between energy and scaling, as Example 1 in Section 3 showed us (two different methods, the one presented here and the one of [14] give the same result actually). The same type of method proved successful for us in two occasions (Sections 3 and 4) whereas Section 5 presented a classical weak-strong uniqueness procedure.

The results presented here raise more questions than they answer: considering the wave equation for instance, is $H^{3/4}$ really an obstacle under which global wellposedness ceases to be true? This is very doubtful, but to this day that is the lowest index yielding global wellposedness; half way between energy and scaling. Similarly as noted above, it is unknown up to now whether weak-strong uniqueness is true in the limit space BMO^{-1}. Again it probably is true (and if it were not, it would probably mean that in some sense BMO^{-1} is not the right space after all).

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