Asymptotics of the Solutions of Hyperbolic Equations with a Skew–Symmetric Perturbation

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Abstract. *Using methods introduced by S. Schochet in [22], we compute the first term of an asymptotic expansion of the solutions of hyperbolic equations perturbated by a skew–symmetric linear operator. That result is first applied to two systems describing the motion of geophysic fluids: the rotating Euler equations and the primitive system of the quasigeostrophic equations. Finally in the last part, we study the slightly compressible Euler equations by application of that same result.*

1 Introduction and statement of results

We are interested in symmetric hyperbolic systems of the type

$$(\mathcal{H}^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t v^\varepsilon + q(v^\varepsilon, v^\varepsilon) + \dfrac{L(v^\varepsilon)}{\varepsilon} & = & 0 \quad \text{in} \quad \mathcal{T}^d \\ v^\varepsilon_{|t=0} & = & v_0, \end{array} \right.$$

where v^ε and v_0 are d'–dimensional vector fields, defined respectively on $\mathbf{R} \times \mathcal{T}^d$ and $\mathcal{T}^d$, where $\mathcal{T}^d$ is a periodic box: functions defined on $\mathcal{T}^d$ are $2\pi b_j$–periodic in each of the d directions of $\mathbf{R}^d$, where the b_j's are strictly positive real numbers. The linear operator L is skew–symmetric, and is defined by

$$L(v) = \mathcal{F}^{-1} \sum_{a=1}^{d'} i\omega^a(n) \left(\mathcal{F}v(n), e^a(n)\right) e^a(n), \tag{1.1}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the hermitian scalar product in $\mathbf{C}^{d'}$. We have written $\mathcal{F}v(n)$ for the discrete Fourier transform of v, and the $\omega^a(n)$ are real numbers and $(e^a(n))_{1\leq a\leq d'}$ is a set of orthonormal vectors for all n. Finally $q(u,v)$ is a bilinear form, of the type

$$q(u,v) = \left(Au \cdot \nabla v + Av \cdot \nabla u\right)/2, \tag{1.2}$$

where A is a linear operator, such that $Au = \left(A^j u\right)_{1\leq j\leq d}$, and for all j, $A^j u$ is a smooth, symmetric matrix. It can also be a form of the type $q(u,v) = P\left(Au \cdot \nabla v + Av \cdot \nabla u\right)/2$, where P is the L^2–orthogonal projector onto the space of divergence free vector fields — that form will be used in the examples of application we shall study, concerning inviscid fluids. In that case, we are not strictly speaking in the setting of hyperbolic equations; however, as we will see, the energy estimates we shall write will be true for (1.2) as well as for that bilinear form. So we shall note in the following

$$q(u,v) = \frac{1}{2}\,\mathcal{F}^{-1} \sum_{k+m=n} P(n) \left(\mathcal{F}A^j u(k) m_j \mathcal{F}v(m) + \mathcal{F}A^j v(k) m_j \mathcal{F}v(m) \right),$$

where $P(n)$ is a homogeneous function of order 0, defined on $\mathbf{Z}^d$. We have used the Einstein convention on the summation over repeated indexes: we have omitted the sum over j in the definition above, and we will keep that convention throughout this paper.

Singular limits in systems such as $(\mathcal{H}^\varepsilon)$ have been studied by a number of authors (among others, one can refer to [12], [14], [15], [16], [19] or [22] for instance). Existence results follow directly from the skew–symmetry of L (as the perturbation disappears in the energy estimates). However, to find the limit of the solutions, one cannot take the limit directly in the system $(\mathcal{H}^\varepsilon)$ (as $\partial_t v^\varepsilon$ is *a priori* not bounded in ε). An idea (as in [1], [12], or [22] for instance) is to "filter" the system by the semi–group $\mathcal{L}(t)$ associated with L, by defining the operator $\mathcal{L}(t)v = \mathcal{F}^{-1} \sum_{a=1}^{d} e^{-i\omega^a(n)t} \left(\mathcal{F}v(n), e^a(n) \right) e^a(n)$, with the same notations as for (1.1), and we consider $u^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon}) v^\varepsilon$, which satisfies

$$(\tilde{\mathcal{H}}^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t u^\varepsilon + \mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) & = & 0 \quad \text{in} \quad \mathcal{T}^d \\ u^\varepsilon_{|t=0} & = & v_0, \end{array} \right.$$

where we have written

$$\mathcal{Q}^\varepsilon(u,v) = \mathcal{L}(-\frac{t}{\varepsilon}) q \left(\mathcal{L}(\frac{t}{\varepsilon}) u, \mathcal{L}(\frac{t}{\varepsilon}) v \right). \tag{1.3}$$

We shall note $\mathcal{Q}(u,u)$ the limit in $\mathcal{D}'$ of $\mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon)$, which, as $\mathcal{L}$ preserves the H^σ norms, is also the limit of $\mathcal{Q}^\varepsilon(u,u)$. The operator $\mathcal{Q}$ satisfies the fundamental property of being independent of time, since by the non stationary phase theorem, $\mathcal{F}\mathcal{Q}(u,v)$ is equal to

$$\frac{1}{2} \sum_{\substack{k+m=n \\ \omega^a(k)+\omega^b(m)=\omega^c(n)}} \left(P(n) A^j u^a(k) m_j v^b(m) \right)^c (n) + \left(P(n) A^j v^a(k) m_j u^b(m) \right)^c (n),$$

where, to simplify the notations, we have written

$$A^j u^a(k) = \mathcal{F}A^j \left(\mathcal{F}^{-1} u^a(\cdot) \right)(k), \quad \text{with} \quad u^a(k) = \left(\mathcal{F}u(k), e^a(k) \right) e^a(k). \tag{1.4}$$

The next theorem is proved in [22]; the space H^σ is the homogeneous Sobolev space of order σ.

Theorem 0 *Consider $v_0 \in H^\sigma(\mathcal{T}^d)$, where $\sigma \geq \frac{d}{2} + 3$, and suppose that the solution u of*

$$(\mathcal{H}_0) \quad \left\{ \begin{array}{rcl} \partial_t u + \mathcal{Q}(u,u) & = & 0 \quad in \quad \mathcal{T}^d \\ u_{|t=0} & = & v_0 \end{array} \right.$$

is defined in $C^0([0,T], H^\sigma(\mathcal{T}^d))$. Then for ε small enough, the solution v^ε of the system $(\mathcal{H}^\varepsilon)$ satisfies $v^\varepsilon \in C^0([0,T], H^\sigma(\mathcal{T}^d))$, and

$$v^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon}) u = o(1) \quad in \quad C^0([0,T], H^{\sigma-1}(\mathcal{T}^d)).$$

The aim of this paper is to analyse more precisely what that $o(1)$ is made of, assuming, if need be, more regularity on the initial data and some additional properties on the perturbation operator L. For instance in the case when $(\mathcal{H}^\varepsilon)$ is the incompressible rotating Euler equation, A. Babin, A. Mahalov, and B. Nicolaenko in [1] find a rate of convergence of u^ε to u in $O(\varepsilon)$, and the proof of that result depends very much on "small divisor estimates" satisfied by the operator L. So let us introduce the following definition.

Definition 1.1 *We will say that L satisfies a small divisor estimate of order $r \geq 2$ and of degree α if there exists a constant $C_{r,\alpha}$ such that for all $(k_1,..,k_r)$ in $\mathbf{Z}^{rd}$, for all $(b_1,..,b_{r+1})$ in $\{1,..,d'\}^{r+1}$, we have*

$$\sum_{i=1}^r \omega^{b_i}(k_i) \neq \omega^{b_{r+1}}(\sum_{i=1}^r k_i) \quad \Rightarrow \quad |\sum_{i=1}^r \omega^{b_i}(k_i) - \omega^{b_{r+1}}(\sum_{i=1}^r k_i)|^{-1} \leq C_{r,\alpha} \prod_{i=1}^r (1+|k_i|)^\alpha .$$

In the following we will use the notation

$$\omega_{k,n}^{a,b,c} \stackrel{def}{=} \omega^a(k) + \omega^b(n-k) - \omega^c(n), \quad \omega_{k',k,n}^{a,b,c,d} \stackrel{def}{=} \omega^a(k') + \omega^b(k-k') + \omega^c(n-k) - \omega^d(n).$$

Remark. Let us note that it is proved in [11] that a large class of perturbations L satisfies such a small divisor estimate for almost all domains $\mathcal{T}^d$.

Theorem 1 *Let us suppose the perturbation L satisfies a second order small divisor estimate of degree $\alpha \geq 0$. Then if $v_0 \in H^{s+\alpha}$, where $s > \frac{d}{2}+4$, and if the solution u of $(\mathcal{H}_0)$ is in the space $C^0([0,T], H^{s+\alpha}(\mathcal{T}^d))$, then the filtered solution u^ε satisfies*

$$u^\varepsilon - u = \varepsilon h - \varepsilon \tilde{R}^\varepsilon_{osc}(u) + o(\varepsilon) \quad in \quad C^0([0,T], H^{s-3}(\mathcal{T}^d)),$$

where h is the solution in $C^0([0,T], H^{s-2}(\mathcal{T}^d))$ of a linearized limit equation

$$(\mathcal{H}_1) \quad \left\{ \begin{array}{rcl} \partial_t h + 2\mathcal{Q}(u,h) & = & R(u) \quad in \quad \mathcal{T}^d \\ h_{|t=0} & = & \tilde{R}^\varepsilon_{osc}(u)_{|t=0}, \end{array} \right.$$

and where $\mathcal{F}\tilde{R}^\varepsilon_{osc}(u)$ and $\mathcal{F}R(u)$ are respectively equal to

$$\sum_{k+m=n} \sum_{\omega_{k,n}^{a,b,c} \neq 0} \frac{i\, e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}}}{\omega_{k,n}^{a,b,c}} \Big(P(n)A^j u^a(k) m_j u^b(m)\Big)^c(n) \qquad and$$

$$\sum_{\substack{k+m=n \\ k'+m'=k}} \sum_{\substack{\omega_{k',k}^{a',b',a} \neq 0 \\ \omega_{k',k,n}^{a',b',b,c}=0}} \frac{i}{\omega_{k',k}^{a',b',a}} \Big(P(n)A^j u^b(m) k_j \Big(P(k)A^\ell u^{a'}(k') m'_\ell u^{b'}(m')\Big)^a(k)\Big)^c(n)$$

$$+ \sum_{\substack{k+m=n \\ k'+m'=k}} \sum_{\substack{\omega_{k',k}^{a',b',a} \neq 0 \\ \omega_{k',k,n}^{a',b',b,c}=0}} \frac{i}{\omega_{k',k}^{a',b',a}} \Big(P(n)A^j \Big(P(k)A^\ell u^{a'}(k') m'_\ell u^{b'}(m')\Big)^a(k) m_j u^b(m)\Big)^c(n),$$

with the notations defined in (1.4).

Remarks. That theorem means that the difference between u^ε and u, at the first order, is measured by a function h independent of ε, and an oscillating function $\tilde{R}^\varepsilon_{osc}(u)$. The function h satisfies a linear equation, where the forcing term is cubic in u; it is due to the fact that in a term such as $\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc}(u), u)$, which appears when one considers the equation satisfied by $\varepsilon^{-1}(u^\varepsilon - u) + \tilde{R}^\varepsilon_{osc}(u)$, the oscillations due to $\tilde{R}^\varepsilon_{osc}(u)$ and $\mathcal{Q}^\varepsilon$ can interfere and cancel out. Let us furthermore note that in [11], a similar result as Theorem 1 is proved in the parabolic case.

The proof of that theorem is written in section 2; it will be found using a more general lemma, Lemma 2.1 proved in section 3, concerning equations where the right–hand side oscillates fast in time. That type of function is the object of the coming definition.

Definition 1.2 *Let $T > 0$, $p \geq 1$ and $\sigma > \frac{d}{2}$ be fixed. We write $\vec{k_q} = (k_1, .., k_q)$, where $k_i \in \mathbf{Z}^d$, and will call $|\vec{k_q}| = \max_{1 \leq i \leq q} |k_i|$. Then a function $R^\varepsilon_{osc}(t)$ will be said to be (p,σ)–oscillating if it can be written as*

$$R^\varepsilon_{osc}(t) = \sum_{q \in \{1,..,p\}} R^\varepsilon_{q,osc}(t), \qquad \textit{where} \tag{1.5}$$

$$R^\varepsilon_{q,osc}(t) = \mathcal{F}^{-1} \sum_{\vec{k_q} \in K^n_q} e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})} r_0(n,\vec{k_q}) f^\varepsilon_1(t,k_1)..f^\varepsilon_q(t,k_q),$$

with $K^n_q = \left\{ \vec{k_q} \in \mathbf{Z}^{dq} \mid \sum_{i=1}^{q} k_i = n \textit{ and } \beta_q(n,\vec{k_q}) \neq 0 \right\}$, *and where r_0 and f^ε_i satisfy*

$$\exists\, (\alpha_i)_{1 \leq i \leq q},\ \alpha_i \geq 0, \qquad \textit{such that} \qquad |r_0(n,\vec{k_q})| \leq C\,(1+|k_1|)^{\alpha_1} .. (1+|k_q|)^{\alpha_q}\ ;$$

$$\forall\, i \in [1,..,q], \qquad \mathcal{F}^{-1}(f^\varepsilon_i(t,\cdot)) \quad \textit{is uniformly bounded in} \quad C^0([0,T], H^{\sigma+\alpha_i})$$

$$\textit{and} \qquad \exists\, \sigma_i > -\sigma \quad \textit{such that} \quad \mathcal{F}^{-1}(\partial_t f^\varepsilon_i(t,\cdot)) \quad \textit{is uniformly bounded in} \quad C^0([0,T], H^{\sigma_i})\,.$$

Remarks. As H^σ is an algebra since $\sigma > \frac{d}{2}$, then a (p,σ)–oscillating function is in particular bounded in $C^0([0,T], H^\sigma)$. For example, we will see that $\tilde{R}^\varepsilon_{osc}(u)$ is a $(2, s-1)$-oscillating function if $v_0 \in H^{s+\alpha}$.

In section 4, we will apply that theorem to two examples: the first example will be the equations governing the motion of inviscid, incompressible, rotating fluids: the tridimensional, divergence free unknown vector field is the velocity, transported *via* the Euler equation. If one considers the perturbation created by the Coriolis force, one obtains the following system, where $B = (0,0,1)$, and where p^ε is the pressure:

$$(RF^\varepsilon) \left\{ \begin{array}{rcl} \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + \dfrac{v^\varepsilon \times B}{\varepsilon} & = & -\nabla p^\varepsilon \quad \text{in} \quad \mathcal{T}^3 \\ \operatorname{div} v^\varepsilon & = & 0 \quad \text{in} \quad \mathcal{T}^3 \\ v^\varepsilon_{|t=0} & = & v_0\,. \end{array} \right.$$

Such a system is namely the object of studies by A. Babin, A. Mahalov, and B. Nicolaenko in [1] and [2]. Let us note that the system (RF^ε) is also studied by T. Colin and P. Fabrie in [7], by E. Grenier in [12], and also in [9]–[11], in the viscous case (*i.e.* in the case of the perturbed Navier–Stokes equations).

Another typical example is that of the primitive system of the quasigeostrophic equations:

$$(PE^\varepsilon)\left\{\begin{array}{rcl} \partial_t U^\varepsilon + v^\varepsilon\cdot\nabla U^\varepsilon + \varepsilon^{-1}LU^\varepsilon & = & \varepsilon^{-1}(-\nabla\Phi^\varepsilon,0) \quad \text{in} \quad \mathcal{T}^3 \\ \operatorname{div} v^\varepsilon & = & 0 \quad \text{in} \quad \mathcal{T}^3 \\ U^\varepsilon_{|t=0} & = & U_0 \end{array}\right.$$

with $L = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}$, where the unknown vector field is $U^\varepsilon = (v^\varepsilon, T^\varepsilon)$, v^ε representing the velocity, and the scalar field T^ε representing the potential temperature. The parameter F^{-1} is a strictly positive constant (see [8]). For the derivation of such a model, one can refer to the work of J.-L. Lions, R. Temam and S. Wang in [18]. In section 4 we will prove, using Theorem 1, the following result.

Theorem 2 *For almost all periodic boxes, the following result is true. Let $s > \frac{5}{2}$ be a real number. If v_0 (resp. U_0) is smooth enough, then the life span T^ε_* of (RF^ε) (resp. (PE^ε)) satisfies $\lim_{\varepsilon\to 0} T^\varepsilon_* = \infty$, and the solution v^ε (resp. U^ε) is such that*

$$\text{for all} \quad T>0, \quad u^\varepsilon = u + \varepsilon h - \varepsilon\tilde{R}^\varepsilon_{osc} + o(\varepsilon), \quad \text{in} \quad C^0([0,T],H^s(\mathcal{T}^d)),$$

$$\text{resp.} \qquad \tilde{U}^\varepsilon = \tilde{U} + \varepsilon h - \varepsilon\tilde{R}^\varepsilon_{osc} + o(\varepsilon), \quad \text{in} \quad C^0([0,T],H^s(\mathcal{T}^d)),$$

with the notations of Theorem 1, in particular where h is the solution of the linearized limit system $(\mathcal{H}_1)$ associated with (RF^ε) (resp. (PE^ε)), and where u (resp. $\tilde{U}$), solution of $(\mathcal{H}_0)$, is the limit of the filtered solution $u^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})v^\varepsilon$ (resp. $\tilde{U}^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})U^\varepsilon$).

Remarks. In [3], T. Beale and A. Bourgeois study and validate the system (PE^ε), and prove a convergence theorem, in the case when the initial data is well prepared: if the initial data converges at $O(\varepsilon)$ speed to an element of the kernel of PL noted $U_{0,QG}$, then the solution of (PE^ε) converges at $O(\varepsilon)$ speed to the solution of $(\mathcal{H}_0)$. Similarly in the case of the whole space, D. Iftimie proves in [13] that the life span of (PE^ε) goes to infinity with ε, and also proves a convergence result in the well prepared case, *i.e.* when the initial data converges to a field $U_{0,QG}$. Finally, in the viscous case, and also in the whole space, J.-Y. Chemin proves in [6] that (PE^ε) is globally well posed when U_0 is close enough to a field $U_{0,QG}$. Here there is no such assumption on the initial data. Similarly, for those examples, convergence and life span results have been proved by A. Babin, A. Mahalov, and B. Nicolaenko in [1] and [2], and by Y. Brenier ([4]), without however such an expansion as that written in Theorem 2.

Moreover, as we will see in section 4, the fact that the result is true for almost all values of the b_j's is linked to two different and independent aspects: the first one is that the small divisor estimates, needed in order to have a rate of convergence in $O(\varepsilon)$, are satisfied for almost all periodic boxes. The second reason is totally independent: we shall see that for almost all values of the b_j's, in both the case of (RF^ε) and (PE^ε), the limit system $(\mathcal{H}_0)$ can be diagonalized in such a way as to guarantee that it is globally well posed, and that naturally leads to the result on T^ε_*. That second limitation on the b_j's defines "non resonant domains".

Let us finally note that in [2], a precise second order small divisor estimate is computed in those particular cases: they find a degree $\alpha = 4 + \varepsilon_0$, but however that only holds when a_3 is outside a set of (small) strictly positive measure: that is made precise in their Theorem 3.2.

Finally in the last section, we will consider the case of slightly compressible, isentropic, inviscid fluids. Writing v^ε for the velocity and ρ^ε for the density of the fluid, where ε is essentially the Mach number, one obtains, after some adimensionalization (see [12], [16], [21], [22]),

$$\left\{\begin{array}{rcl} \partial_t v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + \dfrac{1}{\varepsilon^2\rho^\varepsilon}\nabla p(\rho^\varepsilon) & = & 0 \quad \text{in} \quad \mathcal{T}^d \\ \partial_t \rho^\varepsilon + v^\varepsilon\cdot\nabla\rho^\varepsilon + \rho^\varepsilon \operatorname{div} v^\varepsilon & = & 0 \quad \text{in} \quad \mathcal{T}^d \\ (v^\varepsilon_{|t=0}, \rho^\varepsilon_{|t=0}) & = & (v^\varepsilon_0, \rho^\varepsilon_0), \end{array}\right.$$

where p is related to ρ^ε by $p(\rho^\varepsilon) = \rho^{\varepsilon^\gamma}$, with $\gamma > 1$. That system does not quite have the form $(\mathcal{H}^\varepsilon)$; such a form can be obtained by defining

$$c^\varepsilon = \frac{2}{\gamma-1}\sqrt{\partial_{\rho^\varepsilon} p(\rho^\varepsilon)}\,,$$

which puts the previous system into a symmetric form, and then by writing

$$c^\varepsilon = \bar c_0 + \varepsilon\,\tilde c^\varepsilon,$$

where $\bar c_0$ is a constant. After a few computations, we come up with

$$(\mathcal{CE}^\varepsilon) \quad \left\{\begin{array}{rcl} \partial_t v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + \bar\gamma\tilde c^\varepsilon\nabla\tilde c^\varepsilon + \bar\gamma\bar c_0\,\dfrac{\nabla\tilde c^\varepsilon}{\varepsilon} & = & 0 \quad \text{in} \quad \mathcal{T}^d \\ \partial_t\tilde c^\varepsilon + v^\varepsilon\cdot\nabla\tilde c^\varepsilon + \bar\gamma\tilde c^\varepsilon \operatorname{div} v^\varepsilon + \bar\gamma\bar c_0\,\dfrac{\operatorname{div} v^\varepsilon}{\varepsilon} & = & 0 \quad \text{in} \quad \mathcal{T}^d \\ (v^\varepsilon_{|t=0}, \tilde c^\varepsilon_{|t=0}) & = & (v_0, \tilde c_0), \end{array}\right.$$

where we have written $\bar\gamma = \dfrac{\gamma-1}{2}$, and $c^\varepsilon_{|t=0} = \bar c_0 + \varepsilon\,\tilde c_0$. Let us note that the function $\tilde c^\varepsilon$ satisfies an acoustic wave type equation. If $V^\varepsilon = (v^\varepsilon, \tilde c^\varepsilon)$, then V^ε satisfies an equation of the type $(\mathcal{H}^\varepsilon)$, where $L(v^\varepsilon, \tilde c^\varepsilon) = (\bar\gamma\bar c_0\nabla\tilde c^\varepsilon, \bar\gamma\bar c_0 \operatorname{div} v^\varepsilon)$ is the penalization operator. In the following, we will systematically denote by v_{div} the projection of any vector field v onto the space of divergence free vector fields, and we will define $\tilde v = v - v_{div}$. Similarly, for any $V = (v, c)$, we will write $V_{div} = (v_{div}, 0)$ and $\tilde V = (\tilde v, c)$.

Theorem 3 *For almost all periodic boxes, there exists $\alpha \geq 0$ such that if $V_0 = (v_0, \tilde c_0)$ is in $H^{s+\alpha}(\mathcal{T}^d)$, with $s > \frac{d}{2}+4$, and if $U = (u, c)$ is the solution in $C^0([0,T^*[, H^{s+\alpha}(\mathcal{T}^d))$ of the limit system*

$$(\mathcal{CE}_0) \quad \left\{\begin{array}{rcl} \partial_t u_{div} + u_{div}\cdot\nabla u_{div} & = & -\nabla p \quad in \quad \mathcal{T}^d \\ \partial_t\tilde U + \mathcal{Q}(U, \tilde U) & = & 0 \quad in \quad \mathcal{T}^d \\ U_{div|t=0} & = & (v_{0,div}, 0) \\ \tilde U_{|t=0} & = & (\tilde v_0, \tilde c_0), \end{array}\right.$$

then the solution $V^\varepsilon = (v^\varepsilon, \tilde c^\varepsilon)$ of $(\mathcal{CE}^\varepsilon)$ is such that its filtered associate $U^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})V^\varepsilon$ satisfies, for all $T < T^*$ and for ε small enough, with the notations of Theorem 1,

$$U^\varepsilon = U + \varepsilon h + \varepsilon\tilde R^\varepsilon_{osc} + o(\varepsilon) \quad in \quad C^0([0,T], H^{s-3}(\mathcal{T}^d)).$$

Remarks. Let us note that, contrary to the studies in [16] and [17], that result does not require any assumption on the initial data of the"well prepared" type (*i.e.* we have supposed neither $\operatorname{div} v_0 = 0$ nor $\nabla\tilde c_0 = 0$). That "ill prepared" case is also the subject of studies by E. Grenier in [12] and S. Schochet in [22], but without the computation of the first order terms; such a computation is carried out in [22], but in the well prepared case only. In the case of the whole space, and not necessarily for well prepared data, S. Ukai proves in [23] that the limit of v^ε satisfies the incompressible Euler equation, but the uniform convergence breaks near $t = 0$. That is due to decay estimates, which are not true in the periodic case.

2 Proof of Theorem 1

The proof of Theorem 1 is going to be based essentially on the following lemma, the proof of which is left to section 3.

Lemma 2.1 *Let $T > 0$ and $\sigma > \frac{d}{2} + 2$ be two real numbers, let b^ε be a family of functions, bounded in the space $C^0([0,T], H^\sigma(\mathcal{T}^d))$, and let a_0^ε be a function going to zero with ε in $H^{\sigma-1}(\mathcal{T}^d)$. Let $\mathcal{Q}^\varepsilon$ be as in (1.3), and finally let R^ε_{osc} be a $(p, \sigma-1)$–oscillating function, with $p \geq 1$, and F^ε be a function going to zero with ε in $C^0([0,T], H^{\sigma-1}(\mathcal{T}^d))$. Then the function a^ε, solution of*

$$\left\{ \begin{array}{rcl} \partial_t a^\varepsilon + \mathcal{Q}^\varepsilon(a^\varepsilon, b^\varepsilon) & = & R^\varepsilon_{osc} + F^\varepsilon \quad in \quad \mathcal{T}^d \\ a^\varepsilon_{|t=0} & = & a_0^\varepsilon, \end{array} \right. \tag{2.1}$$

is an $o(1)$ in the space $C^0([0,T], H^{\sigma-1}(\mathcal{T}^d))$.

Moreover we will use constantly the following estimate (see for instance [19], [20]), where we have noted $(\cdot \mid \cdot)_{H^\tau}$ the scalar product in H^τ:

$$\text{for all } \tau > \frac{d}{2}, \quad |(\mathcal{Q}^\varepsilon(a,b) \mid b)_{H^\tau}| \leq C|b|_\tau \|\nabla b\|_{L^\infty}|a|_\tau + C|b|_\tau^2 \left(\|\nabla a\|_{L^\infty} + |a|_{\tau+1}\right), \tag{2.2}$$

as well as the following result:

Lemma 2.2 *If $v_0 \in H^{s+\alpha}$, with $s + \alpha > \frac{d}{2}$, then for all integers $j \in [0, s+\alpha-\frac{d}{2}[$, u is an element of $C^j([0,T], H^{s-j+\alpha})$, and u^ε is bounded in that same space.*

Proof. The function $\partial_t u$ has the smoothness of $\mathcal{Q}(u,u)$, so since H^τ is an algebra if $\tau > \frac{d}{2}$, $\partial_t u$ is an element of the space $C^0([0,T], H^{s-1+\alpha})$. Then an immediate recursive argument shows that as long as $s - j + \alpha > \frac{d}{2}$, if for all $k \in [0, .., j-1]$ we have $\partial_t^k u \in C^0([0,T], H^{s-k+\alpha})$, then $\partial_t^j u$ has the smoothness of $\mathcal{Q}(u, \partial_t^{j-1} u)$, hence is in the space $C^0([0,T], H^{s-j+\alpha})$. The proof for u^ε is identical. $\square$

Then let u^ε and u be the solutions respectively of the systems $(\mathcal{H}^\varepsilon)$ and $(\mathcal{H}_0)$, and let us define the function $w^\varepsilon = u^\varepsilon - u$, which satisfies the equation

$$\partial_t w^\varepsilon + \mathcal{Q}^\varepsilon(w^\varepsilon, w^\varepsilon + 2u) = \mathcal{Q}(u,u) - \mathcal{Q}^\varepsilon(u,u), \tag{2.3}$$

with an initial data equal to zero since $u^\varepsilon_{|t=0} = u_{|t=0} = v_0$. By the non stationary phase theorem, we have

$$\mathcal{Q}^\varepsilon(u,u) - \mathcal{Q}(u,u) = \mathcal{F}^{-1} \sum_{k+m=n} \sum_{\omega^{a,b,c}_{k,n} \neq 0} e^{-i\frac{t}{\varepsilon}\omega^{a,b,c}_{k,n}} \left(P(n) A^j u^a(k) m_j u^b(m), e^c(n)\right) e^c(n),$$

with the notations given in the introduction. Here and throughout the paper, we have omitted to write the dependence on t of u. By Theorem 0, we have

$$w^\varepsilon = o(1) \quad \text{in} \quad C^0([0,T], H^{s-1+\alpha}). \tag{2.4}$$

Before we go any further in the proof of the theorem, let us note that the right–hand side of (2.3) oscillates in time. So let us consider for a moment the following model equation:

$$\partial_t \varphi - F(\varphi) = \cos\frac{t}{\varepsilon}\,.$$

If one defines $\psi = \varphi - \varepsilon \sin\frac{t}{\varepsilon}$, one can note that ψ and φ have the same H^s norm for all s, up to an ε, and that ψ satisfies

$$\partial_t \psi = F\left(\psi + \varepsilon \sin\frac{t}{\varepsilon}\right),$$

which is an equation where, up to an ε, the oscillating part has disappeared. In a similar way to that model problem, and following [22], let us define $\psi^\varepsilon = w^\varepsilon + \varepsilon \tilde{R}^\varepsilon_{osc}(u)$, where, as defined in the introduction,

$$\tilde{R}^\varepsilon_{osc}(u) \stackrel{def}{=} \mathcal{F}^{-1} \sum_{k+m=n} \sum_{\omega^{a,b,c}_{k,n} \neq 0} \frac{i\, e^{-i\frac{t}{\varepsilon}\omega^{a,b,c}_{k,n}}}{\omega^{a,b,c}_{k,n}} \Big(P(n) A^j u^a(k) m_j u^b(m), e^c(n) \Big)\, e^c(n). \tag{2.5}$$

Lemma 2.3 *Under the assumptions of Theorem* 1, $\tilde{R}^\varepsilon_{osc}(u)$ *is a* $(2, s-1)$*-oscillating function.*

Proof. The proof of that lemma is very simple, considering the second order small divisor estimate satisfied by L. Then since we have assumed that $u \in C^0([0,T], H^{s+\alpha})$, the result follows immediately, taking for instance $r_0(n,k,m) = P(n)(\omega^{a,b,c}_{k,n})^{-1}$, $f_1^\varepsilon(t,k) = A^j u^a(t,k)$, and $f_2^\varepsilon(t,m) = m_j u^b(t,m)$, with the notations of Definition 1.2. The fact that $\mathcal{F}^{-1}\partial_t f_i^\varepsilon(t,\cdot)$ is bounded in $C^0([0,T], H^{\sigma_i})$, for some σ_i, is due to Lemma 2.2. $\square$

Then (2.4) and Lemma 2.3 imply that

$$\psi^\varepsilon = o(1) \quad \text{in} \quad C^0([0,T], H^{s-1}). \tag{2.6}$$

Now let us define $\psi_1^\varepsilon = \varepsilon^{-1}\psi^\varepsilon$, which satisfies the equation

$$\partial_t \psi_1^\varepsilon + \mathcal{Q}^\varepsilon(\psi_1^\varepsilon, \psi^\varepsilon + 2u - 2\varepsilon \tilde{R}^\varepsilon_{osc}) = \tilde{R}^{\varepsilon,t}_{osc}(u) + \mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc}, 2u - \varepsilon \tilde{R}^\varepsilon_{osc}), \tag{2.7}$$

where we have defined $\tilde{R}^{\varepsilon,t}_{osc}$ by

$$\tilde{R}^{\varepsilon,t}_{osc}(u) \stackrel{def}{=} \mathcal{F}^{-1} \sum_{k+m=n} \sum_{\omega^{a,b,c}_{k,n} \neq 0} \frac{i\, e^{-i\frac{t}{\varepsilon}\omega^{a,b,c}_{k,n}}}{\omega^{a,b,c}_{k,n}} \partial_t \Big(P(n) A^j u^a(k) m_j u^b(m), e^c(n) \Big)\, e^c(n).$$

Lemma 2.4 *Under the assumptions of Theorem* 1, $\tilde{R}^{\varepsilon,t}_{osc}(u)$ *is a* $(2, s-2)$*–oscillating function.*

Proof. It lies on the fact that L satisfies a second order small divisor estimate of degree α, as well as on Lemma 2.2, which implies that $\partial_t u \in C^0([0,T], H^{s-1+\alpha})$. Then we can take, with the notations of Definition 1.2, $r_0(n,k,m) = P(n)(\omega^{a,b,c}_{k,n})^{-1}$, along with

$$f_1^\varepsilon(t,k) = A^j u^a(t,k) \quad \text{and} \quad f_2^\varepsilon(t,m) = m_j \partial_t u^b(t,m), \quad \text{or symmetrically}$$

$$f_1^\varepsilon(t,k) = A^j \partial_t u^a(t,k) \quad \text{and} \quad f_2^\varepsilon(t,m) = m_j u^b(t,m).$$

Then $\mathcal{F}^{-1} f_i^\varepsilon(t,\cdot) \in C^0([0,T], H^{s-2+\alpha})$ and $\mathcal{F}^{-1}\partial_t f_i^\varepsilon(t,\cdot) \in C^0([0,T], H^{s-3+\alpha})$ by Lemma 2.2, so $\tilde{R}^{\varepsilon,t}_{osc}(u)$ can be written as in (1.5), with $p = 2$ and $\sigma = s-2$. $\square$

Proposition 2.1 *Under the assumptions of Theorem* 1, ψ_1^ε *is bounded in* $C^0([0,T],H^{s-2})$.

Proof. Lemma 2.3 implies that the term $\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc}, 2u-\tilde{R}^\varepsilon_{osc})$ is bounded in $C^0([0,T],H^{s-2})$, and so is the term $\tilde{R}^{\varepsilon,t}_{osc}(u)$, by Lemma 2.4. But $\psi^\varepsilon+2u-2\varepsilon\tilde{R}^\varepsilon_{osc}$ is bounded in $C^0([0,T],H^{s-1})$, using in particular (2.6). Then the estimate recalled in (2.2) leads to the proposition, from classical energy estimates on hyperbolic equations of the type (2.7) (see [19] or [20] for instance): as $s>\frac{d}{2}+4$, we have in particular $H^{s-3}\subset L^\infty$, which leads to

$$\frac{1}{2}\frac{d}{dt}|\psi_1^\varepsilon|^2_{s-2}\leq C|\tilde{R}^{\varepsilon,t}_{osc}+\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc},2u-\tilde{R}^\varepsilon_{osc})|_{s-2}|\psi_1^\varepsilon|_{s-2}+C|\psi_1^\varepsilon|^2_{s-2}|\psi^\varepsilon+2u-2\varepsilon\tilde{R}^\varepsilon_{osc}|_{s-1}. \quad (2.8)$$

Then a Gronwall estimate gives the result:

$$\begin{aligned}|\psi_1^\varepsilon(t)|_{s-2}\leq C\left(|\psi^\varepsilon_{1|t=0}|_{s-2}+\int_0^t|\tilde{R}^{\varepsilon,t}_{osc}+\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc},2u-\tilde{R}^\varepsilon_{osc})|_{s-2}(\tau)\,d\tau\right)\\ \times\quad\exp\int_0^t C|\psi^\varepsilon+2u-2\varepsilon\tilde{R}^\varepsilon_{osc}|_{s-1}(\tau)\,d\tau,\end{aligned}$$

where $\psi^\varepsilon_{1|t=0}=\tilde{R}^\varepsilon_{osc}(u)_{|t=0}$ is in particular bounded in H^{s-2}. The proposition is proved. □

Remark. As a direct consequence, we have

$$\psi^\varepsilon=O(\varepsilon)\quad\text{in}\quad C^0([0,T],H^{s-2}). \quad (2.9)$$

Proposition 2.2 *Under the assumptions of Theorem* 1, *the equation* (2.7) *can be written as*

$$\partial_t\psi_1^\varepsilon+2\mathcal{Q}^\varepsilon(u,\psi_1^\varepsilon)=R^{1,\varepsilon}_{osc}(u)+\varepsilon R^{2,\varepsilon}(u)+R(u), \quad (2.10)$$

where $R^{1,\varepsilon}_{osc}$ *is a* $(3,s-2)$*-oscillating term, and where* $R(u)$ *is independent of* ε *and is an element of* $C^0([0,T],H^{s-2}(\mathcal{T}^d))$. *Finally* $R^{2,\varepsilon}(u)$ *is bounded in the space* $C^0([0,T],H^{s-3}(\mathcal{T}^d))$.

Proof. Lemma 2.3 and Proposition 2.1, associated with (2.9), imply that $\mathcal{Q}^\varepsilon(\psi_1^\varepsilon,\psi^\varepsilon-2\varepsilon\tilde{R}^\varepsilon_{osc})$ is a $O(\varepsilon)$ in $C^0([0,T],H^{s-3})$, and the same goes for the term $\varepsilon\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc},\tilde{R}^\varepsilon_{osc})$, so we can write

$$\mathcal{Q}^\varepsilon(\psi_1^\varepsilon,\psi^\varepsilon-2\varepsilon\tilde{R}^\varepsilon_{osc})+\varepsilon\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc},\tilde{R}^\varepsilon_{osc})=\varepsilon R^{2,\varepsilon}(u),$$

where $R^{2,\varepsilon}(u)$ is bounded in $C^0([0,T],H^{s-3})$. So equation (2.7) becomes

$$\partial_t\psi_1^\varepsilon+2\mathcal{Q}^\varepsilon(u,\psi_1^\varepsilon)=\tilde{R}^{\varepsilon,t}_{osc}(u)+2\mathcal{Q}^\varepsilon(u,R^\varepsilon_{osc})+\varepsilon R^{2,\varepsilon}(u).$$

Moreover, Lemma 2.4 implies that $\tilde{R}^{\varepsilon,t}_{osc}$ can be written in the form $R^{1,\varepsilon}_{osc}$ of Proposition 2.2, so we are left with the study of the term $\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc},u)$. The study is somewhat complicated by the fact that the oscillations due to $\tilde{R}^\varepsilon_{osc}$ can interfere with those of $\mathcal{Q}^\varepsilon$: we have

$$\begin{aligned}2\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc},u)\ &=\ \mathcal{F}^{-1}\sum_{k+m=n}e^{-i\frac{t}{\varepsilon}\omega^{a,b,c}_{k,n}}\left(P(n)A^j(\tilde{R}^{\varepsilon,a}_{osc}(k))m_ju^b(m),e^c(n)\right)e^c(n)\\ &+\ \mathcal{F}^{-1}\sum_{k+m=n}e^{-i\frac{t}{\varepsilon}\omega^{a,b,c}_{k,n}}\left(P(n)A^j(u^b(m))k_j\tilde{R}^{\varepsilon,a}_{osc}(k),e^c(n)\right)e^c(n),\end{aligned}$$

with the notations of the introduction, and $\tilde R^{\varepsilon,a}_{osc}(k)$ is equal to

$$\mathcal{F}^{-1}\sum_{k'+m'=k}\;\sum_{\omega^{a',b',a}_{k',k}\neq 0}\frac{i\,e^{-i\frac{t}{\varepsilon}\omega^{a',b',a}_{k',k}}}{\omega^{a',b',a}_{k',k}}P(k)A^\ell u^{a'}(k')m'_j u^{b'}(m').$$

So in $\mathcal{Q}^\varepsilon(\tilde R^\varepsilon_{osc},u)$ the oscillation phase is $\omega^{a,b,c}_{k,n}+\omega^{a',b',a}_{k',k}=\omega^b(n-k)-\omega^c(n)+\omega^{a'}(k')+\omega^{b'}(k-k')$. So one can consider separately the case when that phase is not zero, which we shall note $\mathcal{Q}^\varepsilon(\tilde R^\varepsilon_{osc},u)_{osc}$, and the case when it is zero: in the latter case there is no more dependence on ε, and that leads to the term $R(u)$, whereas the former corresponds to an oscillating term. So we have the following expression for $\mathcal{F}R(u)$:

$$\sum_{\substack{k+m=n\\ k'+m'=k}}\;\sum_{\substack{\omega^{a',b',a}_{k',k}\neq 0\\ \omega^{a',b',b,c}_{k',k,n}=0}}\frac{i}{\omega^{a',b',a}_{k',k}}\left(P(n)A^j u^b(m)k_j\left(P(k)A^\ell u^{a'}(k')m'_\ell u^{b'}(m')\right)^a(k)\right)^c(n)$$

$$+\sum_{\substack{m+k'+m'=n\\ \omega^{a',b',a}_{k',k}\neq 0\\ \omega^{a',b',b,c}_{k',k,n}=0}}\frac{i}{\omega^{a',b',a}_{k',n-k}}\left(P(n)A^j\left(P(n-m)A^\ell u^{a'}(k')m'_\ell u^{b'}(m')\right)^a(n-m)m_j u^b(m)\right)^c(n),$$

which is an element of $C^0([0,T],H^{s-2})$, since $u\in C^0([0,T],H^{s+\alpha})$. Furthermore, the function $\mathcal{Q}^\varepsilon(\tilde R^\varepsilon_{osc},u)_{osc}$ can be written as $R^\varepsilon_{3,osc}$ of Definition 1.2, with $\sigma=s-2$: for example we can take $r_0(n,k',m',m)$ of the type $P(n)P(n-m)(\omega^{a',b',a}_{k',n-m})^{-1}(n_j-m_j)m'_\ell$, which satisfies

$$|r_0(n,k',m',m)|\le C(1+|k'|)^{\alpha+1}(1+|m'|)^{\alpha+2},$$

and then $f^\varepsilon_1(t,k')=A^\ell u^{a'}(k')$, $f^\varepsilon_2(t,m')=u^{b'}(m')$, $f^\varepsilon_3(t,m)=A^j u^b(m)$. Then we just have to recall that u is an element of $C^0([0,T],H^{s+\alpha})$, as well as the fact that $\partial_t u$ is in $C^0([0,T],H^{s-1+\alpha})$ by Lemma 2.2, and the result follows. So $\mathcal{Q}^\varepsilon(\tilde R^\varepsilon_{osc},u)_{osc}$ can be written as the term $R^{1,\varepsilon}_{osc}$, and Proposition 2.2 is proved. □

Now let us define the functions h and w^ε_1, where $w^\varepsilon_1=\psi^\varepsilon_1-h$ and h is a solution of

$$(\mathcal{H}_1)\quad\left\{\begin{array}{rcl}\partial_t h+2\mathcal{Q}(u,h)&=&R(u)\quad\text{in}\quad\mathcal{T}^d\\ h_{|t=0}&=&\tilde R^\varepsilon_{osc}(u)_{|t=0}.\end{array}\right.$$

By looking at the definition of $\tilde R^\varepsilon_{osc}(u)$, in (2.5), one can note that $h_{|t=0}$ does not depend on ε.

Proposition 2.3 *Under the assumptions of Theorem 1, h is in $C^0([0,T],H^{s-2})$, and w^ε_1 is an $o(1)$ in $C^0([0,T],H^{s-3})$.*

Remark. Let us suppose that Proposition 2.3 has been proved. Then we have

$$\begin{array}{rcl}u^\varepsilon-u&=&\varepsilon\,\psi^\varepsilon_1-\varepsilon\,\tilde R^\varepsilon_{osc}(u)\\ &=&\varepsilon\,h-\varepsilon\,\tilde R^\varepsilon_{osc}(u)+\varepsilon\,w^\varepsilon_1,\end{array}$$

with $w_1^\varepsilon = o(1)$ in $C^0([0,T],H^{s-3})$, so the first order expansion of Theorem 1 is found.

Proof of Proposition 2.3. A classical estimate (of the same type as (2.8)) implies directly, as $R(u)$ is an element of $C^0([0,T],H^{s-2})$ according to Proposition 2.2, that h is in the space $C^0([0,T],H^{s-2})$. So the result on h is proved.

Now let us study $w_1^\varepsilon = \psi_1^\varepsilon - h$, which satisfies the following equation:

$$\partial_t w_1^\varepsilon + 2\mathcal{Q}^\varepsilon(u, w_1^\varepsilon) = R_{osc}^{1,\varepsilon}(u) + 2\mathcal{Q}(h,u) - 2\mathcal{Q}^\varepsilon(h,u) + \varepsilon R^{2,\varepsilon}(u). \tag{2.11}$$

Lemma 2.5 *Under the assumptions of Theorem 1, the right–hand side of* (2.11) *is of the type $R_{osc}^\varepsilon + F^\varepsilon$, where R_{osc}^ε is a $(3, s-3)$–oscillating term, and where F^ε goes to zero with ε, in $C^0([0,T],H^{s-3})$.*

Proof of Lemma 2.5. The term $R_{osc}^{1,\varepsilon}(u)$ is a $(3, s-2)$–oscillating function, according to Proposition 2.2. Furthermore, we have

$$\begin{aligned} 2\mathcal{Q}^\varepsilon(h,u) - 2\mathcal{Q}(h,u) &= \mathcal{F}^{-1} \sum_{k+m=n} \sum_{\omega_{k,n}^{a,b,c} \neq 0} e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} \left(P(n)A^j h^a(k) m_j u^b(m)\right)^c (n) \\ &+ \mathcal{F}^{-1} \sum_{k+m=n} \sum_{\omega_{k,n}^{a,b,c} \neq 0} e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} \left(P(n)A^j u^a(k) m_j h^b(m)\right)^c (n). \end{aligned}$$

But we have seen in the first part of Proposition 2.3 that $h \in C^0([0,T],H^{s-2})$ if u is in the space $C^0([0,T],H^{s+\alpha})$. Moreover $\partial_t h$ is in $C^0([0,T],H^{s-3})$, for the same type of reason as in Lemma 2.2, so $\mathcal{Q}(h,u) - \mathcal{Q}^\varepsilon(h,u)$ is a $(2, s-3)$–oscillating term.

Hence $R_{osc}^{1,\varepsilon}(u) + 2\mathcal{Q}(h,u) - 2\mathcal{Q}^\varepsilon(h,u)$ is a $(3, s-3)$–oscillating term, and Lemma 2.5 is proved, taking $F^\varepsilon = \varepsilon R^{2,\varepsilon}(u)$. □

End of the proof of Proposition 2.3. Now it is just a matter of using Lemma 2.1, with $\sigma = s-2$, since moreover w_1^ε is initially equal to zero. We get directly $w_1^\varepsilon = o(1)$ in $C^0([0,T],H^{s-3})$. So Proposition 2.3 is proved, and with it, Theorem 1. □

Remark. There is *a priori* no reason why one should stop the expansion at order 1; it seems clear that if one supposes more smoothness on the initial data, along with higher order small divisor estimate assumptions on the perturbation, too high an increase in the length of the computations is the only obstacle to writing an expansion of u^ε at any order.

3 Proof of Lemma 2.1

The aim of this section is to prove Lemma 2.1, stated in section 2. It concerns equations where there is a highly oscillating term R_{osc}^ε, in the sense of Definition 1.2, on the right hand side. The difficulty is due to the fact that R_{osc}^ε goes weakly to zero with ε, and not strongly (that is simply due to the non stationary phase theorem). The idea in [22] is to decompose R_{osc}^ε into two parts, cutting at an arbitrary frequency N. We shall show that the high frequency terms converge strongly to zero when N goes to infinity, whereas the low frequency terms can be eliminated, up to an ε, by a change of function.

So let us write equation (2.1) in the form

$$\partial_t a^\varepsilon + \mathcal{Q}^\varepsilon(b^\varepsilon, a^\varepsilon) = R^\varepsilon_{osc,N} + R^{\varepsilon,N}_{osc} + F^\varepsilon,$$

where, writing $\mathbf{1}_X$ for the characteristic function of X, we have defined

$$\begin{aligned} R^\varepsilon_{osc,N} &\stackrel{def}{=} \sum_{q\in\{1,..,p\}} R^\varepsilon_{q,osc,N} \\ &= \sum_{q\in\{1,..,p\}} \mathcal{F}^{-1}\left(\mathbf{1}_{\{|\cdot|,|\vec{k_q}|\le N\}}\mathcal{F}R^\varepsilon_{q,osc}(\cdot)\right), \\ R^{\varepsilon,N}_{osc} &\stackrel{def}{=} R^\varepsilon_{osc} - R^\varepsilon_{osc,N}. \end{aligned}$$

One can notice that there is a finite number of low frequency terms (number which depends on N), which are a finite number of oscillating functions in time. So in a similar way to the model problem $\partial_t\varphi - F(\varphi) = \cos\frac{t}{\varepsilon}$ discussed in section 2, let us define

$$\psi^\varepsilon_N = a^\varepsilon + \varepsilon\tilde{R}^\varepsilon_{osc,N}, \tag{3.1}$$

where $\tilde{R}^\varepsilon_{osc,N}$ is defined by $\tilde{R}^\varepsilon_{osc,N} = \sum\limits_{q\in\{1,..,p\}} \tilde{R}^\varepsilon_{q,osc,N}$, with

$$\mathcal{F}\tilde{R}^\varepsilon_{q,osc,N}(n) \stackrel{def}{=} \sum_{|n|,|\vec{k_q}|\le N}\ \sum_{\vec{k_q}\in K^n_q} \frac{i\, e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})}}{\beta_q(n,\vec{k_q})} r_0(n,\vec{k_q}) f^\varepsilon_1(t,k_1)..f^\varepsilon_q(t,k_q).$$

Lemma 3.1 *There exists a constant $C_{N,T}$ such that $\|\tilde{R}^\varepsilon_{osc,N}\|_{C^0([0,T],H^\sigma)} \le C_{N,T}$.*

Proof. Let us first of all note that each of the $(\beta_q)^{-1}$ is bounded from above by a constant depending on N, as each β_q takes a finite number of non zero values. Then the lemma is simply due to the fact that in each of the quantities estimated here, one sums on low frequencies only, so they are as smooth as needed, provided one pays with enough powers of N. □

That lemma implies that Lemma 2.1 is proved if one shows that ψ^ε_N is an $o(1)$ in the space $C^0([0,T],H^{\sigma-1}(\mathcal{T}^d))$. Hence from now on our study will concentrate on ψ^ε_N.

Proposition 3.1 *Under the assumptions of Lemma* 2.1*, the function ψ^ε_N defined in* (3.1) *satisfies*

$$\left\{\begin{array}{rcl} \partial_t\psi^\varepsilon_N + \mathcal{Q}^\varepsilon(b^\varepsilon,\psi^\varepsilon_N) &=& \varepsilon R^{1,\varepsilon}_{N,osc} + R^{\varepsilon,N}_{osc} + F^\varepsilon \\ \psi^\varepsilon_{N|t=0} &=& \psi^\varepsilon_{N,0}, \end{array}\right. \tag{3.2}$$

where $\psi^\varepsilon_{N,0}$ goes to zero with ε in $H^{\sigma-1}$, $R^{1,\varepsilon}_{N,osc}$ is bounded uniformly in ε, in $C^0([0,T],H^{\sigma-1})$, by a constant depending on N, and $R^{\varepsilon,N}_{osc}$ goes to zero in $C^0([0,T],H^{\sigma-1})$ when N goes to infinity, uniformly in ε.

Proof. Let us first note that $\psi^\varepsilon_{N|t=0} = a^\varepsilon_0 + \varepsilon\tilde{R}^\varepsilon_{osc,N|t=0}$, so the assumption on a^ε_0, as well as Lemma 3.1, imply that $\psi^\varepsilon_{N|t=0}$ goes to zero with ε in $H^{\sigma-1}$.

Furthermore, the change of function introduced above leads to the following equation for ψ_N^ε :

$$\partial_t \psi_N^\varepsilon + \mathcal{Q}^\varepsilon(b^\varepsilon, \psi_N^\varepsilon) = F^\varepsilon + \varepsilon\, \tilde{R}_{osc,N}^{\varepsilon,t} + \varepsilon\, \mathcal{Q}^\varepsilon(\tilde{R}_{osc,N}^\varepsilon, b^\varepsilon) + R_{osc}^{\varepsilon,N}, \tag{3.3}$$

where $\tilde{R}_{osc,N}^{\varepsilon,t}$ is defined in the same way as $\tilde{R}_{osc,N}^\varepsilon$, only where in each function $\tilde{R}_{q,osc,N}^\varepsilon$, one of the f_i^ε's has been replaced by $\partial_t f_i^\varepsilon$: we have $\tilde{R}_{osc,N}^{\varepsilon,t} = \sum_{q\in\{1,..,p\}} \tilde{R}_{q,osc,N}^{\varepsilon,t}$, with

$$\mathcal{F}\tilde{R}_{q,osc,N}^{\varepsilon,t}(n) \stackrel{def}{=} \sum_j \sum_{\substack{\vec{k_q}\in K_q^n \\ |n|,|\vec{k_q}|\le N}} \frac{i\, e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})}}{\beta_q(n,\vec{k_q})} r_0(n,\vec{k_q})\partial_t f_j^\varepsilon(t,k_j)\prod_{i\neq j} f_i^\varepsilon(t,k_i).$$

- The low frequencies :

Lemma 3.2 *The terms $\tilde{R}_{osc,N}^{\varepsilon,t}$ and $\mathcal{Q}^\varepsilon(\tilde{R}_{osc,N}^\varepsilon, b^\varepsilon)$ are bounded in the space $C^0([0,T],H^{\sigma-1})$, uniformly in ε, with an upper bound $C_{N,T}$ which depends on N and T.*

Proof. For $\tilde{R}_{osc,N}^{\varepsilon,t}$, the result is simply due to the fact that all the functions considered are in low frequencies, hence are as smooth as wanted provided $C_{N,T}$ is large enough. Let us recall that we have assumed that each $\mathcal{F}^{-1}\partial_t f_i^\varepsilon$ is bounded in $C^0([0,T],H^{\sigma_i})$, for some σ_i. For the quadratic term $\mathcal{Q}^\varepsilon(\tilde{R}_{osc,N}^\varepsilon, b^\varepsilon)$, the fact that H^s is an algebra if $s > \frac{d}{2}$ gives the result, since b^ε and $\tilde{R}_{osc,N}^\varepsilon$ are bounded in $C^0([0,T],H^\sigma)$, by Lemma 3.1. So Lemma 3.2 is proved, and with it is found the term $R_{N,osc}^{1,\varepsilon}$ of (3.2). □

- The high frequencies :

Lemma 3.3 *For any function $a \in C^0([0,T],H^s(\mathcal{T}^d))$, with $s \in \mathbf{R}$, the high frequency function $a^N \stackrel{def}{=} \mathcal{F}^{-1}\left(\mathbf{1}_{[N,\infty[}\mathcal{F}a\right)$ goes to zero when N goes to infinity, in $C^0([0,T],H^s(\mathcal{T}^d))$.*

Proof. It simply lies on the fact that $|a^N(t)|_s^2 = \sum_{|k|\ge N} |k|^{2s}|\mathcal{F}a(t,k)|^2$, and Dini's theorem implies that $|a^N(t)|_s^2$ goes to zero uniformly in t, as the remainder of the sequence of continuous functions of general term $|k|^{2s}|\mathcal{F}a(t,k)|^2$. □

Lemma 3.4 *The high frequency term $R_{osc}^{\varepsilon,N}$ goes to zero in $C^0([0,T],H^{\sigma-1})$ when N goes to infinity, uniformly in ε.*

Proof. The terms $R_{q,osc}^{\varepsilon,N}$ are such that at least one of the f_i^ε's in the product appears as a high frequency term $f_i^{\varepsilon,N}$. But Lemma 3.3 implies that $\mathcal{F}^{-1}f_i^{\varepsilon,N}$ goes to zero in the space $C^0([0,T],H^{\sigma-1+\alpha_i})$. As the other $\mathcal{F}^{-1}f_j^\varepsilon$'s are bounded in $C^0([0,T],H^{\sigma-1+\alpha_j})$, and $H^{\sigma-1}$ is an algebra since $\sigma-1 > \frac{d}{2}$, Lemma 3.4 is proved and $R_{osc}^{2,\varepsilon,N}$ is found. □

Hence Proposition 3.1 is proved. □

Now we can end the proof of the lemma. Using the estimate recalled in (2.2), we have, since $\sigma > \frac{d}{2} + 2$,

$$\frac{1}{2}\frac{d}{dt}|\psi_N^\varepsilon(t)|_{\sigma-1}^2 \leq C\,|\psi_N^\varepsilon(t)|_{\sigma-1}^2 |b^\varepsilon(t)|_\sigma + C\,|\psi_N^\varepsilon(t)|_{\sigma-1}|(\varepsilon R_{N,osc}^{1,\varepsilon} + R_{osc}^{\varepsilon,N} + F^\varepsilon)(t)|_{\sigma-1}.$$

A classical Gronwall estimate gives

$$|\psi_N^\varepsilon(t)|_{\sigma-1} \leq \left(|\psi_{N,0}^\varepsilon|_{\sigma-1} + C\int_0^t |(\varepsilon R_{N,osc}^{1,\varepsilon} + R_{osc}^{\varepsilon,N} + F^\varepsilon)(\tau)|_{\sigma-1}d\tau\right)\exp CB(t),$$

where $B(t)$ is an upper bound, uniform in ε, of $\|b^\varepsilon\|_{L^1([0,t],H^\sigma)}$. But if one choses N large enough, and then ε small enough, then $|\varepsilon R_{N,osc}^{1,\varepsilon} + R_{osc}^{\varepsilon,N} + F^\varepsilon|_{\sigma-1}$ and $|\psi_{N,0}^\varepsilon|_{\sigma-1}$ are arbitrarily small, so that means that ψ_N^ε, and hence a^ε, is defined on $[0,T]$ for ε small enough and we have $a^\varepsilon = o(1)$ in $C^0([0,T], H^{\sigma-1}(\mathcal{T}^d))$. So Lemma 2.1 is proved. □

4 Application to geophysic fluid equations

4.1 Introduction

The aim of this section is to prove that the life span of the rotating Euler equations and of the primitive equations, defined in the introduction, goes to infinity when ε goes to zero, and to find a first order expansion of the associate filtered solutions. To do so, we shall use Theorem 1, so we have to compute the limit equation $(\mathcal{H}_0)$ in both the case of (RF^ε) and of (PE^ε). If we prove that those limit equations are globally well posed, we will have proved the result on the life span of the solutions of (RF^ε) and (PE^ε). The first order expansion will then be a direct consequence of Theorem 1, once we have checked that the perturbations satisfy a second order small divisor estimate.

4.2 The rotating fluid equations

The following proposition is proved by A. Babin, A. Mahalov and B. Nicolaenko in [1], [2].

Proposition 4.1 *For almost all values of b_3, the limit system $(\mathcal{H}_0)$ corresponding to (RF^ε) can be decomposed into two systems: one is the bidimensional Euler equation*

$$(E2D)\begin{cases} \partial_t\bar{u} + \bar{u}\cdot\nabla_2\bar{u} &= (-\nabla_2\bar{p}, 0) \\ div_2\ \bar{u} &= 0 \\ \bar{u}_{|t=0} &= \bar{v}_0 \end{cases}$$

where $\partial_3\bar{u} = 0$, and the other is linear in the remainder $u_{osc} = u - \bar{u}$:

$$(L_{osc})\begin{cases} \partial_t u_{osc} + 2\mathcal{Q}(\bar{u}, u_{osc}) &= 0 \\ div\ u_{osc} &= 0 \\ u_{osc|t=0} &= v_{0,osc}\,. \end{cases}$$

We have noted $\bar{v}_0$ for the vertical average of v_0 : $\bar{v}_0 = \displaystyle\int_{x_3} v_0\,dx_3$, and $v_{0,osc} = v_0 - \bar{v}_0$.

Remark. The admissible values of b_3 are those that guarantee the absence of three–wave interactions that would prevent such a decoupling in the limit system (see [1], [10]). Those values define the notion of "non resonant domains" for the rotating fluid equations.

That proposition leads immediately to the fact that the limit system is globally well posed: the first equation in the proposition is a bidimensional Euler equation, well–known to be globally well posed (see [5] for example): if $v_0 \in H^s$ and $s > \frac{5}{2}$, then $\bar{u} \in C^0(\mathbf{R}^+, H^s(\mathcal{T}^2))$.

One now just has to look at the second equation: the absence of quadratic terms in u_{osc} enables us to find the result, by a straightforward energy estimate in H^s; however, one can note that in [2] (Theorem 5.3), it is proved that for all s, $(\mathcal{Q}(\bar{u}, u_{osc}) \mid (P\Delta)^s u_{osc}) = 0$. So one has, for all s, $\dfrac{d}{dt}|u_{osc}|_s^2 = 0$, and so we find, for all $t \geq 0$ and for all $s > \frac{5}{2}$, that $|u_{osc}(t)|_s = |v_{0,osc}|_s$.

Hence the limit system is globally well posed in $C^0(\mathbf{R}^+, H^s)$, and one can use Theorem 1 to get the expected result: for any $T > 0$, there exists ε_T small enough so that v^ε is defined on $[0, T]$ for all $\varepsilon < \varepsilon_T$.

In order to compute the first order expansion given in Theorem 2, we just have to use the result proved in [11], which is a classical result in number theory, stating that for almost all values of (b_1, b_2, b_3), the operator L satisfies a second order small divisor estimate at a certain degree. So with that additional restriction on the domain, Theorem 2 is proved by a direct application of Theorem 1. $\square$

4.3 The primitive equations

Here again, as in the case of the rotating fluids, we shall not write the computations leading to the limit system, as they are all written in [2], as well as in [11]. We will just give the following definition, necessary to the statement of Proposition 4.2.

Definition 4.1 *The potential vorticity of $U = (v, T)$ is defined by $\Omega \stackrel{def}{=} \partial_1 v_2 - \partial_2 v_1 - F^{-1}\partial_3 T$. Then the quasigeostrophic part of U is $U_{QG} \stackrel{def}{=} (-\partial_2\Delta^{-1}\Omega, \partial_1\Delta^{-1}\Omega, 0, -F\partial_3\Delta^{-1}\Omega)$, and the oscillating part is $U_{osc} = U - U_{QG}$.*

Proposition 4.2 *For almost all values of (b_1, b_2, b_3), the limit system $(\mathcal{H}_0)$ corresponding to (PE^ε) can be diagonalized in the following way:*

$$\left\{\begin{array}{rcl} \partial_t\Omega + v_{QG}\cdot\nabla\Omega & = & 0 \\ \tilde{U}_{QG|t=0} & = & U_{0,QG} \\ \partial_t\tilde{U}_{osc} + \tilde{\mathcal{Q}}(U_{QG}, \tilde{U}_{osc}) & = & 0 \\ \tilde{U}_{osc|t=0} & = & U_{0,osc}\,. \end{array}\right.$$

Then it is easy to see that the limit system is globally well posed: the first equation is well known to be globally well posed in $C^0(\mathbf{R}^+, H^{s-1})$ if the initial data $U_{0,QG}$ is in H^s, with $s > \frac{5}{2}$, and the second equation is then also globally well posed in $C^0(\mathbf{R}^+, H^s)$, since it is linear, exactly as in the case of the rotating fluids. So we get the expected result on the life span of (PE^ε). To conclude, we just have to notice, as for (RF^ε) and using a result proved in [11], that for almost all periodic boxes $\mathcal{T}^3$, the perturbation L associated with (PE^ε) satisfies a second order small divisor estimate at some degree. Then a direct application of Theorem 1 gives the expected first order expansion, and Theorem 2 is proved. □

Remark. It can be noticed that two different and independent restrictions on the domain have taken place: one is linked to the small divisor estimates satisfied by L, and the other is linked to the possibilities of resonances.

5 The slightly compressible fluid equations

The aim of this final section is to prove Theorem 3, stated in the introduction. In order to do so, we are simply going to apply Theorem 1; so as in the previous case, concerning the geophysical fluid equations, we have to check that the perturbation operator L satisfies a second order small divisor estimate at some degree, and we have to compute the limit system $(\mathcal{H}_0)$ corresponding to the compressible equations $(\mathcal{CE}^\varepsilon)$, noted $(\mathcal{CE}_0)$ in the statement of Theorem 3.

As we have seen in the introduction, the perturbation operator is of the form

$$L(v,c) = (\bar{\gamma}\bar{c}_0 \, \nabla c, \bar{\gamma}\bar{c}_0 \, \mathrm{div}\, v).$$

So after a simple computation, we find that the eigenvalues of the Fourier transform $\mathcal{F}L(n)$ are 0 (of multiplicity $d-1$), and $\pm i\bar{\gamma}\bar{c}_0 \, |n|_\mathcal{T}$, where we have defined $|n|^2_\mathcal{T} = \sum_j \frac{n_j^2}{b_j^2}$. Let us note that terms of the type V_{div}, with the notation defined in the introduction, are elements of the kernel of L. Moreover, a classical result in number theory indicates that L satisfies a second order small divisor estimate at some degree, for almost all periodic boxes (see [11], [22]).

So it follows that Theorem 3 will be proved once we have computed the limit system $(\mathcal{CE}_0)$. The form taken by that limit system follows directly from the general study of $(\mathcal{H}^\varepsilon)$, except for the equation on u_{div}. However since $\tilde{c}^\varepsilon \nabla \tilde{c}^\varepsilon = \frac{1}{2} \nabla {\tilde{c}^\varepsilon}^2$, then v^ε_{div} satisfies

$$\partial_t v^\varepsilon_{div} + P(v^\varepsilon \cdot \nabla v^\varepsilon) = 0,$$

where P is the L^2-orthogonal projection onto divergence free vector fields. But as V^ε_{div} is an element of the kernel of L, we have $U^\varepsilon_{div} = V^\varepsilon_{div}$, so to find the equation satisfied by the limit u_{div} of u^ε_{div}, we just have to find the limit of the equation satisfied by v^ε_{div}. So we just need to compute the limit of $P(v^\varepsilon \cdot \nabla v^\varepsilon)$, and it is easy to see (see [12]) that there exists a function p such that

$$\lim_{\varepsilon \to 0} P(v^\varepsilon \cdot \nabla v^\varepsilon) = u_{div} \cdot \nabla u_{div} + \nabla p.$$

So the equation on u_{div} is found. The equation on $\tilde{U}$ follows directly from the definition of the limit system $(\mathcal{H}_0)$ given in Theorem 0. Then one can apply Theorem 1: the smoothness of U follows from classical results on hyperbolic equations, recalled in section 2. So Theorem 3 is proved. □

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