On global solutions to a defocusing semi-linear wave equation

Isabelle Gallagher

C.N.R.S. U.M.R. 8628, Département de Mathématiques

Université Paris Sud, 91405 Orsay Cedex, France

Isabelle.Gallagher@math.u-psud.fr

Fabrice Planchon

C.N.R.S. U.M.R. 7598, Laboratoire d'Analyse Numérique

Université Paris 6, 75252 Paris Cedex 05, France

fab@ann.jussieu.fr

Abstract

We prove that the 3D cubic defocusing semi-linear wave equation is globally well-posed for data in the Sobolev space $\dot{H}^s$ where $s > \frac{3}{4}\cdot$ This result was obtained in [11] following Bourgain's method ([3]). We present here a different and somewhat simpler argument, inspired by previous work on the Navier-Stokes equations ([4, 7]).

1 Introduction and main theorem

We consider the equation

$$\left\{ \begin{array}{rcl} \partial_t^2\Phi - \Delta\Phi + \Phi^3 & = & 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^3 \\ (\Phi, \partial_t\Phi)_{|t=0} & = & (\Phi_0, \Phi_1), \end{array} \right. \tag{1.1}$$

where Φ is real valued. This equation is sub-critical with respect to the H^1 norm, and, since the nonlinearity is defocusing, local well-posedness in H^1 extends to global well-posedness using the conservation of the Hamiltonian $\|\nabla\Phi\|_{L^2}^2 + \frac{1}{2}\|\Phi\|_{L^4}^4$. While this approach goes back to the 80's (it requires Strichartz estimates, see [8]), it is worth noting that global well-posedness had been known since the sixties (e.g. [10]), through regularization and compactness methods. The problem of local well-posedness for low regularity data was answered later, and definitive results were obtained in [14] in a more general framework. Equation (1.1) turns out to be locally well-posed for initial data in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ and ill-posed below $s = \frac{1}{2}$ (which makes sense from a scaling point of view, as both the equation and the $\dot{H}^{\frac{1}{2}}$ norm are invariant under the same rescaling). A natural question is then whether these local solutions can be extended globally in time, at least for some range $\frac{1}{2} \leq s < 1$. In [3] Bourgain introduced a general framework for obtaining results of this type, and applied it to the 2D cubic Schrödinger equation. For equation (1.1), Bourgain's method yields global well-posedness in $\dot{H}^s$ for $s > \frac{3}{4}$, as proved in [11]. We intend to give a different proof of this result, following a strategy introduced in the context of the Navier-Stokes equations in [4] (see also [7] for a more recent approach). When compared, the two methods appear to be somewhat dual of each other.

Let us start with a simple proof which will yield global well-posedness for $s > \frac{3}{4} + \frac{1}{12} = \frac{5}{6}$. Since the equation is globally well-posed for large data in $\dot{H}^1$ and small data in $\dot{H}^{\frac{1}{2}}$, one may be tempted to follow a general principle of nonlinear interpolation and claim the equation to be globally well-posed in between. To make sense of this heuristic, we proceed in several steps. What follows is an informal proof. Note that from now on, we will never mention the regularity of the time derivative of the solution at time $t = 0$; it is always one derivative less regular than the solution itself.

1. We split the data $\Phi_0 \in \dot{H}^s$: $\Phi_0 = u_0 + v_0$ where $u_0 \in \dot{H}^1$ with a large norm and $v_0 \in \dot{H}^{\frac{1}{2}}$ with a small norm. One may achieve this by splitting in frequency at $|\xi| \sim 2^J$ with large J.

2. We solve the equation (1.1) with small data v_0, getting a global solution v in $C_t(\dot{H}^{\frac{1}{2}}) \cap L^4_t(L^4_x)$, and all norms of v are small and have size less than $2\|v_0\|_{\dot{H}^{\frac{1}{2}}}$ which is of order $2^{(\frac{1}{2}-s)J}$. Remark that the usual smallness assumption on the data forces $2^{(\frac{1}{2}-s)J} \lesssim \varepsilon_0$. For a later step, we remark that any additional regularity is preserved, so that in particular we have (as $v_0 \in \dot{H}^{\frac{1}{2}+\frac{1}{6}}$)

$$\|v\|_{C_t(\dot{H}^{\frac{1}{2}+\frac{1}{6}}) \cap L^3_t(L^6_x)} \approx 2^{(\frac{1}{2}+\frac{1}{6}-s)J}. \tag{1.2}$$

   This will be of use at a later step, as it enables one to estimate v in $L^3_t(L^6_x)$ (note that $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ is smaller than the smoothness we ask for in the theorem, which is $\frac{3}{4}$, while $\frac{2}{3}$ is the smallest smoothness which permits an L^6_x estimate through Strichartz inequalities).

3. To recover a solution of our problem, we solve a perturbed equation,

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u + u^3 + 3u^2 v + 3v^2 u & = & 0 \\ (u, u_t)_{|t=0} & = & (u_0, u_1). \end{array} \right. \tag{1.3}$$

This equation turns out to be locally well-posed in $\dot{H}^1$, on a time interval depending only on $\|u_0\|_{\dot{H}^1}$. This is easily seen through a contraction argument in $C_t(\dot{H}^1)$. It will be enough to prove the nonlinear terms to be $L^1_t(L^2_x)$. Using Sobolev embedding, $u^3 \in C_t(L^2_x)$ and hence is locally L^1_t. On the other hand, $v \in L^3_t(L^6_x)$ for v_0 in $\dot{H}^{\frac{1}{2}+\frac{1}{6}}$ (recall Step 2), which yields $v^2 u \in L^1_t(L^2_x)$ using Hölder. The remaining term $u^2 v$ is controlled by the other two, and denoting by $|||\cdot|||$ the norm in the contraction space, we obtain

$$|||u|||_T \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + 2^{2j(\frac{1}{2}+\frac{1}{6}-s)} T^{\frac{1}{3}} |||u|||_T + T|||u|||^3_T. \tag{1.4}$$

Thus, the linear term on the right can be absorbed on the left, as soon as $T \lesssim 2^{-6j(\frac{1}{2}+\frac{1}{6}-s)}$ and we obtain the desired result, with

$$T \lesssim \inf\left(2^{-6j(\frac{1}{2}+\frac{1}{6}-s)}, \frac{1}{\|u_0\|^2_{\dot{H}^1}}\right). \tag{1.5}$$

4. To extend local solutions to global ones, we then need to obtain an a priori bound on the energy of a solution u. This will be accomplished through the energy inequality, provided one can control the perturbative terms by the energy of u. Indeed, we have the energy estimate

$$\begin{aligned} \|u\|^2_{\dot{H}^1} + \|\partial_t u\|^2_{L^2} + \frac{1}{2}\|u\|^4_{L^4} \quad \leq \quad & \|u_0\|^2_{\dot{H}^1} + \|\partial_t u\|^2_{L^2} + \frac{1}{2}\|u_0\|^4_{L^4} \\ & + 6\int_0^t |u^2 v \partial_t u| ds + 6\int_0^t |v^2 u \partial_t u| ds. \end{aligned} \tag{1.6}$$

Taking the supremum over $t < T$, with $H_T \overset{\text{def}}{=} \sup_{t<T}(\|u\|^2_{\dot{H}^1} + \|\partial_t u\|^2_{L^2} + \|u\|^4_{L^4})$ gives

$$H_T \lesssim H_0 + H_T \int_0^T \|v\|^2_{L^6_x} ds + H_T^{\frac{3}{2}} \int_0^T \|v\|_{L^6_x} ds,$$

using Hölder and Sobolev embeddings for the integrals on the left. Recalling (1.2), we obtain

$$H_T \lesssim 2^{2J(1-s)} + H_T T^{\frac{1}{3}} 2^{2J(\frac{2}{3}-s)} + H_T^{\frac{3}{2}} T^{\frac{2}{3}} 2^{J(\frac{2}{3}-s)}, \tag{1.7}$$

which gives control over H_T for arbitrarily large T as long $s > \frac{5}{6}$ by choosing J accordingly, as one needs

$$T^{\frac{4}{3}} 2^{2J(1-s+\frac{2}{3}-s)} \lesssim 1. \tag{1.8}$$

Note that we have estimated $\|u_0\|^4_{L^4}$ by $2^{2J(1-s)}$: this is always possible by a rescaling in space, as explained below in (2.1).

This last part is really the only non trivial part of the argument, at least if one wants the optimal result $s > \frac{3}{4}$: we will have to refine the estimates on the space time integrals arising from the energy inequality, to lower the $\frac{2}{3} - s$ factor in (1.7) to $\frac{1}{2} - s$. In fact, ignoring all irrelevant epsilons, (1.8) will become

$$T \|u_0\|^2_{\dot{H}^1} \|v_0\|^2_{\dot{H}^{\frac{1}{2}}} \lesssim 1. \tag{1.9}$$

To pick T as large as we want, we have to play with the respective size of u_0 and v_0 and this gives immediately the $s > \frac{3}{4}$ restriction: take $\|v_0\|_{\dot{H}^{\frac{1}{2}}} \approx M^{-1} \|\Phi\|_{\dot{H}^s}$ where M is large. Then $\|u_0\|_{\dot{H}^1} \approx M^{\frac{1-\theta}{\theta}} \|\Phi\|_{\dot{H}^s}$, where $s = \theta + \frac{1}{2}(1-\theta)$. This forces $1 - 2\theta < 0$. This barrier at $\theta = \frac{1}{2}$ between the conservation law (here, $\dot{H}^1$) and the scaling (here, $\dot{H}^{\frac{1}{2}}$) appears to be rooted in both Bourgain's method and ours. Indeed, in all the recent progress for other (technically more complicated) equations like KdV ([6]) or NLS ([5])the same kind of restrictions appear, through for KdV it solves completely the well-posedness question as the equation is ill-posed below half-way to the scaling ([12]).

Let us now state the main theorem.

Theorem 1
Let $(\Phi_0, \Phi_1) \in (\dot{H}^s \cap L^4, \dot{H}^{s-1})$ *with* $s > \frac{3}{4}$. *Then there exists a unique global in time solution to (1.1). Moreover, we have*

$$\|\Phi\|_{\dot{H}^s}(t) \leq C(\|u_0\|_{\dot{H}^s \cap L^4}) t^{\frac{(1-s)(6s-3)}{4s-3}}. \tag{1.10}$$

Before proceeding with the proof, several remarks are in order. The restriction to L^4 data can be disposed of at the expense of working with local Sobolev spaces and use of the finite speed of propagation, as already mentioned in [11]. Alternatively, one could simply replace the homogeneous Sobolev spaces by their inhomogeneous counterpart, as in [14]. One may also state the theorem for other nonlinearities (or Klein-Gordon), and we will comment on that aspect later, as well as on applying the method to other equations like the nonlinear Schrödinger equation.

2 Proof of the theorem

We start with the definition of u_0 and v_0. Since we are working with L^2 based spaces, one may simply cut the data Φ_0 into its low frequency part u_0 and its high frequency part v_0, as we did in the informal proof in the introduction. This gives us an explicit decomposition, with a parameter 2^J which is the frequency at which we cut. We however point out that one needs only an abstract decomposition, as given e.g. by real interpolation. Here we will have

$$\|u_0\|_{\dot{H}^1} \approx 2^{J(1-s)} \|\Phi_0\|_{\dot{H}^s} \quad \text{and} \quad \|v_0\|_{\dot{H}^\gamma} \approx 2^{J(\gamma - s)} \|\Phi_0\|_{\dot{H}^s}, \text{ for all } \gamma \leq s.$$

Let us make a remark concerning the L^4 norms. For any $\lambda > 0$, we have

$$\|\Phi_0(\lambda\cdot)\|_{L^4} = \lambda^{-\frac{3}{4}}\|\Phi_0\|_{L^4} \quad \text{and} \quad \|\Phi_0(\lambda\cdot)\|_{\dot{H}^s} = \lambda^{s-\frac{3}{2}}\|\Phi_0\|_{\dot{H}^s}. \tag{2.1}$$

It follows that as soon as $s > \frac{3}{4}$, the L^4 norm of Φ_0 can be made arbitrarily small compared to the $\dot{H}^s$ norm by rescaling the data. In particular since $\|u_0\|_{L^4} \lesssim \|\Phi_0\|_{L^4}$ we conclude that the quantity $\|u_0\|_{L^4}^4$ can be controlled by $\|u_0\|_{\dot{H}^1}^2$, and we assume this to be the case for the rest of the proof; that will be useful to estimate the Hamiltonian of u in Section 2.3.

Finally, we recall some definitions and properties of the wave equation which will be of use later. First we define the Littlewood–Paley operators S_j and Δ_j.

Definition 1
Let $\Phi \in \mathcal{S}(\mathbb{R}^3)$ be such that $\widehat{\Phi}(\xi) = 1$ for $|\xi| \leq 1$ and $\widehat{\Phi}(\xi) = 0$ for $|\xi| > 2$. Define, for $j \in \mathbb{Z}$, the function $\Phi_j(x) \stackrel{\text{def}}{=} 2^{3j}\Phi(2^j x)$; then the Littlewood–Paley operators are

$$S_j \stackrel{\text{def}}{=} \Phi_j * \cdot \quad \text{and} \quad \Delta_j \stackrel{\text{def}}{=} S_{j+1} - S_j.$$

Then we recall the definition of Besov spaces (the Sobolev space $\dot{H}^s$ being simply the space $\dot{B}^s_{2,2}$).

Definition 2
If $s < \dfrac{3}{p}$, then f belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ if and only if the partial sum $\sum_{-m}^{m} \Delta_j f$ converges towards f as a tempered distribution, and the sequence $\epsilon_j \stackrel{\text{def}}{=} 2^{js}\|\Delta_j f\|_{L^p}$ belongs to $\ell^q(\mathbb{Z})$.

Finally, we have the Strichartz estimates, which we only state in 3D.

Theorem 2 ([9, 14])
Let (p,q) and $(\tilde{p},\tilde{q})$ be admissible pairs, i.e. such that $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{2}$, $p > 2$, and similarly for $(\tilde{p},\tilde{q})$. Let $f(x)$, $F(t,x)$ be two functions localized at frequency $|\xi| \sim 2^j$, and denote $\omega = \sqrt{-\Delta}$. Then

$$\|e^{\pm i\omega t} f(x)\|_{L^p_t(L^q_x)} \lesssim 2^{\frac{2}{p}j}\|f(x)\|_{L^2}, \tag{2.2}$$

and, if $u(x,t) = \Box^{-1}F(x,t)$ is the solution to the inhomogeneous equation with zero Cauchy data, then

$$\|u(x,t)\|_{L^\infty_t(L^2_x)} \lesssim 2^{j(\frac{2}{\tilde{p}}-1)}\|F(x,t)\|_{L^{\tilde{p}'}_t(L^{\tilde{q}'}_x)} \tag{2.3}$$

$$\|u(x,t)\|_{L^p_t(L^q_x)} \lesssim 2^{j(\frac{2}{p}+\frac{2}{\tilde{p}}-1)}\|F(x,t)\|_{L^{\tilde{p}'}_t(L^{\tilde{q}'}_x)} \tag{2.4}$$

where p' denotes the dual exponent of p.

2.1 Global existence for the high frequency part

Let us consider the following equation:

$$\left\{\begin{array}{rcl}\partial_t^2 v - \Delta v + v^3 & = & 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^3 \\ (v,\partial_t v)_{|t=0} & = & (v_0, v_1),\end{array}\right. \tag{2.5}$$

where $(v_0, v_1) \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\gamma} \times \dot{H}^{-\frac{1}{2}} \cap \dot{H}^{\gamma-1}$, with $\gamma \leq s$ to be set later. We have the following result.

Proposition 1
Suppose the initial data satisfies $(v_0, v_1) \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\gamma} \times \dot{H}^{-\frac{1}{2}} \cap \dot{H}^{\gamma-1}$, for any $\frac{1}{2} \leq \gamma \leq s$. There exists a constant ε_0 such that if

$$2^{J(\frac{1}{2}-s)} \lesssim \varepsilon_0, \tag{2.6}$$

then there is a unique, global solution v to (2.5), such that

$$v \in C^0(\mathbb{R}, \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\gamma}), \quad \partial_t v \in C^0(\mathbb{R}, \dot{H}^{-\frac{1}{2}} \cap \dot{H}^{\gamma-1}),$$

and if $\dfrac{2}{p} + \dfrac{2}{q} = 1$, with $p > 2$ and $q < +\infty$, $v \in L^p(\mathbb{R}, \dot{B}^{\gamma-\frac{2}{p}}_{q,2})$, with

$$\|v\|_{L^p(\mathbb{R},\dot{B}^{\gamma-\frac{2}{p}}_{q,2})} \lesssim 2^{J(\gamma-s)}. \tag{2.7}$$

Proof. The proof of that result is straightforward, as using the definition of v_0 we just gave, we have $\|v_0\|_{\dot{H}^{\frac{1}{2}}} \approx 2^{J(\frac{1}{2}-s)}$. The proposition then follows by the global existence theory for small data ([14]). Note that one may even lower assumptions on γ to $\gamma > 0$, as any positive smoothness can be propagated (see [16], [15]).

Let us also notice that one may further refine the estimate given in the proposition, by splitting the solution $v = v_L + w$ into the linear part v_L and the nonlinear part w. Then the nonlinear part w satisfies

$$\|w\|_{L^p(\mathbb{R},\dot{B}^{\gamma-\frac{2}{p}}_{q,2})} \lesssim \|v\|^2_{L^4_{t,x}} \|v\|_{L^4(\mathbb{R},\dot{B}^{\gamma}_{4,2})} \approx 2^{J(1-2s+\gamma-s)}, \tag{2.8}$$

which is an easy consequence of the fixed point procedure.

2.2 Local existence for the low frequency part

In this part we shall study the following equation:

$$\left\{\begin{array}{rcl}\partial_t^2 u - \Delta u + u^3 + 3uv^2 + 3u^2 v & = & 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^3 \\ (u,\partial_t u)_{|t=0} & = & (u_0, u_1),\end{array}\right. \tag{2.9}$$

where v was constructed in the previous part, and with $(u_0, u_1) \in \dot{H}^1 \times L^2$. Recall that

$$E^{\frac{1}{2}}(u_0, u_1) \stackrel{\text{def}}{=} \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} \approx 2^{J(1-s)}.$$

Proposition 2
Suppose the initial data satisfies $(u_0, u_1) \in \dot{H}^1 \cap L^4 \times L^2$. As long as $T \lesssim \dfrac{1}{E(u_0, u_1)}$, there is a solution u satisfying

$$u \in C^0([0,T], \dot{H}^1),\ \partial_t u \in C^0([0,T], L^2) \text{ and } u \in L^p([0,T], \dot{B}_{q,2}^{1-\frac{2}{p}}),$$

with $\dfrac{2}{p}+\dfrac{2}{q} = 1$, $p > 2$. Moreover, the norm of u in that space, noted $|||u|||_T$, is controlled in the following way:

$$|||u|||_T \lesssim E(u_0, u_1).$$

Remark. Let us make a brief comment: since $v \in \dot{H}^s$, one may simply use the procedure described in the introduction. However we give a slightly more complicated proof, which only uses $v_0 \in \dot{H}^{\frac{1}{2}}$. This emphasizes the fact that (2.9) is well-posed whenever the perturbation v has at least $\dot{H}^{\frac{1}{2}}$ regularity.

Proof. Let $\varepsilon > 0$ be a given arbitrarily small real number, and define the norm

$$|||u|||_T \stackrel{\text{def}}{=} \sup_{t\in[0,T]} (\|u(t)\|_{\dot{H}^1} + \|\partial_t u(t)\|_{L^2}) + \|u\|_{L^p([0,T],\dot{B}_{q,2}^{1-\frac{2}{p}})},$$

with $\dfrac{1}{p} = \dfrac{1}{2} - \varepsilon$ and $q = \dfrac{1}{\varepsilon}\cdot$

We will proceed through the usual fixed point argument for the above norm, using the integral formulation of the equation (all Strichartz norms $L^{\widetilde{p}}([0,T], \dot{B}_{\widetilde{q},2}^{1-\frac{2}{\widetilde{p}}})$ with $\widetilde{q} < q$ will be obtained by interpolation with the $L^p([0,T], \dot{B}_{q,2}^{1-\frac{2}{p}})$ and the energy norms). To control $|||u|||_T$ it is enough to control the quantities u^3, uv^2 and u^2v in $L^1([0,T], L^2)$. Obviously the two first control the last, so we shall concentrate on u^3 and uv^2. The first one is the easiest, as

$$\begin{aligned} \|u^3\|_{L^1([0,T],L^2)} &\leq \|u\|^3_{L^3([0,T],L^6)} \\ &\lesssim T\|u\|^3_{L^\infty([0,T],\dot{H}^1)}. \end{aligned}$$

For the second term, remark that $\|u\|_{L^p_T(L^{\frac{3}{\varepsilon}})} \leq |||u|||_T$ by Sobolev embedding, where we have noted $L^p_T(L^q) \stackrel{\text{def}}{=} L^p([0,T], L^q)$. Then notice that if $\dfrac{1}{\eta} = \dfrac{1}{2} + \dfrac{\varepsilon}{3}$ and $\dfrac{1}{\sigma} = \dfrac{1}{2} - \dfrac{\varepsilon}{3}$, then

$$\begin{aligned} \|v^2\|_{L^\eta([0,T],L^\sigma)} &\leq \|v\|^2_{L^{2\eta}([0,T],L^{2\sigma})} \\ &\lesssim \|v_0\|^2_{\dot{H}^{\frac{1}{2}}} \approx 2^{2J(\frac{1}{2}-s)} \end{aligned}$$

by Strichartz' estimates as recalled above. Since $\|\cdot\|_{L^{\frac{2}{1+2\varepsilon}}([0,T])} \leq T^{\frac{2\varepsilon}{3}}\|\cdot\|_{L^{\frac{2}{1+2\varepsilon/3}}([0,T])}$, we get

$$\|uv^2\|_{L^1([0,T],L^2)} \lesssim T^{\frac{2\varepsilon}{3}} 2^{2J(\frac{1}{2}-s)}\|u\|_{L^p([0,T],\dot{B}_{q,2}^{1-\frac{2}{p}})}.$$

Putting those results together we have proved that

$$|||u|||_T \leq \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + CT^{\frac{2\varepsilon}{3}} 2^{2J(\frac{1}{2}-s)}|||u|||_T + T|||u|||_T^3.$$

Under the condition

$$C2^{2J(\frac{1}{2}-s)} \leq \frac{1}{2}, \tag{2.10}$$

a superlinear bootstrap argument (see for instance [1], Lemma 2.2) yields a local solution: indeed as long as $T \lesssim \dfrac{1}{E(u_0,u_1)}$, one finds a solution u satisfying

$$|||u|||_T \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}.$$

In the next section we will push this local solution to an arbitrarily long time by getting an a priori estimate on the energy of u.

2.3 Energy estimate for the low frequency part

In this section we shall prove an estimate for the Hamiltonian of u defined by

$$H(u)(t) \stackrel{\text{def}}{=} \left(\frac{1}{2}\|u(t)\|_{\dot{H}^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 + \frac{1}{4}\|u(t)\|_{L^4}^4\right).$$

We shall denote in the following $H_T(u) \stackrel{\text{def}}{=} \sup_{t\leq T} H(u)(t)$. Similarly we shall call $E(u)(t)$ the energy of u, and $E_T(u) \stackrel{\text{def}}{=} \sup_{t\leq T} E(u)(t)$, with

$$E(u)(t) \stackrel{\text{def}}{=} \left(\frac{1}{2}\|u(t)\|_{\dot{H}^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2\right).$$

PROPOSITION 3

Suppose the initial data satisfies $(u_0,u_1) \in \dot{H}^1 \cap L^4 \times L^2$. Then we have the following estimate: for any $r < +\infty$,

$$\begin{aligned} E_T(u) \leq H_T(u) \quad &\lesssim \quad E(u_0) + T^{\frac{1}{3}}\|v_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{6}}}^2 E_T(u) + T^{\frac{1}{2}+\frac{1}{r}}\|v_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{r}}} E_T(u)^{\frac{3}{2}} \\ &\qquad + T^{\frac{2}{3}}\|v_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{6}}}\|v_0\|_{\dot{H}^{\frac{1}{2}}}^2 E_T(u)^{\frac{3}{2}}. \end{aligned} \tag{2.11}$$

PROOF. Multiplying (1.1) by $\partial_t u$, integrating over x and t, we get

$$\frac{1}{2}\left(\|u(t)\|_{\dot H^1}^2 + \|\partial_t u(t)\|_{L^2}^2\right) + \frac{1}{4}\|u(t)\|_{L^4}^4 \le \frac{1}{2}(\|u_0\|_{\dot H^1}^2 + \|u_1\|_{L^2}^2) + \frac{1}{4}\|u_0\|_{L^4}^4$$
$$+ 3\left|\int_0^t \int_{\mathbb{R}^3} u(s,x)v^2(s,x)\partial_s u \, dxds\right| + 3\left|\int_0^t \int_{\mathbb{R}^3} u^2(s,x)v(s,x)\partial_s u \, dxds\right|.$$

As remarked at the beginning of Section 2 the quantity $\|u_0\|_{L^4}^4$ is negligible compared to the energy of the initial data, so taking the supremum over $t < T$ we get finally

$$H_T(u) \lesssim E(u_0, u_1) \tag{2.12}$$
$$+ 3\int_0^T \left|\int_{\mathbb{R}^3} u(t,x)v^2(t,x)\partial_t u \, dxdt\right| + 3\left|\int_0^T \int_{\mathbb{R}^3} u^2(t,x)v(t,x)\partial_t u \, dxdt\right|.$$

Let us call I and II the two space–time integrals appearing on the right–hand side of the inequality, and let us start by estimating I, which is the easiest. We have

$$\begin{aligned} I &\le \int_0^T \|v(t)\|_{L^6}^2 \|u(t)\|_{L^6} \|\partial_t u(t)\|_{L^2} \, dt \\ &\le E_T(u) \int_0^T \|v(t)\|_{L^6}^2 \, dt, \end{aligned}$$

and by Strichartz' estimates, we can write

$$\begin{aligned} \|v\|_{L_T^2 L^6}^2 &\lesssim T^{\frac{1}{3}} \|v\|_{L_T^3 L^6}^2 \\ &\lesssim T^{\frac{1}{3}} \|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{6}}}^2. \end{aligned}$$

Finally we get

$$I \lesssim T^{\frac{1}{3}} \|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{6}}}^2 E_T(u). \tag{2.13}$$

Now let us estimate the term II. As noticed in the introduction, one could use the same type of estimate as above for the term I, but that would require $s > \dfrac{3}{4} + \dfrac{1}{12}$, which is not the index given by the theorem. To improve the lower bound on s, one needs to improve the estimate on II. We first split II into two different pieces, as one may write $v = v_L + w$ where v_L is its linear part and w the nonlinear part coming from the v^3. The easiest is the second one, as

$$\|w\|_{L_t^3(L_x^6)} \lesssim \|v_0\|_{\dot H^{1/2+1/6}} \|v_0\|_{\dot H^{1/2}}^2$$

as recalled in Section 2.1 (namely (2.8)). Hence

$$\|wu^2\partial_t u\|_{L_T^1 L^1} \lesssim T^{2/3} \|v_0\|_{\dot H^{1/2+1/6}} \|v_0\|_{\dot H^{1/2}}^2 E_T(u)^{\frac{3}{2}}. \tag{2.14}$$

We are left with the remaining part, $II_L = \displaystyle\int_0^T v_L u^2 \partial_t u dt$. Since one can only say $\partial_t u \in L_t^\infty(L_x^2)$, we shall prove some sharper estimates on the term $\|v_L u^2\|_{L_T^1 L_x^2}$. The result is the following.

Lemma 1
Let v_L be a solution of the free wave equation with data in $\dot{H}^{\frac{1}{2}+\frac{1}{r}}$ with $r < \infty$, and let u be such that $E_T(u) < \infty$. Then $u^2 v_L \in L_T^1 L_x^2$ and

$$\|u^2 v_L\|_{L_T^1 L_x^2} \lesssim T^{\frac{1}{2}+\frac{1}{r}} \|v_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{r}}} E_T(u). \tag{2.15}$$

As this result is really the only non trivial part of the proof of Theorem 1, we shall postpone its proof to the end of the article, and first explain how the theorem follows.

2.4 Conclusion

Putting together (2.13), (2.14) and Lemma 1, we find that (2.12) becomes

$$\begin{aligned} E_T(u) &\lesssim E(u_0, u_1) \\ &\quad + T^{\frac{1}{3}} \|v_0\|^2_{\dot{H}^{\frac{1}{2}+\frac{1}{6}}} E_T(u) + E_T(u)^{\frac{3}{2}} \|v_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{r}}} (T^{\frac{1}{2}+\frac{1}{r}} + T^{\frac{2}{3}} \|v_0\|^2_{\dot{H}^{\frac{1}{2}}}). \end{aligned}$$

Now replacing the Sobolev norms of v_0 and the energy of the initial data in terms of the frequency cut 2^J, we get

$$E_T(u) \lesssim 2^{2J(1-s)} + 2^{2J(\frac{1}{2}+\frac{1}{6}-s)} T^{\frac{1}{3}} E_T(u) + 2^{J(\frac{1}{2}+\frac{1}{r}-s)} E_T(u)^{\frac{3}{2}} \left(T^{\frac{1}{2}+\frac{1}{r}} + T^{\frac{2}{3}} 2^{2J(\frac{1}{2}-s)} \right).$$

Now the conclusion is rather straightforward: we start by noticing that since $s > \dfrac{1}{2} + \dfrac{1}{6}$, we can choose J such that, say,

$$2^{2J(\frac{1}{2}+\frac{1}{6}-s)} T^{\frac{1}{3}} \leq \frac{1}{2}. \tag{2.16}$$

Then similarly one can also choose J so that

$$T^{\frac{2}{3}} 2^{2J(\frac{1}{2}-s)} \leq T^{\frac{1}{2}+\frac{1}{r}}. \tag{2.17}$$

So we are left with

$$E_T(u) \lesssim 2^{2J(1-s)} + 2^{J(\frac{1}{2}+\frac{1}{r}-s)} E_T(u)^{\frac{3}{2}} T^{\frac{1}{2}+\frac{1}{r}},$$

and the same superlinear bootstrap argument as that given in the introduction yields the condition:

$$1 - s + \frac{1}{2} + \frac{1}{r} - s < 0,$$

which implies the desired result: for any given T (which may be thought of as very large), as long as $s > \frac{3}{4}$, one may choose J accordingly in order to control E_T and hence have a solution on the time interval $[0,T]$. If one sets $T = 2^K$, then $K = (4s-3)J$ and $E_T \approx 2^{2(1-s)J} \approx T^{2\frac{1-s}{4s-3}}$.

We are left with the $\dot{H}^s$ bound for $\Phi = u + v$ (uniqueness then follows from local existence). The high frequency part v is obviously bounded in $C([0,\infty), \dot{H}^s)$ in terms of $\|v_0\|_{\dot{H}^s}$, since the initial data is in $\dot{H}^s$. As to the low frequency part, the linear part u_L (solution of the free wave equation with data u_0) is of course uniformly bounded in $C^0([0,\infty), \dot{H}^s)$ so we are left with its non linear part u_{NL}. All we need to get is a bound $L^\infty([0,T], L^2)$, and the result will then follow by interpolation with $C^0([0,T], \dot{H}^1)$. For any F, we can write

$$\| \int_0^t \mathcal{F}^{-1}\left(\frac{\sin s|\xi|}{|\xi|}\hat{F}(x,s)\right) ds\|_{L^2} \lesssim \int_0^t s\|F(x,s)\|_{L^2} ds,$$

so that means that

$$\|u_{\mathrm{NL}}\|_{L^2}(T) \lesssim \int_0^T s\left(\|u(\cdot,s)^3\|_{L^2} + \|u^2 v(\cdot,s)\|_{L^2} + \|uv^2(\cdot,s)\|_{L^2}\right) ds.$$

To estimate the two last terms, we use the fact that the initial data v_0 is in $\dot{H}^{\frac{1}{2}+\frac{1}{6}}$, hence as in the introduction, one has $v \in L^3([0,T], L^6)$. It is then easy to see that the first term $\|u(\cdot,s)^3\|_{L^2}$ is of the highest order, and we have

$$\|u_{\mathrm{NL}}\|_{L^2}(T) \lesssim T^2 E_T^{\frac{3}{2}} \approx T^{3\frac{1-s}{4s-3}+2}.$$

Then interpolation between L^2 and $\dot{H}^1$ gives

$$\|u_{\mathrm{NL}}\|_{\dot{H}^s}(T) \lesssim T^{(3\frac{1-s}{4s-3}+2)(1-s)} T^{\frac{1-s}{4s-3}s} \approx T^{\frac{3(1-s)(2s-1)}{2(4s-3)}}.$$

Remark 1 In light of the final part of the argument, one may think that for the quadratic nonlinearity $|\Phi|\Phi$, the situation would be better, as the equivalent of (2.12) would be linear w.r.t. $E_T(u)$. However, in dealing with the space-time integral replacing I, one has to take into account the low regularity of the high frequency part, and even using improved Strichartz estimates, one is not able to do better than $s > 1/4$, which was already obtained in [11].

Remark 2 The approach we developed above extends of course to other equations, but it does not seem to always perform as well as Bourgain's method. For example, one may try the cubic 2D Schrödinger equation, which was the equation considered by Bourgain in [3], where he obtained global well-posedness for $s > 3/5$. With our method, one is led to estimating quantities like $\int_{t,x} u^2 \nabla u \nabla v dt dx$, where u is the low frequency (H^1) part, and v the high frequency part. To go down to $s > 1/2$ which is halfway between

H^1 and scaling, one would need to bound the integral in term of $\|v_0\|_{L^2}$ and $\|u\|^3_{L^\infty_t(H^1)}$, which doesn't seem possible. In fact, one can possibly obtain something with $\|v_0\|_{H^{\frac{1}{2}}}$ using local smoothing (gaining half a derivative). This would lead to well-posedness for $s > 3/4$ which is not as good as Bourgain's result.

3 Proof of Lemma 1

Recall that we want to prove

$$\|vu^2(t)\|_{L^1_T L^2_x} \lesssim T^{\frac{1}{2}+\frac{1}{r}} \|v_0\|^{\dot{H}^{\frac{1}{2}+\frac{1}{r}}} E_T(u),$$

where v is a solution of the free wave equation. Let us decompose vu^2 according to J.-M. Bony's paraproduct algorithm [2]. We get

$$\begin{aligned} vu^2 &= \sum_{j\in\mathbb{Z}} S_{j-2}v\Delta_j(u^2) + \sum_{j\in\mathbb{Z}} \Delta_j \left(\sum_{k>j} \Delta_k v \Delta_k(u^2) \right) + \sum_{j\in\mathbb{Z}} S_{j-2}(u^2)\Delta_j v \qquad (3.18) \\ &= (1) + (2) + (3). \end{aligned}$$

We have

$$\begin{aligned} \|(1)\|_{L^2} &\leq \sup_{j\in\mathbb{Z}} 2^{-j/2} \|S_{j-2}v\|_{L^\infty} \left\| \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} |\Delta_j(u^2)| \right\|_{L^2} \\ &\lesssim \sup_{j\in\mathbb{Z}} 2^{-j/2} \|S_{j-2}v\|_{L^\infty} \|u\|^2_{\dot{H}^1}, \end{aligned}$$

as $u^2 \in \dot{B}^{\frac{1}{2}}_{2,1}$ by product rules in Besov spaces and using Minkowski ($l^1(L^2) \subset L^2(l^1)$). It follows that

$$\|(1)\|_{L^1_T L^2} \lesssim E_T(u) \int_0^T \sup_j 2^{-j/2+3j/r} \|S_j v(t)\|_{L^r}\, dt, \qquad (3.19)$$

using Bernstein as $\|S_{j-2}v\|_{L^\infty} \lesssim 2^{3j/r}\|S_{j-2}v\|_{L^r}$ for all $r < +\infty$. But Strichartz estimates imply that

$$\|\Delta_j v\|_{L^p_T L^r} \lesssim 2^{2j/p} \|\Delta_j v_0\|_{L^2}, \quad \text{with} \quad \frac{2}{p} + \frac{2}{r} = 1.$$

Choosing $s = 1/2 + 1/r$ and noticing that $s - 2/p = -1/2 + 3/r < 0$ we may pass from

$$2^{2j(s-2/p)} \|\Delta_j v\|_{L^p_T L^r} \lesssim 2^{js} \|\Delta_j v_0\|_{L^2} = \|v_0\|_{\dot{H}^s} \mu_j,$$

with $\mu_j \in l^2$, to (summing over low frequencies)

$$2^{2j(s-2/p)}\|S_j v\|_{L^p_T L^r} \lesssim \|v_0\|_{\dot H^s}\tilde\mu_j,$$

where $\tilde\mu_j \in l^2$. We then get from (3.19)

$$\begin{aligned}
\|(1)\|_{L^1_T L^2_x} &\lesssim E_T(u) T^{\frac{p-1}{p}} \left(\int_0^T \sup_j 2^{jp(-1/2+3/r)}\|S_j v(t)\|^p_{L^r}\, dt\right)^{1/p} \\
&\lesssim E_T(u) T^{\frac{p-1}{p}} \left(\sum_{j\in\mathbb{Z}} 2^{2j(-1/2+3/r)}\|S_j v\|^2_{L^p_T L^r}\right)^{1/2},
\end{aligned}$$

since $p > 2$. But $\dfrac{p-1}{p} = \dfrac{1}{2} + \dfrac{1}{r}$, so it follows that

$$\|(1)\|_{L^1_T L^2_x} \lesssim E_T(u) T^{\frac{1}{2}+\frac{1}{r}}\|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{r}}}. \tag{3.20}$$

Now let us estimate the second term of the decomposition (3.18). Let $\overline{r}$ be fixed so that $\dfrac{1}{\overline{r}} = \dfrac{1}{2} + \dfrac{1}{r}\cdot$ Then

$$\begin{aligned}
\|(2)\|_{L^{\overline{r}}} &\leq \sum_{k>j} \|\Delta_k v\|_{L^r}\|\Delta_k(u^2)\|_{L^2} \\
&\lesssim \sup_{k\in\mathbb{Z}} 2^{k(3/r-1/2)}\|\Delta_k v\|_{L^r} \sum_{k>j} 2^{-k(3/r-1/2)}\|\Delta_k(u^2)\|_{L^2} \\
&\lesssim \sup_{k\in\mathbb{Z}} 2^{k(3/r-1/2)}\|\Delta_k v\|_{L^r} 2^{-3j/r} c_j \|u\|^2_{\dot H^1},
\end{aligned}$$

where $(c_j) \in L^\infty([0,T], \ell^2(\mathbb{Z}))$. It follows that

$$\|(2)\|_{\dot B^{3/r}_{\overline{r},2}} \lesssim \sup_{k\in\mathbb{Z}} 2^{k(3/r-1/2)}\|\Delta_k v\|_{L^r}\|u\|^2_{\dot H^1}.$$

By Sobolev embeddings, we are reduced to estimate (3.19) (with Δ_j instead of S_j, which is even easier), and the computations following that estimate yield in an identical way

$$\|(2)\|_{L^1_T L^2_x} \lesssim E_T(u) T^{\frac{1}{2}+\frac{1}{r}}\|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{r}}}. \tag{3.21}$$

We are left with (3), which turns out to be the bad guy, and sharper Strichartz estimates are needed:

Let us recall what these are (with the 3D numerology):

Theorem 3 ([13])
Let $\hat f \in L^2$ be supported in a ball of size 2^k, with center at distance $|\xi| \sim 2^j$, and $k < j$. Then the solution u of the wave equation with data f is such that

$$\|u\|_{L^p_t(L^q_x)} \lesssim 2^{\frac{k-j}{p}+\frac{2j}{p}}\|f\|_2. \tag{3.22}$$

We point out that this estimate is an improvement over what the usual Strichartz estimate would provide, since we gain an additional small factor through the power of $k-j$.

Let us proceed by duality. Let $\phi \in L^2_{t,x}$. Then we seek control of

$$\begin{aligned} I = \int (3)(u, v_L)\phi dx dt &\approx \sum_j \int S_{j-2}(u^2)\Delta_j v_L \Delta_j \phi dx dt, \\ &\approx \sum_k \int \Delta_k(u^2) \sum_{k \lesssim j} \Delta_j v_L \Delta_j \phi dx dt. \end{aligned}$$

Since we know that $u^2 \in L^\infty_T(\dot{B}^{1/2}_{2,1})$, we can see the last sum as the duality between $L^\infty_T(\dot{B}^{1/2}_{2,1})$ and $L^1_T(\dot{B}^{-1/2}_{2,\infty})$, which means we need to prove

$$\int_0^T \sup_k 2^{-k/2} \|\Delta_k \sum_{k \lesssim j} \Delta_j v_L \Delta_j \phi\|_{L^2_x} dt \lesssim \|\phi\|_{L^2_{t,x}}.$$

We will actually prove a slightly better bound: call $f_k = \Delta_k \sum_{k \lesssim j} \Delta_j v_L \Delta_j \phi$, we will estimate $\sum_k f_k$ in $L^1_T(\dot{B}^{-k/2+3k/r}_{\frac{2r}{r+2},1}) \hookrightarrow L^1_T(\dot{B}^{-k/2}_{2,\infty})$. Since we are only interested in low frequencies, one may partition the $|\xi| \sim 2^j$ region into balls of size 2^k. Then we only need to consider in our sum over $k < j$ balls which are (almost) opposite. Denoting by Q and $-Q$ two such opposite balls, we are left with

$$\begin{aligned} \|\Delta_k \sum_{k \lesssim j} \Delta_j v_L \Delta_j \phi\|_{L_x^{\frac{2r}{r+2}}} &\lesssim \sum_{k \lesssim j} \sum_Q \|\Delta_j^Q v_L\|_{L^r} \|\Delta_j^{-Q}\phi\|_{L^2} \\ &\lesssim \sum_{k \lesssim j} \left(\sum_Q \|\Delta_j^Q v_L\|^2_{L^r} \right)^{\frac{1}{2}} \|\phi\|_{L^2}, \end{aligned}$$

using Cauchy-Schwarz on the Q sum and L^2 orthogonality with respect to the Q. Now, at fixed k we have

$$\|f_k\|_{L^1_T(L_x^{\frac{2r}{r+2}})} \lesssim \sum_{k \lesssim j} \left(\sum_Q \|\Delta_j^Q v_L\|^2_{L^2_T L^r} \right)^{\frac{1}{2}} \|\Delta_j \phi\|_{L^2_T L^2}.$$

One can then estimate the Q sum using the precise Strichartz estimates: we get

$$\begin{aligned} \left(\sum_Q \|\Delta_j^Q v_L\|^2_{L^2_T L^r} \right)^{\frac{1}{2}} &\lesssim T^{\frac{1}{2}-\frac{1}{p}} \left(\sum_Q \|\Delta_j^Q v_L\|^2_{L^p_T L^r} \right)^{\frac{1}{2}} \\ &\lesssim T^{\frac{1}{2}-\frac{1}{p}} 2^{(k-j)(\frac{1}{2}-\frac{1}{r})+j\frac{2}{p}} \left(\sum_Q \|\Delta_j^Q v_0\|^2_{L^2} \right)^{\frac{1}{2}} \\ &\lesssim T^{\frac{1}{2}-\frac{1}{p}} 2^{(k-j)(\frac{1}{2}-\frac{1}{r})+j\frac{2}{p}} \|\Delta_j v_0\|_{L^2}. \end{aligned}$$

Since $2^{j(\frac{1}{2}+\frac{1}{r})}\|\Delta_j v_0\|_{L^2}\|\Delta_j\phi\|_{L^2_T L^2}=\mu_j\in l^1$, we have

$$\begin{aligned}\|f_k\|_{L^1_T(L_x^{\frac{2r}{r+2}})} &\lesssim T^{\frac{1}{2}-\frac{1}{p}}\sum_{k\lesssim j}2^{k(\frac{1}{2}-\frac{1}{r})-j(\frac{1}{r}+\frac{1}{r})}\mu_j\\ &\lesssim T^{\frac{1}{2}-\frac{1}{p}}2^{k(\frac{1}{2}-\frac{3}{r})}\tilde{\mu}_k,\end{aligned}$$

which means nothing but

$$\|(3)(u,v_L)\|_{L^2_T L^2}\lesssim T^{\frac{1}{r}}\|u\|^2_{L^\infty_T(\dot H^1)}\|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{r}}},$$

and since we were after the $L^1_T L^2$ norm, one gets an extra $T^{1/2}$ to obtain

$$\|(3)(u,v_L)\|_{L^1_T L^2}\lesssim T^{\frac{1}{2}+\frac{1}{r}}\|v_0\|_{\dot H^{\frac{1}{2}+\frac{1}{r}}}E_T(u).$$

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