A Remark on Smooth Solutions of the Weakly Compressible Periodic Navier-Stokes Equations

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1 Introduction

The aim of this article is to study the asymptotic behaviour of smooth solutions of the periodic, weakly compressible Navier–Stokes equations. A large amount of literature exists on the subject (see [13] for an extensive bibliography), and considerable progress has been made recently, concerning the existence and convergence of weak solutions with various boundary conditions ([3],[4],[14]) as well as concerning the existence of smooth solutions with critical regularity in the whole space and in the periodic case ([2]). Here, we are interested in the convergence of smooth solutions, as well as in long time existence results, in the limit when the Mach number goes to zero, and the density goes to a constant.

So let us consider a compressible, viscous fluid, evolving in a periodic box $\mathbf{T}^d$, where $d \geq 2$ stands for the space dimension. The viscosity of the fluid is $\nu > 0$, its Mach number is noted $\varepsilon > 0$, and the state of the fluid at a time $t \geq 0$ and at a point $x \in \mathbf{T}^d$ is given by its velocity field, noted $v^\varepsilon = (v^{\varepsilon,1}, .., v^{\varepsilon,d})(t,x)$ and its density $\rho^\varepsilon = \rho^\varepsilon(t,x)$. The equations satisfied by v^ε and ρ^ε are the following (see for instance [13]):

$$\left\{ \begin{array}{rcl} \partial_t(\rho^\varepsilon v^\varepsilon) + \mathrm{div}\,(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) - \nu\Delta v^\varepsilon + \dfrac{1}{\varepsilon^2}\nabla p(\rho^\varepsilon) & = & \rho^\varepsilon f \quad \text{in} \quad \mathbf{R}^+ \times \mathbf{T}^d \\ \partial_t \rho^\varepsilon + \mathrm{div}\,(\rho^\varepsilon v^\varepsilon) & = & 0 \quad \text{in} \quad \mathbf{R}^+ \times \mathbf{T}^d \\ (v^\varepsilon_{|t=0}, \rho^\varepsilon_{|t=0}) & = & (v^\varepsilon_0, \rho^\varepsilon_0), \end{array} \right.$$

where $f = (f^1, f^2, f^3)$ represents exterior forces, fixed and independent of ε to avoid unnecessary complications. The fluid is supposed to be isentropic, which means that $p(\rho^\varepsilon) = \rho^{\varepsilon^\gamma}$, where γ is chosen such that $\gamma > 1$. It is well known that this system can be put in a symmetric form, by defining the sound speed

$$c^\varepsilon \stackrel{def}{=} \frac{2}{\gamma - 1}\sqrt{\partial_{\rho^\varepsilon} p(\rho^\varepsilon)}\,,$$

and writing the system in terms of v^ε and c^ε. In this article, the fluid will be also supposed to be weakly compressible, which means that the density is close to a constant, put to 1 here to simplify: $\rho^\varepsilon = 1 + \varepsilon\tilde{\rho}^\varepsilon$. That can also be written as $c^\varepsilon = c_0 + \varepsilon\,\tilde{c}^\varepsilon$, where $c_0 = \dfrac{2\gamma^{1/2}}{\gamma-1}\cdot$ After a few computations, we come up with the following system:

$$(NS^\varepsilon)\quad \left\{\begin{array}{rcl} \partial_t v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + \bar{\gamma}\tilde{c}^\varepsilon\nabla\tilde{c}^\varepsilon - \nu\Delta v^\varepsilon + \bar{\gamma}c_0\,\dfrac{\nabla\tilde{c}^\varepsilon}{\varepsilon} & = & -\nu\Big(1 - c_{\varepsilon,0}^{1/\bar{\gamma}}\Big)\Delta v^\varepsilon + f \\ \partial_t\tilde{c}^\varepsilon + v^\varepsilon\cdot\nabla\tilde{c}^\varepsilon + \bar{\gamma}\tilde{c}^\varepsilon\,\mathrm{div}\,v^\varepsilon + \bar{\gamma}c_0\,\dfrac{\mathrm{div}\,v^\varepsilon}{\varepsilon} & = & 0 \\ (v^\varepsilon_{|t=0}, \tilde{c}^\varepsilon_{|t=0}) & = & (v_0, \tilde{c}_0), \end{array}\right.$$

where we have written $\bar{\gamma} = \dfrac{\gamma-1}{2}$, and $c_{\varepsilon,0} = \dfrac{c_0}{c^\varepsilon}\cdot$ To simplify, we have supposed that the initial data $(v_0, \tilde{c}_0)$ does not depend on ε.

We are interested here in the asymptotic behaviour of the solutions of (NS^ε). Let us notice that (NS^ε) is a system of evolution equations, penalized by a skew–symmetric operator L, defined as

$$L(v,c) \stackrel{def}{=} \bar{\gamma}(\nabla c, \mathrm{div}\,v).$$

Problems around skew–symmetric penalizations have been studied by a number of authors, (among which J.-L. Joly, G. Métivier, and J. Rauch in [9], S. Schochet in [17], E. Grenier in [7]). However, unlike the cases studied by those authors (and also unlike [5],[6]), the system considered here is neither fully hyperbolic, nor fully parabolic. We shall see in the course of the study that this fact leads to additional difficulties compared to the cases referred to here. Let us recall that in the case of fully hyperbolic or fully parabolic systems, long time existence results can be proved by the means of a semi–group method (see [17],[5],[6]): we introduce the semi–group generated by L, $\mathcal{L}(t) = e^{-tL}$, and the "filtered" solution associated with $V^\varepsilon \stackrel{def}{=} (v^\varepsilon, \tilde{c}^\varepsilon)$, defined by $U^\varepsilon \stackrel{def}{=} \mathcal{L}(-\frac{t}{\varepsilon})V^\varepsilon$. In the fully hyperbolic or parabolic case, that new function converges to a function U (see [7]), and it is proved in [17] for the hyperbolic case, in [6] for the parabolic case, that the convergence holds, for $\varepsilon > 0$ small enough, on a lifetime $[0,T]$ for all $T < T_0^*$, where T_0^* is the life span of the "limit system" satisfied by U. In other words, not only does the penalization not destroy the life span of V^ε compared to the non penalized equation (that is due to the skew–symmetry of L), but it expands it up to the life span of U, for $\varepsilon > 0$ small enough. It is this kind of result we are seeking here. We shall prove the following theorem, where we have noted, as in the whole of this text, H^s for the usual homogeneous Sobolev space of order s on $\mathbf{T}^d$: it is defined by the norm $\|u\|_{H^s} = \|\,|k|^s\widehat{u}(k)\,\|_{\ell^2}$, where $\widehat{u}(k)$ is the (discrete) Fourier transform of u, and $k \in \mathbf{Z}^d$ are the Fourier variables.

Theorem 1 *There exists a constant $c > 0$ such that the following holds. Let $(v_0, \tilde{c}_0)$ be an element of $H^s(\mathbf{T}^d)$, with $s > \frac{d}{2} + 3$, and suppose that the exterior force f is an element of the space $C^0(\mathbf{R}^+, H^{s-2}(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{s-1}(\mathbf{T}^d))$. Suppose also that*

$$\|v_0\|_{H^{\frac{d}{2}-1}} \le c\nu \quad \textit{and} \quad \|Pf\|_{L^2(\mathbf{R}^+, H^{\frac{d}{2}-2})} \le c\nu,$$

where P denotes the Leray projector onto divergence free vector fields. Then for all $T > 0$ and for ε small enough, we have

$$V^\varepsilon = (\bar{u}, 0) + \mathcal{L}(\frac{t}{\varepsilon})U_{osc} + o(1) \quad \textit{in} \quad C^0([0,T], H^{s-2}), \tag{1.1}$$

where $\bar{u}$ satisfies the incompressible Navier–Stokes equation, and $U_{osc} = (u_{osc}, \chi_{osc})$ satisfies a parabolic system:

$$(NS_0) \qquad \left\{ \begin{array}{rcl} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} - \nu \Delta \bar{u} & = & -\nabla p + Pf \\ \partial_t U_{osc} + \mathcal{Q}(\bar{u} + U_{osc}, U_{osc}) - \dfrac{\nu}{2}(\nabla div\, u_{osc}, \Delta \chi_{osc}) & = & 0 \\ \bar{u}_{|t=0} = Pv_0, \quad U_{osc|t=0} & = & ((1-P)v_0, \tilde{c}_0). \end{array} \right.$$

The operator $\mathcal{Q}$ is a quadratic form, of the following type (see (2.12) for a precise definition): $\mathcal{Q}(a,b) = \dfrac{1}{2}(Aa \cdot \nabla b + Ab \cdot \nabla a)$, where $Aa = (A^j a)_{1 \leq j \leq d+1}$ is a set of smooth symmetric matrices for all a.

In the case $d = 2$, the result holds under the only assumption that $\|(1-P)v_0\|_{L^2} \leq c\nu$.

Remarks. Let us note that the "limit system" (NS_0) is fully parabolic: some of the viscosity, at the limit, is transferred from the compressible part of the velocity, u_{osc}, to the sound speed, χ_{osc}. This form is specific to the periodic case: in the case of the whole space, the solutions of (NS^ε) converge to the incompressible Navier–Stokes system (see [3]), and that is due to the fact that the fast waves can disappear to infinity in the case of the whole space, due to Strichartz–like estimates. The same phenomenon occurs in the case of the Euler equation: S. Ukai proved in [18] that the solutions of the weakly compressible Euler equations converge to the incompressible Euler equations, whereas in the periodic case, it is proved in [5] that there is an additional, coupled equation in the limit system.

Moreover, that theorem shows that the life span of (NS^ε) is arbitrarily large, for ε small enough, under a smallness condition on the velocity at time $t = 0$. In the bidimensional case, no smallness condition is required for the incompressible part of the initial data, Pv_0. In [8], T. Hagstrom and J. Lorenz prove a very similar result, in the bidimensional case; their result is actually more precise in the sense that the life span is exactly $+\infty$ for ε small enough. However the method followed is completely different to the one used here, since in [8], everything is based on the exponential decay to zero of smooth solutions of the bidimensional, periodic, incompressible Navier–Stokes equations — and of all their derivatives — when $t \to \infty$. Hence, as noted in [8], if one adds an exterior force, or if one considers higher dimensions as it done in our theorem, it is not clear that the proof in [8] still holds.

2 Proof of the theorem

2.1 Notation and preliminary computations

In this section, we are going to transform somewhat the equation (NS^ε), in order to apply more easily the methods of [5] and [6]. It is known that for smooth enough initial data, one can solve (NS^ε) locally in time (see [11] for instance). In the following sections, $s > d/2 + 3$ will be a fixed real number, and we shall place ourselves on a time interval $[0, T]$ such that

$$\|V^\varepsilon\|_{L^\infty([0,T], H^{s-2})} \leq c_\infty, \tag{2.1}$$

where $c_\infty \geq \|V_0\|_{H^{s-2}}$ is a fixed constant. Let us notice that under assumption (2.1), we have, for $\varepsilon < c_0 c_\infty^{-1}$,

$$\|1 - c_{\varepsilon,0}^{1/\bar\gamma}\|_{H^{s-2}} \leq C c_\infty \varepsilon,$$

where C is a "universal" constant, depending only on γ. In the following, we shall note by the same letter C all such universal constants depending only on the dimension, on γ or on other such parameters (and in particular independent of ε).

So finally we have, since $s - 2 > \frac{d}{2}$,

$$\| \left(1 - c_{\varepsilon,0}^{1/\bar\gamma}\right) \Delta v^\varepsilon \|_{H^{s-2}} \leq C c_\infty \varepsilon \|v^\varepsilon\|_{H^s}. \tag{2.2}$$

This immediately implies that if ε is small enough so that $C c_\infty \varepsilon \leq \dfrac{1}{2}$, then the operator defined by

$$a_2^\varepsilon(D) \stackrel{def}{=} -\nu\Delta + \nu \left(1 - c_{\varepsilon,0}^{1/\bar\gamma}\right) \Delta \tag{2.3}$$

satisfies for all v^ε

$$(a_2^\varepsilon(D) v^\varepsilon \mid v^\varepsilon)_{H^{s-2}} \geq C \frac{\nu}{2} \|v^\varepsilon\|_{H^{s-1}}^2, \tag{2.4}$$

where we have noted $(\cdot \mid \cdot)_{H^{s-2}}$ for the scalar product in H^{s-2}. We can re–write equation (NS^ε) as

$$\partial_t V^\varepsilon + q(V^\varepsilon, V^\varepsilon) + (a_2^\varepsilon(D) v^\varepsilon, 0) + \frac{L V^\varepsilon}{\varepsilon} = (f, 0), \tag{2.5}$$

where $q(V^\varepsilon, V^\varepsilon)$ represents all the bilinear terms in (NS^ε). Let us note that q can be written generically as $q(a, b) = 1/2\Big(Aa \cdot \nabla b + Ab \cdot \nabla a\Big)$, where A is a linear operator, such that $Au = (A^j u)_{1 \leq j \leq d+1}$, and for all j, $A^j u$ is a smooth, symmetric matrix.

Notation. Throughout the text, for any vector $V = (v, \tilde c)$ where v has d components and $\tilde c$ is a scalar, we will note, recalling that P is the L^2-orthogonal projector onto divergence free vector fields,

$$\bar v = P v, \quad V_{osc} = V - (\bar v, 0),$$
$$\forall i \in \{1, .., d\}, \quad V^i = v^i \quad \text{and} \quad V^{d+1} = \tilde c.$$

2.2 The filtered equation

Let us recall that we have defined $\mathcal{L}(t) = e^{-tL}$ and $U^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon}) V^\varepsilon$. It is straightforward to see that $U^\varepsilon = (\bar v^\varepsilon, 0) + \mathcal{L}(-\dfrac{t}{\varepsilon}) V_{osc}^\varepsilon$. Then according to (2.5), U^ε satisfies the following equation:

$$\left\{ \begin{array}{rcl} \partial_t U^\varepsilon + \mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) + \mathcal{A}_2^\varepsilon(D) U^\varepsilon &=& \mathcal{L}(-\dfrac{t}{\varepsilon}) f \\ U^\varepsilon_{|t=0} &=& V_0, \end{array} \right.$$

where

$$\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) = \mathcal{L}(-\frac{t}{\varepsilon}) q(\mathcal{L}(\frac{t}{\varepsilon}) U^\varepsilon, \mathcal{L}(\frac{t}{\varepsilon}) U^\varepsilon), \tag{2.6}$$

and where

$$\begin{aligned}\mathcal{A}_2^\varepsilon(D)U^\varepsilon &= -\nu(\Delta\bar{u}^\varepsilon,0) - \nu\mathcal{L}(-\frac{t}{\varepsilon})\left(\Delta\Big(\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon_{osc}\Big)^1,..,\Delta\Big(\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon_{osc}\Big)^d,0\right)\\ &+ \nu\mathcal{L}(-\frac{t}{\varepsilon})\left((1-C_{\varepsilon,0}^{1/\bar\gamma})\Big(\Delta\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon\Big)^1,..,(1-C_{\varepsilon,0}^{1/\bar\gamma})\Big(\Delta\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon\Big)^d,0\right),\end{aligned}$$

where we have defined

$$C_{\varepsilon,0} \stackrel{def}{=} \frac{c_0}{c_0+\varepsilon\left(\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon_{osc}\right)^{d+1}}. \tag{2.7}$$

Notation. We shall call $\mathcal{A}_2^{\varepsilon,1}(D)$ the following operator:

$$\mathcal{A}_2^{\varepsilon,1}(D)V \stackrel{def}{=} -\nu(\Delta\bar{v},0) - \nu\mathcal{L}(-\frac{t}{\varepsilon})\left(\Delta\Big(\mathcal{L}(\frac{t}{\varepsilon})V_{osc}\Big)^1,..,\Delta\Big(\mathcal{L}(\frac{t}{\varepsilon})V_{osc}\Big)^d,0\right), \tag{2.8}$$

and the operator $\mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon,V)$ shall be

$$\mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon,V) \stackrel{def}{=} \nu\mathcal{L}(-\frac{t}{\varepsilon})\left((1-C_{\varepsilon,0}^{1/\bar\gamma})\Big(\Delta\mathcal{L}(\frac{t}{\varepsilon})V\Big)^1,..,(1-C_{\varepsilon,0}^{1/\bar\gamma})\Big(\Delta\mathcal{L}(\frac{t}{\varepsilon})V\Big)^d,0\right). \tag{2.9}$$

In other words, we have written $\mathcal{A}_2^\varepsilon(D)U^\varepsilon = \mathcal{A}_2^{\varepsilon,1}(D)U^\varepsilon + \mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon,U^\varepsilon)$. Let us note that the inequality (2.4) implies the following proposition.

Proposition 2.1 *Under assumption* (2.1)*, the operator* $\mathcal{A}_2^{\varepsilon,1}(D)(\cdot)+\mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon,\cdot)$ *is positive (but not definite): we have for instance*

$$\Big(\mathcal{A}_2^{\varepsilon,1}(D)V+\mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon,V)\mid V\Big)_{H^{s-2}} \geq C\sup_{1\leq i\leq d}\|\Big(\mathcal{L}(\frac{t}{\varepsilon})V\Big)^i\|^2_{H^{s-1}}. \tag{2.10}$$

Now let us prove that U^ε converges to U in $C^0([0,T],H^{s-2})$, when ε is small enough, where U satisfies the following equation:

$$\left\{\begin{array}{rcl}\partial_t U+\mathcal{Q}(U,U)+\mathcal{A}_2(D)U &=& (Pf,0)\\ U_{|t=0} &=& V_0,\end{array}\right. \tag{2.11}$$

where $\mathcal{Q}(U,U)$ and $\mathcal{A}_2(D)U$ are the limits in $\mathcal{D}'$ respectively of $\mathcal{Q}^\varepsilon(U^\varepsilon,U^\varepsilon)$ and $\mathcal{A}_2^\varepsilon(D)U^\varepsilon$, and where $\mathcal{A}_2(D)U$ is second order elliptic.

The coming section consists in the computation of the limit system, and in particular in the proof that $\mathcal{A}_2(D)U$ is second order elliptic. In the last section below (Section 2.4), we combine those results to prove that the limiting system is globally well posed for suitable initial data, and that enables us to prove the theorem stated in the introduction, using methods of [5].

2.3 Computation of the limit system (NS_0)

Proposition 2.2 *The limit of* $\mathcal{Q}^\varepsilon(U^\varepsilon,U^\varepsilon)$ *in* $\mathcal{D}'$ *is* $\mathcal{Q}(U,U)$*, where there exists* p *such that*

$$P\mathcal{Q}(U,U)=\bar{u}\cdot\nabla\bar{u}+\nabla p, \quad \textit{and} \quad (1-P)\mathcal{Q}(U,U)=\mathcal{Q}(\bar{u}+U_{osc},U_{osc}).$$

Moreover, the limit of $(\mathcal{L}(-\frac{t}{\varepsilon})f,0)$ *in* $\mathcal{D}'$ *is* $(Pf,0)$.

Proof of the proposition. The limit of the quadratic term was obtained in [7], and the limit of the filtered forcing term was obtained, by the same method, in [6]. We shall not recall the arguments leading to those results here, as the techniques used are strictly identical to the ones we shall be using in the proof of the following proposition, to obtain the elliptic term $\mathcal{A}_2(D)$. Let us simply recall that

$$\mathcal{Q}(U,U) = \mathcal{F}^{-1} \sum_{\omega_{k,n}^{a,b,c}=0} \Big((AU)^a(n-k) \cdot k\, U^b(k), e^c(n) \Big)\, e^c(n). \tag{2.12}$$

We have noted $(\omega^a(k))_{1\leq a \leq d+1}$ for the eigenvalues of $\mathcal{F}L(k)$, where $\mathcal{F}$ is the (discrete) Fourier transform, and the associate eigenvectors are $(e^a(k))_{1\leq a\leq d+1}$. Finally we have written $\omega_{k,n}^{a,b,c} = \omega^a(k) + \omega^b(n-k) - \omega^c(n)$, and $U^a(k) = \mathcal{F}U(k) \ \cdot\ e^a(k)\, e^a(k)$. □

Proposition 2.3 *Under assumption* (2.1)*, the limit in $\mathcal{D}'$ of $\mathcal{A}_2^\varepsilon(D)U^\varepsilon$ defined in* (2.8), (2.9) *is $\mathcal{A}_2(D)U$, where $\mathcal{A}_2(D)$ is elliptic. More precisely, we have*

$$\mathcal{A}_2(D)(\bar{u},0) = -\nu\Delta(\bar{u},0) \quad \textit{and} \quad \mathcal{A}_2(D)(u_{osc},\chi_{osc}) = -\frac{\nu}{2}(\nabla div\, u_{osc}, \Delta\chi_{osc}).$$

Proof of the proposition. By the same computation as in (2.2), we have for all $s' < s$, by the rules of product in Sobolev spaces,

$$\left\| \mathcal{L}(-\frac{t}{\varepsilon}) \left((1-C_{\varepsilon,0}^{1/\bar\gamma}) \Big(\Delta\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon\Big)^1, .., (1-C_{\varepsilon,0}^{1/\bar\gamma}) \Big(\Delta\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon\Big)^d, 0 \right) \right\|_{H^{s'-4}} \leq C\varepsilon \|U^\varepsilon\|_{H^{s-2}}^2,$$

hence we have, under assumption (2.1),

$$\lim_{\varepsilon\to 0} \mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon, U^\varepsilon) = 0. \tag{2.13}$$

To find the limit of $\mathcal{A}_2^{\varepsilon,1}(D)U^\varepsilon$, in the same way as in [5], [6] and [7], we are going to use the non stationary phase theorem. As $\mathcal{L}(t)$ is unitary, one need only find the limit of $\mathcal{A}_2^{\varepsilon,1}(D)U$. Moreover, we are only interested here in the "compressible part" U_{osc}, of U, since the "incompressible part" of U only appears in the form $-\nu\Delta\bar{u}$, which does not depend on ε. So we are looking for the limit in $\mathcal{D}'$ of $\mathcal{A}_2^{\varepsilon,1}(D)U_{osc}$. Let us compute the eigenvalues and the eigenvectors of $\mathcal{F}L(n)$; in order to simplify the computations, we are going to work in the case $d=3$, the general case is treated exactly in the same way. We have

$$\mathcal{F}L(n) = \bar\gamma \begin{pmatrix} 0 & 0 & 0 & ik_1 \\ 0 & 0 & 0 & ik_2 \\ 0 & 0 & 0 & ik_3 \\ ik_1 & ik_2 & ik_3 & 0 \end{pmatrix},$$

which yields the eigenvalues 0 (of order 2) and $\pm i|k|\bar\gamma$. The eigenvectors associated with the eigenvalue 0 are of course incompressible. Since we are only interested in "compressible–type" vectors here, we shall only compute the last two eigenvectors, respectively associated with $i|k|\bar\gamma$ and $-i|k|\bar\gamma$ and called respectively $e^+(k)$ and $e^-(k)$. We get

$$e^+(k) = \frac{1}{|k|\sqrt{2}} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ |k| \end{pmatrix} \quad \text{and} \quad e^-(k) = \frac{1}{|k|\sqrt{2}} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ -|k| \end{pmatrix}.$$

Now for any "compressible–type" vector B, let us write $\mathcal{F}B(k) = (b^1, b^2, b^3, b^4)$, and

$$\begin{aligned}\mathcal{F}\mathcal{L}(t)B(k) &= \frac{1}{|k|\sqrt{2}} e^{-it|k|\bar\gamma}(k_1b_1 + k_2b_2 + k_3b_3 + |k|b_4)e^+(k)\\ &+ \frac{1}{|k|\sqrt{2}} e^{it|k|\bar\gamma}(k_1b_1 + k_2b_2 + k_3b_3 - |k|b_4)e^-(k).\end{aligned}$$

It follows that

$$\mathcal{F}\left(\left(\Delta\mathcal{L}(t)B\right)^1, .., \left(\Delta\mathcal{L}(t)B\right)^3, 0\right)(k) = \frac{\nu}{2} e^{-it|k|\bar\gamma}(k_1b_1 + k_2b_2 + k_3b_3 + |k|b_4)(k_1, k_2, k_3, 0)$$

$$+ \frac{\nu}{2} e^{it|k|\bar\gamma}(k_1b_1 + k_2b_2 + k_3b_3 - |k|b_4)(k_1, k_2, k_3, 0)$$

Finally we find the following expression, where $U_{osc} = (u_{osc}, \chi_{osc})$: for all j in $\{1, 2, 3\}$,

$$\mathcal{F}\mathcal{A}_2^{\varepsilon,1}(D)U^j_{osc}(k) = \frac{\nu}{2}\left(\left(1 + \cos\frac{2t|k|\bar\gamma}{\varepsilon}\right) k \cdot \widehat{u}_{osc}(k) + \sin\frac{2t|k|\bar\gamma}{\varepsilon} |k|\, \widehat{\chi}_{osc}(k)\right) k^j,$$

$$\text{and}\quad \mathcal{F}\mathcal{A}_2^{\varepsilon,1}(D)U^4_{osc}(k) = \frac{\nu}{2}\left(\left(1 - \cos\frac{2t|k|\bar\gamma}{\varepsilon}\right) |k|^2\widehat{\chi}_{osc}(k) - \sin\frac{2t|k|\bar\gamma}{\varepsilon} |k|\, k \cdot \widehat{u}_{osc}(k)\right).$$

So the non stationary phase theorem yields

$$\lim_{\varepsilon\to0} \mathcal{A}_2^\varepsilon(D)(\bar u, 0) = -\nu\Delta\bar u \quad\text{and}\quad \lim_{\varepsilon\to0} \mathcal{A}_2^\varepsilon(D)(u_{osc}, \chi_{osc}) = -\frac{\nu}{2}(\nabla\mathrm{div}\, u_{osc}, \Delta\chi_{osc}). \qquad (2.14)$$

Finally (2.13) and (2.14) give Proposition 2.3. $\square$

2.4 Proof of the theorem

In the previous section, we have computed the limit system (NS_0). We shall now check that it is globally well posed, under the assumptions of Theorem 1, and we shall prove (1.1).

2.4.1 Global wellposedness of (NS_0)

Proposition 2.4 *Let $s > \frac{d}{2} + 3$ be a fixed real number. Then there exists a constant $c > 0$ such that if $V_0 \in H^s(\mathbf{T}^d)$ and $Pf \in C^0(\mathbf{R}^+, H^{s-2}(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{s-1}(\mathbf{T}^d))$, with*

$$\|v_0\|_{H^{\frac{d}{2}-1}(\mathbf{T}^d)} + \|\tilde c_0\|_{H^{\frac{d}{2}-1}(\mathbf{T}^d)} \leq c\nu, \quad \text{and} \quad \|Pf\|_{L^2(\mathbf{R}^+, H^{\frac{d}{2}-2}(\mathbf{T}^d))} \leq c\nu, \qquad (2.15)$$

then there exists a unique solution U to (NS_0), and

$$U \in C^0(\mathbf{R}^+, H^s(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{s+1}(\mathbf{T}^d)).$$

In the case when $d = 2$, then it is enough to suppose that $\|v_{0,osc}\|_{L^2(\mathbf{T}^2)} + \|\tilde c_0\|_{L^2(\mathbf{T}^2)} \leq c\nu$.

Proof of the proposition. The proof of this proposition follows from classical results concerning parabolic equations (see for instance [1] for the case of the incompressible Navier–Stokes equations): it is well known that if the initial data is in $H^s(\mathbf{T}^d)$, and if $Pf \in C^0(\mathbf{R}^+, H^{s-2}(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{s-1}(\mathbf{T}^d))$, then one can solve (NS_0) locally in time, and we have a unique solution

$$U \in C^0([0,T_0], H^s(\mathbf{T}^d)) \cap L^2([0,T_0], H^{s+1}(\mathbf{T}^d)), \quad \text{for some} \quad T_0 > 0. \tag{2.16}$$

Moreover, assumption (2.15) implies that (NS_0) has a unique solution U such that

$$U \in C^0(\mathbf{R}^+, H^{\frac{d}{2}-1}(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{\frac{d}{2}}(\mathbf{T}^d)),$$

hence (NS_0) has a unique solution

$$U \in C^0([0,T_0], H^s(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{\frac{d}{2}}(\mathbf{T}^d)), \quad \text{for some} \quad T_0 > 0,$$

and Lemma 3.1.1 of [1] implies that

$$U \in C^0(\mathbf{R}^+, H^\sigma(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{\sigma+1}(\mathbf{T}^d)), \quad \text{for all} \quad \sigma \leq \frac{d}{2} \cdot \tag{2.17}$$

One can note that no forcing term is considered in [1], but with the assumptions made on Pf, it can be added to the proofs in [1] with no difficulty. It is then easy to infer, by similar arguments, that

$$U \in C^0(\mathbf{R}^+, H^s(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{s+1}(\mathbf{T}^d)). \tag{2.18}$$

The arguments leading to (2.18) are standard; we re-formulate them here for the convenience of the reader.

We start by choosing $\frac{d}{2} < \sigma' \leq \frac{d}{2} + 1$, for which we have, by an energy estimate in $H^{\sigma'}(\mathbf{T}^d)$,

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{H^{\sigma'}(\mathbf{T}^d)} + \frac{c\nu}{2}\|U(t)\|^2_{H^{\sigma'+1}(\mathbf{T}^d)} \leq \frac{C}{\nu}\|U(t)\|^4_{H^{\sigma'}(\mathbf{T}^d)},$$

which yields that for all $\sigma' \leq d/2+1$ and for all $t \geq 0$,

$$\|U(t)\|^2_{H^{\sigma'}(\mathbf{T}^d)} + \frac{c\nu}{2}\int \|U(\tau)\|^2_{H^{\sigma'+1}(\mathbf{T}^d)}\, d\tau \leq \|U_0\|^2_{H^{\sigma'}(\mathbf{T}^d)} + \frac{C}{\nu}\|U\|^4_{L^4(\mathbf{R}^+, H^{\sigma'}(\mathbf{T}^d))}. \tag{2.19}$$

But (2.17) implies that $U \in L^4(\mathbf{R}^+, H^{\sigma'}(\mathbf{T}^d))$ for all $\sigma' \leq d/2+1/2$, hence (2.19) implies that

$$U \in C^0_b(\mathbf{R}^+, H^\sigma(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{\sigma+1}(\mathbf{T}^d)), \quad \text{for all} \quad \sigma \leq \frac{d+1}{2} \cdot$$

Then $U \in L^4(\mathbf{R}^+, H^{\sigma'}(\mathbf{T}^d))$ for all $\sigma' \leq d/2+1$, so again by (2.19), we have

$$U \in C^0_b(\mathbf{R}^+, H^\sigma(\mathbf{T}^d)) \cap L^2(\mathbf{R}^+, H^{\sigma+1}(\mathbf{T}^d)), \quad \text{for all} \quad \sigma \leq \frac{d}{2} + 1.$$

Finally, for all $\sigma' > \frac{d}{2} + 1$, an energy estimate yields

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{H^{\sigma'}(\mathbf{T}^d)} + \frac{c\nu}{2}\|U(t)\|^2_{H^{\sigma'+1}(\mathbf{T}^d)} \leq \frac{C}{\nu}\|U(t)\|^2_{H^{\sigma'}(\mathbf{T}^d)}\|U(t)\|^2_{H^{\sigma'-1}(\mathbf{T}^d)},$$

hence for all $\sigma' > d/2+1$ and all $t \geq 0$,

$$\|U(t)\|^2_{H^{\sigma'}(\mathbf{T}^d)} + \frac{c\nu}{2}\int \|U(\tau)\|^2_{H^{\sigma'+1}(\mathbf{T}^d)}\,d\tau \leq \|U_0\|^2_{H^{\sigma'}(\mathbf{T}^d)}$$
$$+\frac{C}{\nu}\|U\|^2_{L^\infty(\mathbf{R}^+,H^{\sigma'-1}(\mathbf{T}^d))}\|U\|^2_{L^2(\mathbf{R}^+,H^{\sigma}(\mathbf{T}^d))},$$

and one concludes by recurrence.

So we have the result for $d > 2$. In the bidimensional case, one uses moreover the fact that the bidimensional incompressible Navier–Stokes equations are globally well posed, with no smallness assumption on the initial data, nor on the forcing term, as soon as the initial data is in $L^2(\mathbf{T}^2)$ (see [1]). So there is no smallness condition on the incompressible part of the initial data $\bar{v}_0$, nor on Pf, in that case.

The proposition is proved. □

2.4.2 End of the proof

The function $W^\varepsilon \stackrel{def}{=} U^\varepsilon - U$ satisfies the equation

$$\partial_t W^\varepsilon + \mathcal{Q}^\varepsilon(W^\varepsilon, W^\varepsilon + 2U) + \mathcal{A}_2^{\varepsilon,1}(D)W^\varepsilon + \mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon, W^\varepsilon) = (\mathcal{A}_2(D) - \mathcal{A}_2^{\varepsilon,1}(D))U$$
$$+(\mathcal{Q}-\mathcal{Q}^\varepsilon)(U,U) + \left(\mathcal{L}(-\frac{t}{\varepsilon})f - Pf, 0\right) - \mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon, U), \tag{2.20}$$

with the initial data $W^\varepsilon_{|t=0} = 0$.

Remark. It is not totally clear at this stage why the elliptic operator $\mathcal{A}_2(D)$ alone does not appear on the left–hand side of this equation, as it would imply that W^ε satisfies a paraboliic equation, with additional forcing terms on the right. In the last remark, at the very end of the paper, we explain why the choice above is made, by showing that the additional forcing terms that would appear cannot be controlled properly, contrary to the ones that appear in (2.20).

In the right–hand side of equation (2.20), we find two different types of terms: on the one hand we have the function $\mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon, U)$, defined in (2.9), which can be estimated by

$$\|\mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon, U)\|_{H^{s-2}} \leq C\nu\varepsilon\|U^\varepsilon_{osc}\|_{H^{s-2}}\|U_{osc}\|_{H^s}, \tag{2.21}$$

in the same way as for (2.2). On the other hand, we have three terms,

$$(\mathcal{A}_2(D) - \mathcal{A}_2^{\varepsilon,1}(D))U, \quad (\mathcal{Q}-\mathcal{Q}^\varepsilon)(U,U) \quad \text{and} \quad \left(\mathcal{L}(-\frac{t}{\varepsilon})f - Pf, 0\right).$$

Those terms are going to be treated using a method followed in [5] and [6]. Let us recall that we have defined, in [5], the following notion.

Definition 2.1 (Oscillating functions) *Let $T > 0$, $p \geq 1$ and $\sigma > \frac{d}{2}$ be fixed. We write $\vec{k_q} = (k_1, .., k_q)$, where $k_i \in \mathbf{Z}^d$, and will call $|\vec{k_q}| = \max_{1 \leq i \leq q} |k_i|$. Then a function $R^\varepsilon_{osc}(t)$ will be said to be (p,σ)–oscillating if it can be written as $R^\varepsilon_{osc}(t) = \sum_{q \in \{1,..,p\}} R^\varepsilon_{q,osc}(t)$, where*

$$R^\varepsilon_{q,osc}(t) = \mathcal{F}^{-1} \sum_{\vec{k_q} \in K_q^n} e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})} r_0(n,\vec{k_q}) f_1(t,k_1)..f_q(t,k_q),$$

with $K_q^n = \left\{ \vec{k_q} \in \mathbf{Z}^{dq} \mid \sum_{i=1}^q k_i = n \text{ and } \beta_q(n,\vec{k_q}) \neq 0 \right\}$, *and where r_0 and f_i satisfy*

$$\exists\, (\alpha_i)_{1 \leq i \leq q},\ \alpha_i \geq 0, \qquad \text{such that} \qquad |r_0(n,\vec{k_q})| \leq C\,(1+|k_1|)^{\alpha_1} .. (1+|k_q|)^{\alpha_q}\ ;$$

$$\forall\, i \in [1,..,q], \qquad \mathcal{F}^{-1}(f_i(t,\cdot)) \quad \text{is an element of} \quad C^0([0,T], H^{\sigma+\alpha_i})$$

$$\text{and} \qquad \exists\, \sigma_i > -\sigma \quad \text{such that} \quad \mathcal{F}^{-1}(\partial_t f_i(t,\cdot)) \quad \text{is an element of} \quad C^0([0,T], H^{\sigma_i})\,.$$

In view of that definition, we have the following proposition, which has an obvious proof, using also Proposition 2.4.

Proposition 2.5 *Under the assumptions of Theorem 1, the function $(\mathcal{A}_2(D) - \mathcal{A}_2^{\varepsilon,1}(D))U$ is a $(1, s-2)$–oscillating function, $(\mathcal{Q} - \mathcal{Q}^\varepsilon)(U,U)$ is a $(2, s-1)$–oscillating function, and finally the function $\left(\mathcal{L}(-\frac{t}{\varepsilon})f - Pf, 0\right)$ is a $(1, s-2)$–oscillating function.*

It follows that the function W^ε satisfies the following equation:

$$\partial_t W^\varepsilon + \mathcal{Q}^\varepsilon(W^\varepsilon, W^\varepsilon + 2U) + \mathcal{A}_2^{\varepsilon,1}(D)W^\varepsilon + \mathcal{A}_2^{\varepsilon,2}(D)(U^\varepsilon, W^\varepsilon) = R^\varepsilon_{osc}(f,U) + F^\varepsilon(U),$$

where $R^\varepsilon_{osc}(f,U)$, is a $(2, s-2)$–oscillating function according to Proposition 2.5, with

$$R^{1,\varepsilon}_{osc}(f,U) = (\mathcal{Q} - \mathcal{Q}^\varepsilon)(U,U) + \left(\mathcal{L}(-\frac{t}{\varepsilon})f - Pf, 0\right) + (\mathcal{A}_2(D) - \mathcal{A}_2^{\varepsilon,1}(D))U,$$

and where $F^\varepsilon(U)$ is estimated as follows, according to (2.21):

$$\|F^\varepsilon(U)\|_{H^{s-2}} \leq C\varepsilon c_\infty \|U_{osc}\|_{H^s}.$$

The following lemma, similar to Lemma 2.1 of [5], will enable us to conclude.

Lemma 2.1 *Let $T > 0$ and $\sigma > \frac{d}{2} + 1$ be two real numbers, let b^ε be a family of functions, uniformly bounded in the space $C^0([0,T], H^{\sigma+1}(\mathbf{T}^d))$, and let a_0^ε be a function going to zero with ε in $H^\sigma(\mathbf{T}^d)$. Let $\mathcal{Q}^\varepsilon$ be as in (2.6), and let $A_2^\varepsilon(D)$ be a second order, positive operator in the sense of (2.10). Finally let R^ε_{osc} be a (p,σ)–oscillating function, with $p \geq 1$, and F^ε be a function going to zero with ε in $C^0([0,T], H^\sigma(\mathbf{T}^d))$. Then the function a^ε, solution of*

$$\left\{ \begin{array}{rcl} \partial_t a^\varepsilon + \mathcal{Q}^\varepsilon(a^\varepsilon, b^\varepsilon) + A_2^\varepsilon(D)a^\varepsilon &=& R^\varepsilon_{osc} + F^\varepsilon \quad \text{in} \quad \mathbf{T}^d \\ a^\varepsilon_{|t=0} &=& a_0^\varepsilon, \end{array} \right. \tag{2.22}$$

is an $o(1)$ in the space $C^0([0,T], H^\sigma(\mathbf{T}^d))$.

We shall postpone the proof of that lemma for a while, and finish the proof of the theorem.

Under assumption (2.2), Lemma 2.1 with $\sigma = s-2$, implies immediately that

$$\forall T > 0, \quad \lim_{\varepsilon \to 0} W^\varepsilon = 0 \quad \text{in} \quad C^0([0,T], H^{s-2}(\mathbf{T}^d)).$$

That implies in particular that for all $T > 0$, and for ε small enough,

$$\forall t \in [0,T], \quad \|V^\varepsilon(t)\|_{H^{s-2}} \leq 2.\|U\|_{C^0(\mathbf{R}^+, H^{s-2})} \tag{2.23}$$

So choosing the constant c_∞ larger than $4\|U\|_{C^0(\mathbf{R}^+, H^{s-2})}$, that implies that

$$\|V^\varepsilon(t)\|_{C^0([0,T],H^{s-2})} \leq c_\infty \quad \Rightarrow \quad \|V^\varepsilon(t)\|_{C^0([0,T],H^{s-2})} \leq \frac{c_\infty}{2} \quad \text{for } \varepsilon \text{ small enough},$$

which means finally that W^ε is defined for all times T, for ε small enough.

Hence the theorem is proved. $\Box$

Proof of Lemma 2.1. We shall only give a sketch of the proof, as it is very similar to the proof of Lemma 2.1 in [5]. The only difference is the presence of a bilinear form and of a second order, positive operator on the left–hand side of equation (2.22). We start by re–writing equation (2.22) in the following way:

$$\partial_t a^\varepsilon + \mathcal{Q}^\varepsilon(a^\varepsilon, b^\varepsilon) + A_2^\varepsilon(D) a^\varepsilon = R^\varepsilon_{osc,N} + R^{\varepsilon,N}_{osc} + F^\varepsilon,$$

where, writing $\mathbf{1}_X$ for the characteristic function of X, and with the notation of Definition 2.1, we have defined

$$\begin{aligned} R^\varepsilon_{osc,N} &\stackrel{def}{=} \sum_{q \in \{1,..,p\}} R^\varepsilon_{q,osc,N} \\ &= \sum_{q \in \{1,..,p\}} \mathcal{F}^{-1}\left(\mathbf{1}_{\{|\cdot|, |\vec{k}_q| \leq N\}} \mathcal{F} R^\varepsilon_{q,osc}(\cdot)\right), \\ \text{and} \quad R^{\varepsilon,N}_{osc} &\stackrel{def}{=} R^\varepsilon_{osc} - R^\varepsilon_{osc,N}. \end{aligned}$$

Then, still as in [5], we can define the function

$$\psi^\varepsilon_N = a^\varepsilon + \varepsilon \widetilde{R}^\varepsilon_{osc,N},$$

where $\widetilde{R}^\varepsilon_{osc,N}$ is defined by $\widetilde{R}^\varepsilon_{osc,N} = \sum_{q \in \{1,..,p\}} \widetilde{R}^\varepsilon_{q,osc,N}$, and

$$\mathcal{F}\widetilde{R}^\varepsilon_{q,osc,N}(n) \stackrel{def}{=} \sum_{|n|, |\vec{k}_q| \leq N} \sum_{\vec{k}_q \in K^n_q} \frac{i\, e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k}_q)}}{\beta_q(n,\vec{k}_q)} r_0(n,\vec{k}_q) f^\varepsilon_1(t,k_1)..f^\varepsilon_q(t,k_q).$$

It is obvious that if we prove that for N large enough, uniformly in ε, and for ε small enough, ψ^ε_N is arbitrarily small in $C^0([0,T], H^{\sigma-1}(\mathbf{T}^d))$, then the lemma will be proved.

But the function ψ^ε_N satisfies the following equation:

$$\begin{aligned} \partial_t \psi^\varepsilon_N + \mathcal{Q}^\varepsilon(\psi^\varepsilon_N, \psi^\varepsilon_N + b^\varepsilon + 2\varepsilon \widetilde{R}^\varepsilon_{osc,N}) + A^\varepsilon_2(D)\psi^\varepsilon_N &= R^{\varepsilon,N}_{osc} + F^\varepsilon \\ + \varepsilon\, \mathcal{Q}^\varepsilon(\widetilde{R}^\varepsilon_{osc,N}, 2b^\varepsilon + \varepsilon \widetilde{R}^\varepsilon_{osc,N}) &+ \varepsilon A^\varepsilon_2(D) \widetilde{R}^\varepsilon_{osc,N}. \end{aligned}$$

The following proposition will enable us to conclude, by energy estimates on ψ^ε_N.

Proposition 2.6 *Under the assumptions of Lemma 2.1, the function ψ_N^ε satisfies*

$$\left\{\begin{array}{rcl} \partial_t \psi_N^\varepsilon + \mathcal{Q}^\varepsilon(\psi_N^\varepsilon + b^\varepsilon + 2\varepsilon R_{N,osc}^\varepsilon, \psi_N^\varepsilon) + A_2^\varepsilon(D)\psi_N^\varepsilon & = & \varepsilon R_{N,osc}^{1,\varepsilon} + R_{osc}^{\varepsilon,N} + F^\varepsilon \\ \psi_{N|t=0}^\varepsilon & = & \psi_{N,0}^\varepsilon, \end{array}\right.$$

where $\psi_{N,0}^\varepsilon$ goes to zero with ε in H^σ, $R_{N,osc}^{1,\varepsilon}$ is bounded uniformly in ε, in $C^0([0,T],H^\sigma)$, by a constant depending on N, and $R_{osc}^{\varepsilon,N}$ goes to zero in $C^0([0,T],H^\sigma)$ when N goes to infinity, uniformly in ε.

Proof. We shall not give all the details of the proof here, as it is identical to Proposition 3.1 of [5]: it is based on the fact that the function $\widetilde{R}_{osc,N}^\varepsilon$ is a low frequency term, hence as smooth as one needs (if one pays by enough powers of N), and that on the contrary $R_{osc}^{\varepsilon,N}$ is a high frequency term, hence small if N is large enough, uniformly in ε. So we can write

$$\varepsilon\|A_2^\varepsilon(D)\widetilde{R}_{osc,N}^\varepsilon\|_{L^\infty([0,T],H^\sigma)} + \varepsilon\|\mathcal{Q}^\varepsilon(\widetilde{R}_{osc,N}^\varepsilon, 2b^\varepsilon + \varepsilon\widetilde{R}_{osc,N}^\varepsilon)\|_{L^\infty([0,T],H^\sigma)} \leq \varepsilon c_\sigma(\varepsilon,N),$$

as well as

$$\|R_{osc}^{\varepsilon,N}\|_{L^\infty([0,T],H^\sigma)} \leq c_\sigma(N),$$

where

$$\lim_{N\to\infty} c_\sigma(N) = 0 \quad \text{and} \quad c_\sigma(\varepsilon,N) \quad \text{is uniformly bounded in } N \text{ and in } \varepsilon.$$

The proposition follows. □

Then a classical energy estimate in H^σ yields, since $\sigma > \frac{d}{2}+1$, and since $A_2^\varepsilon(D)$ is positive,

$$\frac{1}{2}\frac{d}{dt}\|\psi_N^\varepsilon(t)\|_{H^\sigma}^2 \leq C\,\|\psi_N^\varepsilon(t)\|_{H^\sigma}^2\|b^\varepsilon(t)\|_{H^{\sigma+1}} + C\,\|\psi_N^\varepsilon(t)\|_{H^\sigma}^3$$
$$+ C\,\|\psi_N^\varepsilon(t)\|_{H^\sigma}\|(\varepsilon R_{N,osc}^{1,\varepsilon} + R_{osc}^{\varepsilon,N} + F^\varepsilon)(t)\|_{H^\sigma}.$$

We have used the fact (see [15] or [16]) that for all vector fields a and b, we have the following estimate:

$$\begin{array}{rcl} \forall \tau > \dfrac{d}{2}, \quad (\mathcal{Q}^\varepsilon(a,b)\mid b)_{H^\tau} & \leq & C\|b\|_{H^\tau(\mathbf{R}^d)}\|a\|_{H^\tau(\mathbf{R}^d)}\|\nabla b\|_{L^\infty(\mathbf{R}^d)} \\ & + & C\|b\|_{H^\tau(\mathbf{R}^d)}^2\left(\|\nabla a\|_{L^\infty(\mathbf{R}^d)} + \|a\|_{H^{\tau+1}(\mathbf{R}^d)}\right), \end{array}$$

$$\text{and} \quad \forall \tau > \frac{d}{2}, \quad (\mathcal{Q}^\varepsilon(b,b)\mid b)_{H^\tau} \leq C\|b\|_{H^\tau(\mathbf{R}^d)}^2\|\nabla b\|_{L^\infty(\mathbf{R}^d)}.$$

Now since the initial data can be chosen arbitrarily small, provided N is large enough and ε is small enough, we can suppose to start with, that it is such that there exists a time T^* satisfying

$$0 < T^* \leq \sup\left\{t\in[0,T] \mid \|\psi_N^\varepsilon(t)\|_{H^\sigma} \leq 1 + B(T)\right\},$$

where $B(T)$ is an upper bound (uniform in ε) for $\|b^\varepsilon\|_{L^\infty([0,T],H^{\sigma+1})}$. Then by a variant of Gronwall's lemma we get, for all $t \leq T^*$:

$$\|\psi_N^\varepsilon(t)\|_{H^\sigma} \leq C\left(\|\psi_{N,0}^\varepsilon\|_{H^\sigma} + \int_0^t \left\|(\varepsilon R_{N,osc}^{1,\varepsilon} + R_{osc}^{\varepsilon,N} + F^\varepsilon)(\tau)\right\|_{H^\sigma} d\tau\right)\exp\left(T(1+B(T))\right). \tag{2.24}$$

We then just have to choose N large enough and ε small enough so that

$$\|\psi^\varepsilon_{N,0}\|_{H^\sigma} + \int_0^t \left\|(\varepsilon R^{1,\varepsilon}_{N,osc} + R^{\varepsilon,N}_{osc} + F^\varepsilon)(\tau)\right\|_{H^\sigma} \, d\tau \leq \frac{(1+B(T))}{2C} \exp\left(-T(1+B(T))\right),$$

to obtain

$$\forall t \leq T^*, \quad \|\psi^\varepsilon_N(t)\|_{H^\sigma} \leq \frac{(1+B(T))}{2}.$$

That means that T^* can be chosen equal to T, and (2.24) implies that

$$a^\varepsilon \to 0 \quad \text{in} \quad C^0([0,T], H^\sigma).$$

The lemma follows. □

Remarks. The fact that in the statement of the theorem, nothing is said about the smallness of $\tilde{c}_0$ is simply due to the fact that one just has to take ε small enough to have $\|\tilde{c}_0\|_{H^s} \leq c\nu$, since $\tilde{c}_0$ is defined by $c^\varepsilon_{|t=0} = c_0 + \varepsilon\, \tilde{c}_0$.

Furthermore, as remarked after equation (2.20) one can wonder at the choice made, to put the positive operator $\mathcal{A}^\varepsilon_2(D)$ on the left–hand side of (2.20), instead of the elliptic part $\mathcal{A}_2(D)$ alone. But it is easy to see that in the latter case, on the right–hand side of the equation satisfied by W^ε would appear the term $(\mathcal{A}^{\varepsilon,1}_2(D) - \mathcal{A}_2(D))U^\varepsilon$. That term is not oscillating in the sense of Definition 2.1, unless in that definition, the functions f_i are replaced by ε–dependent functions f^ε_i, where $\forall\, i \in [1,..,q]$, $\mathcal{F}^{-1}(f^\varepsilon_i(t,\cdot))$ is compact in $C^0([0,T], H^{\sigma+\alpha_i})$. But to ensure that compactness, we must allow for the loss of $\eta > 0$ derivatives when passing from U^ε to W^ε. However in the previous proof, it is crucial that no loss of derivatives occurs between the assumption (2.1) and the result (2.23). In other words, such a method with $\mathcal{A}_2(D)$ on the left–hand side of (2.20) does not enable us to conclude.

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