Ekman boundary layers in rotating fluids

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Introduction

In this paper, we investigate the problem of fast rotating viscous fluids between two plates with Dirichlet boundary conditions. We present the model with so called "turbulent" viscosity. More precisely, we shall study the limit when ε tends to 0 of the system involving the velocity field u^ε and the pressure p^ε

$$(NSC_\varepsilon) \left\{ \begin{array}{rcl} \partial_t u^\varepsilon + \mathrm{div}(u^\varepsilon \otimes u^\varepsilon) - \nu \Delta_h u^\varepsilon - \beta \varepsilon \partial_3^2 u^\varepsilon + \dfrac{e_3 \times u^\varepsilon}{\varepsilon} & = & -\nabla p^\varepsilon \\ \mathrm{div}\, u^\varepsilon & = & 0 \\ u^\varepsilon_{|\partial\Omega} & = & 0 \\ u^\varepsilon_{|t=0} & = & u_0^\varepsilon \in L^2(\Omega) \end{array} \right.$$

where $\Omega = \mathbf{R}^2 \times]0,1[$. We shall use the following notation: if u is a vector on $\mathbf{R}^3$ we state $u = (u^1, u^2, u^3) = (u^h, u^3)$, and we will note $\Delta_h = \partial_1^2 + \partial_2^2$. Moreover, if f is a function on Ω, $\mathcal{F}f$ (and also $\widehat{f}$) will denote the Fourier transform with respect to the horizontal variable x_h. Finally in all that follows, the space L^2 (with no argument) will denote the space $L^2(\Omega)$.

These equations arise in physical contexts when one studies oceanic or atmospheric motions. In general geophysical flows can be considered as incompressible, and with the length and time scales considered, the rotation of the Earth can not be neglected and on the contrary plays a major role. Basically under high rotation, a three dimensional fluid tends to behave like a two dimensional one, and to become invariant in the direction of the rotation. This well known phenomena, called Taylor Proudman theorem is detailled in the monographs [7],[10]. The part of the initial data which depends on the vertical component creates waves, called inertial waves, which propagate with a high speed. Moreover, keeping in mind this anisotropy, it is usual to consider an anisotropic viscosity like in (NSC_ε), the horizontal "turbulent" diffusion being larger than the vertical one. In the periodic case we refer to [1],[6],[8] for mathematical studies. In the presence of horizontal boundaries, bounday layers called Ekman layers appear to match the Dirichlet boundary condition with the tendency of the fluid to become independent on the third variable x_3. It turns out that these layers are always stable in the case of anisotropic viscosity. For previous studies we refer to [5] and [9]. The

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aim of this paper is to combine the study of these layers with the study of the dispersion of the inertial waves (we consider an unbounded domain).

In the sequel, we shall assume that $u_0^\varepsilon \cdot n = u_0^{\varepsilon,3} = 0$ on $\partial\Omega$, and $\operatorname{div} u_0^\varepsilon = 0$. This condition $u_0^{\varepsilon,3} = 0$ implies the following fact: for any L^2 divergence free vector field u, the function $\partial_3 u^3$ is $L^2(]0,1[)$ with respect to the variable x_3 with value in $H^{-1}(\mathbf{R}^2)$. Hence by integration, we get

$$u^3(x_h,1) - u^3(x_h,0) = -\int_0^1 \operatorname{div}_h u^h(x_h,x_3)dx_3 = 0.$$

So the vertical mean value of the horizontal part on the vector field is divergence free as a vector field on $\mathbf{R}^2$. Of course, a quantity will play a key role: this is (twice) the energy defined by

$$\mathcal{E}_\varepsilon(v)(t) \stackrel{\text{def}}{=} \|v(t)\|_{L^2}^2 + 2\nu_h\int_0^t \|\nabla^h v(t')\|_{L^2}^2 dt' + 2\beta\varepsilon\int_0^t \|\partial_3 v(t')\|_{L^2}^2 dt'.$$

The reason why we choose the vertical viscosity of the form $\beta\varepsilon$ will be explained in Section 1 where we shall recall the properties of the "well prepared" case. The relation between this choice and the Ekman pumping will be studied in detail in that next section. First of all we shall study the linear problem in the well prepared case, which means that the function u_0 does not depend on the vertical variable x_3. Then we shall investigate the more delicate case when the initial data u_0 does actually depend on the vertical variable x_3. We shall compute explicitly approximate solutions for the linear problem.

To study the real problem we have to deal with non linear terms. In the well prepared case, we have strong convergence in L^2, as well as in the case when the domain is $\mathbf{T}^2\times]0,1[$ (see [9]). The precise theorem proved in [9] is the following.

Theorem 1 *Let u^ε be a family of weak solutions of (NSC_ε) associated with a family of initial data u_0^ε of divergence free vector fields such that*

$$\lim_{\varepsilon\to 0} u_0^\varepsilon = (\overline{u}_0, 0) \quad \text{in} \quad L^2,$$

where $\overline{u}_0$ is a divergence free vector field in $L^2(\mathbf{R}^2)$. Denoting by $\overline{u}$ the global solution of the two dimensional Navier-Stokes type equations

$$(NSE_{\nu,\beta}) \begin{cases} \partial_t\overline{u} + \operatorname{div}_h(\overline{u}\otimes\overline{u}) - \nu\Delta_h\overline{u} + \sqrt{2\beta}\,\overline{u} &= -\nabla^h\overline{p} \\ \operatorname{div}_h\overline{u} &= 0 \\ \overline{u}_{|t=0} &= \overline{u}_0\,, \end{cases}$$

we have

$$\|u^\varepsilon - (\overline{u},0)\|_{L^\infty(\mathbf{R}_+;L^2)} + \|\nabla_h\left(u^\varepsilon - (\overline{u},0)\right)\|_{L^2(\mathbf{R}_+;L^2)} \to 0 \quad \text{when} \quad \varepsilon\to 0.$$

Here we shall prove the following result.

Theorem 2 *Let u_0 be a divergence free vector field in L^2 such that $u^3_{0|\partial\Omega} = 0$. Let u^ε be a family of weak solutions of (NSC_ε) associated with u_0. Denoting by $\overline{u}$ the global solution of the two–dimensional Navier–Stokes equations*

$$\partial_t\overline{u} + \operatorname{div}_h(\overline{u}\otimes\overline{u}) - \nu\Delta_h\overline{u} + \nabla^h p + \sqrt{2\beta}\,\overline{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^2), \tag{0.1}$$

$$\operatorname{div}_h \overline{u} = 0, \quad \textit{and} \quad \overline{u}_{|t=0} = \int_0^1 u_0(x_h, x_3)dx_3, \tag{0.2}$$

we have

$$\|u^\varepsilon - (\overline{u}, 0)\|_{L^\infty(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^2 \times]0,1[))} + \|\nabla^h (u^\varepsilon - (\overline{u}, 0))\|_{L^2(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^2 \times]0,1[))} \to 0 \quad \textit{when} \quad \varepsilon \to 0.$$

This result is specific to the domain $\mathbf{R}^2 \times]0,1[$. The key point for the proof of this theorem is that the dispersive phenomenon studied in [4] and [3] is not affected by boundary layers. Let us notice that this result is also true when the family (u^ε) is associated with a family of initial data u_0^ε which converges to some u_0 strongly in L^2.

To conclude this introduction, let us state some definitions and notations. As the phenomenon studied here is obviously anisotropic, it is natural to introduce the spaces $H^{s,0}$ which are defined as the closure of smooth compactly supported functions on Ω for the (semi) norm

$$\|f\|_{H^{s,0}} \stackrel{\text{def}}{=} \left(\int_{\mathbf{R}^2 \times]0,1[} |\mathcal{F}f(\xi_h, x_3)|^2 d\xi_h dx_3 \right)^{1/2}.$$

In this spirit we shall introduce

$$\begin{aligned} E_T(v) &\stackrel{\text{def}}{=} \sup_{0 \le t \le T} \Big\{ \|v(t)\|_{L^2}^2 + \int_0^t \|\nabla^h v(t')\|_{L^2}^2 dt' \Big\} \quad \text{and} \\ \overline{E}_T(v) &\stackrel{\text{def}}{=} \sup_{0 \le t \le T} \Big\{ \|v(t)\|_{L^2}^2 + \int_0^t \|\nabla^h v(t')\|_{L^2}^2 dt' + \int_0^t \|v(t')\|_{L^2}^2 dt' \Big\}. \end{aligned}$$

1 The linear problem in the well prepared case

The goal of this section is to recall some results and methods of [9]. The purpose is to have information on approximate solutions of

$$(FRF^\varepsilon) \left\{ \begin{array}{rcl} \partial_t v^\varepsilon - \nu \Delta_h v^\varepsilon - \beta \varepsilon \partial_3^2 v^\varepsilon + \dfrac{e_3 \times v^\varepsilon}{\varepsilon} &=& -\nabla p^\varepsilon + f^\varepsilon \quad \text{in} \quad \mathbf{R}^+ \times \Omega \\ \operatorname{div} v^\varepsilon &=& 0 \quad \text{in} \quad \mathbf{R}^+ \times \Omega \\ v^\varepsilon_{|t=0} &=& v_0^\varepsilon \quad \text{with} \quad v_0^{\varepsilon,3} = 0 \\ v^\varepsilon_{|\partial\Omega} &=& 0 \end{array} \right.$$

with

$$\lim_{\varepsilon \to 0} f^\varepsilon = \overline{f} \quad \text{in} \quad L^2(\mathbf{R}_+; H^{-1,0}) \qquad \text{and} \qquad \lim_{\varepsilon \to 0} v_0^\varepsilon = \overline{v}_0 \quad \text{in} \quad L^2$$

where $\overline{f}$ belongs to $L^2(\mathbf{R}^+; H^{-1}(\mathbf{R}^2))$. In all that follows, we shall denote

$$L^\varepsilon v \stackrel{\text{def}}{=} \partial_t v - \nu \Delta_h v - \beta \varepsilon \partial_3^2 v + \frac{e_3 \times v}{\varepsilon} \cdot$$

In [9], it is proved in particular that

$$\lim_{\varepsilon \to 0} v^\varepsilon = (\overline{v}, 0) \quad \text{in} \quad L^\infty(\mathbf{R}^+; L^2) \cap L^2(\mathbf{R}^+; H^{1,0}),$$

where $\overline{v}$ is the solution of

$$(LE)\begin{cases} \partial_t\overline{v} - \nu\Delta_h\overline{v} + \sqrt{2\beta}\,\overline{v} &= -\nabla^h\overline{p} + \overline{f} \quad \text{in} \quad \mathbf{R}^+\times\mathbf{R}^2\\ \operatorname{div}_h\overline{v} &= 0 \quad \text{in} \quad \mathbf{R}^+\times\mathbf{R}^2\\ \overline{v}_{|t=0} &= \overline{v}_0 \quad \text{with} \quad \overline{v}_0^3 = 0.\end{cases}$$

To simplify notations, we shall note in the following $\overline{v} = (\overline{v}, 0)$.

In fact, much more is done in [9]. We can sum up those results in the following lemma.

Lemma 1 *Let T be in $\overline{\mathbf{R}}^+$ and let $\overline{v}$ be a time dependent divergence free vector field in the space $L^\infty([0,T];L^2(\mathbf{R}^2))$ whose gradient belongs to $L^2([0,T]\times\mathbf{R}^2)$. Let us assume that its Fourier transform is supported in the ball of center 0 and radius N. Then a family of smooth divergence free vector fields $(v^\varepsilon_{app})_{\varepsilon>0}$ whose value is 0 on the boundary of Ω exists such that*

$$L^\varepsilon v^\varepsilon_{app} = \partial_t\overline{v} - \nu\Delta_h\overline{v} + \sqrt{2\beta}\,\overline{v} + O_N(\varepsilon^{\frac12}) \quad \textit{in} \quad L^2([0,T];H^{-1,0}).$$

The vector field v^ε_{app} tends to $\overline{v}$ in the following sense: a constant C_N exists such that

$$E_T(v^\varepsilon_{app} - \overline{v}) \le C_N\varepsilon^{\frac12}E_T(\overline{v}). \tag{1.3}$$

Moreover the family (v^ε_{app}) satisfies the following estimates

$$\int_0^T \sup_{x_3\in]0,1[}\|\nabla^h v^\varepsilon_{app}(t,\cdot,x_3)\|^2_{L^2(\mathbf{R}^2)}dt \le C_N E_T(\overline{v}) \tag{1.4}$$

$$\forall p\in[2,\infty]\,,\ \int_0^T\!\!\int_0^1 d(x_3)\|\partial_3 v^\varepsilon_{app}(t,\cdot,x_3)\|^2_{L^p(\mathbf{R}^2)}dtdx_3 \le C_N\overline{E}_T(\overline{v}) \quad \textit{and} \tag{1.5}$$

$$\int_0^T\!\!\int_0^1 d(x_3)^2\|\partial_3 v^\varepsilon_{app}(t,\cdot,x_3)\|^2_{L^\infty(\mathbf{R}^2)}dtdx_3 \le C_N\varepsilon\overline{E}_T(\overline{v}) \tag{1.6}$$

where $d(x_3)$ denotes the distance from x_3 to the boundary of $]0,1[$.

For the the reader's convenience, let us recall briefly the proof of this lemma. The approximate solution is defined by

$$\begin{aligned} v^\varepsilon_{app} \stackrel{\text{def}}{=}\ & (\overline{v},0) - \varepsilon\sqrt{2\beta}\Big(R\overline{v}, \Big(\frac12 - x_3\Big)\overline{\omega}\Big)\\ & \qquad + \bigg(\Big(M_0\Big(\frac{x_3}{\varepsilon\sqrt{2\beta}}\Big) + M_0\Big(\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\Big) - M_0\Big(\frac{1}{\varepsilon\sqrt{2\beta}}\Big)\Big)\overline{v},0\bigg)\\ & + \varepsilon\sqrt{2\beta}\bigg(\Big(\exp\Big(-\frac{x_3}{\varepsilon\sqrt{2\beta}}\Big) + \exp\Big(-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\Big) - \exp\Big(-\frac{1}{\varepsilon\sqrt{2\beta}}\Big)\Big)R\overline{v},\\ & \qquad \Big(f\Big(\frac{x_3}{\varepsilon\sqrt{2\beta}}\Big) - f\Big(\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\Big) - (1-2x_3)f\Big(\frac{1}{\varepsilon\sqrt{2\beta}}\Big)\Big)\,\overline{\omega}\bigg)\\ & + 2\beta\varepsilon^2\bigg(0, \Big(\exp\Big(-\frac{x_3}{\varepsilon\sqrt{2\beta}}\Big) - \exp\Big(-\frac{1-x_3}{\varepsilon\sqrt{2\beta}}\Big) - (1-2x_3)\exp\Big(-\frac{1}{\varepsilon\sqrt{2\beta}}\Big)\Big)\overline{\omega}\bigg)\\ & + e(\varepsilon\sqrt{2\beta})\Big(-6x_3(x_3-1)R\overline{v}, x_3(x_3-1)(2x_3-1)\overline{\omega}\Big)\end{aligned}$$

with

$$M_0(\zeta) \stackrel{\text{def}}{=} M_0(\zeta) = -e^{-\zeta} R_{-\zeta}\ , \quad f(\zeta) \stackrel{\text{def}}{=} -\frac{1}{2} e^{-\zeta}(\sin\zeta + \cos\zeta) \quad \text{and} \quad \overline{\omega} \stackrel{\text{def}}{=} \operatorname{curl}_h \overline{v}.$$

The function $e(\varepsilon\sqrt{2\beta})$ is exponentially decreasing and can be computed explicitly. From this formula, the proof of estimates (1.4)–(1.5) is straightforward.

2 The linear problem in the ill prepared case

This case is more delicate. The reason why is the following. There is no hope that the term $v_{0,int}$ does not depend on the vertical variable x_3, so we have to deal with very fast time oscillations of the interior solution. This will lead us to consider the equation in a different way. The goal of this section is to construct approximate solutions to

$$(FRF^\varepsilon) \begin{cases} \partial_t v^\varepsilon - \nu\Delta_h v^\varepsilon - \beta\varepsilon\partial_3^2 v^\varepsilon + \dfrac{v^\varepsilon \times e_3}{\varepsilon} & = & -\nabla p^\varepsilon \quad \text{in} \quad \Omega \\ \operatorname{div} v^\varepsilon & = & 0 \quad \text{in} \quad \Omega \\ v^\varepsilon_{|t=0} & = & v_0 \\ v^\varepsilon_{|\partial\Omega} & = & 0. \end{cases}$$

Without loss of generality, we can suppose that $v_0 \in L^1(\Omega)$; this will be useful in the proof of Strichartz estimates in Section 4.

First of all, we shall rewrite the system (FRF^ε) in terms of the Fourier transform of the horizontal divergence and vorticity. To do so, let us decompose the horizontal part of the initial data on the Hilbert basis $(\cos(k\pi x_3))_{k\in\mathbf{N}}$ of $L^2(]0,1[)$. We can write for any horizontal vector field v^h,

$$v^h(x_h, x_3) = \sum_{k\in\mathbf{N}} v^{k,h}(x_h)\cos(k\pi x_3). \tag{2.7}$$

Note that the fact that v is divergence free implies that

$$v_0^3(x_h, x_3) = -\sum_{k\geq 1} \frac{1}{k\pi} \operatorname{div}_h v_0^{k,h}(x_h)\sin(k\pi x_3). \tag{2.8}$$

Let us make some preliminary remarks. As the set of functions whose Fourier transform (on $\mathbf{R}^2$) is supported in rings is dense in L^2, we can assume that the Fourier transform of the initial data is of this type. Moreover for L^2 functions, boundary values do not make any sense. However, as v is divergence free, we have $v^3 \in C([0,1]\,; H^{-1}(\mathbf{R}^2))$, so the quantity $v^3_{|\partial\Omega}$ has a meaning. The case when $k = 0$ corresponds to well-prepared case and has been recalled in the previous section.

The choice of the basis $(\cos(k\pi x_3))_{k\in\mathbf{N}}$ for the horizontal component ensures that the boundary condition ${v^3}_{|\partial\Omega} = 0$ is satisfied since the vertical component is a linear combination of $(\sin(k\pi x_3))_{k\in\mathbf{N}}$ thanks to the divergence free condition. But the terms of type $\cos(k\pi x_3)$ will produce time oscillations. As we shall see later on, the situation is therefore a little bit more complicated than in the well prepared case.

For reasons which will appear clearly when we deal with dispersive phenomena, we need to avoid extreme horizontal frequencies. So we approximate any divergence free vector field v (resp. time dependent) of L^2 (resp. of $L^\infty([0,T];L^2)\cap L^2([0,T];H^{1,0})$) by

$$v_N \stackrel{\text{def}}{=} \mathcal{F}^{-1}\sum_{k=0}^{N}\Big(\mathbf{1}_{\mathcal{C}_N}(\xi_h)\widehat{v}^{k,h}(\xi_h)\cos(k\pi x_3), -\frac{1}{k\pi}\mathbf{1}_{\mathcal{C}_N}(\xi_h)\mathcal{F}\,\mathrm{div}_h\, v_0^{k,h}(\xi_h)\sin(k\pi x_3)\Big)$$

where $\mathcal{C}_N$ denotes the set of all ξ_h in $\mathbf{R}^2$ such that $|\xi_h|\in[N^{-1},N]$. It is obvious that $\lim\limits_{N\to\infty} v_N = v_0$ in L^2 (resp. in $L^2([0,T];H^{1,0})$).

It is in that situation of boundary conditions that the Rossby operator has a good spectral representation; let us be more precise on that point, before stating the key lemma. The first step of the construction of the approximate solution consists in writing the system (FRF^ε) in a simpler way, in fact in terms of the horizontal divergence and horizontal curl of v^h. Let us state the following notation:

$$\widehat{d}^h \stackrel{\text{def}}{=} \mathcal{F}\,\mathrm{div}_h\, v^h\,,\quad \widehat{\omega}^h \stackrel{\text{def}}{=} \mathcal{F}\mathrm{curl}_h v^h\,,\quad \widehat{p} \stackrel{\text{def}}{=} \mathcal{F}p \quad\text{and}\quad W \stackrel{\text{def}}{=} (\widehat{d}^h,\widehat{\omega}^h,\mathcal{F}v^3).$$

For the sake of simplicity in the notation, we drop the ε in $\widehat{d}^h$, $\widehat{\omega}^h$, $\widehat{p}^h$ and W. In the following Π_1 denotes the projection on the first coordinate (recall that the first coordinate of W is the horizontal Fourier transform of the horizontal divergence of the vector field v). Then for vectors of the type

$$\Big(W^{k,h}(t)\cos(k\pi x_3), -\frac{1}{k\pi}\Pi_1 W^k(t)\sin(k\pi x_3),\Big)$$

writing $W^{k,h}=(W^{k,1},W^{k,2})$, and pressures of the type $\widehat{p}^k\cos(k\pi x_3)$ the rotating fluid system is equivalent to the following ordinary differential system

$$\left\{\begin{array}{rcl}
\dfrac{d}{dt}W^{k,1}+\nu|\xi_h|^2W^{k,1}+\beta\varepsilon(k\pi)^2W^{k,1}-\dfrac{1}{\varepsilon}W^{k,2} &=& |\xi_h|^2\widehat{p}^k\\
\dfrac{d}{dt}W^{k,2}+\nu|\xi_h|^2W^{k,2}+\beta\varepsilon(k\pi)^2W^{k,2}+\dfrac{1}{\varepsilon}W^{k,1} &=& 0\\
\dfrac{d}{dt}W^{k,3}+\nu|\xi_h|^2W^{k,3}+\beta\varepsilon(k\pi)^2W^{k,3} &=& k\pi\widehat{p}^k\\
W^{k,1}+k\pi W^{k,3} &=& 0\\
W_{|t=0} &=& W_0.
\end{array}\right.$$

As usual the divergence free condition (which here turns out to be $W^{k,1}+k\pi W^{k,3}=0$) determines the pressure. Here this gives

$$\widehat{p}^k = -\frac{1}{\varepsilon(|\xi_h|^2+(k\pi)^2)}W^{k,2}.$$

The rotating fluid system can now be written

$$(FRF_k^\varepsilon)\left\{\begin{array}{rcl}
\dfrac{d}{dt}W^{k,h}+\nu|\xi_h|^2W^{k,h}+\beta\varepsilon(k\pi)^2W^{k,h}-\dfrac{1}{\varepsilon}R_kW^{k,h} &=& 0\\
{W^{k,h}}_{|t=0} &=& W_0^{k,h}
\end{array}\right.$$

with

$$R_k = \begin{pmatrix} 0 & -\lambda_k^2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \lambda_k \stackrel{\text{def}}{=} \left(\frac{(k\pi)^2}{|\xi_h|^2 + (k\pi)^2} \right)^{\frac{1}{2}}.$$

Now in order to find the interior solution at order zero, and to get rid of the ε^{-1} terms in the equation, let us write

$$W = \sum_{k=1}^{N} \Big(\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k(t)\cos(k\pi x_3), -\frac{1}{k\pi}\Pi_1\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k(t)\sin(k\pi x_3)\Big)$$

where $\mathcal{L}_k$ is a function from $\mathbf{R}^+$ into $\mathcal{L}(\mathbf{R}^2)$ and $\widetilde{W}^k$ is a function from $\mathbf{R}^+$ into $\mathbf{C}^2$, to be determined.

Then looking at the terms of size ε^{-1} in the above equation (FRF_k^ε), we get

$$\dot{\mathcal{L}}_k + R_k\mathcal{L}_k = 0 \quad \text{with} \quad \mathcal{L}_k(0) = \mathrm{Id},$$

so

$$\mathcal{L}_k(\tau) = \begin{pmatrix} \cos\tau_k & \lambda_k\sin\tau_k \\ -\dfrac{1}{\lambda_k}\sin\tau_k & \cos\tau_k \end{pmatrix} \tag{2.9}$$

with, as in all that follows, $\tau_k = \lambda_k\tau$. Let us remark that when $k = 0$, (this corresponds of course to the well prepared case studied in the previous section), we get $\mathcal{L}_k = \mathrm{Id}$ because in this case $\lambda_k = 0$.

To state the approximation lemma in the general case, we need to define the following family of operators $(\mathcal{L}(\tau))_{\tau\in\mathbf{R}}$: for any vector v of the form (2.7,2.8),

$$\begin{aligned}(\mathcal{L}(\tau)v)(x_h, x_3) \stackrel{\text{def}}{=} (v^{0,h}(x_h), 0) + \mathcal{F}^{-1}\sum_{k=1}^{\infty}\Big(&A(\xi_h)\mathcal{L}_k(\tau)A^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\cos(k\pi x_3),\\ &\frac{i}{k\pi}\xi_h\cdot A(\xi_h)\mathcal{L}_k(\tau)A^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\sin(k\pi x_3)\Big)\end{aligned} \tag{2.10}$$

where $\mathcal{L}_k$ is defined by formula (2.9) and with

$$A(\xi_h) \stackrel{\text{def}}{=} \begin{pmatrix} \xi_1|\xi_h|^{-2} & -\xi_2|\xi_h|^{-2} \\ \xi_2|\xi_h|^{-2} & \xi_1|\xi_h|^{-2} \end{pmatrix}.$$

Let us note that $(\mathcal{L}(\tau))_{\tau\in\mathbf{R}}$ is bounded in $\mathcal{L}(H^{s,0})$ for any real number s. Moreover, the restriction of $\mathcal{L}(\tau)$ to functions which do not depend on the third variable (which corresponds to the well prepared case) is Id. This operator $\mathcal{L}(\tau)$ is the Rossby wave operator.

Similarly we shall need the definition of the following "Ekman operator":

$$\begin{aligned}(\mathcal{E}v)(x_h, x_3) \stackrel{\text{def}}{=} \sqrt{2\beta}\,(v^{0,h}(x_h), 0) + \mathcal{F}^{-1}\sum_{k=1}^{\infty}\Big(&A(\xi_h)B_kA^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\cos(k\pi x_3),\\ &\frac{i}{k\pi}\xi_h\cdot A(\xi_h)B_kA^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)\sin k\pi x_3\Big)\end{aligned} \tag{2.11}$$

with

$$B_k \stackrel{\text{def}}{=} \frac{(1-\lambda_k^2)\lambda_k}{4}\begin{pmatrix} \gamma_k^- - \gamma_k^+ & -\lambda_k(\gamma_k^+ + \gamma_k^-) \\ \dfrac{\gamma_k^+ + \gamma_k^-}{\lambda_k} & \gamma_k^- - \gamma_k^+ \end{pmatrix} \quad \text{and with}$$

$$\gamma_k^\pm \stackrel{\text{def}}{=} \left(1 \mp \frac{1}{\lambda_k}\right)\sqrt{\frac{2\beta}{1\pm\lambda_k}}\,.$$

Let us notice that the restriction of $\mathcal{E}$ on functions which do not depend on the third variable (which again corresponds to the well prepared case) is $\sqrt{2\beta}\,\mathrm{Id}$.

The key lemma is the following.

Lemma 2 *Let $(v_N)_{N\in\mathbf{N}}$ be a (time–dependent) bounded sequence of divergence free vector fields in the space*

$$L^\infty([0,T];L^2)\cap L^2([0,T];H^{1,0})$$

of the form

$$v_N \stackrel{\text{def}}{=} \mathcal{F}^{-1}\sum_{k=0}^N \Big(\widehat{v}_N^{k,h}(\xi_h)\cos(k\pi x_3), -\frac{1}{ik\pi}\xi_h\cdot\widehat{v}_N^{k,h}(\xi_h)\sin(k\pi x_3)\Big)$$

with $\mathrm{Supp}\ \widehat{v}_N^{\varepsilon,k,h}(\xi_h)\subset\mathcal{C}_N$. *A sequence of families $(v^\varepsilon_{app,N})_{N\in\mathbf{N}}$ of smooth divergence free vector fields whose value on the boundary of Ω is 0 exists, such that*

$$L^\varepsilon v^\varepsilon_{app,N} = \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Big(\partial_t v_N - \nu\Delta_h v_N + \mathcal{E}v_N\Big) - \nabla p^\varepsilon_{app,N} + R^\varepsilon_N$$

where R^ε_N tends to 0 in the following sense:

$$\forall\eta\,,\ \exists N_1\,,\ \forall N>N_1\,,\ \exists\varepsilon_0\,/\ \forall\varepsilon<\varepsilon_0\,,\ \|R^\varepsilon_N\|_{L^\infty(\mathbf{R}^+;H^{-1,0})\cap L^\infty(\mathbf{R}^+;L^2)}<\eta. \tag{2.12}$$

The vector field $v^\varepsilon_{app,N}$ converges to $\mathcal{L}\left(\frac{t}{\varepsilon}\right)v_N$ in the following sense: a constant C_N exists such that

$$E_T\left(v^\varepsilon_{app,N} - \mathcal{L}\Big(\frac{t}{\varepsilon}\Big)v_N\right) \le C_N\varepsilon E_T(v_N). \tag{2.13}$$

Moreover the family $(v^\varepsilon_{app,N})$ satisfies the following estimates

$$\int_0^T \sup_{x_3\in]0,1[}\|\nabla^h v^\varepsilon_{app,N}(t,\cdot,x_3)\|^2_{L^2(\mathbf{R}^2)}dt \ \le\ C_N E_T(v_N) \tag{2.14}$$

$$\forall p\in[2,\infty]\,,\ \int_0^T\int_0^1 d(x_3)\|\partial_3 v^\varepsilon_{app,N}(t,\cdot,x_3)\|^2_{L^p(\mathbf{R}^2)}dtdx_3 \ \le\ C_N\overline{E}_T(v_N) \quad \text{and} \tag{2.15}$$

$$\int_0^T\int_0^1 d(x_3)^2\left\|\partial_3\left(v^\varepsilon_{app,N} - \mathcal{L}\Big(\frac{t}{\varepsilon}\Big)v_N\right)\right\|^2_{L^\infty(\mathbf{R}^2)} dtdx_3 \ \le\ C_N\varepsilon\overline{E}_T(v_N). \tag{2.16}$$

Remark In the proof, we shall forget the part associated with $\widehat{v}_N^{0,h}$ because this case is nothing but the previous lemma (i.e. the "well-prepared" case).

The proof of this lemma is one of the key points of the paper.

In horizontal divergence and curl formulation, the system (FRF^ε) becomes

$$
(\widetilde{FRF}^\varepsilon)\begin{cases}
\partial_t W^1 + \nu|\xi_h|^2 W^1 - \beta\varepsilon\partial_3^2 W^1 - \dfrac{1}{\varepsilon}W^2 &= |\xi_h|^2\widehat{p}\\
\partial_t W^2 + \nu|\xi_h|^2 W^2 - \beta\varepsilon\partial_3^2 W^2 + \dfrac{1}{\varepsilon}W^1 &= 0\\
\partial_t W^3 + \nu|\xi_h|^2 W^3 - \beta\varepsilon\partial_3^2 W^3 &= -\partial_3\widehat{p}\\
W^1 + \partial_3 W^3 &= 0\\
W_{|t=0} &= W_0\\
W_{|\partial\Omega} &= 0.
\end{cases}
$$

From now on, we shall only consider the above system. Let us search for an approximate solution of the form

$$
\begin{aligned}
W &= W_{0,int} + W_{0,BL} + \varepsilon W_{1,int} + \varepsilon W_{1,BL} + \cdots \quad \text{and}\\
\widehat{p} &= \frac{1}{\varepsilon}\widehat{p}_{-1,int} + \frac{1}{\varepsilon}\widehat{p}_{-1,BL} + \widehat{p}_{0,int} + \widehat{p}_{0,BL} + \cdots
\end{aligned}
$$

where each component of $(v_{j,int}, p_{j,int})$ is a function of the form

$$
f\Big(\frac{t}{\varepsilon}, t, x_3\Big)
$$

and each component of $(v_{j,BL}, p_{j,BL})$ is a function of the form

$$
g\Big(\frac{t}{\varepsilon}, t, \frac{x_3}{\varepsilon}\Big) + h\Big(\frac{t}{\varepsilon}, t, \frac{1-x_3}{\varepsilon}\Big)\cdot
$$

In all that follows, we shall denote $\tau = t/\varepsilon$. First we have to determine the form of $W_{0,int}$ and $\widehat{p}_{-1,int}$. Considering the decomposition of the initial data, we look for $W_{0,int}$ of the form

$$
\sum_{k=1}^N \Big(\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}(t)\cos(k\pi x_3), -\frac{1}{k\pi}\Pi_1\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}(t)\sin(k\pi x_3)\Big)
$$

where $\mathcal{L}_k$ is the function from $\mathbf{R}^+$ into $\mathcal{L}(\mathbf{R}^2)$ defined above in (2.9), $\widetilde{W}^k_{0,int}$ is a function from $\mathbf{R}^+$ into $\mathbf{C}^2$ and recall that Π_1 denotes the projection on the first coordinate. The vector $\widetilde{W}^k_{0,int}$ must be understood as the image of $W^{k,h}_{0,int}$ through the "filtering" operator $\mathcal{L}_k$.

Now let us write

$$
W_{0,int,N} \stackrel{\text{def}}{=} \sum_{k=1}^N \Big(\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}\cos(k\pi x_3), -\frac{1}{k\pi}\Pi^1\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}\sin(k\pi x_3)\Big). \tag{2.17}
$$

We now have to determine the boundary layer of size ε^0. As the third component of the interior solution at order 0 is identically 0, then the vertical component of the boundary layer of size ε^0 vanishes. This implies that $\partial_3\widehat{p}^k_{-1,BL} = 0$. We recover the well known fact that the pressure does not vary in the boundary layer. Now let us study the term of size ε^{-1} for the horizontal component of the boundary layer. As $\cos(k\pi) = (-1)^k$ we look for the boundary layer of the form

$$W^{k,h}_{0,BL} \stackrel{\text{def}}{=} M_k\Big(\frac{x_3}{\varepsilon}\Big)\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int} + (-1)^k M_k\Big(\frac{1-x_3}{\varepsilon}\Big)\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}.$$

The term of size ε^{-1} of

$$\partial_t W^{k,h}_{0,BL} - \beta\varepsilon\partial_3^2 W^{k,h}_{0,BL} + \frac{1}{\varepsilon} R W^{k,h}_{0,BL}$$

must be equal to 0. So we infer that

$$M_k\dot{\mathcal{L}}_k - \beta M_k''\mathcal{L}_k + RM_k\mathcal{L}_k = 0.$$

In the case when $k = 0$, we have $\mathcal{L}_k = \mathrm{Id}$. Now let us assume that $k \geq 1$. As $\dot{\mathcal{L}}_k = -R_k\mathcal{L}_k$, it turns out that the equation on the boundary layer is

$$\left\{\begin{array}{rcl} -\beta M_k'' & = & M_kR_k - RM_k \\ M_k(0) & = & -\mathrm{Id} \\ M_k(+\infty) & = & 0. \end{array}\right.$$

This is a linear differential equation of order 2 with an initial and a final condition. The solution is

$$M_k(\zeta) = -\sum_{\pm} \frac{1}{2}\mu_k^{\pm} \exp(-\zeta_k^{\pm}) M_k^{\pm}(\zeta_k^{\pm})$$

with

$$M_k^{\pm}(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & \mp\lambda_k\sin\theta \\ -\sin\theta & \mp\lambda_k\cos\theta \end{pmatrix}$$

and

$$\zeta_k^{\pm} \stackrel{\text{def}}{=} \frac{\zeta}{\sqrt{2\beta_k^{\pm}}}, \quad \beta_k^{\pm} \stackrel{\text{def}}{=} \frac{\beta}{1 \pm \lambda_k} \quad \text{and} \quad \mu_k^{\pm} \stackrel{\text{def}}{=} 1 \mp \frac{1}{\lambda_k}\cdot$$

So stating

$$E_k^{\pm} \stackrel{\text{def}}{=} 2\varepsilon^2\beta_k^{\pm}$$

we infer by definition of $\mathcal{L}_k$ that

$$\begin{aligned} W^{k,h}_{0,BL} = & -\frac{1}{2}\sum_{\pm}\mu_k^{\pm}\exp\Big(-\frac{x_3}{\sqrt{E_k^{\pm}}}\Big)M_k^{\pm}\Big(\frac{x_3}{\sqrt{E_k^{\pm}}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^k_{0,int} \\ & - (-1)^k\frac{1}{2}\sum_{\pm}\mu_k^{\pm}\exp\Big(-\frac{1-x_3}{\sqrt{E_k^{\pm}}}\Big)M_k^{\pm}\Big(\frac{1-x_3}{\sqrt{E_k^{\pm}}} \mp \frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}. \end{aligned}$$

Let us remark that the eigenvectors of $M_k^{\pm}$ and those of $\mathcal{L}_k$ are different. Moreover, time oscillations introduce a phase shift in the boundary layer.

As in the well prepared case, the fact that the boundary layer must be divergence free implies that we have to introduce a vertical component of the boundary layer of size ε. It is given by the following formula:

$$\varepsilon\partial_3 W^{k,3}_{1,BL} = -\Pi_1 W^{k,h}_{0,BL}.$$

So we get

$$
\begin{aligned}
\varepsilon\partial_3 W^{k,3}_{1,BL} \;=\;& \frac{1}{2}\sum_{\pm}\mu_k^{\pm}\exp\Big(-\frac{x_3}{\sqrt{E_k^{\pm}}}\Big)\bigg(\cos\Big(\frac{x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,1}_{0,int}\\
&\qquad\qquad\mp\lambda_k\sin\Big(\frac{x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,2}_{0,int}\bigg)\\
&+(-1)^k\frac{1}{2}\sum_{\pm}\mu_k^{\pm}\exp\Big(-\frac{1-x_3}{\sqrt{E_k^{\pm}}}\Big)\bigg(\cos\Big(\frac{1-x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,1}_{0,int}\\
&\qquad\qquad\mp\lambda_k\sin\Big(\frac{1-x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,2}_{0,int}\bigg).
\end{aligned}
$$

So with the notation

$$
\mathrm{cs}^{\pm}\stackrel{\mathrm{def}}{=}\cos\pm\sin\quad\text{and}\quad \gamma_k^{\pm}\stackrel{\mathrm{def}}{=}\mu_k^{\pm}\sqrt{2\beta_k^{\pm}}\,,
$$

we get by integration,

$$
\begin{aligned}
W^{k,3}_{1,BL} \;=\;& -\frac{1}{4}\sum_{\pm}\gamma_k^{\pm}\exp\Big(-\frac{x_3}{\sqrt{E_k^{\pm}}}\Big)\bigg(\mathrm{cs}^{-}\Big(\frac{x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,1}_{0,int}\\
&\qquad\qquad\mp\lambda_k\mathrm{cs}^{+}\Big(\frac{x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,2}_{0,int}\bigg)\\
&+(-1)^k\frac{1}{4}\sum_{\pm}\gamma_k^{\pm}\exp\Big(-\frac{1-x_3}{\sqrt{E_k^{\pm}}}\Big)\bigg(\mathrm{cs}^{-}\Big(\frac{1-x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,1}_{0,int}\\
&\qquad\qquad\mp\lambda_k\mathrm{cs}^{+}\Big(\frac{1-x_3}{\sqrt{E_k^{\pm}}}\mp\frac{\lambda_k t}{\varepsilon}\Big)\widetilde{W}^{k,2}_{0,int}\bigg).
\end{aligned}
$$

It is clear that this boundary layer is a sum of a rapidly decreasing function of x_3/ε and of a rapidly decreasing function of $(1-x_3)/\varepsilon$. But it is obvious that those two functions do not vanish respectively in $x_3=0$ and $x_3=1$. More precisely, up to exponentially small terms, we have, stating again $\tau_k=\lambda_k\tau$,

$$
W^{k,3}_{1,BL\,|x_3=0} = -(-1)^k W^{k,3}_{1,BL\,|x_3=1} = -f_k\Big(\frac{t}{\varepsilon}\Big)
$$

with

$$
f_k(\tau)\stackrel{\mathrm{def}}{=}\sum_{\pm}\frac{\gamma_k^{\pm}}{4}\Big(\mathrm{cs}^{\pm}(\tau_k)\widetilde{W}^{k,1}_{0,int}\mp\lambda_k\mathrm{cs}^{\mp}(\tau_k)\widetilde{W}^{k,2}_{0,int}\Big).
$$

Now, let us have a look at the terms of size ε^0 in the interior system. The system of equations is the following.

$$\begin{cases} (\partial_t+\nu|\xi_h|^2)W^h_{0,int,N}+\partial_\tau W^h_{1,int,N}+RW^h_{1,int,N} &= \begin{pmatrix} |\xi_h|^2\widehat{p}_{0,int,N}\\ 0\end{pmatrix}\\ (\partial_t+\nu|\xi_h|^2)W^3_{0,int,N}+\partial_\tau W^3_{1,int,N} &= -\partial_3\widehat{p}_{0,int,N}\\ W^1_{1,int,N}+\partial_3 W^3_{1,int,N} &= 0\\ W^3_{1,int,N|\partial\Omega} &= -W^{k,3}_{1,BL|\partial\Omega}.\end{cases}$$

As usual, we are going to reduce this problem to a problem with a homogeneous Dirichlet boundary condition. To do so, let us first define the function r_ℓ where

$$r_\ell(x_3)=1 \quad \text{when } \ell \text{ is odd}, \quad r_\ell(x_3)=1-2x_3 \quad \text{when } \ell \text{ is even}.$$

Then let us state

$$\underline{W}_{1,int,N} \stackrel{\text{def}}{=} \sum_{\ell=1}^N f_\ell(\tau)\Big(\delta_\ell,0,r_\ell(x_3)\Big) \quad \text{with} \quad \delta_\ell=1+(-1)^\ell$$

and let us look for $W_{1,int,N}$ of the form

$$W_{1,int,N}=\overline{W}_{1,int,N}+\underline{W}_{1,int,N}.$$

Now the above system can be written

$$\begin{cases} (\partial_t+\nu|\xi_h|^2)W^h_{0,int,N}+(\partial_\tau+R)\overline{W}^h_{1,int,N} &= \begin{pmatrix} |\xi_h|^2\widehat{p}_{0,int,N}\\ 0\end{pmatrix}-(\partial_\tau+R)\underline{W}^h_{1,int,N}\\ (\partial_t+\nu|\xi_h|^2)W^3_{0,int,N}+\partial_\tau \overline{W}^3_{1,int,N} &= -\partial_3\widehat{p}_{0,int,N}-\partial_\tau\underline{W}^3_{1,int,N}\\ \overline{W}^1_{1,int,N}+\partial_3 \overline{W}^3_{1,int,N} &= 0\\ \overline{W}^3_{1,int,N|\partial\Omega} &= 0.\end{cases}$$

Still as usual, the original boundary condition appears through a forcing term. Considering this boundary condition on $\overline{W}_{1,int,N}$, it is natural to look for $\overline{W}_{1,int,N}$ and $\widehat{p}_{0,int,N}$ of the form

$$\begin{aligned}\overline{W}_{1,int,N} &= \Big(\sum_{k=0}^N W^{k,h}_{1,int,N}\cos(k\pi x_3),-\sum_{k=1}^N \frac{1}{k\pi}W^{k,1}_{1,int,N}\sin(k\pi x_3)\Big) \quad \text{and}\\ \widehat{p}_{0,int,N} &= \sum_{k=0}^N p^k_{0,int}\cos(k\pi x_3).\end{aligned}$$

Now let us decompose the forcing term in a low and a high vertical frequency part. This part will be small when N tends to ∞. Let us notice that in $L^2(]0,1[)$, we have

$$r_\ell(x_3)=\sum_{k\geq 1} r_{\ell,k}\sin(k\pi x_3) \quad \text{with} \quad r_{\ell,k} \stackrel{\text{def}}{=} \frac{1}{k\pi}(1+(-1)^{k+\ell}). \tag{2.18}$$

So, we write that

$$\underline{W}_{1,int,N} = F_N + R_N^\varepsilon \quad \text{with} \quad F_N \stackrel{\text{def}}{=} \sum_{\ell=1}^{N} f_\ell(\tau)\Big(\delta_\ell, 0, \sum_{k=1}^{N} r_{\ell,k}\sin(k\pi x_3)\Big).$$

Obviously, R_N^ε satisfies the property (2.12). Now let us project on cosine and sine functions, which yields

$$\begin{aligned}\Big(\frac{d}{d\tau}+R\Big)\overline{W}^{0,h}_{1,int,N} &= \sum_{\ell=1}^{N}\Big\{-\frac{df_\ell}{d\tau}(\delta_\ell,0)+f_\ell(\tau)(0,\delta_\ell)\Big\}\\ \frac{d}{d\tau}\overline{W}^{0,3}_{1,int,N} &= 0\end{aligned}$$

and, when $k\neq 0$, considering the form of $W_{0,int,N}$ given by (2.17)

$$\begin{aligned}\mathcal{L}_k(\tau)\Big(\frac{d}{dt}+\nu|\xi_h|^2\Big)\widetilde{W}^k_{0,int}+\Big(\frac{d}{d\tau}+R\Big)\overline{W}^{k,h}_{1,int,N} &= \begin{pmatrix}|\xi_h|^2\widehat{p}^k_{0,int}\\ 0\end{pmatrix}\\ -\Pi_1\frac{1}{k\pi}\mathcal{L}_k(\tau)\Big(\frac{d}{dt}+\nu|\xi_h|^2\Big)\widetilde{W}^k_{0,int}+\frac{d}{d\tau}\overline{W}^{k,3}_{1,int,N} &= -\sum_{\ell\le N} r_{\ell,k}\frac{df_\ell}{d\tau}+k\pi\widehat{p}^k_{0,int}.\end{aligned}$$

Let us solve first the system when $k=0$. As $\overline{W}^{0,3}_{1,int,N}=0$ the divergence free condition implies that $\overline{W}^{0,1}_{1,int,N}=0$. So the system becomes

$$\begin{aligned}-\overline{W}^{0,2}_{1,int,N} &= -\sum_{\ell=1}^{N}\frac{df_\ell}{d\tau}+|\xi_h|^2\widehat{p}^0_{0,int,N}\\ \frac{d}{d\tau}\overline{W}^{0,2}_{1,int,N} &= \sum_{\ell=1}^{N} f_\ell(\tau)\end{aligned}$$

Thus we get

$$W^0_{1,int,N} = \Big(0, \sum_{\ell=1}^{N}\sum_{\pm}\frac{\gamma_\ell^\pm}{4\lambda_\ell}\Big(\mp \mathrm{cs}^\mp(\tau_\ell)\widetilde{W}^{\ell,1}_{0,int}+\lambda_\ell \mathrm{cs}^\pm(\tau_\ell)\widetilde{W}^{\ell,2}_{0,int}\Big), 0\Big).$$

We also get an explicit formula for $p^0_{0,int,N}$ whose computation is left to the reader.

Let us notice that those terms do not depend on the variable x_3 and are bounded in time and time oscillating.

Now let us study the case when $k\neq 0$. As usual, the divergence free condition will determine the pressure by

$$\widehat{p}^k_{0,int} = \frac{1}{|\xi_h|^2+(k\pi)^2}\Big(\sum_{\ell\le N} k\pi r_{\ell,k}\frac{df_\ell}{d\tau}+\overline{W}^{k,2}_{1,int,N}\Big).$$

So now the system becomes

$$\mathcal{L}_k(\tau)\Big(\frac{d}{dt}+\nu|\xi_h|^2\Big)\widetilde{W}^k_{0,int}+\Big(\frac{d}{d\tau}+R_k\Big)\overline{W}^{k,h}_{1,int,N}-\sum_{\ell\le N}\begin{pmatrix}\dfrac{k\pi r_{\ell,k}|\xi_h|^2}{|\xi_h|^2+(k\pi)^2}\dfrac{df_\ell}{d\tau}\\ 0\end{pmatrix}=0. \quad (2.19)$$

Now we have to remember that the f_ℓ are linear maps of $\widetilde{W}^{\ell}_{0,int}$ whose coefficients oscillate in time with the frequency $\frac{\lambda_\ell}{\varepsilon}\cdot$ This leads to the following lemma whose proof is a series of straighforward computations left to the reader.

Lemma 3 *We have the following identity:*

$$-\sum_{\ell\le N}\begin{pmatrix}\dfrac{k\pi r_{\ell,k}|\xi_h|^2}{|\xi_h|^2+(k\pi)^2}\dfrac{df_\ell}{d\tau}\\ 0\end{pmatrix}=\mathcal{L}_k(\tau)\Big(B_k+\sum_{\ell=1}^N B_{k,\ell}(\tau)\Big)\widetilde{W}^k_{0,int}$$

where $B_{k,\ell}(\tau)$ are matrices whose coefficients are cosine or sine functions of $(\lambda_k\pm\lambda_\ell)\tau$ for $\ell\neq k$ and of $\lambda_k\tau$ when $\ell=k$ and where

$$B_k=\frac{(1-\lambda_k^2)\lambda_k}{4}\begin{pmatrix}\gamma_k^- - \gamma_k^+ & -\lambda_k(\gamma_k^+ + \gamma_k^-)\\ \dfrac{\gamma_k^+ + \gamma_k^-}{\lambda_k} & \gamma_k^- - \gamma_k^+\end{pmatrix}.$$

Thus the system splits into the two following ones

$$\begin{aligned}\Big(\frac{d}{dt}+\nu|\xi_h|^2+B_k\Big)\widetilde{W}^k_{0,int} &= 0 \quad \text{and}\\ \Big(\frac{d}{d\tau}+R_k\Big)\overline{W}^{k,h}_{1,int,N} &= -\mathcal{L}_k(\tau)\Big(\sum_{\ell=1}^N B_{k,\ell}(\tau)\Big)\widetilde{W}^k_{0,int}.\end{aligned}$$

Immediately we have that

$$\begin{aligned}\widetilde{W}^k_{0,int}(t) &= \exp(-\nu|\xi_h|^2t-P_kt)\begin{pmatrix}\cos\delta_k t & \lambda_k\sin\delta_k t\\ -\dfrac{1}{\lambda_k}\sin\delta_k t & \cos\delta_k t\end{pmatrix}\widetilde{W}^k_{0,int}(0) \quad \text{with} \quad (2.20)\\ \delta_k &\stackrel{\text{def}}{=} \frac{(1-\lambda_k^2)\lambda_k}{4}(\gamma_k^+ + \gamma_k^-)\,,\\ P_k &\stackrel{\text{def}}{=} \frac{(1-\lambda_k^2)\lambda_k}{4}(\gamma_k^- - \gamma_k^+) \quad \text{and}\\ W^{k,h}_{1,int,N} &= \mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)\sum_{\ell=1}^N C_{k,\ell}\Big(\frac{t}{\varepsilon}\Big)\widetilde{W}^k_{0,int}(t) \qquad (2.21)\end{aligned}$$

where $C_{k,\ell}$ are (2×2 valued) smooth bounded functions of τ whose derivatives are the $B_{k,\ell}$. So applying the usual procedure for the higher order terms for boundary layers, we find the complete expression of the approximate solution.

As far as estimates (2.15) and (2.16) are concerned, the worse terms are obviously $W^{k,h}_{0,BL}$. As the support of $\mathcal{F}W^{k,h}_{0,BL}$ is included in $\mathcal{C}(N^{-1},N)$, it is enough estimate then L^2 norm on $\mathbf{R}^2$.

From the above formula, we have that

$$\|\partial_3 W^{k,h}_{0,BL}(t,\cdot,x_3)\|_{L^2(\mathbf{R}^2)}\le\frac{C_k}{\varepsilon}\sum_{\pm}\Big(\exp\Big(-\frac{x_3}{\varepsilon\sqrt{2\beta_k^\pm}}\Big)+\exp\Big(-\frac{1-x_3}{\varepsilon\sqrt{2\beta_k^\pm}}\Big)\Big)\|\widetilde{W}^{k,h}_{0,int}(t,\cdot)\|_{L^2(\mathbf{R}^2)}.$$

So we infer that

$$
\begin{aligned}
\int_0^1 d(x_3)\|\partial_3 W^{k,h}_{0,BL}(t,\cdot,x_3)\|^2_{L^2(\mathbf{R}^2)} &\leq C_k\|W^{k,h}_{0,int}(t,\cdot)\|^2_{L^2(\mathbf{R}^2)}\int_0^1 \frac{d(x_3)}{\varepsilon}\exp\Big(-2\frac{d(x_3)}{\varepsilon}\Big)dx_3\\
&\leq C_k\|W^{k,h}_{0,int}(t,\cdot)\|^2_{L^2(\mathbf{R}^2)}.
\end{aligned}
$$

Moreover, exactly along the sames lines, we have that

$$
\begin{aligned}
\int_0^1 d(x_3)^2\|\partial_3 W^{k,h}_{0,BL}(t,\cdot,x_3)\|_{L^2(\mathbf{R}^2)} &\leq C_k\|W^{k,h}_{0,int}(t,\cdot)\|^2_{L^2(\mathbf{R}^2)}\int_0^1 \frac{d(x_3)^2}{\varepsilon}\exp\Big(-2\frac{d(x_3)}{\varepsilon}\Big)dx_3\\
&\leq C_k\varepsilon\|W^{k,h}_{0,int}(t,\cdot)\|^2_{L^2(\mathbf{R}^2)}.
\end{aligned}
$$

The result follows.

3 The non linear estimates

This section consists in proving Theorem 2. Let us first cut off the initial data in the frequency space. To do so, let us define

$$
\begin{aligned}
\overline{u}_0(x_h) &\stackrel{\text{def}}{=} \int_0^1 u_0(x_h,x_3)dx_3\,,\\
u_0^{k,h}(x_h) &\stackrel{\text{def}}{=} \frac{1}{2}\int_0^1 u_0(x_h,x_3)\cos(k\pi x_3)dx_3\,,\\
\overline{u}_{0,N} &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\widehat{\overline{u}}_0) \quad \text{and}\\
u^{k,h}_{0,N} &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\widehat{u}^{k,h}_0).
\end{aligned}
$$

For any positive real number η, an integer N_0 exists, depending of course on η and on the initial data u_0 such that

$$
\begin{aligned}
\|u_0-\overline{u}_{0,N}-\widetilde{u}_{0,N}\|_{L^2} &< \frac{\eta}{4} \quad \text{with}\\
\widetilde{u}_{0,N} &= \sum_{k=1}^{N}\Big(u^{k,h}_{0,N}(x_h)\cos(k\pi x_3),-\frac{1}{k\pi}\operatorname{div}_h u^{k,h}_{0,N}(x_h)\sin(k\pi x_3)\Big).
\end{aligned}
$$

Let us define $\overline{u}_N$ as the solution of

$$
(LE_{\nu,\beta})\begin{cases}
\partial_t\overline{u}_N-\nu\Delta_h\overline{u}_N+\sqrt{2\beta}\,\overline{u}_N = -\nabla\overline{p}_N-\mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\mathcal{F}(\operatorname{div}_h(\overline{u}\otimes\overline{u})))\\
\operatorname{div}_h\overline{u}_N = 0\\
\overline{u}_{N|t=0} = \overline{u}_{0,N}.
\end{cases}
$$

A basic energy estimate implies that, for any positive η, an integer N_0 exists such that, for any t in $\mathbf{R}^+$ and any N greater or equal to N_0, we have

$$
\|\overline{u}(t)-\overline{u}_N(t)\|^2_{L^2}+2\int_0^t\Big(\nu\|\nabla^h(\overline{u}(t')-\overline{u}_N(t'))\|^2_{L^2}+\sqrt{2\beta}\,\|\overline{u}(t')-\overline{u}_N(t')\|^2_{L^2}\Big)dt' < \frac{\eta^2}{16}\cdot \quad (3.22)
$$

Sobolev embeddings also imply that $\overline{u}$ belongs to $L^4(\mathbf{R}^+\times\mathbf{R}^2)$ and $\overline{u}_N$ converges to $\overline{u}$ in the space $L^4(\mathbf{R}^+\times\mathbf{R}^2)$. Thus we have

$$\lim_{N\to\infty}\mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_N}\mathcal{F}(\mathrm{div}_h(\overline{u}\otimes\overline{u})))=\lim_{N\to\infty}\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)=\mathrm{div}_h(\overline{u}\otimes\overline{u}) \tag{3.23}$$

in the space $L^2(\mathbf{R}^+;H^{-1}(\mathbf{R}^2))$.

Now we shall use Lemmas 1 and 2 to define the (sequence of) approximate solutions of the system. Let us define $(u^\varepsilon_{wp,N})_{\varepsilon>0}$ as the families given by Lemma 1 applied with $v_N=\overline{u}_N$ solution of $(LE_{\nu,\beta})$; it represents the "well prepared" part pf the solution. The so–called "ill prepared" part is defined as $(u^\varepsilon_{ip,N})$, the families given by Lemma 2 applied with v_N equal to the solution of the linear problem

$$\left\{\begin{array}{rcl}\partial_t v_N-\nu\Delta_h v_N+\mathcal{E}v_N&=&-\nabla p_N\\ \mathrm{div}\,v_N&=&0\\ v_{N|t=0}&=&\widetilde{u}_{0,N}.\end{array}\right. \tag{3.24}$$

Of course, we state

$$u^\varepsilon_{app,N}\stackrel{\mathrm{def}}{=}u^\varepsilon_{wp,N}+u^\varepsilon_{ip,N}. \tag{3.25}$$

Let us derive the equation satisfied by $u^\varepsilon_{app,N}$. Using Lemmas 1, 2 and (3.23), we get that

$$L^\varepsilon u^\varepsilon_{app,N}=R^\varepsilon_N+\nabla p_\varepsilon+\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N). \tag{3.26}$$

Using energy estimates, we get that

$$\begin{aligned}\|u^\varepsilon_{app,N}(t)\|^2_{L^2}+2\nu\int_0^t\|\nabla^h u^\varepsilon_{app,N}(t')\|^2_{L^2}dt'+2\beta\varepsilon\int_0^t\|\partial_3 u^\varepsilon_{app,N}(t')\|^2_{L^2}dt'\\ \leq\|u_0\|^2_{L^2}-\int_0^t(\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt'+\rho^\varepsilon_N\end{aligned}$$

where, as in all that follows, ρ^ε_N denotes generically a scalar quantity such that

$$\forall\eta\,,\ \exists N_0\,,\ \forall N>N_0\,,\ \exists\varepsilon_0\ /\ \forall\varepsilon<\varepsilon_0\,,\ \rho^\varepsilon_N<\eta.$$

In fact, ρ^ε_N will be the sum of a series which tends to 0 uniformly in ε when N goes to ∞ and of terms bounded by $C_N\varepsilon^\alpha$ for some positive α.

Thanks to Inequality (2.13) of Lemma 2 we have

$$\|u^\varepsilon_{app,N}-\mathcal{L}\left(\frac{t}{\varepsilon}\right)v_N\|_{L^2(\mathbf{R}^+;H^{1,0})}\leq C_N\varepsilon^{\frac12}\|u_0\|_{L^2}.$$

So, as $\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)$ belongs to $L^2(\mathbf{R}^+\times\Omega)$, we have

$$\begin{aligned}\int_0^t(\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt'=\int_0^t(\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|\overline{u}_N(t'))_{L^2}dt'\\ +\int_0^t\Big(\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|\mathcal{L}\Big(\frac{t'}{\varepsilon}\Big)v_N(t')\Big)_{L^2}dt'+\rho^\varepsilon_N\end{aligned}$$

But $\overline{u}_N$ (and thus $\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)$) is a horizontal vector field, and since $\mathrm{div}_h\,\overline{u}_N=0$, we have

$$\int_0^t (\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|\overline{u}_N(t'))_{L^2}dt'=0.$$

Moreover, by definition of v_N and $\mathcal{L}(\tau)$, we have

$$\begin{aligned}&(\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t)|\mathcal{L}\Big(\frac{t}{\varepsilon}\Big)v_N(t'))_{L^2}\\&\qquad\sum_{k=1}^{N}\int_\Omega \mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t)\cdot\mathcal{F}^{-1}A(\xi_h)\mathcal{L}_k\Big(\frac{t}{\varepsilon}\Big)A^{-1}(\xi_h)\widehat{v}^{k,h}(\xi_h)(t')\cos(k\pi x_3)dx_hdx_3.\end{aligned}$$

But as $\displaystyle\int_0^1\cos(k\pi x_3)dx_3=0$, if $k\neq 0$, we have

$$\int_0^t (\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|\mathcal{L}\Big(\frac{t'}{\varepsilon}\Big)v_N(t'))_{L^2}dt'=0,$$

so finally

$$\int_0^t (\mathrm{div}_h(\overline{u}_N\otimes\overline{u}_N)(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt'=\rho^\varepsilon_N.$$

As an immediate consequence, it turns out that

$$\|u^\varepsilon_{app,N}(t)\|^2_{L^2}+2\nu\int_0^t\|\nabla^h u^\varepsilon_{app,N}(t')\|^2_{L^2}dt'+2\beta\varepsilon\int_0^t\|\partial_3 u^\varepsilon_{app,N}(t')\|^2_{L^2}dt'\leq\|u_0\|^2_{L^2}+\rho^\varepsilon_N. \tag{3.27}$$

Equation (3.26) can be rewritten as

$$L^\varepsilon u^\varepsilon_{app,N}+u^\varepsilon_{app,N}\cdot\nabla u^\varepsilon_{app,N}=R^\varepsilon_N+\nabla p_\varepsilon+F^\varepsilon_N \tag{3.28}$$

with

$$F^\varepsilon_N\stackrel{\mathrm{def}}{=}u^\varepsilon_{app,N}\cdot\nabla u^\varepsilon_{app,N}-\overline{u}_N\cdot\nabla\overline{u}_N.$$

Now, we use the classical method to prove weak strong type estimates. We are exactly in this situation because we consider a weak solution u^ε without any additionnal regularity and a regular (approximate) solution $u^\varepsilon_{app,N}$. Let us denote by δ^ε the difference $u^\varepsilon-u^\varepsilon_{app,N}$, we write that

$$\begin{aligned}E^\varepsilon(t)\;&\stackrel{\mathrm{def}}{=}\;\|\delta^\varepsilon(t)\|^2_{L^2}+2\nu\int_0^t\|\nabla^h\delta^\varepsilon(t')\|^2_{L^2}dt'+2\beta\varepsilon\int_0^t\|\partial_3\delta^\varepsilon(t')\|^2_{L^2}dt'\\&=\;\|u^\varepsilon(t)\|^2_{L^2}+2\nu\int_0^t\|\nabla^h u^\varepsilon(t')\|^2_{L^2}dt'+2\beta\varepsilon\int_0^t\|\partial_3 u^\varepsilon(t')\|^2_{L^2}dt'\\&\quad+\|u^\varepsilon_{app,N}(t)\|^2_{L^2}+2\nu\int_0^t\|\nabla^h u^\varepsilon_{app,N}(t')\|^2_{L^2}dt'+2\beta\varepsilon\int_0^t\|\partial_3 u^\varepsilon_{app,N}(t')\|^2_{L^2}dt'\\&\qquad-2(u^\varepsilon(t)|u^\varepsilon_{app,N}(t))_{L^2}-4\nu\int_0^t(\nabla^h u^\varepsilon(t')|\nabla^h u^\varepsilon_{app,N}(t'))_{L^2}dt'\\&\qquad\qquad-4\beta\varepsilon\int_0^t(\partial_3 u^\varepsilon(t')|\partial_3 u^\varepsilon_{app,N}(t'))_{L^2}dt'.\end{aligned}$$

As u^ε is a Leray solution of (RF_ε), it satisfies the energy inequality. So thanks to Inequality (3.27) we get

$$\begin{aligned}E^\varepsilon(t) \leq 2\|u_0\|_{L^2}^2 - 2(u^\varepsilon|u^\varepsilon_{app,N})_{L^2} - 4\nu\int_0^t(\nabla^h u^\varepsilon(t')|\nabla^h u^\varepsilon_{app,N}(t'))_{L^2}dt' \\ - 4\beta\varepsilon\int_0^t(\partial_3 u^\varepsilon(t')|\partial_3 u^\varepsilon_{app,N}(t'))_{L^2}dt' + \rho^\varepsilon_N.\end{aligned}$$

As $u^\varepsilon_{app,N}$ is a smooth function such that $u^\varepsilon_{app,N|\partial\Omega} = \partial_t u^\varepsilon_{app,N|\partial\Omega} = 0$, the function

$$t \mapsto (u^\varepsilon(t)|u^\varepsilon_{app,N}(t))_{L^2}$$

is C^1, so we can write, using (3.28) and the fact that $a\times e_3\cdot b + b\times e_3\cdot a = 0$,

$$\begin{aligned}\frac{d}{dt}(u^\varepsilon|u^\varepsilon_{app,N})_{L^2} = \Big(\nu\Delta_h u^\varepsilon + \beta\varepsilon\partial_3^2 u^\varepsilon - u^\varepsilon\cdot\nabla u^\varepsilon - \nabla p^\varepsilon\Big|u^\varepsilon_{app,N}\Big)_{L^2} \\ + \Big(\nu\Delta_h u^\varepsilon_{app,N} + \beta\varepsilon\partial_3^2 u^\varepsilon_{app,N} - u^\varepsilon_{app,N}\cdot\nabla u^\varepsilon_{app,N} - \nabla p^\varepsilon_{app} + F^\varepsilon_N\Big|u^\varepsilon\Big)_{L^2}.\end{aligned}$$

By an integration by parts allowed by the fact that u^ε and $u^\varepsilon_{app,N}$ vanish at the boundary, we have

$$\begin{aligned}\frac{d}{dt}(u^\varepsilon|u^\varepsilon_{app,N})_{L^2} = -2\nu(\nabla^h u^\varepsilon|\nabla^h u^\varepsilon_{app,N})_{L^2} - 2\beta\varepsilon(\partial_3 u^\varepsilon|\partial_3 u^\varepsilon_{app,N})_{L^2} \\ - (u^\varepsilon\cdot\nabla u^\varepsilon|u^\varepsilon_{app,N})_{L^2} - (u^\varepsilon_{app,N}\cdot\nabla u^\varepsilon_{app,N}|u^\varepsilon)_{L^2} + (F^\varepsilon_N|u^\varepsilon)_{L^2}.\end{aligned}$$

The fact that $u^\varepsilon_{app,N}$ is a smooth function and that u^ε and $u^\varepsilon_{app,N}$ are divergence free vector fields that vanish on the boundary imply that

$$(u^\varepsilon\cdot\nabla u^\varepsilon|u^\varepsilon_{app,N})_{L^2} + (u^\varepsilon_{app,N}\cdot\nabla u^\varepsilon_{app,N}|u^\varepsilon)_{L^2} = (\delta^\varepsilon\cdot\nabla\delta^\varepsilon|u^\varepsilon_{app,N})_{L^2}.$$

Using again the fact that $u^\varepsilon_{app,N}$ is a smooth function and that u^ε and $u^\varepsilon_{app,N}$ are divergence free vector fields that vanish on the boundary, we infer that

$$\begin{aligned}\frac{d}{dt}(u^\varepsilon|u^\varepsilon_{app,N})_{L^2} = -2\nu(\nabla^h u^\varepsilon|\nabla^h u^\varepsilon_{app,N})_{L^2} - 2\beta\varepsilon(\partial_3 u^\varepsilon|\partial_3 u^\varepsilon_{app,N})_{L^2} - (\delta^\varepsilon\cdot\nabla\delta^\varepsilon|u^\varepsilon_{app,N})_{L^2} \\ + (F^\varepsilon_N|u^\varepsilon)_{L^2}.\end{aligned}$$

By integration and using Lemmas 1 and 2 we get that

$$\begin{aligned}(u^\varepsilon|u^\varepsilon_{app,N})_{L^2}(t) &= \|u_0\|_{L^2}^2 + \rho^\varepsilon_N \\ &- 2\nu\int_0^t(\nabla^h u^\varepsilon(t')|\nabla^h u^\varepsilon_{app,N}(t'))_{L^2}dt' - 2\beta\varepsilon\int_0^t(\partial_3 u^\varepsilon(t')|\partial_3 u^\varepsilon_{app,N}(t'))_{L^2}dt' \\ &- \int_0^t(\delta^\varepsilon(t')\cdot\nabla\delta^\varepsilon(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt' + \int_0^t(F^\varepsilon_N(t')|u^\varepsilon(t'))_{L^2}dt'.\end{aligned}$$

So we infer that

$$E^\varepsilon(t) \leq 2\int_0^t(\delta^\varepsilon(t')\cdot\nabla\delta^\varepsilon(t')|u^\varepsilon_{app,N}(t'))_{L^2}dt' + 2\int_0^t(F^\varepsilon_N(t')|u^\varepsilon(t'))_{L^2}dt' + \rho^\varepsilon_N. \qquad (3.29)$$

Now let us state the lemmas which will enable us to conclude the proof of Theorem 2.

Lemma 4 *Let u (resp. v) be any vector field (resp. divergence free vector field) in $H^2(\Omega)$ (resp. $H^1_0(\Omega)$). If we define*

$$N(u)^2 \stackrel{\text{def}}{=} \sup_{x_3\in[0,1]} \|\nabla^h u(\cdot,x_3)\|^2_{L^2(\mathbf{R}^2)} + \int_0^1 d(x_3)\|\partial_3 u(\cdot,x_3)\|^2_{L^4(\mathbf{R}^2)}dx_3,$$

then we have

$$(v\cdot\nabla v|u)_{L^2} \leq \frac{\nu}{4}\|\nabla^h v\|^2_{L^2} + \frac{C}{\nu}N(u)^2\|v\|^2_{L^2}.$$

In the following lemma, and subsequently, we note $O_N(\varepsilon^{\frac{1}{2}})$ a quantity of the order of magnitude $\varepsilon^{\frac{1}{2}}$, depending on N.

Lemma 5 *We have*

$$\int_0^t (F^\varepsilon_N(t')|u^\varepsilon(t'))_{L^2}dt' = O_N(\varepsilon^{\frac{1}{2}}).$$

The proof of this lemma uses in a crucial way dispersive estimates which are summarized in the following lemma.

Lemma 6 *Let $p\in[1,+\infty]$ be given, and let us denote by*

$$u^\varepsilon_{0,int,N} \stackrel{\text{def}}{=} \mathcal{L}\left(\frac{t}{\varepsilon}\right)u_N \tag{3.30}$$

where u_N is the solution of

$$\partial_t u_N - \nu\Delta_h u_N + \mathcal{E}u_N = 0$$

with the initial data $u_{0,N}$. Then we have

$$\|u^\varepsilon_{0,int,N} - \overline{u}_N\|_{L^p(\mathbf{R}^+;L^\infty(\Omega))} \leq C_N\varepsilon^{\frac{1}{2}-\frac{1}{4p}}\|u_0\|_{L^2}.$$

The following corollary is the result of an easy interpolation.

Corollary 7 *For any $p\in[1,+\infty]$, any $\alpha>0$ and any $q\in]2,+\infty]$, we have*

$$\|\partial^\alpha(u^\varepsilon_{0,int,N} - \overline{u}_N)\|_{L^p(\mathbf{R}^+;L^q(\Omega))} = \rho^\varepsilon_N.$$

Let us postpone the proof of those lemmas to the next paragraph and to Section 4. They imply that

$$E^\varepsilon(t) \leq \frac{\nu}{2}\int_0^t \|\nabla^h\delta^\varepsilon(t')\|^2_{L^2}dt' + \frac{C}{\nu}\int_0^t N(u^\varepsilon_{app,N}(t'))^2\|\delta^\varepsilon(t')\|^2_{L^2}dt' + \rho^\varepsilon_N.$$

But Lemmas 1 and 2 imply in particular that there is a constant C_N such that for any ε,

$$\int_0^\infty N(u^\varepsilon_{app,N}(t))^2dt \leq C_N\|u_0\|^2_{L^2}.$$

So a Gronwall lemma implies that

$$\sup_{t\geq 0}\|u^\varepsilon(t) - u^\varepsilon_{app,N}(t)\|^2_{L^2} + 2\nu\int_0^\infty \|\nabla^h(u^\varepsilon(t') - u^\varepsilon_{app,N}(t'))\|^2_{L^2}dt' = \rho^\varepsilon_N. \tag{3.31}$$

Now we can conclude the proof of Theorem 2. From (2.13) we infer that

$$\sup_{t\geq 0}\|u^\varepsilon(t)-u^\varepsilon_{0,int,N}(t)\|^2_{L^2}+2\nu\int_0^\infty\|\nabla^h(u^\varepsilon-u^\varepsilon_{0,int,N})(t')\|dt'=\rho^\varepsilon_N.$$

Thanks to the above corollary, we infer that for any compact subset K of Ω, we have

$$\sup_{t\geq 0}\|u^\varepsilon(t)-\overline{u}_N(t)\|^2_{L^2}+2\nu\int_0^\infty\|\nabla^h(u^\varepsilon-\overline{u}_N)(t')\|dt'=\rho^\varepsilon_N.$$

The result follows from the fact that $\overline{u}_N$ converges towards $\overline{u}$ in the energy space, as N goes to infinity.

Now let us prove the lemmas we have admitted above, starting with Lemma 4. We have to estimate

$$I_{j,k}\stackrel{\text{def}}{=}\int_\Omega v^j(x)v^k(x)\partial_k u^j(x)dx.$$

The case when $k\neq 3$ is easier. Let us write that

$$|I_{j,k}|\leq\int_0^1\|v(\cdot,x_3)\|^2_{L^4(\mathbf{R}^2)}\|\nabla^h u(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}dx_3.$$

Using the fact that $\|a\|^2_{L^4(\mathbf{R}^2)}\leq\|a\|_{L^2(\mathbf{R}^2)}\|\nabla^h a\|_{L^2(\mathbf{R}^2)}$, we have by definition of $N(u)$,

$$\begin{aligned}|I_{j,k}| &\leq \int_0^1\|v(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}\|\nabla^h v(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}\|\nabla^h u(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}dx_3\\ &\leq N(u)\int_0^1\|v(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}\|\nabla^h v(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}dx_3\\ &\leq N(u)\|v\|_{L^2}\|\nabla^h v\|_{L^2}.\end{aligned}$$

So Lemma 4 is proved for $I_{j,k}$ which $k\neq 3$. When $k=3$, we shall use the following lemma.

Lemma 8 *Let v be a divergence free vector field in $H^1_0(\Omega)$. Then we have for almost every $x_h\in\mathbf{R}^2$*

$$\sup_{x_3\in]0,1[}\frac{|v^3(x_h,x_3)|}{d(x_3)^{\frac12}}\leq\|\operatorname{div}_h v^h(x_h,\cdot)\|_{L^2(]0,1[)}.$$

Let us prove Lemma 8. As v is a smooth divergence free vector field whose value on the boundary is 0, we have

$$\begin{aligned}v^3(x_h,x_3) &= \int_0^{x_3}\partial_3 v^3(x_h,y_3)dy_3\\ &= -\int_0^{x_3}\operatorname{div}_h v^h(x_h,y_3)dy_3.\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$|v^3(x_h,x_3)|^2\leq x_3\int_0^1|\operatorname{div}_h v^h(x_h,y_3)|^2dy_3.$$

As the same is true for the upper boundary, Lemma 8 is proved.

Lemma 8 implies that

$$\forall x_3 \in]0,1[\,,\ \|v(\cdot,x_3)\|_{L^2(\mathbf{R}^2)} \leq d(x_3)^{\frac{1}{2}} \|\operatorname{div}_h v^h\|_{L^2}.$$

Now let us write that

$$\begin{aligned}
|I_{j,3}| &\leq \int_0^1 \|v(\cdot,x_3)\|_{L^4(\mathbf{R}^2)} \|v^3(\cdot,x_3)\|_{L^2(\mathbf{R}^2)} \|\partial_3 u(\cdot,x_3)\|_{L^4(\mathbf{R}^2)} dx_3 \\
&\leq \|\operatorname{div}_h v^h\|_{L^2} \int_0^1 \|v(\cdot,x_3)\|_{L^4(\mathbf{R}^2)} d(x_3)^{\frac{1}{2}} \|\partial_3 u(\cdot,x_3)\|_{L^4(\mathbf{R}^2)} dx_3.
\end{aligned}$$

Using Cauchy-Schwarz and Sobolev inequalities, that inequality implies that

$$\begin{aligned}
|I_{j,3}| &\leq \|\operatorname{div}_h v^h\|_{L^2} \left(\int_0^1 \|v(\cdot,x_3)\|^2_{L^4(\mathbf{R}^2)} dx_3\right)^{\frac{1}{2}} \left(\int_0^1 d(x_3) \|\partial_3 u(\cdot,x_3)\|^2_{L^4(\mathbf{R}^2)} dx_3\right)^{\frac{1}{2}} \\
&\leq \|\nabla^h v\|^{\frac{3}{2}}_{L^2} \|v\|^{\frac{1}{2}}_{L^2} N(u).
\end{aligned}$$

Using a log-concavity inequality, Lemma 4 is proved.

Now let us prove Lemma 5. To begin with let us decompose F^ε_N as

$$\begin{aligned}
F^\varepsilon_N &= F^\varepsilon_{N,1} + F^\varepsilon_{N,2} \quad \text{with} \\
F^\varepsilon_{N,1} &\stackrel{\text{def}}{=} u^\varepsilon_{app,N} \cdot \nabla u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N} \cdot \nabla u^\varepsilon_{0,int,N} \quad \text{and} \\
F^\varepsilon_{N,2} &\stackrel{\text{def}}{=} u^\varepsilon_{0,int,N} \cdot \nabla u^\varepsilon_{0,int,N} - \overline{u}_N \cdot \nabla \overline{u}_N.
\end{aligned}$$

These two terms will be estimated in two different ways. The estimate on $F^\varepsilon_{N,1}$ does not use any dispersion estimates but the one on $F^\varepsilon_{N,2}$ does indeed.

To estimate $F^\varepsilon_{N,1}$, let us write

$$\begin{aligned}
F^\varepsilon_{N,1} &= F^{\varepsilon,h}_{N,11} + F^{\varepsilon,h}_{N,12} + F^{\varepsilon,3}_{N,11} + F^{\varepsilon,3}_{N,12} \quad \text{with} \\
F^{\varepsilon,h}_{N,11} &\stackrel{\text{def}}{=} (u^{\varepsilon,h}_{app,N} - u^{\varepsilon,h}_{0,int,N}) \cdot \nabla^h u^\varepsilon_{app,N} \\
F^{\varepsilon,h}_{N,12} &\stackrel{\text{def}}{=} u^{\varepsilon,h}_{0,int,N} \cdot \nabla^h (u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N}) \\
F^{\varepsilon,3}_{N,11} &\stackrel{\text{def}}{=} (u^{\varepsilon,3}_{app,N} - u^{\varepsilon,3}_{0,int,N}) \partial_3 u^\varepsilon_{app,N} \quad \text{and} \\
F^{\varepsilon,3}_{N,12} &\stackrel{\text{def}}{=} u^{\varepsilon,3}_{0,int,N} \partial_3 (u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N}).
\end{aligned}$$

As usual, terms which do not contain vertical derivatives are easier to deal with. Using Cauchy-Schwarz and Sobolev inequalities, we get

$$\begin{aligned}
\left|(F^{\varepsilon,h}_{N,11} | u^\varepsilon)_{L^2}\right| &\leq \sup_{x_3 \in]0,1[} \|\nabla^h u^\varepsilon_{app,N}\|_{L^2(\mathbf{R}^2)} \\
&\qquad \times \int_0^1 \|(u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N})(\cdot,x_3)\|_{L^4(\mathbf{R}^2)} \|u^\varepsilon(\cdot,x_3)\|_{L^4(\mathbf{R}^2)} dx_3 \\
&\leq N(u^\varepsilon_{app,N}) \int_0^1 \|\nabla^h (u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N})(\cdot,x_3)\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)} \|u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N}(\cdot,x_3)\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)} \\
&\qquad \times \|\nabla^h u^\varepsilon(\cdot,x_3)\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)} \|u^\varepsilon(\cdot,x_3)\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)} dx_3.
\end{aligned}$$

Using estimates (2.13) and (2.14) of Lemma 2, we have that

$$\int_0^\infty |(F^{\varepsilon,h}_{N,11}(t)|u^\varepsilon(t))_{L^2}|dt \leq C_N \varepsilon^{\frac{1}{2}} \|u_0\|^2_{L^2}.$$

Note that $E(T)$ is bounded uniformly in time. The estimate about $F^{\varepsilon,h}_{N,12}$ will be proved following the same lines. We have

$$\begin{aligned}
|(F^{\varepsilon,h}_{N,12}|u^\varepsilon)_{L^2}| &\leq \sup_{x_3\in]0,1[} \|u^\varepsilon_{0,int,N}(\cdot,x_3)\|_{L^4(\mathbf{R}^2)} \\
&\qquad \times \int_0^1 \|\nabla^h(u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N})(\cdot,x_3)\|_{L^2(\mathbf{R}^2)} \|u^\varepsilon\|_{L^4(\mathbf{R}^2)} dx_3 \\
&\leq C_N \|u_0\|_{L^2} N(u^\varepsilon_{0,int,N})^{\frac{1}{2}} \|\nabla^h(u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N})\|_{L^2} \|\nabla^h u^\varepsilon\|^{\frac{1}{2}}_{L^2}.
\end{aligned}$$

So it turns out that

$$\int_0^\infty |(F^{\varepsilon,h}_{N,12}(t)|u^\varepsilon(t))_{L^2}|dt \leq C_N \varepsilon^{\frac{1}{2}} \|u_0\|^2_{L^2}.$$

As it is now usual, terms which contain vertical derivatives are estimated in another way. Using Lemma 8, we have

$$\|(u^{\varepsilon,3}_{app,N} - u^{\varepsilon,3}_{0,int,N})(\cdot,x_3)\|_{L^2(\mathbf{R}^2)} \leq d(x_3)^{\frac{1}{2}} \|\nabla^h(u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N})\|_{L^2}.$$

So we have

$$\begin{aligned}
|(F^{\varepsilon,3}_{N,11}|u^\varepsilon)_{L^2}| &\leq \|\nabla^h(u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N})\|_{L^2} \\
&\qquad \times \int_0^1 d(x_3)^{\frac{1}{2}} \|\partial_3 u^\varepsilon_{app,N}(\cdot,x_3)\|_{L^\infty(\mathbf{R}^2)} \|u^\varepsilon(\cdot,x_3)\|_{L^2(\mathbf{R}^2)} dx_3.
\end{aligned}$$

By Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
|(F^{\varepsilon,3}_{N,11}|u^\varepsilon)_{L^2}| &\leq \|\nabla^h(u^\varepsilon_{app,N} - u^\varepsilon_{0,int,N})\|_{L^2} \\
&\qquad \times \left(\int_0^1 d(x_3) \|\partial u^\varepsilon_{app,N}(\cdot,x_3)\|_{L^\infty(\mathbf{R}^2)} dx_3\right)^{\frac{1}{2}} \|u^\varepsilon\|_{L^2}.
\end{aligned}$$

So applying Inequalities (2.13) and (2.15) of Lemma 2 we get that

$$\int_0^\infty |(F^{\varepsilon,3}_{N,11}(t)|u^\varepsilon(t))_{L^2}|dt \leq C_N \varepsilon^{\frac{1}{2}} \|u_0\|^2_{L^2}.$$

To estimate the term $F^{\varepsilon,3}_{N,12}$, let us observe that

$$u^{\varepsilon,3}_{0,int,N}(x_h,x_3) = -\sum_{k=1}^N \frac{1}{k\pi} \operatorname{div}_h u^{\varepsilon,k,h}_{0,int,N}(x_h) \sin(k\pi x_3).$$

So it turns out that

$$\|u^{\varepsilon,3}_{0,int,N}(\cdot,x_3)\|_{L^2(\mathbf{R}^2)} \leq C_N d(x_3) \|\nabla^h u^\varepsilon_{0,int,N}\|_{L^2}.$$

So we find

$$
\begin{aligned}
|(F^{\varepsilon,3}_{N,12}|u^\varepsilon)_{L^2}| \leq \|\nabla^h u^\varepsilon_{0,int,N}\|_{L^2}& \\
\times \int_0^1 d(x_3)\|\partial_3(u^\varepsilon_{app,N}-u^\varepsilon_{0,int,N})(\cdot,x_3)\|_{L^\infty(\mathbf{R}^2)}&\|u^\varepsilon(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}dx_3.
\end{aligned}
$$

So by Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|(F^{\varepsilon,3}_{N,12}|u^\varepsilon)_{L^2}| \leq \|\nabla^h u^\varepsilon_{0,int,N}\|_{L^2}\|u^\varepsilon\|_{L^2}& \\
\times \int_0^1 d(x_3)^2\|\partial_3(u^\varepsilon_{app,N}-u^\varepsilon_{0,int,N})(\cdot,x_3)\|^2_{L^\infty(\mathbf{R}^2)}&dx_3.
\end{aligned}
$$

The estimate (2.16) allows to conclude that

$$
\int_0^\infty |(F^{\varepsilon,3}_{N,12}(t)|u^\varepsilon(t))_{L^2}|dt \leq C_N\varepsilon^{\frac{1}{2}}\|u_0\|^2_{L^2}.
$$

So we have that

$$
\int_0^\infty |(F^{\varepsilon}_{N,1}(t)|u^\varepsilon(t))_{L^2}|dt \leq C_N\varepsilon^{\frac{1}{2}}\|u_0\|^2_{L^2}.
$$

Let us notice that no dispersion effects have been used to prove this estimate. But we shall use them to estimate

$$
\int_0^\infty |(F^{\varepsilon}_{N,2}(t)|u^\varepsilon(t))_{L^2}|dt.
$$

As $\overline{u}^3_N=0$, let us write that

$$
\begin{aligned}
F^\varepsilon_{N,2} &= F^\varepsilon_{N,21}+F^\varepsilon_{N,22} \quad \text{with}\\
F^\varepsilon_{N,21} &\stackrel{\text{def}}{=} \sum_{j=1}^2 \partial_j(u^{\varepsilon,j}_{0,int,N}u^\varepsilon_{0,int,N}-\overline{u}^j_N\overline{u}_N) \quad \text{and}\\
F^\varepsilon_{N,22} &\stackrel{\text{def}}{=} \sum_{j=1}^2 \partial_3((u^{\varepsilon,3}_{0,int,N}-\overline{u}^3_N)u^\varepsilon_{0,int,N}).
\end{aligned}
$$

As $u^\varepsilon_{0,int,N}$ is a (finite) sum of products of a function of the horizontal variable those Fourier transform is included in the ball of center 0 and radius N by a function $\cos(k\pi x_3)$ or $\sin(k\pi x_3)$ we have, for $j\in\{1,2\}$,

$$
\|F^\varepsilon_{N,2j}(t)\|_{L^2}\leq C_N\|(u^\varepsilon_{0,int,N}-\overline{u}_N)(t)\|_{L^\infty}(\|u^\varepsilon_{0,int,N}(t)\|_{L^2}+\|\overline{u}_N(t)\|_{L^\infty}).
$$

So using Lemma 6 and of course the conservation of energy for (NSE) and the fact that $\mathcal{L}(\tau)$ is uniformly bounded on L^2, we can write that

$$
\begin{aligned}
\int_0^\infty |(F^{\varepsilon}_{N,2}(t)|u^\varepsilon(t))_{L^2}|dt &\leq C_N\|u^\varepsilon_{0,int,N}-\overline{u}_N\|_{L^1(\mathbf{R}^+;L^\infty)}\|u_0\|^2_{L^2}\\
&\leq C_N\varepsilon^{\frac{1}{4}}\|u_0\|^2_{L^2}.
\end{aligned}
$$

Lemma 5 is now proved and thus Theorem 2 also.

4 Proof of the Strichartz type inequality

In this final section, we shall prove Lemma 6, stated in the previous section. This type of estimate in the case of rotating fluids has been investigated in [3] in various situations. For the convenience of the reader, we give here a self contained proof.

Thanks to the definition of $\mathcal{L}$ (see (2.10)) we have that

$$\begin{aligned} u^{\varepsilon}_{0,int,N}-\overline{u}_N &= \sum_{k=1}^{N}\Big(A_k\star\mathcal{I}^{\varepsilon}_k(t,x_h)\cos(k\pi x_3),-\frac{1}{k\pi}\,\mathrm{div}_h\,A_{k,N}\star\mathcal{I}^{\varepsilon}_k(t,x_h)\sin(k\pi x_3)\Big)\quad\text{with}\\ \mathcal{I}^{\varepsilon}_k(t,x_h) &\stackrel{\text{def}}{=} \int_{\mathbf{R}^2} e^{i(x_h|\xi_h)+i\frac{k\pi}{((k\pi)^2+|\xi_h|^2)^{1/2}}\frac{t}{\varepsilon}-\nu t|\xi_h|^2}\,\widehat{\gamma}_k(\xi_h)d\xi_h\end{aligned}$$

where A_k is a L^1 function on $\mathbf{R}^2$ and γ_k is a linear function of $\widehat{d}_k$ and $\widehat{\omega}_k$. Now let φ_N be a radial function in $\mathcal{D}(\mathbf{R}^2\setminus\{0\})$ whose value is 1 near $\mathcal{C}_N$. We have

$$\begin{aligned}\mathcal{I}^{\varepsilon}_k &= \mathcal{K}_k\left(\frac{t}{\varepsilon},t,\cdot\right)\star\gamma_k\quad\text{with}\\ \mathcal{K}_k(\tau,t,x_h) &\stackrel{\text{def}}{=} \int_{\mathbf{R}^2} e^{i(x_h|\xi_h)+i\frac{k\pi}{((k\pi)^2+|\xi_h|^2)^{1/2}}\tau-\nu t|\xi_h|^2}\varphi_N(\xi_h)d\xi_h.\end{aligned}$$

As usual in the proof of Strichartz type estimates, we start by proving a so called dispersive estimate which here is the following.

Lemma 9 *There exist constants C_N and c_N such that for any $k\leq N$, we have*

$$\|\mathcal{K}_k(\tau,t,\cdot)\|_{L^\infty(\mathbf{R}^2)}\leq\frac{C_N}{\tau^{\frac{1}{2}}}e^{-c_N t}.$$

Let us prove Lemma 9. Using the rotation invariance on $\mathbf{R}^2$, we may assume that $x_h=(x_1,0)$. So we have to estimate

$$\mathcal{K}_k(\tau,t,x_1,0)=\int_{\mathbf{R}} e^{ix_1\xi_1}\int_{\mathbf{R}} e^{i\frac{k\pi}{((k\pi)^2+\xi_1^2+\xi_2^2)^{1/2}}\tau-\nu t(\xi_1^2+\xi_2^2)}\varphi_N(\xi_1,\xi_2)d\xi_2d\xi_1.$$

Now let us define the function α_k by

$$\alpha_k(\xi_2)\stackrel{\text{def}}{=}\partial_{\xi_2}\frac{k\pi}{((k\pi)^2+\xi_1^2+\xi_2^2)^{1/2}}=\frac{\xi_2 k\pi}{((k\pi)^2+\xi_1^2+\xi_2^2)^{3/2}}.$$

We notice that

$$|\alpha_k(\xi_2)|\geq C_N|\xi_2|. \tag{4.32}$$

Similarly we define the vector field X_k by

$$X_k a\stackrel{\text{def}}{=}\frac{1}{1+\tau\alpha_k^2(\xi_2)}(1+i\alpha_k(\xi_2)\partial_{\xi_2}a).$$

This vector field acts only on the ξ_2 variable and satisfies obviously

$$X_k e^{i\tau\frac{k\pi}{((k\pi)^2+\xi_1^2+\xi_2^2)^{1/2}}}=e^{i\tau\frac{k\pi}{((k\pi)^2+\xi_1^2+\xi_2^2)^{1/2}}}.$$

An integration by parts allows us to obtain

$$\mathcal{K}_k(\tau,t,x_1,0)=\int_{\mathbf{R}} e^{ix_1\xi_1}\int_{\mathbf{R}} e^{i\frac{k\pi}{((k\pi)^2+\xi_1^2+\xi_2^2)^{1/2}}\tau}\,({}^tX_k)\left(\varphi_N(\xi_1,\xi_2)e^{-\nu t(\xi_1^2+\xi_2^2)}\right)d\xi_2 d\xi_1.$$

But easy computations yield

$$\begin{aligned}{}^tX_k\left(\varphi_N(\xi_1,\xi_2)e^{-\nu t(\xi_1^2+\xi_2^2)}\right)=&\left(\frac{1}{1+\tau\alpha_k^2(\xi_2)}-i(\partial_{\xi_2}\alpha_k)\frac{1-\tau\alpha_k^2(\xi_2)}{(1+\tau\alpha_k^2(\xi_2))^2}\right)\\&-\frac{i\alpha_k(\xi_2)}{1+\tau\alpha_k^2(\xi_2)}\partial_{\xi_2}\left(\varphi_N(\xi_1,\xi_2)e^{-\nu t(\xi_1^2+\xi_2^2)}\right).\end{aligned}$$

Using (4.32), it turns out that

$$\left|{}^tX_k\left(\varphi_N(\xi_1,\xi_2)e^{-\nu t(\xi_1^2+\xi_2^2)}\right)\right|\le C_N\frac{e^{-c_Nt}}{1+\tau|\xi_2|^2},$$

so going back to $\mathcal{K}_k$, we get

$$\begin{aligned}|\mathcal{K}_k(\tau,t,x_h)| &\le C_Ne^{-c_Nt}\int_{\mathcal{C}_N}\frac{d\xi_1d\xi_2}{1+\tau|\xi_2|^2}\\ &\le \frac{C_N}{\tau^{1/2}}e^{-c_Nt},\end{aligned}$$

which proves Lemma 9.

Now let us finish the proof of Lemma 6. As an immediate corollary of the result above, we have

$$\|\mathcal{K}_k\left(\frac{t}{\varepsilon},t,\cdot\right)\|_{L^\infty(\mathbf{R}^2)}\le\frac{C_N\varepsilon^{1/2}}{t^{1/2}}e^{-c_Nt}\|\gamma_k\|_{L^1(\mathbf{R}^2)}. \tag{4.33}$$

Now we shall use a duality argument, otherwise known as the TT^* argument. Once we have observed that

$$\|a\|_{L^1(\mathbf{R}^+;L^\infty(\mathbf{R}^2))}=\sup_{\varphi\in\mathcal{B}}\int_{\mathbf{R}^+\times\mathbf{R}^2}a(t,x_h)\varphi(t,x_h)\,dx_hdt$$

with $\mathcal{B}\stackrel{\text{def}}{=}\{\varphi\in\mathcal{D}(\mathbf{R}^+\times\mathbf{R}^2),\ \|\varphi\|_{L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^2))}\le1\}$, we can write

$$\begin{aligned}\|\mathcal{K}_k\left(\frac{t}{\varepsilon},t,\cdot\right)*\gamma_k\|_{L^1(\mathbf{R}^+;L^\infty(\mathbf{R}^2))}&=\sup_{\varphi\in\mathcal{B}}\int_{\mathbf{R}^+\times\mathbf{R}^4}\mathcal{K}_k\left(\frac{t}{\varepsilon},t,x-y\right)\gamma_k(y)\varphi(t,x)\,dxdydt\\&=\sup_{\varphi\in\mathcal{B}}\int_{\mathbf{R}^+\times\mathbf{R}^2}\gamma_k(y)\left(\int_{\mathbf{R}^2}\mathcal{K}_k\left(\frac{t}{\varepsilon},t,x-y\right)\varphi(t,x)\,dx\right)\,dydt.\end{aligned}$$

A Cauchy Schwarz inequality then yields

$$\|\mathcal{K}_k\left(\frac{t}{\varepsilon},t,\cdot\right)*\gamma_k\|_{L^1(\mathbf{R}^+;L^\infty(\mathbf{R}^2))}\le\|\gamma_k\|_{L^2(\mathbf{R}^2)}\sup_{\varphi\in\mathcal{B}}\left\|\int_{\mathbf{R}^+}\check{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,\cdot\right)*\varphi_k(t,\cdot)dt\right\|_{L^2(\mathbf{R}^2)}.$$

By Fourier–Plancherel, we have

$$\left\|\int_{\mathbf{R}^+}\check{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,\cdot\right)*\varphi(t,\cdot)dt\right\|_{L^2(\mathbf{R}^2)}^2=(2\pi)^2\left\|\int_{\mathbf{R}^+}\widehat{\check{\mathcal{K}}_k}\left(\frac{t}{\varepsilon},t,\cdot\right)\widehat{\varphi}(t,\cdot)dt\right\|_{L^2(\mathbf{R}^2)}^2$$

and we can write

$$\left\|\int_{\mathbf{R}^+} \overline{\mathcal{K}_k}\left(\frac{t}{\varepsilon},t,\cdot\right) * \widehat{\varphi}(t,\cdot)dt\right\|^2_{L^2(\mathbf{R}^2)} \leq C\int_{\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^2} \widehat{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,-\xi_h\right)\widehat{\varphi}(t,\xi_h)$$
$$\times\overline{\widehat{\mathcal{K}}_k}\left(\frac{s}{\varepsilon},s,\xi_h\right)\widehat{\varphi}(s,\xi_h)d\xi_h dtds.$$

But by definition of $\mathcal{K}$, we have

$$\widehat{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,-\xi_h\right)\overline{\widehat{\mathcal{K}}_k}\left(\frac{s}{\varepsilon},s,\xi_h\right) = \widehat{\mathcal{K}}_k\left(\frac{t-s}{\varepsilon},t+s,-\xi_h\right).$$

It follows that

$$\left\|\int_{\mathbf{R}^+} \check{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,\cdot\right) * \varphi(t,\cdot)dt\right\|^2_{L^2(\mathbf{R}^2)} \leq C\int_{\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^2} \mathcal{F}\left(\check{\mathcal{K}}_k\left(\frac{t-s}{\varepsilon},t+s,\cdot\right) * \varphi(t,\cdot)\right)$$
$$\times\overline{\widehat{\varphi}}(s,\xi_h)d\xi_h dtds.$$

Now we use Fourier–Plancherel again to get

$$\left\|\int_{\mathbf{R}^+} \check{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,\cdot\right) * \varphi(t,\cdot)dt\right\|^2_{L^2(\mathbf{R}^2)} \leq C\int_{\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^2} \left(\check{\mathcal{K}}_k\left(\frac{t-s}{\varepsilon},t+s,\cdot\right) * \varphi(t,\cdot)\right)(x)$$
$$\times\varphi(s,-x)dxdtds$$
$$\leq C\int_{\mathbf{R}^+\times\mathbf{R}^+} \left\|\check{\mathcal{K}}_k\left(\frac{t-s}{\varepsilon},t+s,\cdot\right) * \varphi(t,\cdot)\right\|_{L^\infty(\mathbf{R}^2)} \|\varphi(s,\cdot)\|_{L^1(\mathbf{R}^2)}dtds.$$

Using the dispersion estimate (4.33), we get

$$\left\|\int_{\mathbf{R}^+} \check{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,\cdot\right) * \varphi(t,\cdot)dt\right\|^2_{L^2(\mathbf{R}^2)} \leq C_N\int_{\mathbf{R}^+\times\mathbf{R}^+} \frac{\varepsilon^{1/2}}{(t-s)^{1/2}}e^{-c_N(t+s)}\|\varphi(t,\cdot)\|_{L^1(\mathbf{R}^2)}$$
$$\times\|\varphi(s,\cdot)\|_{L^1(\mathbf{R}^2)}dsdt,$$

so

$$\begin{aligned}\left\|\int_{\mathbf{R}^+} \check{\mathcal{K}}_k\left(\frac{t}{\varepsilon},t,\cdot\right) * \varphi(t,\cdot)dt\right\|^2_{L^2(\mathbf{R}^2)} &\leq C_N\varepsilon^{1/2}\|\varphi\|^2_{L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^2))}\int_{\mathbf{R}^+\times\mathbf{R}^+}\frac{e^{-c_N(t+s)}}{(t-s)^{1/2}}dsdt\\ &\leq C_N\varepsilon^{1/2}\|\varphi\|^2_{L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^2))}.\end{aligned}$$

Lemma 6 is proved.

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