THE CAUCHY PROBLEM FOR QUASI-LINEAR PARABOLIC SYSTEMS REVISITED - ERRATUM

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Abstract. In [1, p. 1666], the proof of [1, Theorem 6] had a misprint, which turned into a wrong argument that we correct here.

Let us first recall a few useful notations. We use here the Besov spaces $B^s_{p,1}$ defined in [1, Appendix B]. For any given $T>0$ and $p\in[1,\infty)$, the analogue of the solution space E^s_T in the Besov setting is:

$$\mathbb{E}^s_T := \mathcal{C}^0([0,T];B^s_{p,1})\cap \mathrm{L}^1(0,T;B^{s+2}_{p,1})\,,$$

while the analogue of the exterior force space Y^s_T is:

$$\mathbb{Y}^s_T := \mathrm{L}^1(0,T;B^s_{p,1})\,.$$

Subsection 7.4 of [1] intended to define a fixed-point procedure leveraging on [1, Theorem 7]. We propose below a complete rewriting of this subsection.

7.4. **Conclusion.** To conclude the proof we shall use the linear estimate provided by [1, Theorem 7] to implement a fixed point argument. Let us set

$$\mathbb{G}^p_T := \mathbb{E}^{\frac{d}{p}}_T \cap \mathcal{C}^0(Q_T)\,.$$

Given $(U^0,F)\in B^{\frac{d}{p}}_{p,1}\cap \mathbb{Y}^{\frac{d}{p}}_T$, we consider the following map

$$\begin{aligned}\Theta : \mathbb{G}^{\frac{d}{p}}_T &\longrightarrow \mathbb{E}^{\frac{d}{p}}_T\\ U &\longmapsto U^\star,\end{aligned}$$

where $U^\star$ is the only element of $\mathbb{E}^{\frac{d}{p}}_T$ solving $L_{A(U)}U^\star = F$ with $U(0)=U^0$. We define as before the reference solution $U_F := \Theta(0)$.

Before starting the fixed point procedure, let us check that Θ maps $\mathbb{G}^p_T$ onto itself. We recall that $U^\star := \Theta(U)$ solves

$$\partial_t U^\star = \sum_k \partial_k\big(A(U)\partial_k U^\star\big) + F$$

and we know that F belongs to $\mathbb{Y}^{\frac{d}{p}}_T$. Let us prove that the same information holds for $\partial_k\big(A(U)\big)\partial_k U^\star\big)$. We write

$$\partial_k\big(A(U)\partial_k U^\star\big) = A(U)\partial_k^2 U^\star + A'(U)\partial_k U \partial_k U^\star$$

and we know that smooth functions are continuous over $B^{\frac{d}{p}}_{p,1}$, so since $B^{\frac{d}{p}}_{p,1}$ is an algebra, it follows that

$$\big\|A(U)\partial_k^2 U^\star\big\|_{\mathbb{Y}^{\frac{d}{p}}_T} \lesssim \big\|A(U)\big\|_{L^\infty_T(B^{\frac{d}{p}}_{p,1})}\|\partial_k^2 U^\star\|_{\mathbb{Y}^{\frac{d}{p}}_T} \lesssim \Phi(\|U\|_{L^\infty_T(B^{\frac{d}{p}}_{p,1})})\|U^\star\|_{\mathbb{Y}^{\frac{d}{p}+2}_T}$$

with Φ smooth and increasing. Similarly

$$\big\|A'(U)\partial_k U\partial_k U^\star\big\|_{\mathbb{Y}_T^{\frac{d}{p}}} \lesssim \big\|A'(U)\big\|_{L^\infty_T(B^{\frac{d}{p}}_{p,1})}\|\partial_k U\|_{L^2_T(B^{\frac{d}{p}+1}_{p,1})}\|\partial_k U^\star\|_{L^2_T(B^{\frac{d}{p}+1}_{p,1})}\,.$$

We thus find that $\partial_t U^\star$ belongs to $\mathrm{L}^1(0,T;B^{\frac{d}{p}}_{p,1})$, which implies the expected result since $B^{\frac{d}{p}}_{p,1}$ is embedded in the space of continuous functions. In the following we restrict our attention to the space $\mathbb{E}_T^{\frac{d}{p}}$, as the estimates can then be extended to the full space $\mathbb{G}_T^p$ by the same argument on the time derivative.

Consider U_1 in the $\mathbb{E}_T^{\frac{d}{p}}$ closed ball of radius r centered at U_F and set $V := U_1^\star - U_F$. The same argument as in [1, Corollary 2.5] allows to replace the continuity constant $[A(U_F)]_0$ by $\|U_F\|_{\mathbb{G}_T^p}$ so that

$$\|V\|_{\mathbb{E}_T^{\frac{d}{p}}} \lesssim_{T,\|U_F\|_{\mathbb{G}_T^p},\omega_{U_F}} \|V(0)\|_{B^{\frac{d}{p}}_{p,1}} + \|L_{A(U_F)}V\|_{\mathbb{E}_T^{\frac{d}{p}}}\,.$$

Since

$$L_{A(U_F)}V = L_{A(U_1)}V + \sum_k \partial_k\big(A(U_1)-A(U_F)\big)\partial_k V$$

we find thanks to [1, Proposition B.2] that

$$\begin{aligned}\|V\|_{\mathbb{E}_T^{\frac{d}{p}}} \lesssim_{T,\|U_F\|_{\mathbb{G}_T^p},\omega_{U_F}} &\|L_{A(U_1)}V\|_{\mathbb{E}_T^{\frac{d}{p}}} + \|A(U_1)-A(U_F)\|_{\mathrm{L}^\infty(Q_T)}\|V\|_{\mathbb{Y}_T^{\frac{d}{p}+2}}\\ &+\int_0^T \big\|\big[A(U_1)-A(U_F)\big](t)\big\|_{B^{\frac{d}{p}+2}_{p,1}}\|V(t)\|_{B^{\frac{d}{p}}_{p,1}}\,\mathrm{d}t\,.\end{aligned}$$

It follows that if r is small enough (depending only on T and U_F) then

$$\|V\|_{\mathbb{E}_T^{\frac{d}{p}}} \lesssim_{T,\|U_F\|_{\mathbb{G}_T^p},\omega_{U_F}} \|L_{A(U_1)}V\|_{\mathbb{E}_T^{\frac{d}{p}}} + \int_0^T \big\|\big[A(U_1)-A(U_F)\big](t)\big\|_{B^{\frac{d}{p}+2}_{p,1}}\|V(t)\|_{B^{\frac{d}{p}}_{p,1}}\,\mathrm{d}t\,.$$

But

$$L_{A(U_1)}V = \sum_k \partial_k\big(A(U_1)-A(0)\big)\partial_k U_F$$

so again thanks to [1, Proposition B.2]

$$\begin{aligned}\|V\|_{\mathbb{E}_T^{\frac{d}{p}}} \lesssim_{T,\|U_F\|_{\mathbb{G}_T^p},\omega_{U_F}} &\|A(U_1)-A(0)\|_{\mathrm{L}^\infty(Q_T)}\|U_F\|_{\mathbb{Y}_T^{\frac{d}{p}+2}}\\ &+\int_0^T \big\|\big[A(U_1)-A(0)\big](t)\big\|_{B^{\frac{d}{p}+1}_{p,1}}\|U_F(t)\|_{B^{\frac{d}{p}+1}_{p,1}}\,\mathrm{d}t\\ &+\int_0^T \big\|\big[A(U_1)-A(U_F)\big](t)\big\|_{B^{\frac{d}{p}+2}_{p,1}}\|V(t)\|_{B^{\frac{d}{p}}_{p,1}}\,\mathrm{d}t\,.\end{aligned}$$

In the right-hand side, the two first lines are small as soon as both T and r are small: for the first one we can split $A(U_1)-A(0)$ into $A(U_1)-A(U_F)+A(U_F)-A(0)$ and use the triangular inequality while for the second one, we use the embedding of $\mathbb{E}_T^{d/p}$ into $\mathrm{L}^2([0,T];B^{d/p+1}_{p,1})$ by interpolation. Then, the last line is handled thanks to a Gronwall estimate and we can find T small enough so that $\|V\|_{\mathbb{E}_T^{d/p}} \le r$.

Similarly, we can prove that Θ is Lipschitz on the ball of $\mathbb{E}_T^{\frac{d}{p}}$ centered at U_F and of radius r with a Lipschitz constant smaller than one (for r and T small enough). This follows the same lines as the above computations: we fix U_1 and U_2 such that $\|U_i - U_F\|_{\mathbb{E}_T^{\frac{d}{p}}} \leq r$ and we note that thanks to [1, Proposition B.2]

$$
\begin{aligned}
\|U_1^\star - U_2^\star\|_{\mathbb{E}_T^{\frac{d}{p}}} &\lesssim_{T,\|U_1\|_{\mathbb{G}_T^p},\omega_{U_1}} \int_0^T \big\|(A(U_1) - A(U_2)(t))\big\|_{B_{p,1}^{\frac{d}{p}+1}} \|U_2^\star(t)\|_{B_{p,1}^{\frac{d}{p}+1}}\,\mathrm{d}t \\
&\quad + \int_0^T \big\|(A(U_1) - A(U_2)(t))\big\|_{\mathrm{L}^\infty} \|U_2^\star(t)\|_{B_{p,1}^{\frac{d}{p}+2}}\,\mathrm{d}t \\
&\lesssim_{T,r,\|U_F\|_{\mathbb{G}_T^p},\omega_{U_F}} \|U_1 - U_2\|_{\mathbb{E}_T^{\frac{d}{p}}}(r + \|U_F\|_{L_T^1 B_{p,1}^{\frac{d}{p}+2}}) + \|U_1 - U_2\|_{\mathbb{E}_T^{\frac{d}{p}}}(r + \|U_F\|_{L_T^2 B_{p,1}^{\frac{d}{p}+1}})
\end{aligned}
$$

where we have used that since $\|U_1 - U_F\|_{\mathbb{E}_T^{d/p}} \leq r$ then in particular for any (t,x) and (t',x') in Q_T there holds

$$
|U_1(t,x) - U_1(t',x')| \leq 2r + |U_F(t,x) - U_F(t',x')|
$$

whence

$$
\omega_{U_1} \leq r + \omega_{U_F}\,.
$$

The result follows choosing r and T small enough. □

References

[1] Isabelle Gallagher and Ayman Moussa. « The Cauchy problem for quasi-linear parabolic systems revisited ». en. In: *Journal de l'École polytechnique — Mathématiques* 12 (2025), pp. 1633–1676. DOI: 10.5802/jep.319. URL: https://jep.centre-mersenne.org/articles/10.5802/jep.319/.

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