ON PRESSURELESS GASES DRIVEN BY A STRONG INHOMOGENEOUS MAGNETIC FIELD

ISABELLE GALLAGHER AND LAURE SAINT-RAYMOND

Abstract. We are interested in the life span and the asymptotic behaviour of the solutions to a system governing the motion of a pressureless gas, submitted to a strong, inhomogeneous magnetic field $\varepsilon^{-1}B(x)$, of variable amplitude but fixed direction — this is a first step in the direction of the study of rotating Euler equations. This leads to the study of a multi–dimensional Burgers type system on the velocity field u_ε, penalized by a rotating term $\varepsilon^{-1}u_\varepsilon \wedge B(x)$. We prove that the unique, smooth solution of this Burgers system exists on a uniform time interval $[0,T]$. We also prove that the phase of oscillation of u_ε is an order one perturbation of the phase obtained in the case of a pure rotation (with no nonlinear transport term), $\varepsilon^{-1}B(x)t$. Finally going back to the pressureless gas system, we obtain the asymptotics of the density as ε goes to zero.

Sur des gaz sans pression soumis à un fort champ magnétique inhomogène

Résumé. On s'intéresse au temps d'existence et au comportement asymptotique des solutions d'un système modélisant le mouvement d'un gaz sans pression, soumis à un fort champ magnétique $\varepsilon^{-1}B(x)$, d'intensité variable mais de direction fixe — cela étant un premier pas dans la compréhension des équations d'Euler en rotation rapide. Cela conduit à l'étude d'un système de type Burgers multi–dimensionnel sur le champ de vitesse u_ε, pénalisé par un terme de rotation $\varepsilon^{-1}u_\varepsilon \wedge B(x)$. On démontre que la solution unique régulière de ce système de Burgers existe sur un intervalle de temps uniforme $[0,T]$. On montre aussi que la phase d'oscillation de u_ε est une perturbation au premier ordre de la phase obtenue dans le cas d'une rotation pure (sans terme de transport non linéaire), $\varepsilon^{-1}B(x)t$. Enfin en revenant au système des gaz sans pression, on obtient le comportement asymptotique de la densité quand ε tend vers zéro.

1. Introduction

The aim of this paper is to study the asymptotic behaviour of a fluid submitted to a strong external inhomogeneous magnetic field.

The case when the field is constant has been studied by a number of authors, both for compressible and incompressible models of fluids (see for instance [1], [3] or [7] for incompressible fluids, and [4] or [6] for rarefied plasmas). In that case, one can not only derive the asymptotic average motion (which is given by the weak limit of the velocity field), but one can also describe all the oscillations in the system and possibly their coupling : the filtering techniques used for that rely on explicit computations in Fourier space.

2000 *Mathematics Subject Classification.* Primary 35B40; Secondary 76U05,76W05.
Key words and phrases. rotating pressureless gas, asymptotic behaviour, oscillations.

In the case when the magnetic field is inhomogeneous, those methods are not relevant any more. Weak compactness and compensated compactness arguments allow nevertheless to determine the average motion (see [6] in the case of a rarefied plasma governed by the Vlasov-Poisson system, and [5] in the case of a viscous incompressible fluid). In order to describe the oscillating component of the motion, one has to understand the interaction between the penalization and the nonlinear term of transport : indeed one expects that the flow modifies substantially the phase of oscillation (which is of course inhomogeneous).

We propose here to analyse this interaction for a simplified model of magneto-hydro-dynamics, the so-called Euler system of pressureless gas dynamics.

1.1. **A simple model for magneto-hydro-dynamics.** We consider the following system of partial differential equations :

$$
\begin{aligned}
&\partial_t \rho + \nabla\cdot (\rho u) = 0, \quad x \in \mathbf{R}^3, t > 0\\
&\partial_t(\rho u) + \nabla\cdot (\rho u \otimes u) = \rho u \wedge B, \quad x \in \mathbf{R}^3, t > 0\\
&\rho(t=0) \equiv \rho_0, \quad u(t=0) \equiv u_0 \quad x \in \mathbf{R}^3,
\end{aligned}
\tag{1.1}
$$

where ρ denotes the density of the fluid, u its mean velocity and B is the external magnetic field ($\nabla\cdot B = 0$). The first equation expresses the local conservation of mass, while the second one gives the local conservation of momentum provided that there is no internal force (no pressure). This assumption is relevant only in some particular regimes (corresponding to sticky particles [2]). From a physical point of view, this may seem a strong restriction, but it allows to perform a first mathematical study of that type of inhomogeneous singular perturbation problem : indeed in this special case a major simplification arises since the equation on the mean velocity can be (at least formally) decoupled from the rest of the system :

$$\partial_t u + (u \cdot \nabla) u = u \wedge B \quad x \in \mathbf{R}^3, t > 0.$$

We then obtain a system of Burgers' type, that is a prototype of hyperbolic system. A work in progress should extend the present results to more realistic models, in particular to the 3D incompressible Euler system.

In order to further simplify the analysis, we assume that the direction of the field B is constant

$$B(x) \equiv \frac{1}{\varepsilon} b(x_1, x_2) e_3, \quad (x_1, x_2) \in \mathbf{R}^2 \text{ and } e_3 = {}^t(0,0,1)$$

which allows to get rid of the geometry of the field lines (for detailed comments on this subject see for instance [5], Remark 1.4). Any solution to the system (1.1) has then uniform regularity with respect to the variable x_3. To isolate the phenomenon of inhomogeneous oscillations with instantaneous loss of regularity, we restrict therefore our attention to the 2D singular perturbation problem, in the plane orthogonal to the magnetic field. We finally have :

$$
\begin{aligned}
&\partial_t \rho + \nabla\cdot (\rho u) = 0, \quad x \in \mathbf{R}^2, t > 0\\
&\partial_t(\rho u) + \nabla\cdot (\rho u \otimes u) = \frac{b}{\varepsilon} \rho u^\perp, \quad x \in \mathbf{R}^2, t > 0\\
&\rho(t=0) \equiv \rho_0, \quad u(t=0) \equiv u_0, \quad x \in \mathbf{R}^2,
\end{aligned}
\tag{1.2}
$$

where $u^\perp$ denotes the vector field with components $(u_2, -u_1)$, and the intensity b of the magnetic field satisfies the following assumptions :

$$b \in C^\infty(\mathbf{R}^2) \cap W^{2,\infty}(\mathbf{R}^2), \tag{$H0$}$$

$$\inf_{x\in\mathbf{R}^2} b(x) = b_- > 0. \tag{$H1$}$$

A standard fixed point argument then allows to prove the local well-posedness of (1.2). The result is the following.

Theorem 1. *Consider a function b satisfying assumptions $(H0)(H1)$. Let ρ_0 be a nonnegative function and u_0 be a vector-field in $H^s(\mathbf{R}^2)$ $(s > 2)$. Then, for all $\varepsilon > 0$, there exist $T_\varepsilon \in]0, +\infty]$ and a unique solution of (1.2), $(\rho_\varepsilon, u_\varepsilon) \in L^\infty_{loc}([0, T_\varepsilon[, H^s(\mathbf{R}^2))$.*

Note that the lifespan T_ε of the solution depends on ε, and that the lower bound on T_ε coming from the Duhamel formula goes to zero as $\varepsilon \to 0$. The first difficulty to study the asymptotics $\varepsilon \to 0$ consists then in understanding why the magnetic penalization does not destabilize the system, and in proving that the solution $(\rho_\varepsilon, u_\varepsilon)$ exists on a uniform interval of time.

1.2. **Formal analysis.** Before stating more precise results on the lifespan of the solutions and on the asymptotics $\varepsilon \to 0$, we have chosen to give some simple observations about the problem to guide intuition. In this first approach we restrict our attention to the analysis of the equation governing the velocity.

The first step of the formal analysis consists in determining the mean behaviour of the velocity field, that is the weak limit of u_ε. We have

$$u_\varepsilon = \frac{\varepsilon}{b} \left(\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon\right)^\perp .$$

As b is bounded from below, if we are able to establish convenient a priori bounds on u_ε, this will imply

$$u_\varepsilon \rightharpoonup 0$$

in some weak sense. This means that we expect the velocity to oscillate at high frequency (on vanishing temporal or spatial scales).

Another way to get an idea of the asymptotic behaviour of the velocity is to study the simple case when b is constant. The group of oscillations generated by the magnetic penalization is then homogeneous :

$$R\left(\frac{t}{\varepsilon}\right) u = u \cos\left(\frac{bt}{\varepsilon}\right) - u^\perp \sin\left(\frac{bt}{\varepsilon}\right) ,$$

which corresponds to the rotation with frequency $2\pi b/\varepsilon$. As the coefficients are constant, this group is not perturbed by the transport. Classical filtering methods (see namely [7],[8]) can then be applied : setting

$$v_\varepsilon \stackrel{\text{def}}{=} R\left(-\frac{t}{\varepsilon}\right) u_\varepsilon$$

leads to

$$\partial_t v_\varepsilon + Q\left(\frac{t}{\varepsilon}, v_\varepsilon, v_\varepsilon\right) = 0$$

where $Q\left(\frac{t}{\varepsilon}, ., .\right)$ is a quadratic form with bounded coefficients depending on t/ε. As there is only one oscillation frequency, there is no resonance, which implies that

$$v_\varepsilon \to u_0$$

in some strong sense, provided that convenient a priori bounds on u_ε (and consequently on v_ε) hold. This means that we can describe completely the oscillations and get a strong convergence result. Of course, we get as a corollary that the lifespan T_ε is uniformly bounded from below, and we even expect that $T_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

The case we consider here is much more complicated. The group of oscillations generated by the magnetic penalization is again very easy to describe :

$$R\left(\frac{t}{\varepsilon}, x\right) u = u \cos\left(\frac{b(x)t}{\varepsilon}\right) - u^\perp \sin\left(\frac{b(x)t}{\varepsilon}\right),$$

but it is non homogeneous, which entails

• a loss of regularity ($R\left(\frac{t}{\varepsilon}, x\right) u$ blows up in all Sobolev norms $H^s(\mathbf{R}^2)$ for $s > 0$);

• an interaction with the transport operator (with the same definition of $v_\varepsilon = R\left(-\frac{t}{\varepsilon}, x\right) u_\varepsilon$ as previously, we do not expect $\partial_t v_\varepsilon$ to be bounded in any space of distributions).

The stake behind this model problem is to understand how to overcome these difficulties. The first step is to explain how the phase of oscillations is modified by the flow : note that even a small correction on the phase changes strongly the vector field. Then we have to establish a strong convergence result using a new method : classical energy methods fail because of the lack of regularity on approximate solutions. Here an appropriate rewriting of the system by means of characteristics associated with the flow allows to understand the underlying structure and to answer both questions : in particular we will see that the spaces which are well adapted for this type of study are constructed on $L^\infty(\mathbf{R}^2)$. In the case of incompressible dynamics, the analysis will be therefore much more difficult since the transport is replaced by a non-local pseudo-differential operator.

1.3. **Main results.** As long as the solution $(\rho_\varepsilon, u_\varepsilon)$ of system (1.2) is smooth, the velocity u_ε satisfies the equation of Burgers type

$$\begin{aligned} \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \frac{b}{\varepsilon} u_\varepsilon^\perp = 0, \quad x \in \mathbf{R}^2, t > 0 \\ u_\varepsilon(t=0) = u^0, \quad x \in \mathbf{R}^2. \end{aligned} \tag{1.3}$$

Using refined a priori estimates on this last equation, we can prove that for all $\varepsilon > 0$ it admits a smooth solution on a uniform time $T > 0$. We will prove the following result.

Theorem 2. *Consider a function b satisfying assumptions $(H0)(H1)$. Let ρ_0 be a non-negative function in $W^{s-1,\infty}(\mathbf{R}^2)$, and u_0 be a vector-field in $W^{s,\infty}(\mathbf{R}^2)$ ($s \geq 1$). Then, there exists $T^* \in]0, +\infty]$ such that, for all $T < T^*$ and all $\varepsilon \leq \varepsilon_T$, there is a unique $(\rho_\varepsilon, u_\varepsilon) \in L^\infty([0,T], W^{s-1,\infty}(\mathbf{R}^2) \times W^{s,\infty}(\mathbf{R}^2))$ solution of (1.2) (which is nevertheless not uniformly bounded in $L^\infty([0,T], W^{s-1,\infty}(\mathbf{R}^2) \times W^{s,\infty}(\mathbf{R}^2))$ for $s > 0$).*

Remark 1. *The proof of Theorem 2 shows actually that the supremum life time T_ε^* of the solution $(\rho_\varepsilon, u_\varepsilon)$ (corresponding to the first crossing of characteristics) converges to T^{**} as ε goes to 0 with*

$$T^{**} = C\|u_0\|_{L^\infty}^{-1}\|\nabla b\|_{L^\infty}^{-1}.$$

In this framework, it is relevant to consider the asymptotics $\varepsilon \to 0$ on the time interval $[0, T]$. The same type of computations as previously allows to prove that the velocity field behaves almost as in the constant case (with slight modifications of the phase of oscillations).

Theorem 3. *Consider a function b satisfying assumptions $(H0)(H1)$. Let u_0 be a vector-field in $W^{s,\infty}(\mathbf{R}^2)$ $(s \geq 1)$. For all $T \leq T^*$ as in Theorem 2 and all $\varepsilon \leq \varepsilon_T$, denote by u_ε the solution of (1.3) in $L^\infty([0, T], W^{s,\infty}(\mathbf{R}^2))$. Then,*

$$u_\varepsilon(t, x) - \Big(u_0(x)\cos\theta_\varepsilon(t, x) - u_0^\perp(x)\sin\theta_\varepsilon(t, x)\Big)$$

converges strongly to 0 in $L^\infty([0, T] \times \mathbf{R}^2)$, where the phase θ_ε is defined by the following equation

$$\begin{aligned}\theta_\varepsilon(t, x) = \frac{b(x)t}{\varepsilon} &- tu_0(x)\cdot\nabla\log b(x)\sin\theta_\varepsilon(t, x)\\ &+tu_0^\perp(x)\cdot\nabla\log b(x)\cos\theta_\varepsilon(t, x).\end{aligned} \tag{1.4}$$

Rewriting the equation on the density ρ_ε with a transport term and a penalization term (coming from the divergence of u_ε which is of order $1/\varepsilon$)

$$\begin{aligned}\partial_t\rho_\varepsilon + (u_\varepsilon\cdot\nabla)\rho_\varepsilon + \rho_\varepsilon\nabla\cdot u_\varepsilon = 0, &\quad x \in \mathbf{R}^2, t > 0\\ \rho_\varepsilon(t = 0) = \rho^0, &\quad x \in \mathbf{R}^2,\end{aligned} \tag{1.5}$$

we can then determine the global asymptotics of the Euler system of pressureless gases (1.2).

Theorem 4. *Consider a function b satisfying assumptions $(H0)(H1)$. Let ρ_0 be a nonnegative function in $W^{s-1,\infty}(\mathbf{R}^2)$, and u_0 be a vector-field in $W^{s,\infty}(\mathbf{R}^2)$ $(s \geq 1)$. For all $T \leq T^*$ as in Theorem 2 and all $\varepsilon \leq \varepsilon_T$, denote by $(\rho_\varepsilon, u_\varepsilon)$ the solution of (1.2) in $L^\infty([0, T], W^{s-1,\infty}(\mathbf{R}^2) \times W^{s,\infty}(\mathbf{R}^2))$. Then,*

$$\rho_\varepsilon(t, x) - \rho_0(x)\Big(1 + tu_0\cdot\nabla\log(x)\cos\theta_\varepsilon(t, x) - tu_0^\perp\cdot\nabla\log b(x)\sin\theta_\varepsilon(t, x)\Big)$$

converges strongly to 0 in $L^\infty([0, T] \times \mathbf{R}^2)$, where the phase θ_ε is defined as previously by (1.4).

Remark 2. *It should be noted that assumption $(H1)$ is not merely a technical artefact which could be removed with some work. In fact all the uniform estimates in this paper blow up if the field B is allowed to have a level curve where $B = 0$ — which could occur in many physical cases — and new mathematical difficulties arise in that case.*

Let us comment a little on the proof of those theorems, and give the structure of the paper.

It is quite clear that energy methods will not enable us to have a good control on the asymptotics of $(\rho_\varepsilon, u_\varepsilon)$, since as soon as we want a control on derivatives of u_ε unbounded terms will appear. So the most appropriate way to study System (1.2) is to rewrite it using the charcateristics of the flow and to study those characteristics precisely.

Section 2 is therefore devoted to rewriting System (1.2) in characteristic form, and in the derivation of a few a priori estimates.

In order to establish the existence of a solution $(\rho_\varepsilon, u_\varepsilon)$ to system (1.2) on a uniform time interval $[0,T]$, it is enough to see that the solution is well-defined (and smooth) as long as the flow generates a diffeomorphism $X_\varepsilon(t,.)$

$$\frac{dX_\varepsilon}{dt}(t,x) = u_\varepsilon(t, X_\varepsilon(t,x))$$

hence to prove that the characteristics cannot cross before time T. The precise estimates on DX_ε leading to Theorem 2 are performed in Section 3, they use in a crucial way some results of non-stationary phase type.

The asymptotic behaviour of $u_\varepsilon(t, X_\varepsilon(t,.))$ and $\rho_\varepsilon(t, X_\varepsilon(t,.))$ is then simply obtained from the explicit approximation of the characteristics X_ε, using Taylor expansions for the various fields. In order to establish the convergence results stated in Theorems 3 and 4, the main difficulty is therefore to get a precise description of the inverse characteristics $X_\varepsilon^{-1}(t,.)$, which is done in Section 4.

2. Appropriate formulation of the system

As pointed out in the introduction, energy estimates do not seem to be the right angle of attack for our problem. We shall therefore in this short section present a new formulation of System (1.2), by means of characteristics (Paragraph 2.1). In that way some a priori estimates can be deduced immediately (see Paragraph 2.2).

To simplify notation, from now on we shall drop the index ε in u_ε and simply write u (and similarly for any other ε-dependent function).

2.1. **Trajectories associated with the flow.** Let us write System (1.2) in the following form:

$$\begin{gathered}\frac{dX}{dt} = u(t,X), \quad X_{|t=0} = x\\ \frac{d}{dt}(\rho(t,X)) + \rho\nabla\cdot u(t,X) = 0, \quad \rho_{|t=0} = \rho_0\\ \frac{d}{dt}(u(t,X)) + \frac{b(X)}{\varepsilon}u^\perp(t,X) = 0, \quad u_{|t=0} = u_0.\end{gathered} \tag{2.1}$$

As seen in Theorem 1, there is a solution to System (1.2) for a time depending on ε, and as long as the trajectories do not intersect we can write in particular

$$u(t,X(t,x)) = u_0(x)\cos\left(\frac{\phi(t,x)}{\varepsilon}\right) - u_0^\perp(x)\sin\left(\frac{\phi(t,x)}{\varepsilon}\right), \tag{2.2}$$

where we have defined the functions

$$\phi(t,x) = \int_0^t \beta(s,x)\,ds, \quad \beta(t,x) \stackrel{\text{def}}{=} b(X(t,x))$$

(these functions are well defined as long as the characteristics do not cross each other).

If u is smooth enough, then ρ is uniquely defined by the transport equation it satisfies. So from now on we can concentrate on u (and X). As one of the aims of this article is to prove

Theorem 2 (which will be achieved in the next section), we shall from now on call T^ε the largest time before which no characterestic intersect; one of our goals is to prove that T^ε is uniformly bounded from below as ε goes to zero.

In the next paragraph we are going to derive from (2.1) and (2.2) some easy a priori estimates for times $0 \leq t \leq T^\varepsilon$, which will help us prove Theorem 2 in the following Section 3, and Theorems 3 and 4 in Section 4.

2.2. **A priori estimates.** Formula (2.2) immediately enables us to deduce the following a priori estimate:

$$\|u\|_{L^\infty([0,T^\varepsilon[\times \mathbf{R}^2)} \leq 2\|u_0\|_{L^\infty}\ , \tag{2.3}$$

which implies that

$$\left\|\frac{dX}{dt}\right\|_{L^\infty([0,T^\varepsilon[\times \mathbf{R}^2)} \leq 2\|u_0\|_{L^\infty}. \tag{2.4}$$

In particular $X - x$ remains bounded in space for all times $0 \leq t < T^\varepsilon$, and we have

$$\forall t \in [0, T^\varepsilon[, \quad \|X(t,\cdot) - x\|_{L^\infty(\mathbf{R}^2)} \leq 2t\|u_0\|_{L^\infty(\mathbf{R}^2)}. \tag{2.5}$$

Since $\beta(t, x) = b(X(t, x))$, we have

$$\begin{aligned} \|\partial_t\beta\|_{L^\infty([0,T^\varepsilon[\times \mathbf{R}^2)} &\leq \|\nabla b\|_{L^\infty(\mathbf{R}^2)} \left\|\frac{dX}{dt}\right\|_{L^\infty([0,T^\varepsilon[\times \mathbf{R}^2)} \\ &\leq 2\|\nabla b\|_{L^\infty(\mathbf{R}^2)}\|u_0\|_{L^\infty}, \end{aligned} \tag{2.6}$$

as well as

$$\forall t \in [0, T^\varepsilon[, \quad \forall x \in \mathbf{R}^2, \quad b_- \leq \beta(t, x) \leq \|b\|_{L^\infty(\mathbf{R}^2)} \tag{2.7}$$

with b_- defined in $(H1)$.

Now we are going to look for an approximation of X: integrating formula (2.2) in time yields

$$X(t, x) = x + u_0(x) \int_0^t \cos\left(\frac{\phi(s, x)}{\varepsilon}\right) ds - u_0^\perp(x) \int_0^t \sin\left(\frac{\phi(s, x)}{\varepsilon}\right) ds \tag{2.8}$$

recalling that $\phi(t, x) = \displaystyle\int_0^t b(X(s, x))\, ds$. The following section will be devoted to a precise study of the trajectories X, which will enable us to infer Theorem 2.

3. Study of the trajectories

Formulation (2.1) of System (1.3) shows that the study of the Euler system of pressureless gases with magnetic penalization comes down to a precise analysis of the characteristics, and in particular of their invertibility.

In this section we will establish that the trajectories defined by (2.8) are invertible on a time interval $[0, T^\varepsilon[$ with

$$\lim_{\varepsilon \to 0} T^\varepsilon = T^* > 0,$$

where T^* depends on the magnetic field b and on the initial velocity field u_0. This result is based on an asymptotic expansion of the Jacobian

$$J(t,x) \stackrel{\text{def}}{=} |\det(DX(t,x))|,$$

which implies that

$$\forall t \in [0,T^*[, \quad \liminf_{\varepsilon\to 0} J(t,x) > 0.$$

The asymptotic expansions of X and DX (Paragraphs 3.2 and 3.4) are obtained using some results of non-stationary phase type and the L^∞-bounds established in Paragraphs 3.1 and 3.3.

3.1. **Bounds on** $X(t,\cdot)$**.** The first step of the analysis consists in showing that for any point $x \in \mathbf{R}^2$, the characteristic stemming from x stays in a ball of size $O(\varepsilon)$ around x. This shows that the rotation has a drastic influence over the transport by u.

We have the following proposition.

Proposition 1. *Let $x \in \mathbf{R}^2$ be given, and let $X(\cdot,x)$ be the trajectory starting from x at time 0, defined by (2.8). As long as it is defined, it satisfies*

$$\forall t < \min(T,T^\varepsilon), \quad |X(t,x)-x| \leq 4\frac{\varepsilon}{b_-}\|u_0\|_{L^\infty}(1+T\frac{\|\nabla b\|_{L^\infty}\|u_0\|_{L^\infty}}{b_-}).$$

Proof of Proposition 1. The proof is an immediate application of the non-stationary phase theorem. As we will be using such arguments many times in the following, let us state and prove the following lemma, which will be invoked systematically in the next sections.

Lemma 1. *Let T be a given real number, possibly depending on ε. Let F be a function uniformly bounded in $W^{1,\infty}([0,T],L^\infty(\mathbf{R}^2))$, and let β be a positive function, also uniformly bounded in $W^{1,\infty}([0,T],L^\infty(\mathbf{R}^2))$, and bounded by below by b_-. Then for all $t\in[0,T]$ and all $x \in \mathbf{R}^2$, the following bounds hold:*

$$\left|\int_0^t F(s,x)\cos\left(\int_0^s \frac{\beta(s',x)}{\varepsilon}\,ds'\right)\,ds\right| \leq \varepsilon\left(\frac{\|F(t,\cdot)\|_{L^\infty(\mathbf{R}^2)}}{b_-} + t\left\|\partial_s\frac{F(s,\cdot)}{\beta(s,\cdot)}\right\|_{L^\infty([0,t]\times\mathbf{R}^2)}\right),$$

and

$$\left|\int_0^t F(s,x)\sin\left(\int_0^s \frac{\beta(s',x)}{\varepsilon}\,ds'\right)\,ds\right| \leq \varepsilon\left(\frac{\|F(t,\cdot)\|_{L^\infty(\mathbf{R}^2)}+\|F(0,\cdot)\|_{L^\infty(\mathbf{R}^2)}}{b_-} + t\left\|\partial_s\frac{F(s,\cdot)}{\beta(s,\cdot)}\right\|_{L^\infty([0,t]\times\mathbf{R}^2)}\right).$$

Proof of Lemma 1. The proof is a simple application of the nonstationary phase theorem: an integration by parts leads to

$$\int_0^t F(s,x)\cos\left(\int_0^s \frac{\beta(s',x)}{\varepsilon}\,ds'\right)\,ds = \varepsilon\frac{F(t,x)}{\beta(t,x)}\sin\left(\int_0^t\frac{\beta(s,x)}{\varepsilon}\,ds\right) - \varepsilon\int_0^t \partial_s\left(\frac{F(s,x)}{\beta(s,x)}\right)\sin\left(\int_0^s\frac{\beta(s',x)}{\varepsilon}\,ds'\right)\,ds,$$

and similarly

$$\int_0^t F(s,x)\sin\left(\int_0^s \frac{\beta(s',x)}{\varepsilon}\,ds'\right)\,ds = \varepsilon\frac{F(0,x)}{\beta(0,x)} - \varepsilon\frac{F(t,x)}{\beta(t,x)}\cos\left(\int_0^t \frac{\beta(s,x)}{\varepsilon}\,ds\right)$$
$$+\varepsilon\int_0^t \partial_s\left(\frac{F(s,x)}{\beta(s,x)}\right)\cos\left(\int_0^s \frac{\beta(s',x)}{\varepsilon}\,ds'\right)\,ds.$$

The result follows immediately.

Now let us go back to the proof of Proposition 1. Recalling formula (2.8), we simply apply Lemma 1 to the case $F(t,x) = u_0(x)$ to get

$$|X(t,x) - x| \le 4\varepsilon\frac{\|u_0\|_{L^\infty}}{b_-} + 2\varepsilon t\left\|u_0\frac{\partial_s\beta(s,\cdot)}{\beta^2(s,\cdot)}\right\|_{L^\infty([0,t]\times\mathbf{R}^2)}.$$

Estimates (2.6) and (2.7) immediately yield Proposition 1.

3.2. **Asymptotics of $X(t,\cdot)$.** The same type of computations based on the non-stationary phase theorem allows actually to obtain an explicit approximation of the characteristic X at any order with respect to ε (in fact we will stop at order 2 but the argument can be pushed as far as wanted if necessary).

Lemma 2. *For any point $x \in \mathbf{R}^2$ and any time $t \le \min(T, T^\varepsilon)$, the following approximation of the trajectories defined in (2.8) holds:*

$$\left|X(t,x) - x - \varepsilon\frac{u_0(x)}{b(x)}\sin\left(\frac{\phi(t,x)}{\varepsilon}\right) + \varepsilon\frac{u_0^\perp(x)}{b(x)}\left(1-\cos\left(\frac{\phi(t,x)}{\varepsilon}\right)\right) + \varepsilon t v(x)\right| \le C_T\varepsilon^2,$$

where the drift velocity is given by

$$v(x) = \frac{1}{2b^2(x)}\Big((u_0^\perp\cdot\nabla b)u_0(x) - (u_0\cdot\nabla b)u_0^\perp(x)\Big),$$

and C_T denotes a constant depending only on T, u_0 and b.

Proof of Lemma 2. Let us write the following expression for $X(t,x)$, obtained from (2.8): we have

$$X(t,x) = x + R^\varepsilon(t,x),$$

with

$$\begin{aligned} R^\varepsilon(t,x) &\stackrel{\text{def}}{=} u_0(x)\int_0^t \cos\left(\frac{\phi(s,x)}{\varepsilon}\right)ds - u_0^\perp(x)\int_0^t \sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds \\ &= R_1^\varepsilon(t,x) + R_2^\varepsilon(t,x). \end{aligned} \tag{3.1}$$

We shall only compute the approximation for $R_1^\varepsilon(t,x)$, and we leave $R_2^\varepsilon(t,x)$ to the reader. By an integration by parts we have

$$R_1^\varepsilon(t,x) = \varepsilon\frac{u_0(x)}{\beta(t,x)}\sin\left(\frac{\phi(t,x)}{\varepsilon}\right) - \varepsilon u_0(x)\int_0^t \partial_s\left(\frac{1}{\beta(s,x)}\right)\sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds. \tag{3.2}$$

The first term is easy to approximate : we have, due to Proposition 1

$$\begin{aligned} \left|\frac{1}{\beta(t,x)} - \frac{1}{b(x)}\right| &\le \frac{\|\nabla b\|_{L^\infty}}{b_-^2}|X(t,x)-x| \\ &\le \frac{4\|\nabla b\|_{L^\infty}\|u_0\|_{L^\infty}}{b_-^3}\left(1 + T\frac{\|\nabla b\|_{L^\infty}\|u_0\|_{L^\infty}}{b_-}\right), \end{aligned} \tag{3.3}$$

for all $t \leq \min(T, T^\varepsilon)$ and all $x \in \mathbf{R}^2$. So $\beta(t,x)$ can be replaced by $b(x)$ in the first term of R_1^ε in (3.2), up to a remainder $\varepsilon \mathcal{R}^\varepsilon$ with $\|\mathcal{R}^\varepsilon\|_{L^\infty([0,T]\times\mathbf{R}^2)} \leq C_T$.

Now we need to approximate the second term. Using the fact that

$$\partial_s \beta(s,x) = (u \cdot \nabla b)(s, X(s,x))$$

with

$$u(s, X(s,x)) = u_0(x) \cos\left(\frac{\phi(s,x)}{\varepsilon}\right) - u_0^\perp(x) \sin\left(\frac{\phi(s,x)}{\varepsilon}\right),$$

we can therefore write

$$\begin{aligned} -\int_0^t \partial_s \left(\frac{1}{\beta(s,x)}\right) \sin\left(\frac{\phi(s,x)}{\varepsilon}\right) ds &= \int_0^t u_0(x) \cdot \frac{\nabla b(X(s,x))}{2b^2(X(s,x))} \sin\left(\frac{2\phi(s,x)}{\varepsilon}\right) ds \\ &\quad -\int_0^t u_0^\perp(x) \cdot \frac{\nabla b(X(s,x))}{2b^2(X(s,x))} \left(1 - \cos\left(\frac{2\phi(s,x)}{\varepsilon}\right)\right) ds. \end{aligned} \tag{3.4}$$

Note that similar computations lead to the following formula, which is useful to estimate R_2^ε:

$$\begin{aligned} \int_0^t \partial_s \left(\frac{1}{\beta(s,x)}\right) \cos\left(\frac{\phi(s,x)}{\varepsilon}\right) ds &= \int_0^t u_0^\perp(x) \cdot \frac{\nabla b(X(s,x))}{2b^2(X(s,x))} \sin\left(\frac{2\phi(s,x)}{\varepsilon}\right) ds \\ &\quad -\int_0^t u_0(x) \cdot \frac{\nabla b(X(s,x))}{2b^2(X(s,x))} \left(1 + \cos\left(\frac{2\phi(s,x)}{\varepsilon}\right)\right) ds. \end{aligned} \tag{3.5}$$

Both formulas (3.4) and (3.5) show that new harmonics have been created by the coupling in the equation.

Let us go back to the estimate of the right-hand side in (3.4). To estimate the oscillating terms, we use Lemma 1 with

$$F_1(s,x) = u_0(x) \cdot \frac{\nabla b(X(s,x))}{2b^2(X(s,x))} \quad \text{and} \quad F_2(s,x) = u_0^\perp(x) \cdot \frac{\nabla b(X(s,x))}{2b^2(X(s,x))}.$$

We get

$$\begin{aligned} \left| \int_0^t F_1(s,x) \sin\left(\frac{2\phi(s,x)}{\varepsilon}\right) ds \right| &\leq 2\varepsilon \|u_0\|_{L^\infty} \frac{\|\nabla b\|_{L^\infty}}{b_-^3} \\ &\quad + \varepsilon t \|u_0\|_{L^\infty} \left\| \partial_s \frac{\nabla b(X(s,\cdot))}{b^2(X(s,\cdot))} \right\|_{L^\infty([0,t]\times\mathbf{R}^2)}, \end{aligned}$$

and similarly

$$\begin{aligned} \left| \int_0^t F_2(s,x) \cos\left(\frac{2\phi(s,x)}{\varepsilon}\right) ds \right| &\leq \varepsilon \|u_0\|_{L^\infty} \frac{\|\nabla b\|_{L^\infty}}{b_-^3} \\ &\quad + \varepsilon t \|u_0\|_{L^\infty} \left\| \partial_s \frac{\nabla b(X(s,\cdot))}{b^2(X(s,\cdot))} \right\|_{L^\infty([0,t]\times\mathbf{R}^2)}. \end{aligned}$$

By (2.4) and (2.6) we have

$$\left\| \partial_s \frac{\nabla b(X(s,\cdot))}{b^2(X(s,\cdot))} \right\|_{L^\infty([0,t]\times\mathbf{R}^2)} \leq 2\|D^2 b\|_{L^\infty} \frac{\|u_0\|_{L^\infty}}{b_-^2} + 4\|\nabla b\|_{L^\infty}^2 \frac{\|u_0\|_{L^\infty}}{b_-^3}.$$

Plugging that estimate along with (3.3) into the definition of R_1^ε in (3.2), we get finally

$$R_1^\varepsilon(t,x) = \varepsilon\frac{u_0(x)}{b(x)}\sin\left(\int_0^t \frac{\beta(s,x)}{\varepsilon}\,ds\right) + \varepsilon^2\mathcal{R}^\varepsilon(t,x) - \varepsilon u_0(x)\int_0^t \frac{u_0^\perp(x)\cdot\nabla b(X(s,x))}{2b^2(X(s,x))}\,ds.$$

Since the trajectories lie in balls of size ε,

$$|\nabla b(X(\tau,x)) - \nabla b(x)| \le C_T\|D^2 b\|_{L^\infty}\varepsilon, \tag{3.6}$$

for all $x \in \mathbf{R}^2$ and all $\tau \le t \le \min(T^\varepsilon, T)$. So we can approximate $\dfrac{\nabla b(X(s,x))}{b^2(X(s,x))}$ by $\dfrac{\nabla b(x)}{b(x)}$ up to a remainder $\varepsilon\mathcal{R}^\varepsilon$, and we have

$$-\varepsilon u_0(x)\int_0^t \frac{u_0^\perp(x)\cdot\nabla b(X(s,x))}{2b^2(X(s,x))}\,ds = -\varepsilon t\frac{u_0^\perp(x)\cdot\nabla b(x)}{2b^2(x)}u_0(x) + \varepsilon^2\mathcal{R}^\varepsilon.$$

The estimate of $R_2^\varepsilon(t,x)$ is similar and left to the reader. This ends the proof of Lemma 2.

3.3. **A priori estimates on $DX(t,\cdot)$.** A necessary and sufficient condition for $X(t,\cdot)$ to be invertible is that

$$J(t,x) \stackrel{\text{def}}{=} |\det(DX(t,x))|$$

does not cancel. In order to obtain a lower bound on the time T^ε (before which the characteristics do no cross each other), we therefore need to study the behaviour of the derivatives $DX(t,\cdot)$. First of all we derive a uniform L^∞-bound which will allow to neglect some terms in the asymptotic expansion.

Lemma 3. *Let $x \in \mathbf{R}^2$ be given, and let $X(\cdot,x)$ be the trajectory starting from x at time 0, defined by (2.8). As long as it is defined, it satisfies*

$$\forall t < \min(T, T^\varepsilon), \quad \forall x \in \mathbf{R}^2, \quad \|DX(t,x)\| \le C_T,$$

where C_T denotes a constant depending only on b, u_0 and T.

Proof of Lemma 3. Differentiating (2.8), leads to

$$\begin{aligned} DX(t,x) - Id = {} & Du_0(x)\int_0^t \cos\left(\frac{\phi(s,x)}{\varepsilon}\right)ds - Du_0^\perp(x)\int_0^t \sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds \\ & -\frac{1}{\varepsilon}u_0(x)\int_0^t D\phi(s,x)\sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds - \frac{1}{\varepsilon}u_0^\perp(x)\int_0^t D\phi(s,x)\cos\left(\frac{\phi(s,x)}{\varepsilon}\right)ds, \end{aligned}$$

with

$$D\phi(s,x) \stackrel{\text{def}}{=} \int_0^s DX(\tau,x)\cdot\nabla b(X(\tau,x))d\tau.$$

Applying the Fubini theorem to both last terms, we can set this identity in a suitable form to get a Gronwall estimate

$$\begin{aligned} DX(t,x) - Id = {} & Du_0(x)\int_0^t \cos\left(\frac{\phi(s,x)}{\varepsilon}\right)ds - Du_0^\perp(x)\int_0^t \sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds \\ & -\frac{1}{\varepsilon}u_0(x)\int_0^t (DX(\tau,x)\cdot\nabla)b(X(\tau,x))\int_\tau^t \sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds\,d\tau \\ & -\frac{1}{\varepsilon}u_0^\perp(x)\int_0^t (DX(\tau,x)\cdot\nabla)b(X(\tau,x))\int_\tau^t \cos\left(\frac{\phi(s,x)}{\varepsilon}\right)ds\,d\tau. \end{aligned} \tag{3.7}$$

From formula (2.8) we deduce that

$$u_0(x)\int_\tau^t \sin\left(\frac{\phi(s,x)}{\varepsilon}\right)\,ds + u_0^\perp(x)\int_\tau^t \cos\left(\frac{\phi(s,x)}{\varepsilon}\right)\,ds = (X(t,x)-X(\tau,x))^\perp.$$

Plugging this identity back into (3.7) leads to

$$\begin{aligned} DX(t,x)-Id = \;& Du_0(x)\int_0^t \cos\left(\frac{\phi(s,x)}{\varepsilon}\right)\,ds - Du_0^\perp(x)\int_0^t \sin\left(\frac{\phi(s,x)}{\varepsilon}\right)\,ds \\ & -\frac{1}{\varepsilon}\int_0^t (X(t,x)-X(\tau,x))^\perp \otimes (DX(\tau,x)\cdot\nabla)b(X(\tau,x))\,d\tau \end{aligned} \tag{3.8}$$

As in the proof of Proposition 1, Lemma 1 yields the following estimate : for all $t\le\min(T,T^\varepsilon)$,

$$\left|Du_0(x)\int_0^t \cos\left(\frac{\phi(s,x)}{\varepsilon}\right)\,ds - Du_0^\perp(x)\int_0^t \sin\left(\frac{\phi(s,x)}{\varepsilon}\right)\,ds\right| \le C_T\varepsilon\|\nabla u_0\|_{L^\infty}, \tag{3.9}$$

with

$$C_T = \frac{4}{b_-}\left(1+T\frac{\|\nabla b\|_{L^\infty}\|u_0\|_{L^\infty}}{b_-}\right).$$

From (3.8) we then deduce an inequality of Gronwall type

$$\|DX(t,\cdot)\|_{L^\infty} \le 1 + C_T\varepsilon\|\nabla u_0\|_{L^\infty} + \int_0^t \|DX(\tau,\cdot)\|_{L^\infty}\|\nabla b\|_{L^\infty}\left\|\frac{1}{\varepsilon}(X(\tau,x)-X(t,x))\right\|_{L^\infty} d\tau \tag{3.10}$$

By Proposition 1,

$$\forall t\le\min(T,T^\varepsilon),\quad \left\|\frac{1}{\varepsilon}(X(t,\cdot)-X(\tau,\cdot))\right\|_{L^\infty} \le 2C_T\|u_0\|_{L^\infty},$$

hence

$$\|DX(t,\cdot)\|_{L^\infty} \le (1+C_T\varepsilon\|\nabla u_0\|_{L^\infty})\exp\left(2C_T\|\nabla b\|_{L^\infty}\|u_0\|_{L^\infty}t\right),$$

which is the expected estimate, proving Lemma 3.

3.4. **Asymptotics of DX.** In view of the results established in Lemma 2, we expect actually the derivatives $\partial_i X(t,x)$ to behave asymptotically as

$$\lambda(t,x)+\mu(t,x)\cos\left(\frac{\phi(t,x)}{\varepsilon}\right)+\nu(t,x)\sin\left(\frac{\phi(t,x)}{\varepsilon}\right)$$

where λ, μ and ν denote some functions which do not depend on ε. Such an asymptotics can be justified using the same techniques as in the previous paragraph: let us prove the following lemma.

Lemma 4. *Let $x\in\mathbf{R}^2$ be given, and let $X(\cdot,x)$ be the trajectory starting from x at time 0, defined by (2.8). Then, for all $t\le\min(T^\varepsilon,T)$ and for all $x\in\mathbf{R}^2$,*

$$\left\|DX(t,x)-Id-tu_0\otimes\nabla\log b\cos\left(\frac{\phi(t,x)}{\varepsilon}\right)+tu_0^\perp\otimes\nabla\log b\sin\left(\frac{\phi(t,x)}{\varepsilon}\right)\right\| \le C_T\varepsilon,$$

where C_T denotes a constant depending only on b, u_0 and T.

Proof of Lemma 4. Denote by g the function defined on $[0,T]\times\mathbf{R}^2$ by

$$g(t,x) \stackrel{\text{def}}{=} DX(t,x) - Id - tu_0 \otimes \nabla \log b \cos\left(\frac{\phi(t,x)}{\varepsilon}\right) + tu_0^\perp \otimes \nabla \log b \sin\left(\frac{\phi(t,x)}{\varepsilon}\right)\cdot$$

In view of (3.10), we expect g to satisfy a Gronwall inequality of the following type

$$\|g(t,x)\| \le \int_0^t \|g(\tau,x)\| \left\|\frac{1}{\varepsilon}(X(t,\cdot)-X(\tau,\cdot))\right\|_{L^\infty} \|\nabla b\|_{L^\infty} d\tau + C_T\varepsilon, \tag{3.11}$$

for all $t \le \min(T^\varepsilon, T)$, where C_T denotes a constant depending only on T, u_0 and b.

Let us postpone the proof of this inequality for a while, and show how it enables us to infer Lemma 4. It is easy to see that

$$g(0,x) = 0.$$

Applying the Gronwall lemma and using Proposition 1 as in (3.10) leads to

$$\forall t \le \min(T^\varepsilon,T), \quad \forall x \in \mathbf{R}^2, \quad \|g(t,x)\| \le C_T\varepsilon.$$

Now let us go back to the proof of (3.11). We first compute

$$A(t,x) \stackrel{\text{def}}{=} \int_0^t \left(\frac{1}{\varepsilon}(X(t,x)-X(\tau,x))\right)^\perp \otimes (g(\tau,x)\cdot\nabla b(X(\tau,x)))\, d\tau. \tag{3.12}$$

By Lemma 2,

$$\begin{aligned}\frac{1}{\varepsilon}(X(t,x)-X(\tau,x))^\perp = \;& \frac{u_0^\perp}{b}(x)\left(\sin\left(\frac{\phi(t,x)}{\varepsilon}\right) - \sin\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)\\ & -\frac{u_0}{b}(x)\left(\cos\left(\frac{\phi(t,x)}{\varepsilon}\right) - \cos\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)\\ & -\frac{t-\tau}{2b^2(x)}\Big((u_0^\perp\cdot\nabla b)u_0^\perp(x) + (u_0\cdot\nabla b)u_0(x)\Big) + \varepsilon\mathcal{R}^\varepsilon(t,\tau,x),\end{aligned}$$

where $\mathcal{R}^\varepsilon$ is uniformly bounded in $L^\infty([0,T]^2\times\mathbf{R}^2)$. Plugging this formula back into the integral (3.12) leads to

$$\int_0^t \left(\frac{1}{\varepsilon}(X(t,x)-X(\tau,x))\right)^\perp \otimes (g(\tau,x)\cdot\nabla b(X(\tau,x)))\, d\tau = A_1(t,x) - A_2(t,x), \tag{3.13}$$

with

$$A_1(t,x) \stackrel{\text{def}}{=} \int_0^t \left(\frac{1}{\varepsilon}(X(t,x)-X(\tau,x))\right)^\perp \otimes (DX(\tau,x)\cdot\nabla b)(X(\tau,x))d\tau$$

and

$$\begin{aligned}A_2(t,x) \stackrel{\text{def}}{=} \int_0^t & \left[\frac{u_0^\perp}{b}(x)\left(\sin\left(\frac{\phi(t,x)}{\varepsilon}\right) - \sin\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)\right.\\ & -\frac{u_0}{b}(x)\left(\cos\left(\frac{\phi(t,x)}{\varepsilon}\right) - \cos\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)\\ & \left.-\frac{1}{2b^2(x)}(t-\tau)\Big((u_0^\perp\cdot\nabla b)u_0^\perp(x) + (u_0\cdot\nabla b)u_0(x)\Big) + \varepsilon\mathcal{R}^\varepsilon(t,\tau,x)\right]\\ & \otimes\left[\nabla b(X(\tau,x)) + (u_0(x)\cdot\nabla b(X(\tau,x)))\nabla\log b(x)\tau\cos\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right.\\ & \left.-(u_0^\perp(x)\cdot\nabla b(X(\tau,x)))\nabla\log b(x)\tau\sin\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right]\, d\tau.\end{aligned}$$

From (3.8) and (3.9) we deduce that

$$A_1(t,x) = Id - DX(t,x)$$

up to terms of order ε.

In order to estimate the second term $A_2(t,x)$, we use again a non-stationnary phase theorem. We can use (3.6) again, and as $\partial_s\beta$ is uniformly bounded according to (2.6), Lemma 1 shows that

$$\begin{aligned}
A_2(t,x) &= \left[t\frac{u_0^\perp}{b}(x)\sin\left(\frac{\phi(t,x)}{\varepsilon}\right) - t\frac{u_0}{b}(x)\cos\left(\frac{\phi(t,x)}{\varepsilon}\right) - \frac{t^2}{2}v^\perp(x)\right]\otimes\nabla b(x)\\
&+\int_0^t \left(\frac{u_0^\perp}{b}(x)\sin\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)\otimes\left((\tau u_0^\perp(x)\cdot\nabla b(x))\nabla\log b(x)\sin\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)d\tau\\
&+\int_0^t \left(\frac{u_0}{b}(x)\cos\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)\otimes\left((\tau u_0(x)\cdot\nabla b(x))\nabla\log b(x)\cos\left(\frac{\phi(\tau,x)}{\varepsilon}\right)\right)d\tau + \varepsilon\mathcal{R}^\varepsilon(t,x)
\end{aligned}$$

Using the identities

$$\cos^2\phi = \frac{1}{2}(1+\cos(2\phi)), \quad \sin^2\phi = \frac{1}{2}(1-\cos(2\phi)),$$

we then obtain that

$$A_2(t,x) = \left(t\frac{u_0^\perp}{b}(x)\sin\left(\frac{\phi(t,x)}{\varepsilon}\right) - t\frac{u_0}{b}(x)\cos\left(\frac{\phi(t,x)}{\varepsilon}\right)\right)\otimes\nabla b(x)$$

up to terms of order ε.

Then (3.13) can be rewritten

$$\int_0^t \left(\frac{1}{\varepsilon}(X(t,x)-X(\tau,x))\right)^\perp \otimes (g(\tau,x)\cdot\nabla b(X(\tau,x)))\,d\tau = -g(t,x) + \varepsilon\mathcal{R}^\varepsilon(t,x),$$

which implies immediately (3.11) and yields Lemma 4 as explained above.

3.5. **Existence on a uniform time interval.** As an immediate corollary of Lemma 4 we obtain that $X(t,\cdot)$ is a diffeomophism of $\mathbf{R}^2$ on a uniform time interval. Indeed, DX is invertible as long as

$$\|DX - Id\|_{L^\infty} < 1.$$

Corollary 1. *Consider a function b satisfying assumptions $(H0)(H1)$. Let (ρ_0,u_0) be respectively a nonnegative function of $W^{s-1,\infty}(\mathbf{R}^2)$ and a vector-field of $W^{s,\infty}(\mathbf{R}^2)$ $(s\geq 1)$. Then, for all $T < \|u_0\|_{L^\infty}^{-1}\|\nabla b\|_{L^\infty}^{-1}$, there exists $\varepsilon_T > 0$ such that System (1.2) admits a unique solution $(\rho_\varepsilon,u_\varepsilon)\in L^\infty([0,T],W^{s-1,\infty}(\mathbf{R}^2)\times W^{s,\infty}(\mathbf{R}^2))$ for all $\varepsilon\leq\varepsilon_T$.*

Proof of Corollary 1. By Lemma 4, the trajectories defined by (2.8) are continuously differentiable and satisfy for all $t\leq T$ and all $x\in\mathbf{R}^2$,

$$\left\|DX(t,x) - Id - tu_0\otimes\nabla\log b\cos\left(\frac{\phi(t,x)}{\varepsilon}\right) + tu_0^\perp\otimes\nabla\log b\sin\left(\frac{\phi(t,x)}{\varepsilon}\right)\right\| \leq C_T\varepsilon.$$

This implies in particular the following estimate on the Jacobian $J(t,x)\stackrel{\text{def}}{=}|\det(DX(t,x))|$:

$$\left|J(t,x) - 1 - tu_0\cdot\nabla\log b\cos\left(\frac{\phi(t,x)}{\varepsilon}\right) + tu_0^\perp\cdot\nabla\log b\sin\left(\frac{\phi(t,x)}{\varepsilon}\right)\right| \leq C_T\varepsilon.$$

Then for $T < \|u_0\|_{L^\infty}^{-1}\|\nabla b\|_{L^\infty}^{-1}$, there exists ε_T such that

$$\forall \varepsilon \leq \varepsilon_T, \quad \forall t \in [0,T], \quad \sup_{x\in\mathbf{R}^2} |J(t,x) - 1| < 1,$$

which means that X is a C^1-diffeomorphism of $\mathbf{R}^2$.

Moreover, from formula (2.8) we can deduce by induction that $X(t,.)$ (and consequently its inverse $X^{-1}(t,\cdot)$) is smooth, its regularity being the same as the regularity of the initial velocity field u_0. Then the vector field u given by

$$u(t,x) = u_0(X^{-1}(t,x)) \cos\left(\frac{\phi(t, X^{-1}(t,x))}{\varepsilon}\right) - u_0^{\perp}(X^{-1}(t,x)) \cos\left(\frac{\phi(t, X^{-1}(t,x))}{\varepsilon}\right)$$

belongs to $L^\infty([0,T], W^{s,\infty}(\mathbf{R}^2))$ and it is easy to check that it satisfies System (1.3) in strong sense.

The density ρ is then obtained as the strong solution of the linear transport equation

$$\partial_t \rho + u \cdot \nabla\rho + \rho\nabla \cdot u = 0$$

whose coefficients belong to $L^\infty([0,T], W^{s-1,\infty}(\mathbf{R}^2))$, with initial data in $W^{s-1,\infty}(\mathbf{R}^2)$. It therefore stays in $L^\infty([0,T], W^{s-1,\infty}(\mathbf{R}^2))$. We emphasize once again that no uniform bound on (ρ,u) is available in $L^\infty([0,T], W^{s-1,\infty}(\mathbf{R}^2) \times W^{s,\infty}(\mathbf{R}^2))$.

Theorem 2 is proved.

Remark 3. *The supremum of the life span of the solutions corresponds to a crossing phenomenon, to be compared with the caustic in geometrical optics. Beyond this time, the differential system*

$$\begin{cases} \dot{X} = \xi \\ \dot{\xi} = \xi \wedge b \end{cases}$$

with initial data $(x, u_0(x))_{x\in\mathbf{R}^2}$ still admits a unique smooth solution, but the application $(X(t,x), \xi(t,x)) \mapsto X(t,x)$ is no longer injective, it cannot be lifted. The hyperbolic system (1.3) no longer has a solution.

4. Study of the asymptotics of u_ε and ρ_ε

Let $T < T^* = \|u_0\|_{L^\infty}^{-1}\|\nabla b\|_{L^\infty}^{-1}$ be fixed. Then, for any $\varepsilon \leq \varepsilon_T$ as in Corollary 1, the solution (ρ,u) of System (1.2) with initial data $(\rho_0,u_0) \in W^{s-1,\infty}(\mathbf{R}^2) \times W^{s,\infty}(\mathbf{R}^2)$ belongs to $L^\infty([0,T], W^{s-1,\infty}(\mathbf{R}^2) \times W^{s,\infty}(\mathbf{R}^2))$. Then it makes sense to study their asymptotic behaviour as $\varepsilon \to 0$, and the aim of this section is to prove Theorems 3 and 4.

Paragraph 4.1 is devoted to the asymptotics of $u(t,X(t,x))$ ad $\rho(t,X(t,x))$. The last paragraph consists in inverting the characteristics in order to infer Theorems 3 and 4.

4.1. **Asymptotics of $u(t,X)$ and $\rho(t,X)$.** From the characteristic formulation of System (1.2) and the asymptotic expansion of $X(t,\cdot)$ we immediately deduce the asymptotic behaviour of $u(t,X(t,\cdot))$ and $\rho(t,X(t,\cdot))$.

Proposition 2. *Consider a function b satisfying assumptions $(H0)(H1)$. Let u_0 be a vector-field in $W^{s,\infty}(\mathbf{R}^2)$ $(s\geq 1)$. For all $T<T^*$ and $\varepsilon\leq\varepsilon_T$ as in Theorem 2, denote by u the solution of (1.3) in $L^\infty([0,T],W^{s,\infty}(\mathbf{R}^2))$. Then*

$$u(t,X(t,x))-\Big(u_0(x)\cos(\widetilde{\phi}_\varepsilon(t,x))-u_0^\perp(x)\sin(\widetilde{\phi}_\varepsilon(t,x))\Big)$$

converges strongly to 0 in $L^\infty([0,T]\times\mathbf{R}^2)$, at speed $O(\varepsilon)$, where the phase $\widetilde{\phi}_\varepsilon$ is defined by

$$\widetilde{\phi}_\varepsilon(t,x)=\frac{b(x)t}{\varepsilon}-t\left(u_0^\perp(x)\cdot\nabla\right)\log b(x). \tag{4.1}$$

Proof of Proposition 2. Let us first recall that

$$u(t,X(t,x))=u_0(x)\cos\left(\frac{\phi(t,x)}{\varepsilon}\right)-u_0^\perp(x)\sin\left(\frac{\phi(t,x)}{\varepsilon}\right),$$

where the phase ϕ is given by

$$\phi(t,x)=\int_0^t b(X(s,x))ds.$$

Then in order to establish Proposition 2, we have to approximate the phase. By Lemma 2,

$$\begin{aligned}b(X(t,x))=b(x)+\frac{\varepsilon u_0(x)}{b(x)}\sin\left(\frac{\phi(t,x)}{\varepsilon}\right)\cdot\nabla b(x)&\\ -\frac{\varepsilon u_0^\perp(x)}{b(x)}\left(1-\cos\left(\frac{\phi(t,x)}{\varepsilon}\right)\right)\cdot\nabla b(x)+\varepsilon^2\mathcal{R}^\varepsilon(t,x),&\end{aligned}$$

noticing that $v\cdot\nabla b=0$. It follows that

$$\begin{aligned}u(t,X(t,x))=u_0(x)\cos\Big(\frac{b(x)t}{\varepsilon}+\frac{u_0(x)\cdot\nabla b(x)}{b(x)}\int_0^t\sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds&\\ -\frac{u_0^\perp(x)\cdot\nabla b(x)}{b(x)}\int_0^t\left(1-\cos\left(\frac{\phi(s,x)}{\varepsilon}\right)\right)ds+\varepsilon\mathcal{R}^\varepsilon(t,x)\Big)&\\ -u_0^\perp(x)\sin\Big(\frac{b(x)t}{\varepsilon}+\frac{u_0(x)\cdot\nabla b(x)}{b(x)}\int_0^t\sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds&\\ -\frac{u_0^\perp(x)\cdot\nabla b(x)}{b(x)}\int_0^t\left(1-\cos\left(\frac{\phi(s,x)}{\varepsilon}\right)\right)ds+\varepsilon\mathcal{R}^\varepsilon(t,x)\Big)&\end{aligned}$$

Finally remembering that due to Lemma 1

$$\left|\int_0^t\sin\left(\frac{\phi(s,x)}{\varepsilon}\right)ds\right|+\left|\int_0^t\cos\left(\frac{\phi(s,x)}{\varepsilon}\right)ds\right|\leq\varepsilon\mathcal{R}^\varepsilon(t,x),$$

with the usual uniform bounds on $\mathcal{R}^\varepsilon$, yields Proposition 2.

The asymptotic behaviour of ρ is obtained in a similar way using the fact that ρ is proportionnal to the Jacobian $J(t,x)=|\det DX(t,x)|$.

Proposition 3. *Consider a function b satisfying assumptions $(H0)(H1)$. Let ρ_0 be a nonnegative function in $W^{s-1,\infty}(\mathbf{R}^2)$, and u_0 be a vector-field in $W^{s,\infty}(\mathbf{R}^2)$ ($s \geq 1$). For all $T < T^*$ and $\varepsilon \leq \varepsilon_T$ as in Theorem 2, denote by (ρ, u) the solution of (1.2) in $L^\infty([0,T], W^{s-1,\infty}(\mathbf{R}^2))$ and $L^\infty([0,T], W^{s,\infty}(\mathbf{R}^2))$ respectively. Then*

$$\rho(t, X(t,x)) - \rho_0(x)\left(1 + tu_0 \cdot \nabla \log b(x) \sin(\widetilde{\phi}_\varepsilon(t,x)) - tu_0^\perp \cdot \nabla \log b(x) \cos(\widetilde{\phi}_\varepsilon(t,x))\right)$$

converges strongly to 0 in $L^\infty([0,T] \times \mathbf{R}^2)$, where the phase $\widetilde{\phi}_\varepsilon$ is defined as previously by (4.1).

Proof of Proposition 3. As long as the solution of (1.2) is regular, the equation governing ρ can be rewritten

$$\frac{d}{dt}(\log \rho) = \nabla \cdot u,$$

where $\frac{d}{dt}$ denotes as usual the derivative along the trajectories associated with the flow. Of course, the Liouville theorem implies that the equation on the Jacobian of the flow states

$$\frac{d}{dt}(J) = \nabla \cdot u.$$

Then, for all $\varepsilon \leq \varepsilon_T$, all $t \in [0,T]$ and all $x \in \mathbf{R}^2$,

$$\rho(t, X(t,x)) = \rho_0(x) J(t,x),$$

since $J_0(t,x) = \det(Id) = 1$.

From Lemma 4 we then deduce that

(4.2)
$$\left|\rho(t, X(t,x)) - \rho_0(x)\left(1 + tu_0 \cdot \nabla \log b(x) \sin\left(\frac{\phi(t,x)}{\varepsilon}\right) - tu_0^\perp \cdot \nabla \log b(x)\left(\frac{\phi(t,x)}{\varepsilon}\right)\right)\right| \leq C_T \varepsilon.$$

Plugging the approximation of the phase obtained previously

$$\frac{\phi(t,x)}{\varepsilon} = \widetilde{\phi}_\varepsilon + \varepsilon \mathcal{R}^\varepsilon(t,x) \tag{4.3}$$

back into formula (4.2) leads then to the expected asymptotics.

4.2. **Inversion of the characteristics.** In this section we shall prove Theorems 3 and 4. From now on T^* is the time given by Theorem 2, and we will call T any time smaller than T^* (in the following we will also suppose $\varepsilon \leq \varepsilon_T$ as given in Theorem 2).

Let $X^{-1}(t,x)$ be the point at time 0 of the trajectory reaching x at time t. By Proposition 2, we have

$$u(t,x) = u_0(X^{-1}(t,x)) \cos\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x))\right) - u_0^\perp(X^{-1}(t,x)) \sin\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x))\right) + \varepsilon \mathcal{R}^\varepsilon(t,x)$$

with the usual uniform bounds on $\mathcal{R}^\varepsilon$. That remainder function $\mathcal{R}^\varepsilon(t,x)$ is liable to change from line to line in this paragraph.

By Proposition 1 there is a constant C_T (depending on T, u_0 and b), such that

$$\forall x \in \mathbf{R}^2, \quad \forall t \in [0,T], \quad |X^{-1}(t,x) - x| \leq C_T \varepsilon, \tag{4.4}$$

so we can write rather

$$u(t,x) = u_0(x) \cos\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x))\right) - u_0^\perp(x) \sin\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x))\right) + \varepsilon \mathcal{R}^\varepsilon(t,x).$$

By definition of $\widetilde{\phi}_\varepsilon$ in (4.1), we have, using again (4.4),

$$\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x)) = \frac{b(x)t}{\varepsilon} + \frac{t}{\varepsilon}(X^{-1}(t,x) - x) \cdot \nabla b(x) - tu_0^\perp(x) \cdot \nabla \log b(x) + \varepsilon \mathcal{R}^\varepsilon(t,x)$$

hence defining

$$\widetilde{\theta}_\varepsilon(t,x) \stackrel{\text{def}}{=} \frac{t}{\varepsilon}(X^{-1}(t,x) - x) \cdot \nabla b(x) - tu_0^\perp(x) \cdot \nabla \log b(x)$$

we have

$$\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x)) = \frac{b(x)t}{\varepsilon} + \widetilde{\theta}_\varepsilon(t,x) + \varepsilon \mathcal{R}^\varepsilon(t,x). \tag{4.5}$$

Now we shall try to make $\widetilde{\theta}_\varepsilon$ more precise. According to Lemma 2 and the approximation for the phase derived in the previous paragraph, we have

$$\begin{aligned} x - X^{-1}(t,x) = {} & \varepsilon \frac{u_0(x)}{b(x)} \sin\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x)) + \varepsilon \mathcal{R}^\varepsilon(t,x)\right) \\ & - \varepsilon \frac{u_0^\perp(x)}{b(x)} \left(1 - \cos\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x)) + \varepsilon \mathcal{R}^\varepsilon(t,x)\right)\right) - \varepsilon t v(x) + \varepsilon^2 \mathcal{R}^\varepsilon(t,x) \end{aligned}$$

where again we have used (4.4). So we obtain, using the fact that $v \cdot \nabla b = 0$,

$$\widetilde{\theta}_\varepsilon(t,x) = -tu_0(x) \cdot \nabla \log b(x) \sin\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x)) + \varepsilon \mathcal{R}^\varepsilon(t,x)\right)$$

$$+ tu_0^\perp(x) \cdot \nabla \log b(x) \cos\left(\widetilde{\phi}_\varepsilon(t, X^{-1}(t,x)) + \varepsilon \mathcal{R}^\varepsilon(t,x)\right) + \varepsilon \mathcal{R}^\varepsilon(t,x)$$

which by (4.5) yields directly the result (1.4), defining $\theta_\varepsilon(t,x) = \widetilde{\theta}_\varepsilon(t,x) + \dfrac{b(x)t}{\varepsilon} \cdot$

Theorem 3 is proved.

The proof of Theorem 4 is now immediate: we use the formula obtained in Proposition 3 and replace $\rho_\varepsilon(t, X(t,x))$ by $\rho_\varepsilon(t,x)$ using the above formulation of $X^{-1}(t,x)$. The result follows.

References

[1] A. Babin, A. Mahalov, and B. Nicolaenko, Regularity and integrability of 3D Euler and Navier-Stokes equations for rotating fluids, *Asymptot. Anal.* **15** (1997), no. 2, pages 103–150.

[2] Y. Brenier and E. Grenier, Sticky particles and scalar conservation laws, *SIAM Journal of Numerical Analysis*, **35**, 6, 1998, pages 2317–2328.

[3] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Anisotropy and dispersion in rotating fluids, *Nonlinear Partial Differential Equations and their applications, Collège de France Seminar*, Studies in Mathematics and its Applications, **31**, pages 171–191.

[4] E. Frénod and E. Sonnendrücker, Homogeneization of the Vlasov equation and the Vlasov–Poisson system with a strong external magnetic field, *Asymptotic Analysis*, **18**, 3, 1998, pages 193–213.

[5] I. Gallagher and L. Saint-Raymond, Weak convergence results for inhomogeneous rotating fluid equations, *submitted*, *Preprint of the Centre de Mathématiques de l'École polytechnique*, 2003.

[6] F. Golse and L. Saint-Raymond, The Vlasov-Poisson system with strong magnetic field, *Journal de Mathématiques Pures et Appliquées*, **78**, 1999, pages 791–817.

[7] E. Grenier, Pseudodifferential energy estimates of singular perturbations, *Communications in Pure and Applied Mathematics*, **50**, 9, 1997, pages 821–865.

[8] S. Schochet, Fast singular limits of hyperbolic PDEs. *J. Diff. Equ.* **114**, 1994, pages 476 − 512.

(I. Gallagher) Centre de Mathématiques Laurent Schwartz UMR 7640, Ecole Polytechnique, 91128 Palaiseau, FRANCE

E-mail address: Isabelle.Gallagher@math.polytechnique.fr

(L. Saint-Raymond) Laboratoire J.-L. Lions UMR 7598, Université Paris VI, 175, rue du Chevaleret, 75013 Paris, FRANCE

E-mail address: saintray@ann.jussieu.fr