Profile decomposition for the wave equation outside a convex obstacle

by

Isabelle Gallagher

Patrick Gérard

Département de Mathématiques
Université de Paris-Sud
91405 Orsay Cedex, France

UMR 8628 du CNRS

Isabelle.Gallagher@math.u-psud.fr
Patrick.Gerard@math.u-psud.fr

Abstract. We study the nonlinear wave equation

(1) $$\Box u + |u|^4 u = 0 \text{ in } \mathbb{R}_t \times \Omega,$$

with Dirichlet boundary conditions, where Ω is the exterior of a strictly convex domain of $\mathbb{R}^3$. We first prove a structure theorem for bounded energy sequences of solutions to the linear wave equation in Ω, following a method introduced by H. Bahouri and the second author in [BG1]. The proof requires a non-concentration theorem for such sequences, the proof of which involves semi-classical measures. We then infer, using Strichartz estimates proved by H. Smith and C. Sogge in [SmSo1], the description of bounded energy sequences of solutions of (1), up to remainder terms, small both in energy and in Strichartz norms. As corollaries, we obtain an a priori bound for the Strichartz norms in terms of the energy, as well as a Lipschitz estimate for the nonlinear evolution group.

Key words. Nonlinear wave equations, Strichartz inequalities, semiclassical measures.

AMS Subject Classification Index. 35 L 40, 35 L 25.

I. Introduction and statement of results

The aim of this article is to study the following problem :

(1) $$\begin{cases} \Box u_n + |u_n|^4 u_n = 0 \text{ in } \mathbb{R}_t \times \Omega, \ \ u_n|_{\mathbb{R}_t \times \partial\Omega} = 0 \\ (u_n, \partial_t u_n)|_{t=0} = (\varphi_n, \psi_n), \end{cases}$$

where Ω is the exterior of a compact, strictly convex, smooth domain of $\mathbb{R}^3$, and where (φ_n, ψ_n) is a sequence of functions, bounded in the energy space $E(\Omega) \stackrel{\text{def}}{=} \dot{H}^1_0(\Omega) \times L^2(\Omega)$. We have written $\dot{H}^1_0(\Omega)$ for the closure of $C_0^\infty(\Omega)$ for the norm $||\nabla \cdot||_{L^2(\mathbb{R}^3)}$, in $\dot{H}^1(\mathbb{R}^3)$. The space $E(\Omega)$ is endowed with the Hilbert norm

$$||(\varphi, \psi)||_E = \left(||\nabla\varphi||^2_{L^2(\Omega)} + ||\psi||^2_{L^2(\Omega)} \right)^{1/2}.$$

Unless specified otherwise, we shall identify elements of $E(\Omega)$ with their extensions by 0 in $\dot H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$.

Before stating the main results of the paper, let us start by recalling a few known facts concerning (1), starting with the case $\Omega=\mathbb{R}^3$. In that case, global existence of solutions for smooth data is known since W. Strauss [Str] for small data, J. Rauch [R] for small energy data, M. Struwe [S] for radially symmetric data, and finally M. Grillakis ([Gr1], [Gr2]) for the general case. When the data is in the energy space $\dot H^1\times L^2(\mathbb{R}^3)$, J. Shatah and M. Struwe proved in [ShSt] existence and uniqueness of solutions in the class

$$u\in C^0(\mathbb{R}_t,\dot H^1(\mathbb{R}^3))\cap L^5_{\rm loc}(\mathbb{R}_t,L^{10}(\mathbb{R}^3)),\ \ \partial_t u\in C^0(\mathbb{R}_t,L^2(\mathbb{R}^3)). \tag{2}$$

Note that the space $L^5([0,T],L^{10}(\mathbb{R}^3))$ corresponds to a so-called "Strichartz norm", defined more generally as $L^q([0,T],L^r(\mathbb{R}^3))$, with $\frac1q+\frac3r=\frac12$: in [GV], it is proved that for any function f such that $(f,\partial_t f)\in C^0([0,T],\dot H^1\times L^2(\mathbb{R}^3))$,

$$||f||_{L^q([0,T],L^r(\mathbb{R}^3))}\leq C_r\Big(||\nabla f|_{t=0}||_{L^2(\mathbb{R}^3)}+||\Box f||_{L^1([0,T],L^2(\mathbb{R}^3))}\Big), \tag{3}$$

with $\frac1q+\frac3r=\frac12$, and $q>2$. A simple scaling argument shows that C_r does not depend on T.

Using in a crucial way that Strichartz estimate, still in the case where $\Omega=\mathbb{R}^3$, H. Bahouri and the second author proved in [BG1] a structure theorem for the solution of (1), up to small remainder terms, both in energy and in Strichartz norms. Our aim is to prove a similar result in the case of a domain $\Omega\subset\mathbb{R}^3$. In the case where Ω is the exterior of a strictly convex domain of $\mathbb{R}^3$, which is our case, H. Smith and C. Sogge proved in [SmSo1] existence and uniqueness of smooth solutions to (1) for smooth initial data. The non-concentration result proved in [SmSo1] (lemma 3.3), along with the method followed by [ShSt] when $\Omega=\mathbb{R}^3$, implies in fact in an easy way existence and uniqueness of solutions when the data are in the energy space : we shall not write the details here. In [SmSo1] is proved the following fundamental Strichartz estimate, which is at the basis of our study : for any $f\in C^0([0,T],\dot H^1_0(\Omega))\cap \dot C^1([0,T],L^2(\Omega))$,

$$||f||_{L^q([0,T],L^r(\Omega))}\leq C_{T,r}\Big(||\nabla_{t,x} f|_{t=0}||_{L^2(\Omega)}+||\Box f||_{L^1([0,T],L^2(\Omega))}\Big), \tag{4}$$

with $\frac1q+\frac3r=\frac12$ and $q>2$. Note that a recent paper by H. Smith and C. Sogge ([SmSo2]) proves that $C_{T,r}$ does not depend on T. That fact will not be used in the sequel.

In order to study the solution of (1),we shall be led, following [BG1], to studying the linear equation with the same Cauchy data, and we shall start by proving a structure theorem for the following equation :

$$\begin{cases}\Box v_n=0\ \text{ in }\ \mathbb{R}_t\times\Omega,\ \ v_n|_{\mathbb{R}_t\times\partial\Omega}=0\\ (v_n,\partial_t v_n)|_{t=0}=(\varphi_n,\psi_n).\end{cases} \tag{5}$$

Definition I.1. *Let $(\varphi,\psi)\in\dot H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, and let $(h_n,x_n,t_n)\in(\mathbb{R}^+\times\mathbb{R}^3\times\mathbb{R})^{\mathbb{N}}$ be such that for some $(x_\infty,t_\infty)\in\mathbb{R}^3\times\mathbb{R}$,*

$$\lim_{n\to\infty}(h_n,x_n,t_n)=(0,x_\infty,t_\infty).$$

Then the linear concentrating wave associated with $(\varphi, \psi, h_n, x_n, t_n)$ is the solution of the following linear wave equation :

$$\begin{cases} \Box p_n = 0 \ \text{ in } \mathbb{R}_t \times \Omega, \quad p_n|_{\mathbb{R}_t\times\partial\Omega} = 0 \\ (p_n; \partial_t p_n)|_{t=t_n} = \Big(\frac{1}{h_n^{1/2}} \mathcal{P}_\Omega \Big(\varphi\Big(\frac{\cdot - x_n}{h_n}\Big)\Big) ; \frac{1}{h_n^{3/2}} \mathbf{1}_\Omega(\cdot)\, \psi\Big(\frac{\cdot - x_n}{h_n}\Big)\Big), \end{cases}$$

where $\mathcal{P}_\Omega$ is the orthogonal projection $\dot H^1(\mathbb{R}^3)$ onto $\dot H^1_0(\Omega)$.

Remarks I.2.

1) Unless specified otherwise, in this paper a linear concentrating wave will always solve the linear wave equation in the domain Ω.

2) Up to a subsequence and to a translation in time, one can assume that the sequences (h_n) and (t_n) of definition I.1 satisfy

$$\frac{t_n}{h_n} \xrightarrow[n\to\infty]{} +\infty, -\infty \ \text{ or } \ 0. \tag{6}$$

In the following, we shall call concentrating data quantities such as $(\varphi, \psi, h_n, x_n, t_n)$ appearing in definition I.1. We shall say that (h_n, x_n, t_n) and $(\tilde h_n, \tilde x_n, \tilde t_n)$ are orthogonal, as in [BG1], if

$$\text{either } \Big|\log\Big(\frac{\tilde h_n}{h_n}\Big)\Big| \xrightarrow[n\to\infty]{} +\infty, \ \text{ or } \tilde h_n = h_n \quad \text{and} \quad \frac{|(x_n,t_n) - (\tilde x_n, \tilde t_n)|}{h_n} \xrightarrow[n\to\infty]{} +\infty.$$

We recall that the energy of any function u is defined by

$$E_0(u,t) \stackrel{\text{def}}{=} \int_\Omega (|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2)\, dx, \tag{7}$$

and is independent of t when u satisfies the linear wave equation.

In the whole of this paper, we shall assume that the initial data is compact at infinity, in the sense that

$$\overline{\lim_{n\to\infty}} \int_{|x|\geq R} \big(|\nabla \varphi_n(x)|^2 + |\psi_n(x)|^2\big)\, dx \xrightarrow[R\to\infty]{} 0. \tag{8}$$

We are now ready to state our first result, concerning (5).

Theorem 1. *Let v_n be the solution of (5), where (φ_n, ψ_n) is bounded in $E(\Omega)$ and satisfies (8). Then there exist a finite energy solution to the linear wave equation, v, orthogonal concentrating data $(\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)})$, for $j \in \mathbb{N}^*$, such that v_n can be decomposed as follows, up to the extraction of a subsequence : for any $\ell \in \mathbb{N}^*$,*

$$v_n(t,x) = v(t,x) + \sum_{j=1}^{\ell} p_n^{(j)}(t,x) + w_n^{(\ell)}(t,x), \tag{9}$$

where $p_n^{(j)}$ is the linear concentrating wave associated with $(\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)})$, where the remainder $w_n^{(\ell)}$ satisfies, for every $T > 0$,

$$\overline{\lim_{n\to\infty}} \|w_n^{(\ell)}\|_{L^\infty([-T,T], L^6(\Omega))} \xrightarrow[\ell\to\infty]{} 0, \tag{10}$$

and finally where the energies are orthogonal, in the sense that, for any ℓ,

$$E_0(v_n) = E_0(v) + \sum_{j=1}^{\ell} E_0(p_n^{(j)}) + E_0(w_n^{(\ell)}) + o(1)\,, \quad n \to +\infty\,. \tag{11}$$

Now let us consider the nonlinear equation (1). We shall start by proving a theorem comparing a linear concentrating wave and the solution of the nonlinear wave equation with the same Cauchy data.

Definition I.3. *Let $(\varphi, \psi, h_n, x_n, t_n)$ be concentrating data, and p_n the associate linear concentrating wave. Then the nonlinear concentrating wave associated with p_n is the solution of the following equation :*

$$\begin{cases} \Box q_n + |q_n|^4 q_n = 0 \ \text{ in } \mathbb{R}_t \times \Omega\,, \ \ q_n|_{\mathbb{R}_t \times \partial\Omega} = 0 \\ (q_n, \partial_t q_n)|_{t=0} = (p_n, \partial_t p_n)|_{t=0}\,. \end{cases}$$

Before stating the theorem, let us define the rescaled domain

$$\Omega_n \stackrel{\text{def}}{=} \frac{\Omega - x_n}{h_n}\,. \tag{12}$$

We shall see in section II.1, that one can define a limit domain to Ω_n, as n goes to infinity.

Definition I.4. *Let M_n and M be two open subsets of $\mathbb{R}^3$. We shall say that M_n converges towards M as n goes to infinity if the two following properties hold :*

(i) $\forall K$ compact subset of M, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $K \subset M_n$,

(ii) $\forall K'$ compact subset of ${}^c\bar{M}$, $\exists N' \in \mathbb{N}$, $\forall n \geq N'$, $K' \subset {}^c\bar{M}_n$.

Notice that a sequence of open subsets may converge towards different limits – for instance the sequence of balls of radius n centered at 0 converges towards any dense open subset of $\mathbb{R}^3$. However, the closure of these limits will be the same.

We shall prove the following result in section II.1.

Proposition I.5. *Let (h_n, x_n) be defined as in definition I.1, and let Ω_n be as defined in (12). Then as n goes to infinity, up to a subsequence, Ω_n takes one of the three following limits : $\emptyset$, $\mathbb{R}^3$ or a half-space H, depending on the position of x_n with respect to $\partial\Omega$.*

Finally we define the following scattering operator : let us recall that in [BG1]-[BG2] is studied a scattering operator S, as well as wave operators $W^{\pm}$. Using that notation, and analogously to [BG1], formula (1.17), we shall define the following linear concentrating waves associated with a linear concentrating wave (p_n).

Definition I.6. *Let p_n be a linear concentrating wave, associated with concentrating data $(\varphi, \psi, h_n, x_n, t_n)$. Let Ω_∞ be the limit of Ω_n, as defined in definition I.4. If Ω_∞ is a half-space, we denote by R_{Ω_∞} the skew-symmetric reflection with respect* *to* $\partial\Omega_\infty$*, and by* P_{Ω_∞} *the operator defined by* $P_{\Omega_\infty}(f,g) \stackrel{\text{def}}{=} \big(\mathcal{P}_{\Omega_\infty} f, \mathbf{1}_{\Omega_\infty} g\big)$ *for any* $(f,g) \in \dot H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

Then we associate with (p_n) *two linear concentrating waves, denoted* $p_n^\pm$*, associated with concentrating data* $(\varphi^\pm, \psi^\pm, h_n, x_n, t_n)$ *in the following way :*

$$(\varphi^-, \psi^-) \stackrel{\text{def}}{=} \begin{cases} (\varphi, \psi) & \text{if } \dfrac{t_n}{h_n} \xrightarrow[n\to\infty]{} +\infty \\ P_{\Omega_\infty} W^- R_{\Omega_\infty}(\varphi,\psi) & \text{if } \dfrac{t_n}{h_n} \xrightarrow[n\to\infty]{} 0 \\ P_{\Omega_\infty} S^{-1} R_{\Omega_\infty}(\varphi,\psi) & \text{if } \dfrac{t_n}{h_n} \xrightarrow[n\to\infty]{} -\infty\,, \end{cases}$$

$$(\varphi^+, \psi^+) \stackrel{\text{def}}{=} \begin{cases} P_{\Omega_\infty} S R_{\Omega_\infty}(\varphi,\psi) & \text{if } \dfrac{t_n}{h_n} \xrightarrow[n\to\infty]{} +\infty \\ P_{\Omega_\infty} W^+ R_{\Omega_\infty}(\varphi,\psi) & \text{if } \dfrac{t_n}{h_n} \xrightarrow[n\to\infty]{} 0 \\ (\varphi, \psi) & \text{if } \dfrac{t_n}{h_n} \xrightarrow[n\to\infty]{} -\infty\,. \end{cases}$$

Theorem 2. *Let* p_n *be the linear concentrating wave associated with concentrating data* $(\varphi_n, \psi, h_n, x_n, t_n)$*, and let* q_n *be the associate nonlinear concentrating wave, in the sense of definition I.3. Then* (q_n) *is bounded in* $L^5_{\text{loc}}(\mathbb{R}; L^{10}(\Omega))$*, and satisfies the following properties, with the notation of definition I.6 : for any* $T > 0$*, we have*

$$\overline{\lim_{n\to\infty}} \left(\sup_{-T \le t \le t_n - \lambda h_n} E_0(p_n^- - q_n, t)^{1/2} + \|p_n^- - q_n\|_{L^5([-T, t_n - \lambda h_n], L^{10}(\Omega))} \right) \xrightarrow[\lambda\to\infty]{} 0\,;$$

$$\overline{\lim_{n\to\infty}} \left(\sup_{t_n + \lambda h_n \le t \le T} E_0(p_n^+ - q_n, t)^{1/2} + \|p_n^+ - q_n\|_{L^5([t_n + \lambda h_n, T], L^{10}(\Omega))} \right) \xrightarrow[\lambda\to\infty]{} 0\,.$$

We shall actually be proving a stronger result than that stated in theorem 2, with a precise analysis of the behavior of q_n around $t = t_n$, i.e. for times $t \in [t_n - \lambda h_n, t_n + \lambda h_n]$. That will enable us to infer, using the methods of [BG1], the following theorem.

Theorem 3. *Let* $(\varphi_n, \psi_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $E(\Omega)$ *satisfying (8), and let* u_n *be the solution of (1) associated with* (φ_n, ψ_n)*. Then with the notation of theorem 1, and up to the extraction of a subsequence, we can write, for any* $\ell \in \mathbb{N}^*$,

$$u_n(t,x) = u(t,x) + \sum_{j=1}^{\ell} q_n^{(j)}(t,x) + w_n^{(\ell)}(t,x) + r_n^{(\ell)}(t,x)\,,$$

where u *is a solution of the nonlinear wave equation, where* $q_n^{(j)}$ *is the nonlinear concentrating wave associated with* $p_n^{(j)}$*, and where for any* $T > 0$,

$$\overline{\lim_{n\to\infty}} \left(\sup_{-T \le t \le T} E_0(r_n^{(\ell)}, t)^{1/2} + \|r_n^{(\ell)}\|_{L^5([-T,T], L^{10}(\Omega))} \right) \xrightarrow[\ell\to\infty]{} 0\,.$$

Remarks I.7.

1) The results in this paper can be understood, in terms of geometrical optics, in the following way : outside strictly convex obstacles, nonlinear geometrical optics have been reduced to linear geometrical optics, associated with scattering in the whole space $\mathbb{R}^3$ or in half-spaces. Notice that, contrary to the case of the whole space ($\Omega = \mathbb{R}^3$, see [BG1]), our results are only local in time. However, using the recent results of [SmSo2], it is likely theorem 3 can be extended to a global in time result, as in [BG1], up to enlarging our collection of scattering operators.

2) The proofs of theorems 2 and 3 are based on the Strichartz estimate (4), which is the reason why we are restricted to the case where Ω is the exterior of a strictly convex domain : that is the only situation for which, to our knowledge, such an estimate is available. However, supposing (4) held in more general cases, problems would still remain. For instance, the time-orthogonality of the linear concentrating waves is proved using the following non-concentration result, for which the form of Ω has an important role to play. Its proof involves semi-classical measures (see [GL], [GMMP], [LP]), and is given in section II.3.

Proposition I.8. *Let (p_n) be the linear concentrating wave associated with concentrating data $(\varphi, \psi, h_n, x_n, t_n = 0)$. Then for any bounded interval $I \subset \mathbb{R}$, for any sequence (ε_n) such that $\lim_{n\to\infty} \frac{\varepsilon_n}{h_n} = +\infty$, we have*

$$\lim_{n\to\infty} ||p_n||_{L^\infty(I\setminus[-\varepsilon_n,\varepsilon_n],L^6(\Omega))} = 0\,.$$

3) As a corollary to theorem 3, we shall prove the following *a priori* estimate.

Corollary 1. *Let $T > 0$ be fixed. There exists a non-decreasing function, $A : [0,\infty[\to [0,\infty[$, such that if u satisfies (1), then*

$$||u||_{L^5([0,T],L^{10}(\Omega))} \leq A(E_0(u))\,.$$

In the global case $\Omega = \mathbb{R}^3$, that *a priori* estimate was proved in [BG1], and, by different methods, by K. NAKANISHI in [N]. However it is not clear whether the methods of [N] can be easily adapted to the case of a domain, as they are closely linked to Besov spaces and dyadic decompositions.

4) Using corollary 1 and theorem 3, a Lipschitz estimate can be proved.

Corollary 2. *Let F map initial data $(\varphi, \psi) \in E(\Omega)$ to the solution u of (1). Then F is Lipschitz on bounded subsets of $E(\Omega)$.*

Such a result does not hold in the supercritical case (see [Le2]).

The structure of the paper is as follows. In section II, we prove some preliminary results which will be used in the following sections : section II.1 is devoted to some analysis on linear concentrating waves and on the rescaled domain Ω_n defined in (12), section II.2 to the notion and propagation of strictly $(h_n)-$oscillatory functions, and finally section II.3 to the proof of proposition I.8 stated above. In section III, we prove theorem 1, whereas theorems 2 and 3 are proved respectively in sections IV.1 and IV.2. Finally corollaries 1 and 2 are proved in the appendix.

II. Preliminary analysis

In this preliminary section, we are going to present some notation we shall be using throughout this study, as well as preliminary results. We will start by defining more precisely the structure of linear concentrating waves in a domain Ω of $\mathbb{R}^3$; then we will address the question of $(h_n)-$oscillatory sequences and their propagation ; finally, we will prove a crucial non-concentration property for linear concentrating waves outside a convex domain, stated in proposition I.8.

II.1. Linear concentrating waves and rescaled domains

We gave in definition I.1 the definition of a linear concentrating wave ; let us note that if p_n is such a wave, associated with concentrating data $(\varphi, \psi, h_n, x_n, t_n)$, then the rescaled function

$$P_n(s,y) \stackrel{\text{def}}{=} h_n^{1/2} p_n(t_n + h_n s, x_n + h_n y)\,, \quad (s,y) \in \mathbb{R} \times \mathbb{R}^3\,, \tag{1.1}$$

satisfies the following system :

$$\begin{cases} \Box P_n = 0 \ \text{ in } \ \mathbb{R}_s \times \Omega_n\,, \quad P_n|_{\mathbb{R}_s \times \partial\Omega_n} = 0 \\ (P_n, \partial_s P_n)|_{s=0} = (\mathcal{P}_{\Omega_n}\varphi, \mathbf{1}_{\Omega_n}\psi) \end{cases}$$

where Ω_n is defined by

$$\Omega_n \stackrel{\text{def}}{=} \frac{\Omega - x_n}{h_n}\,. \tag{1.2}$$

It will be useful in the following to have some information on the behavior of Ω_n as n goes to infinity : let us recall that we have defined in definition I.4 the notion of a limit domain to Ω_n. We are going to prove proposition I.5 in this section.

Let us first introduce some notation : we have defined $x_\infty \in \mathbb{R}^3$ as the limit of the sequence (x_n). In the case where $x_\infty \in \partial\Omega$, let us denote by ϕ a defining function of Ω near x_∞ :

$$\Omega = \{x \in \mathbb{R}^3 | \phi(x) > 0\}\,, \quad \nabla\phi \neq 0 \ \text{ on } \ \partial\Omega\,. \tag{1.3}$$

From now on, when $x_\infty \in \partial\Omega$, we shall write

$$\omega \stackrel{\text{def}}{=} \nabla\phi(x_\infty) \ \text{ and } \ \alpha \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{\phi(x_n)}{h_n}\,, \tag{1.4}$$

limit which exists in $\overline{\mathbb{R}}$ up to extracting a subsequence.

Then we define the sets

$$H^{\alpha,\omega} \stackrel{\text{def}}{=} \{y \in \mathbb{R}^3 | y \cdot \omega > -\alpha\} \ \text{ if } \ \alpha \in \mathbb{R}, \ \ H^{+\infty,\omega} = \mathbb{R}^3 \ \text{ and } \ H^{-\infty,\omega} = \emptyset\,. \tag{1.5}$$

Finally two concentrating data $(\varphi, \psi, h_n, x_n, t_n)$ and $(\tilde\varphi, \tilde\psi, \tilde h_n, \tilde x_n, \tilde t_n)$ will be said to be equivalent if the difference of the associate linear concentrating waves goes to zero in energy as n goes to infinity.

Proposition II.1.1. *Let (h_n, x_n) and Ω_n be defined as in definition I.1 and (1.2). Then as n goes to infinity, Ω_n takes, up to a subsequence, one of the following limits Ω_∞ :*

(i) If $x_\infty \in \Omega$, then $\Omega_\infty = \mathbb{R}^3$ and for any $(\varphi, \psi) \in \dot{H}^1 \times L^2(\mathbb{R}^3)$, for any sequence (t_n), the linear concentrating wave p_n associated with $(\varphi, \psi, h_n, x_n, t_n)$ satisfies

$$E_0(p_n) \xrightarrow[n\to\infty]{} ||\nabla\varphi||^2_{L^2(\mathbb{R}^3)} + ||\psi||^2_{L^2(\mathbb{R}^3)} . \tag{1.6}$$

(ii) If $x_\infty \in^c \overline{\Omega}$, then $\Omega_\infty = \emptyset$ and all concentrating data are equivalent to zero.

(iii) If $x_\infty \in \partial\Omega$, then $\Omega_\infty = H^{\alpha,\omega}$, and for any sequence (t_n), all concentrating data $(\varphi, \psi, h_n, x_n, t_n)$ are equivalent to $(\mathcal{P}_{H^{\alpha,\omega}}\varphi, \mathbf{1}_{H^{\alpha,\omega}}\psi, h_n, x_n, t_n)$.

Note that the proposition above holds for any open, C^1 domain, bounded in $\mathbb{R}^3$ or the exterior of any such domain.

Proof of the proposition. We restrict our attention to case (iii). The arguments for (i) and (ii) are easier and left to the reader.

Let us start by recalling that by definition,

$$y \in \Omega_n \iff \phi(x_n + h_n y) > 0 . \tag{1.7}$$

Let us suppose $\alpha = +\infty$, and let K be a compact subset of $\mathbb{R}^3$. Since $x_n \to x_\infty$, we have

$$\forall y \in K , \quad \frac{\phi(x_n + h_n y)}{h_n} = \frac{\phi(x_n)}{h_n} + \nabla\phi(x_\infty)\cdot y + o(1) , \quad n \to \infty \tag{1.8}$$

by Taylor's formula, uniformly in $y \in K$. So since $\alpha = +\infty$, we have for n large enough, $\phi(x_n + h_n y) > 0$, hence $\Omega_\infty = \mathbb{R}^3$. Conversely if $\alpha = -\infty$, the same formula implies that $\Omega_\infty = \emptyset$. Now if $\alpha \in \mathbb{R}$, let K be a compact subset of $H^{\alpha,\omega}$. Then

$$\forall y \in K , \quad \nabla\phi(x_\infty)\cdot y = -\alpha + m(y) \ \text{ with } \ m(y) > 0 , \tag{1.9}$$

hence for n large enough, uniformly in $y \in K$, we have $\phi(x_n + h_n y) > 0$. Conversely, if K is a compact subset of ${}^c\overline{H}^{\alpha,\omega}$, then for any $y \in K$, for n large enough uniformly in y, we have $\phi(x_n + h_n y) < 0$. So the result follows : $\Omega_\infty = H^{\alpha,\omega}$.

To conclude the proof of the proposition, let $(\varphi, \psi, h_n, x_n, t_n)$ be concentrating data, with associate linear concentrating wave p_n, and let $p_n^{\alpha,\omega}$ be the linear concentrating wave associated with $(\varphi^{\alpha,\omega}, \psi^{\alpha,\omega}, h_n, x_n, t_n)$, where

$$(\varphi^{\alpha,\omega}, \psi^{\alpha,\omega}) \stackrel{\text{def}}{=} (\mathcal{P}_{H^{\alpha,\omega}}\varphi, \mathbf{1}_{H^{\alpha,\omega}}\psi) . \tag{1.10}$$

Conservation of energy yields

$$\begin{aligned} E_0(p_n - p_n^{\alpha,\omega}) &= h_n^{-3}\int_{\mathbb{R}^3}\left|\nabla\left(\mathcal{P}_\Omega\left(\varphi^{\alpha,\omega}\left(\frac{x-x_n}{h_n}\right)\right) - \mathcal{P}_\Omega\left(\varphi\left(\frac{x-x_n}{h_n}\right)\right)\right)\right|^2 dx \\ &\quad + h_n^{-3}\int_{\mathbb{R}^3}\left|\mathbf{1}_\Omega(x)\,\psi^{\alpha,\omega}\left(\frac{x-x_n}{h_n}\right) - \mathbf{1}_\Omega(x)\,\psi\left(\frac{x-x_n}{h_n}\right)\right|^2 dx \\ &= \int_{\mathbb{R}^3}\left|\nabla\mathcal{P}_{\Omega_n}\varphi^{\alpha,\omega}(y) - \nabla\mathcal{P}_{\Omega_n}\varphi(y)\right|^2 + \int_{\mathbb{R}^3}\left|\mathbf{1}_{\Omega_n}(y)\,\psi^{\alpha,\omega}(y) - \mathbf{1}_{\Omega_n}(y)\,\psi(y)\right|^2 dy . \end{aligned}$$

Then all we need to prove is the strong convergence of $\mathcal{P}_{\Omega_n} f$ towards $\mathcal{P}_{\Omega_\infty} f$ in $\dot H^1(\mathbb{R}^3)$, as well as that of $\mathbf{1}_{\Omega_n} g$ towards $\mathbf{1}_{\Omega_\infty} g$ in $L^2(\mathbb{R}^3)$, for any $f \in \dot H^1(\mathbb{R}^3)$ and any $g \in L^2(\mathbb{R}^3)$.

Lemma II.1.2. *Let Ω_n converge towards Ω_∞, in the sense of definition I.4, where Ω_∞ is a half-space, $\emptyset$ or $\mathbb{R}^3$. Then the following properties hold :*

(i) $\dot H^1_0(\Omega_\infty) = \big\{u \in \dot H^1(\mathbb{R}^3) \,|\, \operatorname{supp} u \subset \overline{\Omega}_\infty\big\}$;

(ii) $\forall f \in \dot H^1(\mathbb{R}^3)$, $\mathcal{P}_{\Omega_n} f \xrightarrow[n\to+\infty]{} \mathcal{P}_{\Omega_\infty} f$ *in* $\dot H^1(\mathbb{R}^3)$;

(iii) $\forall g \in L^2(\mathbb{R}^3)$, $\mathbf{1}_{\Omega_n} g \xrightarrow[n\to+\infty]{} \mathbf{1}_{\Omega_\infty} g$ *in* $L^2(\mathbb{R}^3)$.

Proof of the lemma. The cases $\Omega_\infty = \emptyset$ or $\mathbb{R}^3$ are obvious, so let us suppose that Ω_∞ is a half-space, defined for instance by

$$\Omega_\infty \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 | x_3 > 0\}. \tag{1.11}$$

(i) Let us consider $u \in \dot H^1(\mathbb{R}^3)$, supported in $\overline{\Omega}_\infty$, and let us prove that $u \in \dot H^1_0(\Omega_\infty)$. We can suppose that u has compact support; now let $\rho \in C_0^\infty(\mathbb{R}^3)$ be such that $\operatorname{supp} \rho \subset \{x \in \mathbb{R}^3 | x_3 > 0\}$. If ρ^ε is the smoothing kernel $\rho^\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^3}\rho\big(\frac{x}{\varepsilon}\big)$, then the function

$$u^\varepsilon \stackrel{\text{def}}{=} u * \rho^\varepsilon \tag{1.12}$$

satisfies $u^\varepsilon \in C_0^\infty(\Omega_\infty)$, and converges towards u in $\dot H^1(\mathbb{R}^3)$. That proves (i).

(ii) Let us notice that for any domain $M \subset \mathbb{R}^3$, the function $v \stackrel{\text{def}}{=} \mathcal{P}_M f$ is the solution of the following equation :

$$\Delta v = \Delta f \ \text{ in } \ M\,, \quad v \in \dot H^1_0(M).$$

We have indeed, for any $\varphi \in \dot H^1_0(M)$,

$$\int_{\mathbb{R}^3} \nabla(v-f)\cdot\nabla\overline{\varphi}\,dx = \int_{\mathbb{R}^3} (-\Delta f\overline{\varphi} - \nabla f\cdot\nabla\overline{\varphi})\,dx = 0.$$

Now let $v_n \stackrel{\text{def}}{=} \mathcal{P}_{\Omega_n} f$, which is bounded in $\dot H^1(\mathbb{R}^3)$, hence has weak limit points in $\dot H^1(\mathbb{R}^3)$. Let v be such a limit point. We claim that $v = \mathcal{P}_{\Omega_\infty} f$: if $\varphi \in C_0^\infty(\Omega_\infty)$, then for n large enough, we have $\varphi \in C_0^\infty(\Omega_n)$, hence

$$\int_{\mathbb{R}^3} \Delta v_n \overline{\varphi}\,dx = \int_{\mathbb{R}^3} \Delta f\,\overline{\varphi}\,dx\,, \tag{1.13}$$

which implies that $\Delta v = \Delta f$ in Ω_∞. Moreover, if $\varphi \in C_0^\infty({}^c\overline{\Omega}_\infty)$, then for n large enough, $\varphi \in C_0^\infty({}^c\overline{\Omega}_n)$, hence

$$\int_{\mathbb{R}^3} v_n \overline{\varphi}\,dx = 0\,, \tag{1.14}$$

so we have $v|_{{}^c\bar{\Omega}_\infty} = 0$, namely $v \in \dot H^1_0(\Omega_\infty)$ in view of (i).

The strong convergence of v_n towards v is due to the following identity :

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx = \int_{\mathbb{R}^3} \nabla v_n \cdot \nabla \bar{f} \, dx \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^3} \nabla v \cdot \nabla \bar{f} \, dx = \int_{\mathbb{R}^3} |\nabla v|^2 \, dx \, .$$

(iii) Let $g_n \stackrel{\text{def}}{=} \mathbf{1}_{\Omega_n} g$, which converges weakly in $L^2(\mathbb{R}^3)$ towards a function $\tilde{g}$. Let us prove that $\tilde{g} = \mathbf{1}_{\Omega_\infty} g$: if $\varphi \in C_0^\infty(\Omega_\infty)$, then we have

$$\int_{\mathbb{R}^3} (g - \tilde{g}) \overline{\varphi} \, dx = \int_{\mathbb{R}^3} g \overline{\varphi} \, dx - \lim_{n \infty} \int_{\mathbb{R}^3} \mathbf{1}_{\Omega_n} g \overline{\varphi} \, dx \, .$$

But for n large enough, we have $\varphi \in C_0^\infty(\Omega_n)$, hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \mathbf{1}_{\Omega_n} \, g \overline{\varphi} \, dx = \int_{\mathbb{R}^3} g \overline{\varphi} \, dx \, ,$$

hence $\tilde{g} = g$ on Ω_∞. If $\varphi \in C_0^\infty({}^c\bar{\Omega}_\infty)$ similar arguments imply that $\tilde{g} = 0$ in ${}^c\bar{\Omega}_\infty$. So $\tilde{g} = \mathbf{1}_{\Omega_\infty} g$.

Finally the strong convergence is simply due to the fact that

$$\int_{\mathbb{R}^3} |g_n|^2 \, dx = \int_{\mathbb{R}^3} \mathbf{1}_{\Omega_n} g \overline{g} \, dx \, ,$$

hence $\int_{\mathbb{R}^3} |g_n|^2 \, dx$ converges towards $\int_{\mathbb{R}^3} \tilde{g} \overline{g} \, dx = \int_{\mathbb{R}^3} |\mathbf{1}_{\Omega_\infty} g|^2 \, dx$.

So the lemma is proved, and the proof of proposition II.1.1 is complete. ∎

Finally, the next proposition will be used many times in the rest of the paper.

Proposition II.1.3. *Let Ω_n be an open domain in $\mathbb{R}^3$, converging towards Ω_∞ as n goes to infinity, where Ω_∞ is a half space, $\emptyset$ or $\mathbb{R}^3$. Consider (f_n^0, f_n^1), a bounded sequence in $\dot{H}^1 \times L^2(\mathbb{R}^3)$, supported in Ω_n, and converging weakly (resp. strongly) towards (f^0, f^1) in $\dot{H}^1 \times L^2(\mathbb{R}^3)$.*

Then the solution of

$$\text{(1.15)} \qquad \begin{cases} \Box f_n = 0 \ \text{ in } \ \mathbb{R}_t \times \Omega_n \, , \quad f_n|_{\mathbb{R}_t \times \partial\Omega_n} = 0 \\ (f_n, \partial_t f_n)|_{t=0} = (f_n^0, f_n^1) \end{cases}$$

converges weakly (resp. strongly) towards f in $L^\infty_{\text{loc}}(\mathbb{R}_t; \dot{H}^1(\mathbb{R}^3))$, and $\partial_t f_n$ converges weakly (resp. strongly) towards $\partial_t f$ in $L^\infty_{\text{loc}}(\mathbb{R}_t; L^2(\mathbb{R}^3))$, where

$$\begin{cases} \Box f = 0 \ \text{ in } \ \mathbb{R}_t \times \Omega_\infty \, , \quad f|_{\mathbb{R}_t \times \partial\Omega_\infty} = 0 \\ (f, \partial_t f)|_{t=0} = (f^0, f^1) \, . \end{cases}$$

Proof of the proposition. Let $\varphi, \psi \in C_0^\infty(\Omega_\infty)$. In view of equation (1.15) and definition I.4, the sequences of functions $t \mapsto (f_n(t,\cdot), \varphi)_{\dot{H}^1(\mathbb{R}^3)}$ and $t \mapsto (\partial_t f_n(t,\cdot), \psi)_{L^2(\mathbb{R}^3)}$ are locally equicontinuous. Using lemma II.1.2, these properties can be extended by density to arbitrary functions $\varphi \in \dot{H}^1(\mathbb{R}^3)$, $\psi \in L^2(\mathbb{R}^3)$. The rest of the proof is very similar to that of lemma II.1.2, and we omit it. ∎

II.2. Propagation of $h-$oscillatory sequences

First we introduce an abstract definition.

Definition II.2.1. *Let $\mathcal{H}$ be a Hilbert space, A be a selfadjoint (unbounded) operator on $\mathcal{H}$, and (h_n) a sequence of positive numbers converging to 0. A bounded sequence (u_n) in $\mathcal{H}$ is said $(h_n)-$oscillatory with respect to A if*

$$\overline{\lim_{n\to\infty}} \, ||\mathbf{1}_{|A|\geq \frac{R}{h_n}} u_n|| \xrightarrow[R\to\infty]{} 0 . \tag{2.1}$$

Sequence (u_n) is said strictly $(h_n)-$oscillatory with respect to A if it satisfies (2.1) and moreover

$$\overline{\lim_{n\to\infty}} \, ||\mathbf{1}_{|A|\leq \frac{\varepsilon}{h_n}} u_n|| \xrightarrow[\varepsilon\to 0]{} 0 . \tag{2.2}$$

Remarks II.2.2.

a) In the case $\mathcal{H} = L^2(\mathbb{R}^d)$, $A = |D_x|$, the above notions were considered in [GL], [GMMP], [G1,2], [BG1,2]. In that case, we shall speak simply of $(h_n)-$oscillatory sequences.

b) It is easy to check that a sequence (u_n) is $(h_n)-$oscillatory with respect to A if and only if, for every $\delta > 0$, there exists a sequence (u_n^δ) in $D(A)$ such that

$$\sup_n h_n ||Au_n^\delta|| < +\infty \text{ and } \overline{\lim_{n\to\infty}} \, ||u_n - u_n^\delta|| \leq \delta . \tag{2.3}$$

c) Given a bounded sequence (u_n) in $\mathcal{H}$, the spectral theorem for A implies that, for every n, the functional

$$\mu_n : f \in \mathcal{C}_0(\mathbb{R}) \longmapsto (f(h_n A)u_n, u_n) \tag{2.4}$$

is a nonnegative Radon measure with total mass $||u_n||^2$. Hence, by Helly's lemma, a subsequence (μ_{n_k}) is convergent to some measure μ for the weak -$*$ topology of measures. Then (u_n) is $(h_n)-$oscillatory with respect to A if and only if (μ_n) is tight, or, equivalently, if $||u_n||^2 \to \mu(\mathbb{R})$ for any such μ. Moreover, under those conditions, (u_n) is strictly $(h_n)-$oscillatory if and only if $\mu(\{0\}) = 0$ for any such μ. As a consequence, a $(h_n)-$oscillatory sequence (u_n) is strictly $(h_n)-$oscillatory if and only if

$$\overline{\lim_{n\to\infty}} \; ||(i + Rh_n A)^{-1} u_n||^2 \xrightarrow[R\to+\infty]{} 0 . \tag{2.5}$$

The purpose of this section is to investigate how those notions change when the operator A changes. Let us start with a general result.

Proposition II.2.3. *Let $\Lambda : \mathcal{H}_1 \to \mathcal{H}_2$ be a continuous linear map between Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Let A_1 be a selfadjoint operator on $\mathcal{H}_1$, A_2 be a selfadjoint operator on $\mathcal{H}_2$. Assume there exists $C > 0$ such that $\Lambda(D(A_1)) \subset D(A_2)$, $\Lambda^*(D(A_2)) \subset D(A_1)$ and, for any $u \in D(A_1)$, $v \in D(A_2)$,*

$$||A_2 \Lambda u|| \leq C\big(||A_1 u|| + ||u||\big) \tag{2.6}$$

$$(2.6)' \qquad ||A_1\,\Lambda^* v|| \le C\big(||A_2 v|| + ||v||\big)\,.$$

If a bounded sequence (u_n) in $\mathcal{H}_1$ is (strictly) $(h_n)-$oscillatory with respect to A_1, then (Λu_n) is (strictly) $(h_n)-$oscillatory with respect to A_2.

PROOF. Assume (u_n) is $(h_n)-$oscillatory with respect to A_1. By remark II.2.2. b), for every $\delta > 0$, let (u_n^δ) be a sequence in $D(A_1)$ such that $\overline{\lim\limits_{n\to\infty}}\, ||u_n - u_n^\delta|| \le \delta$ and $h_n ||A_1\, u_n^\delta|| \le C(\delta)$. Then $\overline{\lim\limits_{n\to\infty}}\, ||\Lambda u_n - \Lambda u_n^\delta|| \le K\delta$, since Λ is continuous, and, by estimate (2.6),

$$\begin{aligned} h_n ||A_2\,\Lambda\, u_n^\delta|| &\le \big(C(\delta) + h_n\, ||u_n^\delta||\big) \\ &\le \tilde{C}(\delta) \end{aligned}$$

since (u_n) is bounded. Therefore $(\Lambda\, u_n^\delta)$ is $(h_n)-$oscillatory with respect to A_2. Assume moreover that (u_n) is strictly $(h_n)-$oscillatory with respect to A_1. From the obvious inequality

$$||h_n\, A_2\, \mathbf{1}_{h_n|A_2|\le\varepsilon}(\Lambda u_n)|| \le \varepsilon$$

and estimate $(2.6)'$, we obtain, setting $w_n = \Lambda^*\, \mathbf{1}_{h_n|A_2|\le\varepsilon}\, \Lambda\, u_n\,,$

$$||h_n\, A_1\, w_n|| = O(\varepsilon + h_n)\,,$$

hence

$$(2.7) \qquad \big|\big|\mathbf{1}_{h_n|A_1|\le\sqrt{\varepsilon}} w_n - w_n\big|\big| = \big|\big|\mathbf{1}_{h_n|A_1|>\sqrt{\varepsilon}} w_n\big|\big| = O\Big(\sqrt{\varepsilon} + \frac{h_n}{\sqrt{\varepsilon}}\Big)\,.$$

On the other hand,

$$(2.8) \quad \big|\big|\mathbf{1}_{h_n|A_2|\le\varepsilon}(\Lambda u_n)\big|\big|^2 = (w_n, u_n) = \big(w_n - \mathbf{1}_{h_n|A_1|\le\sqrt{\varepsilon}} w_n, u_n\big) + \big(w_n, \mathbf{1}_{h_n|A_1|\le\sqrt{\varepsilon}} u_n\big)\,.$$

Putting together (2.7) and (2.8) through the Schwarz inequality, we conclude

$$\big|\big|\mathbf{1}_{h_n|A_2|\le\varepsilon}(\Lambda u_n)\big|\big|^2 = O\Big(\sqrt{\varepsilon} + \frac{h_n}{\sqrt{\varepsilon}} + \big|\big|\mathbf{1}_{h_n|A_1|\le\sqrt{\varepsilon}} u_n\big|\big|\Big)\,,$$

therefore (Λu_n) is strictly $(h_n)-$oscillatory with respect to A_2. ∎

Now we come to the main result of this section. Let $\Omega \subset \mathbb{R}^d$ be a smooth open subset with compact boundary. On the Hilbert space $\mathcal{H} = \dot{H}_0^1(\Omega) \times L^2(\Omega)$, define a selfadjoint operator A by

$$\begin{cases} D(A) = \left\{U = \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \in \dot{H}_0^1(\Omega) \times H_0^1(\Omega)\,,\, \Delta u \in L^2(\Omega)\right\} \\ A\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = i\begin{pmatrix} \dot{u} \\ \Delta u \end{pmatrix}\,. \end{cases}$$

It is well known that A is the infinitesimal generator of the wave group with Dirichlet boundary conditions.

Proposition II.2.4. *A bounded sequence $(U_n) = \begin{pmatrix} u_n \\ \dot{u}_n \end{pmatrix}$ in $\dot{H}_0^1(\Omega) \times L^2(\Omega)$ is (strictly) $(h_n)-$oscillatory with respect to A if and only if the sequences $(\nabla \underline{u}_n)$, $(\underline{\dot{u}}_n)$ are (strictly) $(h_n)-$oscillatory with respect to $|D|$ on $L^2(\mathbb{R}^d)$.*

PROOF. It requires several steps.

Step 1. Localization.

Let χ be a smooth function on $\mathbb{R}^d$ such that χ is constant outside a compact subset. Observe that the map $\Lambda : U \mapsto \chi U$ defined by $\chi \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \chi u \\ \chi \dot{u} \end{pmatrix}$ is bounded on $\mathcal{H} = \dot{H}^1_0(\Omega) \times L^2(\Omega)$. Moreover, one checks easily that Λ satisfies assumptions of proposition II.2.3 with $A_1 = A_2 = A$.

Therefore, we conclude that $(h_n)-$oscillation and strict $(h_n)-$oscillation are preserved by χ.

Let χ_j, $1 \leq j \leq N$ be C_0^∞ functions on $\mathbb{R}^d$ such that each χ_j is supported into a coordinate patch for $\partial\Omega$ and $\sum_{j=1}^{N} \chi_j = 1$ near $\partial\Omega$. Set $\chi_{\text{int}} = \big(1 - \sum_{j=1}^{N} \chi_j\big) \mathbf{1}_\Omega$; observe that χ_{int} is C^∞, constant outside a compact subset of $\mathbb{R}^d$, and $\operatorname{dist}\big(\operatorname{supp}(\chi_{\text{int}}), \Omega^c\big) > 0$. Since $U = \chi_{\text{int}} U + \sum_{j=1}^{N} \chi_j U$ for every $U \in \mathcal{H}$, the stability result we have just proved implies that the (strict) $(h_n)-$oscillation of a sequence (U_n) in $\mathcal{H}$ is equivalent to the (strict) $(h_n)-$oscillation of the sequences $(\chi_{\text{int}} U_n)$, $(\chi_j U_n)$, $j = 1, \ldots, N$. Let B be the selfadjoint operator $i \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$ on $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. We claim that the (strict) $(h_n)-$oscillations of $(\chi_{\text{int}} U_n)$ with respect to A and with respect to B occur simultaneously. Indeed, let $\chi'_{\text{int}} \in C^\infty(\mathbb{R}^d)$ be such that $\chi'_{\text{int}} \chi_{\text{int}} = \chi_{\text{int}}$ and

$$\operatorname{dist}\big(\operatorname{supp}(\chi'_{\text{int}}), \Omega^c\big) > 0 .$$

Then, for every $U \in \mathcal{H}$, $\chi'_{\text{int}} U \in D(A)$ if and only if $\chi'_{\text{int}} U \in D(B)$ and, in this case, $A(\chi'_{\text{int}} U) = B(\chi'_{\text{int}} U)$. Since the estimate $||\chi_{\text{int}} U_n - V_n^\delta|| \leq \delta$ implies $||\chi_{\text{int}} U_n - \chi'_{\text{int}} V_n^\delta|| \leq C\delta$, this proves the claim concerning $(h_n)-$oscillation. Moreover, the elementary estimates

$$\Big|\Big|(i + \varepsilon A)^{-1} \chi_{\text{int}} - \chi'_{\text{int}} (i + \varepsilon B)^{-1} \chi_{\text{int}}\Big|\Big| = O(\varepsilon)$$

$$\Big|\Big|(1 - \chi'_{\text{int}})(i + \varepsilon B)^{-1} \chi_{\text{int}}\Big|\Big|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} = O(\varepsilon)$$

prove the claim concerning strict $(h_n)-$oscillation. Finally, by use of Fourier transform, one checks easily that $|B| = \begin{pmatrix} |D| & 0 \\ 0 & |D| \end{pmatrix}$, hence a sequence $(V_n) = \begin{pmatrix} v_n \\ \dot{v}_n \end{pmatrix}$ in $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ is (strictly) $(h_n)-$oscillatory with respect to B if and only if the sequences (∇v_n), $(\dot{v}_n)$ are (strictly) $(h_n)-$oscillatory with respect to $|D|$ in $L^2(\mathbb{R}^d)$.

As a conclusion to this first step, we are reduced to proving proposition II.2.4 for U_n supported in a coordinate patch for $\partial\Omega$.

Step 2. Changing variables.

Let $\theta : \tilde{\omega} \to \omega$ be a diffeomorphism between a neighbourhood of 0 in $\mathbb{R}^d$ and a neighbourhood in $\mathbb{R}^d$ of some point $x_0 \in \partial\Omega$, such that $\theta(\{y_1 > 0\} \cap \tilde{\omega}) = \Omega \cap \omega$. Observe that if u is a function on ω, $(\Delta u) \circ \theta = L(u \circ \theta)$, where $L = \operatorname{div}(a\nabla) + b \cdot \nabla$ is an elliptic differential operator on $\tilde{\omega}$, with smooth coefficients a valued in positive definite matrices and b valued in vectors. Let K be a compact subset of ω and $\tilde{K} = \theta^{-1}(K)$. Let $\tilde{a}$ be a smooth positive definite matrix-valued function on $\mathbb{R}^d$ such that $\tilde{a}$ is constant outside a compact subset and $\tilde{a} = a$ in a neighbourhood of $\tilde{K}$. We set $\widetilde{\mathcal{H}} = \dot{H}_0^1(\{y_1 > 0\}) \cap L^2(\{y_1 > 0\})$ endowed with the Hilbert norm

$$||V||^{\sim} = \left[(\tilde{a}\nabla v, \nabla v)_{L^2} + ||\dot{v}||_{L^2}^2\right]^{1/2}, \;\; V = \begin{pmatrix} v \\ \dot{v} \end{pmatrix},$$

and we define

$$\begin{cases} \tilde{A}\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = i\begin{pmatrix} \dot{v} \\ \operatorname{div}(a\nabla v) \end{pmatrix} \\ D(\tilde{A}) = \left\{ \begin{pmatrix} v \\ \dot{v} \end{pmatrix} \in \widetilde{\mathcal{H}}, \dot{v} \in H_0^1(\{y_1 > 0\}), \operatorname{div}(a\nabla v) \in L^2 \right\}. \end{cases}$$

Then it is easy to check the following estimates, for every $U \in \mathcal{H}$ supported in K,

$$\frac{1}{C}||U|| \leq ||U \circ \theta||^{\sim} \leq C||U||$$

$$\frac{1}{C}\left(||\tilde{A}(U \circ \theta||^{\sim} - ||U||\right) \leq ||AU \circ \theta||^{\sim} \leq C\left(||\tilde{A}(U \circ \theta)||^{\sim} + ||U||\right)$$

$$\left|\left|\chi\left[(i + \varepsilon A)^{-1}(U) \circ \theta\right] - (i + \varepsilon\tilde{A})^{-1}(U \circ \theta)\right|\right|^{\sim} \leq C\varepsilon|U||$$

where $\chi \in C_0^\infty(\tilde{\omega})$ equals 1 near $\tilde{K}$.

Using these inequalities and remarks II.2.2. b) and II.2.2. c), we infer that a sequence (U_n) of $\mathcal{H}$ supported in a fixed compact subset of ω is (strictly) $(h_n)-$oscillatory with respect to A if and only if the sequence $(U_n \circ \theta)$ is (strictly) $(h_n)-$oscillatory with respect to $\tilde{A}$ in $\widetilde{\mathcal{H}}$.

Finally, one can get rid of matrix $\tilde{a}$ as follows. Let $\mathcal{H}' = \dot{H}_0^1(\{y_1 > 0\}) \times L^2(\{y_1 > 0\})$ endowed with the usual norm

$$||V||' = \left(||\nabla v||_{L^2}^2 + ||\dot{v}||_{L^2}^2\right)^{1/2}, \;\; V = \begin{pmatrix} v \\ \dot{v} \end{pmatrix},$$

and let

$$\begin{cases} A'\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = i\begin{pmatrix} \dot{v} \\ \Delta v \end{pmatrix} \\ D(A') = \left\{ \begin{pmatrix} v \\ \dot{v} \end{pmatrix} \in \mathcal{H}', \dot{v} \in H_0^1(\{y_1 > 0\}), \Delta v \in L^2 \right\}. \end{cases}$$

Then usual elliptic regularity estimates for the Dirichlet problem in a half-space (see for instance [GT]) imply that the identity map $\Lambda : \widetilde{\mathcal{H}} \to \mathcal{H}'$ and operators $\tilde{A}, A'$ satisfy the assumptions of proposition II.2.3, as well as Λ^{-1}, A', $\tilde{A}$. As a consequence, the (strict) $(h_n)-$oscillation of (U_n) with respect to A is equivalent to the (strict) $(h_n)-$oscillation of $(U_n \circ \theta)$ with respect to A'.

Step 3. Getting rid of the boundary.

We are reduced to proving the analog of proposition II.2.4 when Ω is the half-space $\{y_1 > 0\}$ in $\mathbb{R}^d$. Again we consider the selfadjoint operator $B = i\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$ on $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, and we define

$$\begin{aligned} \Lambda : \dot{H}_0^1(\{y_1 > 0\}) \times L^2(\{y_1 > 0\}) &\longrightarrow \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \\ \begin{pmatrix} u \\ \dot{u} \end{pmatrix} &\longrightarrow \begin{pmatrix} v \\ \dot{v} \end{pmatrix}, \end{aligned}$$

by the formula $v(y_1,y') = -u(-y_1,y')$, $\dot v(y_1,y') = -\dot u(-y,y')$.

Notice that $2^{-1/2}\Lambda$ is an isometry onto the $B-$invariant subspace $\mathcal{H}_{\rm odd}$ of odd functions with respect to y_1 in $\dot H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$, and that

$$\forall U\in D(A')\,,\ \ \Lambda(U)\in D(B)\ \text{ and }\ B\Lambda(U)=\Lambda A'(U)\,.$$

In view of proposition II.2.3, we infer that (U_n) is (strictly) $(h_n)-$oscillatory with respect to A' if and only if $(\Lambda(U_n))$ is (strictly) $(h_n)-$oscillatory with respect to B. Recall from the last part of step 1 that a sequence $\begin{pmatrix} v_n\\ \dot v_n\end{pmatrix}$ satisfies the latter property if and only if the sequences (∇v_n), $(\dot v_n)$ are (strictly) $(h_n)-$oscillatory with respect to $|D|$ in $L^2(\mathbb{R}^d)$. Our argument is completed by the following lemma.

Lemma II.2.5. *For every $f\in L^2(\mathbb{R}^d)$, define*

$$\tilde f(y_1,y') = \mathrm{sgn}(y_1)\, f(y_1,y')\,.$$

Then the (strict) $(h_n)-$oscillations with respect to $|D|$ of (f_n) and $(\tilde f_n)$ are equivalent.

PROOF. In view of definition (2.1), it is enough to prove

$$\Big|\Big|\mathbf{1}_{|D|\geq b}\,\mathrm{sgn}(y_1)\,\mathbf{1}_{|D|\leq a}\Big|\Big|_{L^2\to L^2}\longrightarrow 0 \tag{2.9}$$

as $\frac{a}{b}\to 0$. Indeed, (2.9) implies that $(h_n)-$oscillation is conserved by the action of $\mathrm{sgn}(y_1)$, while the conservation of strict $(h_n)-$oscillation is a consequence of the adjoint estimate. We prove (2.9) by means of Fourier transform. If $g=\mathbf{1}_{|D|\geq b}\,\mathrm{sgn}(y_1)\,\mathbf{1}_{|D|\leq a}f$ and $3a<b$, we have

$$\hat g(\xi) = c\,\mathbf{1}_{|\xi|\geq b}\int_{|(\eta_1,\xi')|\leq a}\frac{\hat f(\eta_1,\xi')}{\xi_1-\eta_1}\,d\eta_1\,.$$

Notice that $|\xi_1|\geq b-|\xi'|\geq b-a$ and $|\eta_1|\leq a$, hence $|\xi_1|\geq 2|\eta_1|$ and $|\xi_1-\eta_1|\geq\frac{|\xi_1|}{2}$. Therefore, by the Plancherel formula,

$$\begin{aligned}||g||_{L^2} &\leq 2|c|\Big(\int_{|\eta_1|\leq a,\,|\xi_1|\geq b-a}\frac{d\eta_1\,d\xi_1}{\xi_1^2}\Big)^{1/2}||f||_{L^2}\\ &\leq 4|c|\Big(\frac{a}{b-a}\Big)^{1/2}||f|_{L^2}\,,\end{aligned}$$

whence (2.9). ∎

In order to complete the proof of the analog of proposition II.2.4 for $\Omega=\{y_1>0\}$, we just need to observe that $\mathbf{1}_{y_1>0}f=\frac12(f+\tilde f)$, hence, if (f_n) is a sequence of even or odd functions with respect to y_1, the (strict) $(h_n)-$oscillations of (f_n) and of $(\mathbf{1}_{y_1>0}f_n)$ are equivalent.

Conclusion. We conclude the proof of proposition II.2.4 by coming back to $x-$coordinates : if $(v_n,\dot v_n)$ is bounded in $\dot H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ and supported in a fixed compact subset of ω — with the notation of step 2 — then the (strict) $(h_n)-$oscillations of $(\nabla v_n,\dot v_n)$ and of $(\nabla(v_n\circ\theta),\dot v_n\circ\theta)$ are equivalent. This can be done either by mimicking the argument of step 2 — this time in the whole space — or by using the invariance of semiclassical measures by change of coordinates (see [GL]). ■

Since (strictly) h_n-oscillatory sequences with respect to an operator A are conserved by bounded functions of A, we infer from proposition II.2.4 the following result.

Proposition II.2.6. *Let (φ_n, ψ_n) be a bounded sequence of $\dot H^1_0(\Omega) \times L^2(\Omega)$, such that $(\nabla\varphi_n, \psi_n)$ is (strictly) $(h_n)-$oscillatory in $L^2(\mathbb{R}^3)$. Let v_n be the solution of*

$$\begin{cases} \Box v_n = 0 \ \text{ in } \ \mathbb{R}_t \times \Omega\,, \ \ v_n|_{\mathbb{R}_t\times\partial\Omega} = 0 \\ (v_n, \partial_t v_n)|_{t=0} = (\varphi_n, \psi_n)\,. \end{cases}$$

Then $(\nabla v_n(t), \partial_t v_n(t))$ is (strictly) $(h_n)-$oscillatory in $L^2(\mathbb{R}^3)$ for every $t \in \mathbb{R}$, uniformly in t.

In [BG1, lemma 3.2 (iii)] it is proved that every bounded sequence (f_n) in $L^2(\mathbb{R}^d)$ has a subsequence $(f_{n(j)})$ which can be written as

$$f_{n(j)} = g_j + s_j\,, \tag{2.10}$$

where g_j is strictly $(h_{n(j)})-$oscillatory, and s_j is $(h_{n(j)})-$singular, namely

$$\Big|\Big|\mathbf{1}_{a\le h_{n(j)}|D|\le b}\, s_j\Big|\Big|_{L^2} \xrightarrow[j\to\infty]{} 0\,, \quad \forall b > a > 0\,. \tag{2.11}$$

Given the subsequence $(f_{n(j)})$, such a sequence (g_j) is unique up to a sequence going strongly to 0, and is called the strictly $h_{n(j)}-$oscillatory component of the subsequence $(f_{n(j)})$. Notice that, in view of decomposition (2.10), a bounded sequence (f_n) in $L^2(\mathbb{R}^d)$ is $(h_n)-$singular if and only if, for every strictly $(h_n)-$oscillatory sequence (g_n), the scalar product $(f_n, g_n)_{L^2}$ goes to 0.

Coming back to proposition II.2.6, we obtain, in view of the conservation of the energy scalar product, the following result.

Corollary II.2.7. *Let (v_n), $(\tilde v_n)$ be sequences of solutions of $\Box v = 0$, $v|_{\mathbb{R}\times\partial\Omega} = 0$, such that $(\nabla v_n(0), \partial_t v_n(0))$ is bounded in $L^2(\Omega)$, and $(\nabla\tilde v_n(0), \partial_t\tilde v_n(0))$ is its strictly $(h_n)-$oscillatory component. Then, for every sequence (t_n) of real numbers, $(\nabla\tilde v_n(t_n), \partial_t\tilde v_n(t_n))$ is the strictly $(h_n)-$oscillatory component of $(\nabla v_n(t_n), \partial_t v_n(t_n))$.*

II.3. A non-concentration property

The goal of this section is to prove the following result. Here Ω is the complement of a smooth strictly convex obstacle, and $(\varphi, \psi) \in \dot H(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

Proposition II.3.1. *Let p_n be the linear concentrating wave associated with a concentrating data $(\varphi, \psi, h_n, x_n, t_n = 0)$, namely*

$$\begin{cases} \Box p_n = 0 \ \text{ in } \ \mathbb{R}_t \times \Omega\,, \ \ p_n|_{\mathbb{R}_t\times\partial\Omega} = 0 \\ (p_n, \partial_t p_n)|_{t=0} = \Big(\dfrac{1}{h_n^{1/2}}\mathcal{P}_\Omega\Big(\varphi\Big(\dfrac{\cdot - x_n}{h_n}\Big)\Big),\ \dfrac{1}{h_n^{3/2}}\mathbf{1}_\Omega\,\psi\Big(\dfrac{\cdot - x_n}{h_n}\Big)\Big)\,. \end{cases} \tag{3.1}$$

Then, for every bounded interval I, for every sequence (ε_n) such that $\frac{\varepsilon_n}{h_n} \xrightarrow[n\to\infty]{} \infty$, we have

$$||p_n||_{L^\infty(I\setminus[-\varepsilon_n,\varepsilon_n],L^6(\Omega))} \xrightarrow[n\to\infty]{} 0. \tag{3.2}$$

PROOF. We may assume that $x_\infty \in \overline{\Omega}$. We shall distinguish between the cases $x_\infty \in \Omega$ and $x_\infty \in \partial\Omega$. In both cases, we shall prove the following two facts separately,

a) for every sequence θ_n such that $\theta_n \to t \neq 0$, $||p_n(\theta_n)||_{L^6(\Omega)} \to 0$ as $n \to \infty$,

b) for every sequence ε_n such that $\varepsilon_n \to 0$, $\frac{\varepsilon_n}{h_n} \to \infty$, $||p_n(\varepsilon_n)||_{L^6(\Omega)} \to 0$ as $n \to \infty$.

We shall see that b) is a degenerate version of a), which can be obtained in a similar – in fact simpler – way after rescaling the coordinates. Let us briefly explain our strategy to prove part a), already used in [G1]. Introduce the time-dependent energy density on Ω,

$$e_n(t) = (|\partial_t p_n(t,x)|^2 + |\nabla_x p_n(t,x)|^2)\, dx\,, \tag{3.3}$$

which satisfies in $\mathbb{R}\times\Omega$

$$\partial_t e_n = \mathrm{div}_x(\partial_t p_n \nabla_x p_n)\,. \tag{3.4}$$

Notice that, since $\partial_t p_n|_{\mathbb{R}\times\partial\Omega} = 0$, equation (3.4) still holds in $\mathbb{R}\times\mathbb{R}^3$ if we extend e_n and p_n by 0 outside Ω. As a consequence, e_n is equicontinuous as a measure-valued function of t and the limit points e_∞ of e_n are continuous measure-valued functions of t. Therefore property a) above will be a consequence of

$$e_\infty(t)(\{x_0\}) = 0\,, \quad \forall x_0 \in \overline{\Omega}\,, \quad \forall t \neq 0\,, \tag{3.5}$$

through the concentration-compactness principle (see for instance [L2, lemma I.1]). In order to prove (3.5), we shall calculate $e_\infty(t)$ using semiclassical Wigner measures. Recall (see for instance [GL], [LP], [GMMP]) that the correlation functions

$$c_n(t,x,k,\ell) = \sum_{a=0}^{4} \partial_a p_n\Big(t - h_n\frac{k}{2}, x - h_n\frac{\ell}{2}\Big)\, \partial_a \overline{p}_n\Big(t + h_n\frac{k}{2}, x + h_n\frac{\ell}{2}\Big) \tag{3.6}$$

where $\partial_0 = \partial_t$, $\partial_j = \partial_{x_j}$ for $1 \le j \le 3$, satisfy, up to the extraction of a subsequence,

$$\forall f \in C_0^\infty(\mathbb{R}_t \times \Omega_x \times \mathbb{R}_k \times \mathbb{R}^3_\ell)\,, \quad \int f c_n \xrightarrow[n\to\infty]{} \int \hat{f}(t,x,\tau,\xi)\, d\mu(t,x,\tau,\xi)\,, \tag{3.7}$$

where $\hat{f}$ denotes the Fourier transform of f with respect to variables (k,ℓ), and μ is a positive Radon measure. The following properties are classical :

$$\int_{\mathbb{R}_\tau\times\mathbb{R}^3_\xi} \mu(t,x,d\tau,d\xi) \le e_\infty(t)\, dt \tag{3.8}$$

$$(\tau^2 - |\xi|^2)\mu = 0 \quad \text{(characteristic equation)}\,, \tag{3.9}$$

$$(\tau\,\partial_t - \xi\cdot\nabla_x)\mu = 0 \quad \text{(transport equation)}\,. \tag{3.10}$$

Suppose we are able to exhibit some measure μ_* such that

$$\mu \geq \mu_* \,, \tag{3.11}$$

$$\int \mu_*(t,x,d\tau,d\xi) = e_*(t)\,dt \tag{3.12}$$

where e_* is continuous for $t \neq 0$, valued in measures on Ω and satisfying, for every $t \neq 0$,

$$e_*(t)(\Omega) = ||\nabla \mathcal{P}_{\Omega_\infty}\varphi||^2_{L^2} + ||\mathbf{1}_{\Omega_\infty}\psi||^2_{L^2} \,. \tag{3.13}$$

We observe that the right hand side of (3.13) is the total mass of $e_\infty(0)$. Moreover, by conservation of energy and finite propagation speed, the total mass of $e_\infty(t)$ is independent of t. On the other hand, by (3.8), (3.11) and continuity with respect to $t \neq 0$, we have

$$e_*(t) \leq e_\infty(t) \,, \quad t \neq 0 \,. \tag{3.14}$$

Since (3.13) implies that the total masses of these measures are equal, (3.14) is in fact an equality. If μ_* is explicit, it will therefore be possible to check property (3.5). Our main task in the sequel will be to find explicit measures μ_* satisfying (3.11), (3.12), (3.13).

3.1. The case $x_\infty \in \Omega$

Let us denote by u_n the extension of p_n by 0 outside Ω, and introduce the semiclassical measure $\overline{\mu}$ of $\nabla_{t,x} u_n$, defined similarly to (3.7) by

$$\begin{aligned}\Big\langle \nabla_{t,x} u_n\Big(t - h_n \frac{k}{2}, x - h_n \frac{\ell}{2}\Big), \nabla_{t,x} u_n\Big(t + h_n \frac{k}{2}\ , x + h_n \frac{\ell}{2}\Big)\Big\rangle \\ \xrightarrow[n\to\infty]{} \int_{\mathbb{R}\times\mathbb{R}^3} e^{-i(k\tau+\ell\cdot\xi)}\,\overline{\mu}(t,x,d\tau,d\xi) \,.\end{aligned} \tag{3.15}$$

Despite the singularity induced by the extension by 0, it is still possible to prove that $\overline{\mu}$ is supported by the characteristic set $\{\tau^2 = |\xi|^2\}$ (see for instance [GL] for the stationary case, where the proof is similar, or [Bu]). Moreover, we have trivially $\overline{\mu} = 0$ if $x \in \overline{\Omega}^c$. The problem is to understand the behavior of $\overline{\mu}$ as x approaches $\partial\Omega$. For $x \in \partial\Omega$, denote by $N(x)$ the unitary normal vector to $\partial\Omega$ at point x, pointing outward with respect to Ω. Then, in view of equation (3.10), measure $\overline{\mu}$ has a trace $\overline{\mu}_\partial$ as (t,x,τ,ξ) approaches the set $\{x \in \partial\Omega, \xi \cdot N(x) \neq 0\}$ from $\{x \in \Omega\}$ (see for instance [H, theorem 4.4.8′]). The distribution $\overline{\mu}_\partial$ inherits the positivity of $\overline{\mu}$, hence is a positive measure. Therefore, outside the set $\{x \in \partial\Omega, \xi \cdot N(x) = 0\}$, (3.10) can be continuated as

$$\tau\,\partial_t \overline{\mu} - \xi \cdot \nabla_x \overline{\mu} = (\xi \cdot N)\,\overline{\mu}_\partial \otimes \delta_{\partial\Omega} \,. \tag{3.16}$$

Observe that, near the set $\{x \in \partial\Omega, \xi \cdot N(x) \neq 0, \tau^2 = |\xi|^2\}$, the wave operator $\partial_t^2 - \Delta_x$ is strictly hyperbolic with respect to the normal coordinate to $\partial\Omega$. Moreover $u_n|_{\mathbb{R}\times\partial\Omega} = 0$ by the Dirichlet condition. Consequently (see for instance [GL], [Bu], [M]), we have

Lemma II.3.2. *For $x \in \partial\Omega$, denote by j_x the symmetry with respect to the tangent plane $T_x(\partial\Omega)$, given by*

$$j_x(\xi) = \xi - 2(\xi \cdot N(x))\,N(x)$$

and define J on $\mathbb{R}\times\partial\Omega\times\mathbb{R}\times\mathbb{R}^3$ by

$$J(t,x,\tau,\xi)=(t,x,\tau,j_x(\xi))\,.$$

Then

$$J(\overline{\mu}_\partial)=\overline{\mu}_\partial\,. \tag{3.17}$$

We now collect the above informations on $\overline{\mu}$ to obtain an integral representation of $\overline{\mu}$ using the billiards flow above Ω.

Given $x\in\overline{\Omega}$, $\tau\neq 0$, $\xi\in\mathbb{R}^3$, the set

$$S(x,\tau,\xi)=\Big\{t\in\mathbb{R},\ x-t\frac{\xi}{\tau}\notin\overline{\Omega}\Big\}$$

is an open bounded interval, since $\overline{\Omega}^c$ is an open bounded convex set in $\mathbb{R}^3$. Denote by $I(x,\tau,\xi)$ the component of $S(x,\tau,\xi)^c$ which contains 0, and by $t_0(x,\tau,\xi)$, if it exists, the finite endpoint of $I(x,\tau,\xi)$. Time $t_0(x,\tau,\xi)$ is therefore the nearest time to 0 such that $x-t\frac{\xi}{\tau}\in\partial\Omega$. We set, for every real number t,

$$\begin{cases} G_t(x,\tau,\xi)=\Big(x-t\dfrac{\xi}{\tau},\tau,\xi\Big)\ \text{ if } t\in I(x,\tau,\xi)\\ G_t(x,\tau,\xi)=\Big(x-t_0\dfrac{\xi}{\tau}-(t-t_0)\Big)\dfrac{\xi'}{\tau},\tau,\dfrac{\xi'}{\tau}\Big),\ \text{ where } \xi'=j_{x-t_0\frac{\xi}{\tau}}(\xi)\,,\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } t\notin I(x,\tau,\xi)\,. \end{cases} \tag{3.18}$$

Observe that $(G_t)_{t\in\mathbb{R}}$ is a (time-discontinuous) flow on $\{\xi\in\overline{\Omega}\}\times\{\tau\neq 0\}\times\{\xi\in\mathbb{R}^3\}$. Among the points (x,τ,ξ) such that $S(x,\tau,\xi)=\emptyset$, a special role is played by those such that the ray $\{x-t\frac{\xi}{\tau},t\in\mathbb{R}\}$ hits the boundary $\partial\Omega$ tangentially. We denote by $\mathcal{D}$ the set of such "diffractive" points, near which the analysis is a little more intricate (see for instance [BuG]) – but we shall not need this analysis here.

Lemma II.3.3. *Let $f\in C_0^\infty\big((\Omega_x\times\mathbb{R}_\tau\setminus\{0\}\times\mathbb{R}^3_\xi)\setminus\mathcal{D}\big)$. Then, for every $t\in\mathbb{R}$,*

$$\int f(G_{-t}(x,\tau,\xi))\,\overline{\mu}(t,dx,d\tau,d\xi)=\int f(x,\tau,\xi)\,d\mu_0(x,\tau,\xi)\,, \tag{3.19}$$

where

$$d\mu_0(x,\tau,\xi)=\frac{1}{2}\delta(x-x_\infty)\sum_{\pm}\delta(\tau\mp|\xi|)|\hat{\psi}(\xi)\pm i|\xi|\,\hat{\varphi}(\xi)|^2\,\frac{d\xi}{(2\pi)^3}\,. \tag{3.20}$$

PROOF. Let us calculate the distributional derivative of the left hand side of (3.19). Observe that $f(y,\tau,\eta)=0$ if y lies near $\partial\Omega$, hence $f(G_{-t}(x,\tau,\xi))=0$ if t lies near $-t_0(x,\tau,\xi)$. As a consequence, in view of (3.18), $t\mapsto f(G_{-t}(x,\tau,\xi))$, is smooth and

$$\begin{aligned}\frac{\partial}{\partial t}f(G_{-t}(x,\tau,\xi))=&\ \frac{\xi}{\tau}\cdot\nabla_x f\Big(x+t\frac{\xi}{\tau},\tau,\xi\Big)\,\mathbf{1}_{-t\in I(x,\tau,\xi)}\\ &+\frac{\xi'}{\tau}\cdot\nabla_x f\Big(x-t_0\frac{\xi}{\tau}+(t+t_0)\frac{\xi'}{\tau},\tau,\frac{\xi'}{\tau}\Big)\,\mathbf{1}_{-t\notin I(x,\tau,\xi)}\,.\end{aligned} \tag{3.21}$$

On the other hand, since f is supported outside $\mathcal{D}$, so is $f \circ G_{-t}$. The point is that, by the implicit function theorem, t_0 is a C^∞ function outside $\mathcal{D}$, and

$$\nabla_x t_0(x,\tau,\xi) = \frac{\tau N(x - t_0 \frac{\xi}{\tau})}{\xi \cdot N(x - t_0 \frac{\xi}{\tau})}, \tag{3.22}$$

in particular $\frac{\xi}{\tau} \cdot \nabla_x t_0 = 1$. Consequently,

$$\frac{\xi}{\tau} \cdot \nabla_x \Big[N\Big(x - t_0 \frac{\xi}{\tau}\Big)\Big] = 0 \text{ and } \frac{\xi}{\tau} \cdot \nabla_x (\xi') = 0 .$$

We conclude, in view of (3.18),

$$\begin{aligned} \frac{\xi}{\tau} \cdot \nabla_x f(G_{-t}(x,\tau,\xi)) = & \frac{\xi}{\tau} \cdot \nabla_x f\Big(x + t\frac{\xi}{\tau}, \tau, \xi\Big) \mathbf{1}_{-t\in I} \\ & + \frac{\xi'}{\tau} \cdot \nabla_x f\Big(x - t_0 \frac{\xi}{\tau} + (t+t_0)\frac{\xi'}{\tau}, \tau, \frac{\xi'}{\tau}\Big) \mathbf{1}_{-t \notin I} , \end{aligned} \tag{3.23}$$

which is the same as the right hand side of (3.21). Using (3.16), we conclude

$$\begin{aligned} \frac{\partial}{\partial t} \int f(G_{-t}((x,\tau,\xi))\, \mu(t, dx\, d\tau\, d\xi) = & \int \frac{\xi \cdot N}{\tau} f(G_{-t}(x,\tau,\xi))\, \mu_\partial \otimes \delta_{\partial\Omega}(t, dx\, d\tau\, d\xi) \\ = & \int_{\{t\, \frac{\xi}{\tau}\cdot N(x)>0\}} \frac{\xi\cdot N}{\tau} f\Big(x + t\frac{j_x(\xi)}{\tau}, \tau, j_x(\xi)\Big) \mu_\partial \otimes \delta_{\partial\Omega}(t, dx\, d\tau\, d\xi) \\ & + \int_{\{t\, \frac{\xi}{\tau}\cdot N(x)<0\}} \frac{\xi\cdot N}{\tau} f\Big(x + t\frac{\xi}{\tau}, \tau, \xi\Big) \mu_\partial \otimes \delta_{\partial\Omega}(t, dx\, d\tau\, d\xi) , \end{aligned}$$

which is 0 by using $\xi \cdot N(x) = -j_x(\xi)\cdot N(x)$ and lemma II.3.2.

It remains to prove (3.19) for $|t|$ small enough. Let f_n be the solution to the free wave equation,

$$\begin{cases} \Box f_n = 0 , \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3 , \\ f_n(0,x) = \dfrac{1}{\sqrt{h_n}} \varphi\Big(\dfrac{x-x_n}{h_n}\Big) \\ \partial_t f_n(0,x) = \dfrac{1}{h_n^{3/2}} \psi\Big(\dfrac{x-x_n}{h_n}\Big) . \end{cases} \tag{3.24}$$

By finite propagation speed, it is easy to prove that, for $|t| < d(x_\infty, \partial\Omega)$,

$$||\nabla_{t,x}(p_n - f_n)(t)||_{L^2(\Omega)} \xrightarrow[n\to\infty]{} 0 . \tag{3.25}$$

Therefore $\overline{\mu}$ coincides for $|t| < d(x_\infty, \partial\Omega)$ with the semiclassical measure of $\nabla_{t,x} f_n$, which is well known (see for instance [G1]) and satisfies (3.19), (3.20). This completes the proof of lemma II.3.3. ■

It is now easy to carry out the program sketched in (3.11), (3.12), (3.13) with

$$\mu_*(t,x,\tau,\xi) = \mathbf{1}_{\mathcal{O}}(x,\tau,\xi)\, G_t(\mu_0)(x,\tau,\xi)\, dt ,$$

$$\mathcal{O} = \Big\{ (t, G_t(x,\tau,\xi)) , (x,\tau,\xi) \in (\Omega\times\mathbb{R}\setminus\{0\}\times\mathbb{R}^3) \setminus \mathcal{D} \Big\} .$$

Indeed, (3.13) holds because $\Omega_\infty = \mathbb{R}^3$ and the set of directions ξ such that the line $\{x_\infty + s\xi, s \in \mathbb{R}\}$ is tangent to $\partial\Omega$ is a null set for the Lebesgue measure by Sard's theorem – or by a direct computation, since Ω^c is a convex set here. We conclude

$$e_\infty(t)(\{x_0\}) = \mu_0\Big(\big\{(x,\tau,\xi) \in \Omega \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3,\, p_x G_t(x,\tau,\xi) = x_0\big\}\Big)\,, \tag{3.26}$$

where p_x denotes the projection onto the x space. Using (3.20), (3.26) becomes

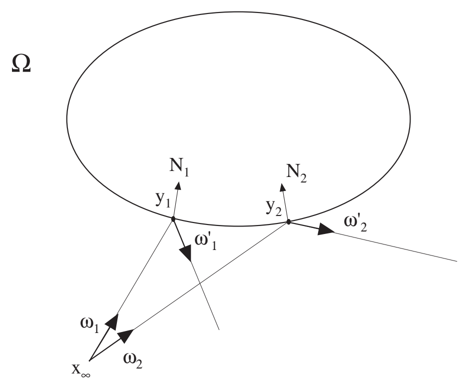
$$e_\infty(t)(\{x_0\}) = \sum_\pm \int_{E_\pm(t)} \Gamma_\pm(\xi)\, d\xi\,, \tag{3.27}$$

when $\Gamma_\pm \in L^1(\mathbb{R}^3)$ and

$$E_\pm(t) = \Big\{\xi \in \mathbb{R}^3 / p_x G_t(x_\infty, \pm|\xi|, \xi) = x_0\Big\}\,. \tag{3.28}$$

Observe that $p_x G_t(x_\infty, \pm|\xi|, \xi) = F_t\big(\mp\frac{\xi}{|\xi|}\big)$ where $F_t(\omega) = x_\infty + t\omega$ if $[x_\infty, x_\infty + t\omega] \subset \overline{\Omega}$, and $F_t(\omega) = x_\infty + t_0(\omega)\omega + (t - t_0(\omega))\omega'$ if $[x_\infty, x_\infty + t_0(\omega)\omega]$ is the component of x_∞ in $[x_\infty, x_\infty + t\omega] \cap \overline{\Omega}$, $\omega' = j_{x_\infty + t_0(\omega)\omega}(\omega)$.

We claim that, for every $t \neq 0$, the map $F_t : S^2 \to \overline{\Omega}$ is one-to-one. This implies that $E_\pm(t)$ is either empty, or is a half-line in $\mathbb{R}^3$; in any case $E_\pm(t)$ is a null set and (3.27) implies (3.5).



The fact that F_t is one-to-one for all $t \neq 0$ is an elementary geometrical fact. Let us prove it for the convenience of the reader. Assume

$$F_t(\omega_1) = F_t(\omega_2)\,. \tag{3.29}$$

We have to discuss whether the segments $[x_\infty, x_\infty + t\omega_j]$ meet $\overline{\Omega}^c$ or not. If both are contained in $\overline{\Omega}$, $F_t(\omega_j) = x_\infty + t\omega_j$, and $\omega_1 = \omega_2$. If one of the segments meets $\overline{\Omega}^c$, the triangle inequality shows that the other one must meet $\overline{\Omega}^c$ too.

Set $t_0(\omega_j) = t_j$, $y_j = x_\infty + t_j\omega_j \in \partial\Omega$, and $N_j = N(y_j)$. Up to changing ω_j into $-\omega_j$, we may assume $t > 0$, hence $t_j > 0$. Set

$$F_t(\omega_j) = y_j + \lambda_j \omega_j'\,, \quad \lambda_j > 0\,.$$

Then, by (3.29),

$$(y_1 - y_2)\cdot(\omega_1' - \omega_2') = (-\lambda_1\,\omega_1' + \lambda_2\,\omega_2')\cdot(\omega_1' - \omega_2') \tag{3.30}$$
$$= -(\lambda_1 + \lambda_2)(1 - \omega_1'\cdot\omega_2') \leq 0\,.$$

On the other hand,

$$\begin{aligned}(y_1 - y_2)\cdot(\omega_1' - \omega_2') = {} & (t_1\,\omega_1 - t_2\,\omega_2)\cdot(\omega_1 - \omega_2) \\ & - 2(y_1 - y_2)\cdot N_1(\omega_1\cdot N_1) \\ & + 2(y_1 - y_2)\cdot N_2(\omega_2\cdot N_2)\,.\end{aligned} \tag{3.31}$$

The last two terms of the right hand side of (3.31) are nonnegative because of the convexity of Ω^c. Therefore (3.31) implies

$$(y_1 - y_2)\cdot(\omega_1' - \omega_2') \geq (t_1 + t_2)(1 - \omega_1\cdot\omega_2) > 0$$

unless $\omega_1 = \omega_2$. This completes the proof of property a) if $x_\infty \in \Omega$. ■

Remark II.3.4. Notice that, in the last part of the proof above, the geometry of Ω played a crucial role – indeed, proposition II.3.1 is clearly wrong in the case of other geometries.

Let us close this subsection by proving part b) when $x_\infty \in \Omega$. Coming back to the solution f_n to the free problem (3.24), we observe that convergence (3.25) is uniform for $|t| \leq \frac{1}{2}d(x_\infty, \partial\Omega)$, say, hence (3.25) holds with $t = \varepsilon_n$.

Hence we are led to proving

$$||f_n(\varepsilon_n)||_{L^6(\mathbb{R}^3)} \xrightarrow[n\to\infty]{} 0\,, \tag{3.32}$$

which for instance can be done similarly to the case $|t|$ small enough above, by computing the semi-classical measure of $\nabla_{s,y}g_n$, where

$$g_n(s,y) = \varepsilon_n^{1/2}\, f_n(\varepsilon_n s, \varepsilon_n y)\,, \tag{3.33}$$

associated with the scale $\delta_n = \frac{h_n}{\varepsilon_n} \to 0$.

3.2. The case $x_\infty \in \partial\Omega$

We begin with proving part a), namely (3.5). Of course we may assume, say, $t > 0$. The strategy sketched before subsection 3.1 will be completely carried out if we prove the following result.

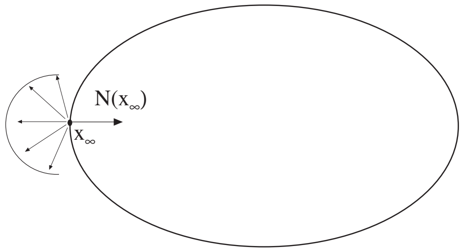
Lemma II.3.5. *The semiclassical measure μ of $\nabla_{t,x}p_n$ above $\{x \in \Omega, \tau \neq 0\}$ satisfies, for all $t > 0$,*

$$\mu(t,x,\tau,\xi) \geq \delta\Big(x - x_\infty + t\,\frac{\xi}{\tau}\Big)\,\lambda_0(\tau,\xi)\,, \tag{3.34}$$

where λ_0 is a measure, absolutely continuous with respect to the Lebesgue measure on the cone $\{\tau^2 = |\xi|^2,\, \frac{\xi}{\tau}\cdot N(x_\infty) > 0\}$, with total mass given by

$$TM(\lambda_0) = ||\nabla\,\mathcal{P}_{\Omega_\infty}\varphi||_{L^2}^2 + ||\mathbf{1}_{\Omega_\infty}\psi||_{L^2}^2\,. \tag{3.35}$$

Observe that lemma II.3.5 makes sense since the right hand side of (3.34) is a measure above $\{x \in \Omega\}$, in view of the specific geometry of Ω.



PROOF OF LEMMA II.3.5. Since we already know that μ satisfies $\tau\partial_t\mu - \xi\cdot\nabla_x\mu = 0$ above $\{x \in \Omega\}$, (3.34) is true for every $t > 0$ if and only if it is true for some $t > 0$. Hence we are led to proving (3.34) for small $t > 0$. As a consequence we only need to study μ above a convenient neighbourhood of $t = 0$, $x = x_\infty$ intersected by $]0, +\infty[\times\Omega$.

In order to accomplish this program, we introduce the following normal coordinates near x_∞. By rotational invariance, we may assume

$$N(x_\infty) = (0,0,1)\,.$$

Then there exists a local parametrization of $\partial\Omega$, $m : \{|z| < 1\} \subset \mathbb{R}^2 \to \partial\Omega \subset \mathbb{R}^3$, such that

$$m(0) = x_\infty\,,\quad \frac{\partial m}{\partial z_1}(0) = (1,0,0)\,,\quad \frac{\partial m}{\partial z_2}(0) = (0,1,0)\,. \tag{3.36}$$

Set $N(z) = N(m(z))$ for simplicity. For $r_0 > 0$ small enough, the mapping

$$\{|z| < 1\} \times \{|r| < r_0\} \subset \mathbb{R}^2 \times \mathbb{R} \longrightarrow m(z) - rN(z) \in \mathbb{R}^3 \tag{3.37}$$

induces local coordinates near x_∞ in $\mathbb{R}^3$, so that $\Omega = \{r > 0\}$. We denote by v_n the local extension of p_n in a neighbourhood of $x = x_\infty$ by the reflexion formula

$$v_n(t,x') = -v_n(t,x) \ \text{ if } \ z(x') = z(x),\ \ r(x') = -r(x)\,. \tag{3.38}$$

Then v_n satisfies a wave equation with variable coefficients,

$$\rho\,\partial_t^2 v_n - \operatorname{div}(A\nabla v_n) = 0 \tag{3.39}$$

in a neighbourhood of $t = 0$, $x = x_\infty$, where ρ, A are continuous piecewise smooth functions with interface on $\partial\Omega$. Of course, if $x \in \Omega$, $\rho(x) = 1$ and A(x) is the identity matrix. Next we introduce the Wigner correlation functions,

$$C_n^{ab}(t,x,k,\ell) = U_n^a\Big(t - h_n\frac{k}{2}, x - h_n\frac{\ell}{2}\Big)\,\overline{U}_n^b\Big(t + h_n\frac{k}{2}, x + h_n\frac{\ell}{2}\Big)\,, \tag{3.40}$$

where $U_n = (\sqrt{\rho}\,\partial_t v_n, \sqrt{A}\,\nabla_x v_n)$, and index $a = 0$ corresponds to coordinate t. Notice that the above expression of U_n is motivated by the energy associated to (3.39), namely $||\sqrt{\rho}\,\partial_t v_n(t)||_{L^2}^2 + ||\sqrt{A}\,\nabla_x v_n||_{L^2}^2$. Using [GMMP], we have, up to a subsequence,

$$C_\eta^{ab}(t,x,k,\ell) \rightharpoonup \int_{\mathbb{R}\times\mathbb{R}^3} e^{-i(\tau k + \xi\cdot\ell)}\,\mathcal{V}^{ab}(t,x,d\tau,d\xi) \tag{3.41}$$

in the distributional sense, where $(\mathcal{V}^{ab})_{0\le a,b\le 3}$ is a positive matrix-valued measure. Observe that the local feature of the Wigner transform implies that the restriction of $\nu = \mathrm{tr}(\mathcal{V})$ to $\{x \in \Omega\}$ is exactly μ. Next we prove that, outside $\{\tau = \xi = 0\}$, $\mathcal{V}$ is determined by ν. For simplicity, introduce the following notation,

$$\partial_{(t,x)} := \partial\,,\;\; \partial_{(k,\ell)} := \dot{\partial}\,, \tag{3.42}$$

$$f_n^{\pm}(t,x,k,\ell) := f_n\Big(t \pm h_n\,\frac{k}{2}\,, x \pm h_n\,\frac{\ell}{2}\Big)\,, \tag{3.43}$$

for every sequence $f_n = f_n(t,x)$.

Set $V_n = (\partial_t\, v_n, \nabla_x\, v_n)$, $D_n^{ab} = (V_n^a)^-\,(\overline{V}_n^b)^+$, and

$$P = \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & \sqrt{A} \end{pmatrix}\,. \tag{3.44}$$

Then

$$C_n = P^-\, D_n\, P^+ \tag{3.45}$$

and, by the Schwarz identity,

$$\Big(\frac{1}{2}\,h_n\,\partial_c - \dot{\partial}_c\Big)\, D_n^{ab} = \Big(\frac{1}{2}\,h_n\,\partial_a - \dot{\partial}_a\Big)\, D_n^{cb}\,. \tag{3.46}$$

Passing to the limit in (3.45), (3.46), we conclude

$$\mathcal{V}(t,x,\tau,\xi) = \frac{(\sqrt{\rho}\,\tau, \sqrt{A}\,\xi)\otimes(\sqrt{\rho}\,\tau, \sqrt{A}\,\xi)}{\rho\tau^2 + A\xi\cdot\xi}\,\nu(t,x,\tau,\xi) \tag{3.47}$$

outside $\{\tau = \xi = 0\}$.

Therefore we can derive the localization law for ν : from (3.39) we have

$$\Big(\frac{1}{2}\,h_n\,\partial_t - \partial_k\Big)(\rho^-\,\partial_t\, v_n^-\,\partial_t\, v_n^+) - \Big(\frac{1}{2}\,h_n\,\mathrm{div}_x - \mathrm{div}_\ell\,\Big)(A^-\,\nabla v_n^-\,\partial_t\, v_n^+) = 0\,. \tag{3.48}$$

Passing to the limit in (3.48), we obtain, outside $\{\tau = 0\}$, in view of (3.47),

$$(\rho\tau^2 - A\xi\cdot\xi)\,\nu = 0\,. \tag{3.49}$$

Finally we have to derive a transport equation analogous to (3.10).

Observe that $c_n = \mathrm{tr}(C_n)$ satisfies

$$c_n = \rho^-\,\partial_t\, v^-\,\partial_t\,\overline{v}^+ + A^-\,\nabla v^-\cdot\nabla\overline{v}^+ + E_n\,,$$

where E_n goes to 0 weakly in k, x, ℓ, uniformly in t. Therefore a straightforward calculation leads to the identity – where we dropped subscripts n for simplicity —

$$\begin{aligned}\frac{\partial c_n}{\partial t} - \frac{\partial E_n}{\partial t} = \mathrm{div}_x\,\Big[A^-\,\nabla v^-\,\partial_t\overline{v}^+ + \frac{\rho^-}{\rho^+}\,\partial_t\, v^-\,A^+\,\nabla\overline{v}^+\Big] - \partial_t\, v^-\,\nabla\Big(\frac{\rho^-}{\rho^+}\Big)\cdot\nabla\overline{v}^+ \\ -\Big(\frac{1}{2}\,h_n\,\partial_t - \partial_k\Big)\Big[\frac{\ell\cdot\widetilde{\nabla\rho}}{\rho^+}\,\nabla v^-\cdot A^+\,\nabla\overline{v}^+ - \ell\cdot\widetilde{\nabla A}\,\nabla v^-\cdot\nabla\overline{v}^+\Big]\end{aligned} \tag{3.50}$$

with the notation

$$\tilde{g}(x,\ell) = \int_{-1/2}^{1/2} g(x + h_n\theta\ell)\,d\theta\,.$$

From (3.50) we deduce several pieces of information. First, since A, ρ are Lipschitz functions, we observe that, for some sequence $\tilde{E}_n$ going to 0 weakly in x, k, ℓ and uniformly in t, the quantity $\frac{\partial}{\partial t}(c_n - \tilde{E}_n)$ is bounded in t, valued in $\mathcal{D}'_{x,k,\ell}$.

We conclude that the sequence c_n is equicontinuous in t, in particular ν is continuous in t, and its value ν_0 at $t = 0$ is the inverse Fourier transform of the limit of $c_n(t = 0)$.

Next we multiply (3.50) by $\varepsilon g\big(\frac{r(x)}{\varepsilon}\big)$, where $g \in C_0^\infty(\mathbb{R})$ satisfies $g'(0) = 1$. Passing to the limit as $n \to \infty$, then as $\varepsilon \to 0$, and taking the inverse Fourier transform in (k,ℓ), we obtain, for $\tau \neq 0$,

$$\mathbf{1}_{r=0}\ \nabla r(x)\cdot A(x)\xi\ \ \nu = 0\,, \tag{3.51}$$

which, since $\{r = 0\} = \{x \in \partial\Omega\}$ and $A(x) = I$ on $\partial\Omega$, means exactly

$$\nu\big(\{x \in \partial\Omega, \xi\cdot N(x) \neq 0, \tau \neq 0\}\big) = 0\,. \tag{3.52}$$

In view of (3.52), it is now possible to pass to the limit in (3.50), using the following lemma.

Lemma II.3.6. *Let f_n, g_n be bounded sequences in $L^2_{\rm loc}(D)$, where D is an open domain in $\mathbb{R}^d$. Let a be a Lipschitz function on D, C^1 outside a closed subset F of D and let $b \in L^\infty(D)$, continuous outside F. Assume the Wigner measure $\mathcal{V}$ of (f_n, g_n) satisfies*

$$\mathrm{tr}(\mathcal{V})((F\times\mathbb{R}^d)\cap G) = 0\,, \tag{3.53}$$

where G is some open subset of $T^*\mathbb{R}^d$. Then

$$\frac{a(y + h_n\frac{z}{2}) - a(y - h_n\frac{z}{2})}{h_n}\, f_n\Big(y - h_n\frac{z}{2}\Big)\,\overline{g}_n\Big(y + h_n\frac{z}{2}\Big) \rightharpoonup \int_{\mathbb{R}^d} e^{-iz\cdot\eta}\,\lambda(y, d\eta) \tag{3.54}$$

where the distribution λ satisfies

$$\lambda|_G = -i\ \mathrm{div}_\eta[\nabla_y a\,\mathcal{V}^{12}]|_G\,.$$

Similarly, we have

$$b\Big(y + h_n\frac{z}{2}\Big)\, f_n\Big(y - h_n\frac{z}{2}\Big)\,\overline{g}_n\Big(y + h_n\frac{z}{2}\Big) \rightharpoonup \int_{\mathbb{R}^d} e^{-iz\cdot\eta}\, b(y)\,\mathcal{V}^{12}(y, d\eta)\,. \tag{3.54$'$}$$

We postpone the proof of lemma II.3.6 and show how to get lemma II.3.5. By (3.52), we may apply lemma II.3.6 to components of U_n, with $F = \partial\Omega$ and G is the complement of $\{\tau = 0\} \cup \{x \in \partial\Omega, \xi\cdot N(x) = 0\}$. Therefore (3.50) leads to the following equation in G,

$$\tau\,\frac{\partial\nu}{\partial t} = \mathrm{div}_x\Big[\frac{A\xi}{\rho}\,\nu\Big] + \frac{1}{2}\,\mathrm{div}_\xi\Big[\Big(\frac{\nabla\rho}{\rho^2}\,A\xi\cdot\xi - \frac{1}{\rho}(\nabla A)\,\xi\cdot\xi\Big)\,\nu\Big]\,, \tag{3.55}$$

which is the transport equation along the hamiltonian vector field associated with the wave symbol $\frac{1}{2}\left(\tau^2 - \frac{1}{\rho} A\xi \cdot \xi\right)$. The problem is that coefficients $\nabla\rho$, ∇A are discontinuous. One can resolve this difficulty by means of the following elementary lemma.

Lemma II.3.7. *Let b be a piecewise smooth vector field on a domain $D \subset \mathbb{R}^N$ with interface localized on a hypersurface $S \subset D$. We assume that the normal component of b to S is continuous through S and does not vanish. Then, for every $X_0 \in \mathbb{R}^N$, there exists a unique maximal solution to the differential equation*

$$\dot{X}(t) = b(X(t)), \quad X(0) = X_0, \tag{3.56}$$

interpreted in the distributional sense in the t variable. The maps $G_t : X_0 \mapsto X(t)$ constitute a Lipschitz local flow on D, and, for every positive measure ν on $[0,T] \times D$ satisfying

$$\begin{cases} \dfrac{\partial \nu}{\partial t} + \operatorname{div}(b\nu) = 0 \;\; on \;\;]0,T[\times D, \\ \nu([0,T] \times S) = 0 \\ \nu|_{t=0} \geq \nu_* \end{cases} \tag{3.57}$$

such that, $\forall X_0 \in \operatorname{supp}(\nu_*)$, $G_t(X_0)$ exists for $t \in [0,T[$, we have, for every $t \in [0,T[$,

$$\nu(t) \geq G_t(\nu_*). \tag{3.58}$$

The proof of lemma II.3.7 is based on the observation that, if $X_0 \in S$, $X(t)$ cannot stay in S since $b(X)$ is transverse to S. The whole proof is an easy adaptation of the usual arguments for Lipschitz vector fields and is left to the reader.

In order to apply lemma II.3.7 to our context, we need some information about $\nu|_{t=0}$, which, as already observed, is the inverse Fourier transform of the weak limit of the correlation function c_n at $t = 0$.

Denote by $v_{n,\pm}$ the solution to

$$\begin{cases} \rho\, \partial_t^2 v_{n,+} - \operatorname{div}(A\nabla v_{n,+}) = 0 \\ v_{n,+}(0,x) = \mathcal{P}_\Omega\left(\dfrac{1}{\sqrt{h_n}}\varphi\left(\dfrac{x-x_n}{h_n}\right)\right) \\ \partial_t v_{n,+}(0,x) = \mathbf{1}_\Omega(x)\, \dfrac{1}{h_n^{3/2}}\, \psi\left(\dfrac{x-x_n}{h_n}\right), \end{cases} \tag{3.59}$$

and set

$$v_{n,-}(t,x) = v_{n,+}(t,x'). \tag{3.60}$$

Then we have $v_n = v_{n,+} - v_{n,-}$. Arguing as in proposition II.1.3, one proves easily that the function $\phi_{n,+}$ defined by

$$\phi_{n,+}(k,\ell) = h_n^{1/2}\, v_{n,+}(h_n k, x_n + h_n \ell) \tag{3.61}$$

converges strongly for the energy norm to the solution ϕ_+ of

$$\Box\phi_+ = 0, \;\; \phi_+|_{k=0} = \mathcal{P}_{\Omega_\infty}\varphi, \;\; \partial_k \phi_+|_{k=0} = \mathbf{1}_{\Omega_\infty}\psi, \tag{3.62}$$

locally uniformly with respect to k. We have a similar result for $v_{n,-}$ and the corresponding solution ϕ_- of $\Box\phi_- = 0$, so that

$$v_n(h_n k, x) = \frac{1}{\sqrt{h_n}}\Big[\phi_+\Big(k, \frac{x-x_n}{h_n}\Big) - \phi_-\Big(k, \frac{x-x'_n}{h_n}\Big)\Big] + o(1) \tag{3.63}$$

in the energy norm. Plugging (3.63) into (3.40) with $t=0$, and passing to the limit as $n \to \infty$, we obtain, after a straightforward calculation,

$$\nu_0(x,\tau,\xi) = \delta(x - x_\infty)\sum_{\pm} \delta(\tau \mp |\xi|)\, H_\pm(\xi)\, d\xi\,, \tag{3.64}$$

where

$$\tag{3.64$'$} \begin{cases} H_\pm(\xi) = \dfrac{1}{2}\Big(\Big|\hat\psi(\xi) \pm i|\xi|\,\hat\varphi(\xi)\Big|^2 + \Big|\hat\psi(j(\xi)) \pm i|\xi|\,\hat\varphi(j(\xi))\Big|^2\Big) \ \text{ if } \Omega_\infty = \mathbb{R}^3\,, \\ H_\pm(\xi) = \dfrac{1}{2}\Big|\widehat{\mathbf{1}_{\Omega_\infty}\psi}(\xi) - \widehat{\mathbf{1}_{\Omega_\infty}\psi}(j(\xi))\, e^{2i\alpha\xi\cdot N} \\ \qquad\qquad \pm\, i|\xi|\Big(\widehat{\mathcal{P}_{\Omega_\infty}\varphi}(\xi) - \widehat{\mathcal{P}_{\Omega_\infty}\varphi}(j(\xi))\, e^{2i\alpha\xi\cdot N}\Big)\Big|^2 \ \text{ if } \Omega_\infty = \{\xi\cdot N > \alpha\}\,. \end{cases}$$

In both cases, $j = j_{x_\infty}$, $N = N(x_\infty)$.

We set $\lambda_0(\xi,\tau) = \mathbf{1}_{\frac{\xi}{\tau}\cdot N > 0}\sum_{\pm}\delta(\tau \mp |\xi|)\, H_\pm(\xi)\, d\xi$. Then an elementary calculation yields (3.35).

Finally, we observe that, if $\frac{\xi}{\tau}\cdot N > 0$, then $G_t(x_\infty,\tau,\xi) = \big(x_\infty - t\frac{\xi}{\tau}, \tau, \xi\big)$ stays above Ω for $t > 0$, hence we can apply lemma 3.7 with

$$\nu_*^\varepsilon = \mathbf{1}_{\frac{\xi}{\tau}\cdot N > \varepsilon}\,\nu_0$$

for all ε, and get finally inequality (3.34). This completes the proof of lemma II.3.5. ■

Now we come back to the proof of lemma II.3.6. We prove only formula (3.54), the arguments for (3.54)$'$ being similar. Up to cutting off functions f_n, g_n, we may assume that f_n, g_n are bounded in $L^2(\mathbb{R}^d)$. Let $\chi_\varepsilon \in C^\infty(\mathbb{R}^d)$ such that $\chi_\varepsilon(x) = 1$ for $(x,F) \le \varepsilon$, and $\chi_\varepsilon(x) = 0$ for $d(x,F) \ge 2\varepsilon$. We denote by $\Lambda_n(y,z)$ the left hand side of (3.54). For fixed ε, we have, in $\mathcal{S}'(\mathbb{R}^d_y \times \mathbb{R}^d_z)$,

$$(1 - \chi_\varepsilon^2(y))\,\Lambda_n(y,z) \underset{n\to\infty}{\rightharpoonup} z\cdot(1-\chi_\varepsilon^2)k\nabla_y a\,\hat{\mathcal{V}}^{12}(y,z) \tag{3.65}$$

since $(1-\chi_\varepsilon^2)\nabla_y a$ is continuous.

On the other hand, for every $k \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\int k(y,z)\,\chi_\varepsilon^2(y)\,\Lambda_n(y,z)\,dy\,dz = \int k(y,z)\,\frac{a_+ - a_-}{h_n}\,(\chi_\varepsilon f_n)^-\,(\overline{\chi_\varepsilon g_n})^+\,dy\,dz + o(1) \tag{3.66}$$

as n goes to infinity. Let Q_n be the operator of kernel $\frac{1}{h_n^d}k\big(\frac{y+y'}{2}, \frac{y-y'}{h_n}\big)$. Then (3.66) reads as

$$\int k\,\chi_\varepsilon^2\,\Lambda_n\,dy\,dz = \Big(\frac{1}{h_n}[a,Q_n]\chi_\varepsilon f_n, \chi_\varepsilon g_n\Big)_{L^2}. \tag{3.67}$$

Choose $q = \hat{k} \in C_0^\infty(G)$ and $\tilde{q} \in C_0^\infty(G)$ such that $\tilde{q} = 1$ near $\mathrm{supp}(q)$. If $\tilde{Q}_n$ denotes the operator corresponding to $\tilde{q}$, the pseudodifferential calculus implies

$$Q_n - Q_n \tilde{Q}_n = O(h_n^\infty) \ \text{ in } \ B(L^2)\,. \tag{3.68}$$

On the other hand

$$\begin{aligned}\int k\,\chi_\varepsilon^2\,\Lambda_n\,dy\,dz = {} & \Big(\frac{1}{h_n}[a, Q_n]\tilde{Q}_n(\chi_\varepsilon f_n), \chi_\varepsilon g_n\Big)_{L^2} \\ & + \Big(\frac{1}{h_n}[a, Q_n - Q_n\tilde{Q}_n]\chi_\varepsilon f_n, \chi_\varepsilon g_n\Big)_{L^2} \\ & + \frac{1}{h_n}\big(Q_n[a, \tilde{Q}_n](\chi_\varepsilon f_n), \chi_\varepsilon g_n\big)_{L^2}\,.\end{aligned} \tag{3.69}$$

In the right hand side of (3.69), the first term is bounded by $C\,||\tilde{Q}_n(\chi_\varepsilon f_n)||_{L^2}\,||\chi_\varepsilon g_n||_{L^2}$, the second term goes to 0 in view of (3.68), while the third term is bounded by $||\chi_\varepsilon f_n||_{L^2}\,||Q_n^*(\chi_\varepsilon g_n)||_{L^2}$. Passing to the limit as $n \to \infty$, we obtain

$$\overline{\lim_{n\to\infty}} \left| \int k\,\chi_\varepsilon^2\,\Lambda_n\,dy\,dz \right| \le \tilde{C}\,\sup(|q| + |\tilde{q}|)\left(\int_G |\chi_\varepsilon|^2\, d\,\mathrm{tr}(\mathcal{V}) \right)^{1/2} \tag{3.70}$$

which goes to 0 as $\varepsilon \to 0$ in view of assumption (3.53). Combining (3.70) and (3.65) completes the proof of lemma II.3.6. ∎

We close this section by indicating briefly how to modify the above proof in order to prove part b) in the case $x_\infty \in \partial\Omega$. Denote by $\tilde{v}_n(t, z, r)$ the expression of v_n in the normal coordinates (3.37) near x_∞, and set

$$w_n(s, y) = \varepsilon_n^{1/2}\,\tilde{v}_n(\varepsilon_n s, z_n + \varepsilon_n y', \varepsilon_n y_3)\,, \tag{3.71}$$

where (z_n, r_n) are the normal coordinates of x_n. Our goal will be attained if we prove

$$||w_n(1)||_{L^6_y} \xrightarrow[n\to\infty]{} 0\,, \tag{3.72}$$

which we prove once again using Wigner measures corresponding to the scale $\delta_n = \frac{h_n}{\varepsilon_n}$. Indeed, w_n is solution to a wave equation

$$\rho_n\,\partial_t^2 w_n - \mathrm{div}(A_n \nabla w_n) = 0\,, \tag{3.73}$$

where ρ_n, A_n are sequences of continuous piecewise smooth functions of y with interface on $\{y_3 = 0\}$, and which satisfy moreover

$$|\rho_n - 1| + |A_n - I| \longrightarrow 0\,, \quad |\nabla\rho_n| + |\nabla A_n| \longrightarrow 0 \tag{3.74}$$

uniformly on compact subsets.

Then one can copy formulae (3.40) $\to$ (3.50) in this context, the limiting process being much simplified by (3.74). One finds that the Wigner measure $\dot{\nu}$ associated with $(\nabla_{s,y} w_n)$ satisfies

$$(\sigma^2 - |\eta|^2)\,\dot{\nu} = 0\,, \quad (\sigma\partial_s - \eta\cdot\partial_y)\,\dot{\nu} = 0\,. \tag{3.75}$$

Moreover the computation of $\dot{\nu}|_{s=0}$ can be done similarly to formulae (3.59) $\rightarrow$ (3.63). This allows one to conclude as before, using conservation of energy and the concentration-compactness principle. We leave the easy details to the reader. ∎

Remark II.3.8. An alternative strategy in order to deal with measures of solutions to the Dirichlet problem for the wave equation could be to use measures on the Melrose compressed cotangent bundle as introduced by Lebeau in [Le1] (see also the recent paper by Burq-Lebeau [BuL]).

III. The linear equation

The aim of this section is to prove theorem 1 stated in the introduction. For the convenience of the reader, let us recall the statement. We consider a sequence (v_n) of finite energy solutions to the following problem :

$$\begin{cases} \Box v_n = 0 \ \text{ in } \ \mathbb{R}_t \times \Omega\,, \quad v_n|_{\mathbb{R}_t\times\partial\Omega} = 0 \\ (v_n, \partial_t v_n)|_{t=0} = (\varphi_n, \psi_n)\,, \end{cases}$$

where (φ_n, ψ_n) is bounded in $E(\Omega)$, and compact at infinity (see the introduction, equation (8)). We shall prove that up to an extraction, v_n can be decomposed as follows :

$$v_n(t,x) = v(t,x) + \sum_{j=1}^{\ell} p_n^{(j)}(t,x) + w_n^{(\ell)}(t,x)\,, \tag{1}$$

where v is the weak limit of v_n, and where the $p_n^{(j)}$'s and $w_n^{(\ell)}$ satisfy the following properties :

- $p_n^{(j)}$ is the linear concentrating wave associated with concentrating data $\big(\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)}\big)$, where if $j \neq k$,

$$\left|\log\Big(\frac{h_n^{(j)}}{h_n^{(k)}}\Big)\right| \xrightarrow[n\to\infty]{} \infty \quad \text{or} \quad h_n^{(j)} = h_n^{(k)} \ \text{ and } \ \frac{\big|(x_n^{(j)}, t_n^{(j)}) - (x_n^{(k)}, t_n^{(k)})\big|}{h_n^{(j)}} \xrightarrow[n\to\infty]{} \infty\,.$$

- The remainder $w_n^{(\ell)}$ satisfies, for every $T > 0$

$$\varlimsup_{n\to\infty} ||w_n^{(\ell)}||_{L^\infty([-T,T];L^6(\Omega))} \xrightarrow[\ell\to\infty]{} 0\,; \tag{2}$$

- The energies are orthogonal, in the following sense :

$$E_0(v_n) = E_0(v) + \sum_{j=1}^{\ell} E_0(p_n^{(j)}) + E_0(w_n^{(\ell)}) + o(1)\,, \quad n \to \infty\,. \tag{3}$$

Without loss of generality we can suppose that the initial data satisfies

$$(\varphi_n, \psi_n) \underset{n\to\infty}{\rightharpoonup} (0,0) \ \text{ in } \ E(\Omega)\,, \tag{4}$$

hence that $v \equiv 0$.

The structure of the proof is as follows : in the first part, we extract the scales $h_n^{(j)}$ of concentration, using results proved in [G2]. In the second part, we consider a fixed scale

h, and decompose a strictly $h-$oscillatory function into a sum of concentrating waves, with data at time $t_h^{(j)}$ concentrated at a point $x_h^{(j)}$, and satisfying the properties required in the theorem. The combination of both steps yields the result.

III.1. Extraction of scales

Let us prove the following proposition.

Proposition III.1. *Under the assumptions of theorem 1, the function v_n can, up to an extraction, be decomposed in the following way : for any $\ell \in \mathbb{N}^*$,*

$$v_n(t,x) = \sum_{j=1}^{\ell} v_n^{(j)}(t,x) + \rho_n^{(\ell)}(t,x)\,, \tag{1.1}$$

where each $v_n^{(j)}$ is a strictly $(h_n^{(j)})-$oscillatory solution of the linear wave equation in Ω, where $h_n^{(j)} \xrightarrow[n\to\infty]{} 0$, and where

$$\Big|\log \frac{h_n^{(j)}}{h_n^{(k)}}\Big| \xrightarrow[n\to\infty]{} +\infty \;\text{ if } j \neq k\,. \tag{1.2}$$

Moreover, we have, for every $T>0$,

$$\overline{\lim_{n\to\infty}}\; ||\rho_n^{(\ell)}||_{L^\infty([-T,T],L^6(\Omega))} \xrightarrow[\ell\to\infty]{} 0 \tag{1.3}$$

as well as

$$E_0(v_n) = \sum_{j=1}^{\ell} E_0(v_n^{(j)}) + E_0(\rho_n^{(\ell)}) + o(1)\,, \quad n\to\infty \tag{1.4}$$

and

$$\forall j\neq k\,, \quad ||v_n^{(j)}\, v_n^{(k)}||_{L^\infty(\mathbb{R}_t,L^3(\Omega))} \xrightarrow[n\to\infty]{} 0\,. \tag{1.5}$$

PROOF OF THE PROPOSITION. The proof is an adaptation of [G2], using the propagation of $h-$oscillation proved in proposition II.2.6. Indeed, the functions $\nabla\varphi_n$ and ψ_n are bounded in $L^2(\mathbb{R}^3)$, hence can be (see [G2]) decomposed in the following way, up to the extraction of a subsequence :

$$\forall \ell\in\mathbb{N}^*\,, \quad (\varphi_n,\psi_n) = \sum_{j=1}^{\ell} (\varphi_n^{(j)},\psi_n^{(j)}) + (\Phi_n^{(\ell)},\Psi_n^{(\ell)})\,,$$

where the $\varphi_n^{(j)}$'s and the $\psi_n^{(j)}$'s are strictly $(h_n^{(j)})-$oscillatory, $h_n^{(j)} \xrightarrow[n\to\infty]{} 0$, and where

$$\overline{\lim_{n\to\infty}} \left(||\nabla\Phi_n^{(\ell)}||_{\dot B^0_{2,\infty}} + ||\Psi_n^{(\ell)}||_{B^0_{2,\infty}} \right) \xrightarrow[\ell\to\infty]{} 0\,. \tag{1.6}$$

Let us recall that for any function f,

$$||f||_{\dot B^0_{2,\infty}} \overset{\text{def}}{=} \sup_{k\in\mathbb{Z}} \left(\int_{2^k\le|\xi|\le 2^{k+1}} |\hat f(\xi)|^2\, d\xi \right)^{1/2}, \tag{1.7}$$

where $\hat{f}$ is the Fourier transform of f.

Moreover we have the orthogonality property :

$$\forall \ell \in \mathbb{N}^*\,, \quad ||(\varphi_n, \psi_n)||_E^2 = \sum_{j=1}^{\ell} ||(\varphi_n^{(j)}, \psi_n^{(j)})||_E^2 + ||(\Phi_n^{(\ell)}, \Psi_n^{(\ell)})||_E^2 + o(1)\,, \quad n \to \infty$$

and $h_n^{(j)}$ is orthogonal to $h_n^{(k)}$ for all $j \neq k$, in the sense that

$$\Big| \log \frac{h_n^{(j)}}{h_n^{(k)}} \Big| \xrightarrow[n\to\infty]{} \infty \text{ if } j \neq k\,. \tag{1.8}$$

Now we know from proposition II.2.6, that strict $(h_n)-$oscillation is propagated by the wave equation in Ω. That implies that, up to the extraction of a subsequence, we can write

$$v_n(t,x) = \sum_{j=1}^{\ell} v_n^{(j)}(t,x) + \rho_n^{(\ell)}(t,x)\,, \tag{1.9}$$

where each $v_n^{(j)}$ is strictly $(h_n^{(j)})-$oscillatory, satisfies the equation

$$\begin{cases} \Box v_n^{(j)} = 0 \text{ in } \mathbb{R}_t \times \Omega\,, \quad v_n^{(j)}|_{\mathbb{R}_t \times \partial\Omega} = 0 \\ (v_n^{(j)}, \partial_t v_n^{(j)})|_{t=0} = (\varphi_n^{(j)}, \psi_n^{(j)})\,, \end{cases} \tag{1.10}$$

and conservation of energy yields

$$E_0(v_n) = \sum_{j=1}^{\ell} E_0(v_n^{(j)}) + E_0(\rho_n^{(\ell)}) + o(1)\,, \quad n \to \infty\,. \tag{1.11}$$

Now let us prove that for any $T > 0$,

$$\overline{\lim_{n\to\infty}} \; ||\rho_n^{(\ell)}||_{L^\infty([-T,T],L^6(\Omega))} \xrightarrow[\ell\to\infty]{} 0\,. \tag{1.12}$$

Let $\varepsilon > 0$ be fixed. According to (1.6), we have for ℓ large enough,

$$\overline{\lim_{n\to\infty}} \left(||\nabla \Phi_n^{(\ell)}||_{\dot{B}^0_{2,\infty}} + ||\Psi_n^{(\ell)}||_{\dot{B}^0_{2,\infty}} \right) \leq \varepsilon\,. \tag{1.13}$$

Now let (h_n) be a sequence of scales, and let $(\nabla \tilde{\Phi}_n^{(\ell)}, \tilde{\Psi}_n^{(\ell)})$ be the strictly $(h_n)-$oscillatory component of $(\nabla \Phi_n^{(\ell)}, \Psi_n^{(\ell)})$. According to [BG1], equality (3.14), we get from (1.13) that

$$\overline{\lim_{n\to\infty}} \left(||\nabla \tilde{\Phi}_n^{(\ell)}||_{L^2(\mathbb{R}^3)} + ||\tilde{\Psi}_n^{(\ell)}||_{L^2(\mathbb{R}^3)} \right) \leq C\,\varepsilon\,, \tag{1.14}$$

where C is a universal constant.

Now we use corollary II.2.7 : we infer that for ℓ large enough, for any sequence (h_n) of scales and (t_n) of real numbers, the strictly $(h_n)-$oscillatory component of $(\nabla \rho_n^{(\ell)}(t_n), \partial_t \rho_n^{(\ell)}(t_n))$, noted $(\tilde{f}_n^{(\ell)}, \tilde{g}_n^{(\ell)})$, satisfies by conservation of energy

$$\overline{\lim_{n\to\infty}} \left(||\tilde{f}_n^{(\ell)}||_{L^2(\mathbb{R}^3)} + ||\tilde{g}_n^{(\ell)}||_{L^2(\mathbb{R}^3)} \right) \leq C\,\varepsilon\,. \tag{1.15}$$

Using again [BG1], equality (3.14), we infer that

$$\overline{\lim_{n\to\infty}}\, ||\nabla\rho_n^{(\ell)}||_{L^\infty(\mathbb{R},\dot B^0_{2,\infty})} \xrightarrow[\ell\to\infty]{} 0.. \tag{1.16}$$

Now it is just a matter of using the refined Sobolev inequality proved in [GMO] (see also [BG1]), which states that

$$\forall f\,,\quad ||f||_{L^6(\mathbb{R}^3)} \leq C\,||\nabla f||^{1/3}_{L^2(\mathbb{R}^3)}\,||\nabla f||^{2/3}_{\dot B^0_{2,\infty}}\,, \tag{1.17}$$

which proves (1.12).

We are finally left with the proof of the orthogonality property (1.5). It is due to the following lemma.

Lemma III.2. *Let h_n and $\tilde h_n$ be two orthogonal scales, and let (f_n) and $(\tilde f_n)$, be two sequences such that ∇f_n (resp. $\nabla \tilde f_n$) is strictly (h_n) (resp. $\tilde h_n$)$-$oscillatory. Then we have*

$$\overline{\lim_{n\to\infty}}\, ||f_n\,\tilde f_n||_{L^3(\mathbb{R}^3)} = 0.. \tag{1.18}$$

Proof of the lemma. We suppose for instance that $\frac{\tilde h_n}{h_n} \xrightarrow[n\to\infty]{} 0$. Let us start by assuming that the Fourier transforms of f_n and $\tilde f_n$ are both localized in a ring of $\mathbb{R}^3$, of the type

$$\mathcal{C}_n = \{\xi\in\mathbb{R}^3 \mid a\leq h_n|\xi|\leq b\}\,,\quad b>a>0\,. \tag{1.19}$$

Then we have, by Hausdorff-Young's inequality,

$$\begin{aligned} ||f_n\,\tilde f_n||_{L^3(\mathbb{R}^3)} &\leq C\,||\hat f_n * \widehat{\tilde f}_n||_{L^{3/2}(\mathbb{R}^3)}\\ &\leq C\,||\hat f_n||_{L^1(\mathbb{R}^3)}||\widehat{\tilde f}_n||_{L^{3/2}(\mathbb{R}^3)}\,, \end{aligned}$$

which implies that

$$\begin{aligned} ||f_n\,\tilde f_n||_{L^3(\mathbb{R}^3)} &\leq C\Big(\frac{1}{h_n}\Big)^{3/2}||f_n||_{L^2(\mathbb{R}^3)}\Big(\frac{1}{\tilde h_n}\Big)^{1/2}||\tilde f_n||_{L^2(\mathbb{R}^3)}\\ &\leq C\ \frac{\tilde h_n^{1/2}}{h_n^{1/2}}\,, \end{aligned}$$

since $||f_n||_{L^2(\mathbb{R}^3)} \leq a^{-1}h_n||\nabla f_n||_{L^2(\mathbb{R}^3)}$ by the spectral localization assumption on f_n, and similarly for $\tilde f_n$. So the result is proved in that case.

If f_n (resp. $\tilde f_n$) is not spectrally localized, then the strict $(h_n)-$oscillatory assumption enables one to approximate f_n by f_n^δ with

$$\overline{\lim_{n\to\infty}}\, ||\nabla(f_n - f_n^\delta)||_{L^2(\mathbb{R}^3)} \leq \delta\,, \tag{1.20}$$

and supp $\hat f_n^\delta \subset \mathcal{C}_n$. Then it is just a matter of using Hölder's and Sobolev's inequalities, which yield

$$\begin{aligned} ||(f_n - f_n^\delta)\,\tilde f_n||_{L^2(\mathbb{R}^3)} &\leq ||f_n - f_n^\delta||_{L^6(\mathbb{R}^3)}\,||\tilde f_n||_{L^6(\mathbb{R}^3)}\\ &\leq C\,\delta\,||\nabla\tilde f_n||_{L^2(\mathbb{R}^3)}\,, \end{aligned}$$

and the lemma is proved, and with it, proposition III.1. ■

III.2. Extraction of times and cores of concentration

Each function $v_n^{(j)}$ obtained in the previous section is going to be decomposed into a sum of orthogonal concentrating waves. To simplify the notation, we shall consider a strictly $h-$oscillatory family of functions, noted $v_h(t,x)$.

Proposition III.3. *Let v_h be a strictly $h-$oscillatory family of functions. Then up to an extraction, there exist linear concentrating waves $p_h^{(k)}$, associated with concentrating data $(\varphi^{(k)},\psi^{(k)},h,x_h^{(k)},t_h^{(k)})$, such that for any $\ell\in\mathbb{N}^*$, and up to a subsequence,*

$$v_h(t,x)=\sum_{k=1}^{\ell}p_h^{(k)}(t,x)+\overline{w}_h^{(\ell)}(t,x)\,; \tag{2.1}$$

$$\forall j\neq k\,,\quad \frac{|(x_h^{(j)},t_h^{(j)})-(x_h^{(k)},t_h^{(k)})|}{h}\xrightarrow[h\to 0]{}+\infty\,; \tag{2.2}$$

$$E_0(v_h)=\sum_{k=1}^{\ell}E_0(p_h^{(k)})+E_0(\overline{w}_h^{(\ell)})+o(1)\,,\quad h\to 0\,; \tag{2.3}$$

$$\forall T>0\,,\quad \overline{\lim_{h\to 0}}\,||\overline{w}_h^{(\ell)}||_{L^\infty([-T,T],L^6(\Omega))}\xrightarrow[\ell\to\infty]{}0\,. \tag{2.4}$$

Remark III.4. It is clear that proposition III.3, combined with proposition III.1, proves theorem 1.

Proof of the proposition. The proof is inspired by the method introduced in [MS]. Let us start by defining some notation. For any sequence $(x_h^{(k)},t_h^{(k)})$, we define the operator $D_h^{(k)}$, which acts on any f defined on $\mathbb{R}_t\times\mathbb{R}^3$ in the following way :

$$D_h^{(k)}f(y)\stackrel{\text{def}}{=}\Big(h^{1/2}f(t_h^{(k)},x_h^{(k)}+hy),h^{3/2}\partial_t f(t_h^{(k)},x_h^{(k)}+hy)\Big)\,. \tag{2.5}$$

If f satisfies the linear wave equation in Ω, then we define the function

$$\mathcal{W}_h^{(k)}f(s,y)\stackrel{\text{def}}{=}h^{1/2}f(t_h^{(k)}+hs,x_h^{(k)}+hy)\,, \tag{2.6}$$

which satisfies the linear wave equation in

$$\Omega_{1/h}^{(k)}\stackrel{\text{def}}{=}\frac{\Omega-x_h^{(k)}}{h}\,, \tag{2.7}$$

and we have $(\mathcal{W}_h^{(k)}f,\partial_s\mathcal{W}_h^{(k)}f)|_{s=0}=D_h^{(k)}f$. Finally, we define the exhaustion function δ by

$$\forall\underline{u}\stackrel{\text{def}}{=}(u_h)_{h>0}\,,\quad \delta(\underline{u})\stackrel{\text{def}}{=}\sup_{\underline{x},\underline{t}}\Big\{||\nabla\varphi||^2_{L^2(\mathbb{R}^3)}+||\psi||^2_{L^2(\mathbb{R}^3)}\,,$$
$$D_hu_h\rightharpoonup(\varphi,\psi),\ \text{up to a subsequence}\Big\}\,. \tag{2.8}$$

Let us prove the following result.

Lemma III.5. *For any bounded energy sequence $(u_h)_{h>0}$ of functions, we have*

$$\forall T>0\,,\quad \overline{\lim_{h\to 0}}\ \|u_h\|_{L^\infty([-T,T],L^6(\mathbb{R}^3))} \leq C\,\delta(\underline{u})^{1/3}\ \overline{\lim_{h\to 0}}\ E_0(u_h)^{1/6}\,. \tag{2.9}$$

PROOF OF THE LEMMA. It is based on the result proved in [G2]; let t_h be an arbitrary sequence of real numbers, and let us write, up to an extraction and according to [G2], proposition 4.1,

$$u_h(t_h,x)=\sum_{j=1}^{\ell}\ \frac{1}{h^{1/2}}\ \overline{\varphi}^{(j)}\Big(\frac{x-x_h^{(j)}}{h}\Big)+W_h^{(\ell)}(x)\,,\quad \forall \ell\in\mathbb{N}^*\,, \tag{2.10}$$

with, $\forall \ell\in\mathbb{N}^*$,

$$E_0(u_h)=\sum_{j=1}^{\ell}\ E_0(\overline{\varphi}^{(j)})+E_0(W_h^{(\ell)})+o(1)\,,\quad h\to 0\,, \tag{2.11}$$

and

$$\overline{\lim_{h\to\infty}}\ \|W_h^{(\ell)}\|_{L^6(\mathbb{R}^3)}\ \xrightarrow[\ell\to\infty]{}\ 0\,. \tag{2.12}$$

It is easy to see that up to an extraction (see [G2], remark 1.2 b),

$$\overline{\lim_{h\to 0}}\ \|u_h(t_h,\cdot)\|^6_{L^6(\mathbb{R}^3)}=\sum_{j=1}^{\infty}\ \|\overline{\varphi}^{(j)}\|^6_{L^6(\mathbb{R}^3)}\,, \tag{2.13}$$

and also

$$\overline{\lim_{h\to 0}}\ \|\nabla u_h(t_h,\cdot)\|^2_{L^2(\mathbb{R}^3)}\geq\sum_{j=1}^{\infty}\ \|\nabla\overline{\varphi}^{(j)}\|^2_{L^2(\mathbb{R}^3)}\,. \tag{2.14}$$

By the embedding of $\dot H^1(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, we get

$$\begin{aligned}\overline{\lim_{h\to 0}}\ \|u_h(t_h,\cdot)\|^6_{L^6(\mathbb{R}^3)}&\leq C\ \ \overline{\lim_{h\to 0}}\ \|\nabla u_h(t_h,\cdot)\|^2_{L^2(\mathbb{R}^3)}\ \sup_j\ \|\nabla\overline{\varphi}^{(j)}\|^4_{L^2(\mathbb{R}^3)}\\ &\leq C\ \ \overline{\lim_{h\to 0}}\ E_0(u_h)\,\delta(u_h)^2\,.\end{aligned}$$

The lemma is proved. ∎

Let us now proceed to the extraction of the $x_h^{(k)}$'s and $t_h^{(k)}$'s : we can suppose that $\delta(\underline{v})>0$, otherwise lemma III.5 shows that there is nothing to be proved. Now let us choose $(\varphi^{(1)},\psi^{(1)})\in\dot H^1\times L^2(\mathbb{R}^3)$ and $(x_h^{(1)},t_h^{(1)})_{h>0}$ such that

$$\|\nabla\varphi^{(1)}\|^2_{L^2(\mathbb{R}^3)}+\|\psi^{(1)}\|^2_{L^2(\mathbb{R}^3)}\geq\frac{1}{2}\,\delta(\underline{v})\,, \tag{2.15}$$

and

$$D_h^{(1)}v_h\ \underset{h\to 0}{\rightharpoonup}\ (\varphi^{(1)},\psi^{(1)})\,. \tag{2.16}$$

Then $p_h^{(1)}$ is the linear concentrating profile associated with $(\varphi^{(1)}, \psi^{(1)}, h, x_h^{(1)}, t_h^{(1)})$.

Lemma III.6. *Let* $W_h^{(1)} \stackrel{\text{def}}{=} v_h - p_h^{(1)}$. *Then*

$$E_0(v_h) = E_0(p_h^{(1)}) + E_0(w_h^{(1)}) + o(1)\,, \quad h \to 0\,.$$

PROOF OF THE LEMMA. It is an easy computation. Let B be the bilinear form associated with the energy :

$$B(a,b) \stackrel{\text{def}}{=} \int_\Omega \big(\partial_t a \cdot \partial_t \overline{b}(x) + \nabla a \cdot \nabla \overline{b}(x)\big)\, dx\,. \tag{2.17}$$

All we need to prove is that

$$B(v_h, p_h^{(1)}) = E_0(p_h^{(1)}) + o(1)\,, \quad h \to 0\,, \tag{2.18}$$

which is a straightforward consequence of conservation of energy combined with proposition II.1.1 and lemma II.1.2. ■

Now the expansion of v_h given in proposition III.3 follows by induction : let us suppose that

$$v_h(t,x) = \sum_{k=1}^{K} p_h^{(k)}(t,x) + w_h^{(K)}(t,x)\,, \tag{2.19}$$

where

$$E_0(v_h) = \sum_{k=1}^{K} E_0(p_h^{(k)}) + E_0(w_h^{(k)}) + o(1)\,, \quad h \to 0\,, \tag{2.20}$$

and where $p_h^{(k)}$ is a linear concentrating wave, associated with data $(\varphi^{(k)}, \psi^{(k)}, h, x_h^{(k)}, t_h^{(k)})$ where the $(x_h^{(k)}, t_h^{(k)})_{k\le K}$ are orthogonal :

$$\forall j \neq k\,,\ \ (j,k) \in \{1, \ldots, K\}^2\,, \quad \frac{|(x_h^{(k)}, t_h^{(k)}) - (x_h^{(j)}, t_h^{(j)})|}{h} \xrightarrow[h\to 0]{} \infty\,. \tag{2.21}$$

Exactly as previously, we can suppose that $\delta(\underline{w}^{(k)}) > 0$, and we can define $(\varphi^{(K+1)}, \psi^{(K+1)}, x_h^{(K+1)}, t_h^{(K+1)})$ such that

$$||\nabla \varphi^{(k+1)}||_{L^2}^2 + ||\psi^{(k+1)}||_{L^2}^2 \geq \frac{1}{2}\,\delta(\underline{w}^{(K)})\,, \tag{2.22}$$

$$D_h^{(K+1)} W_h^{(K)} \underset{h\to 0}{\rightharpoonup} (\varphi^{(K+1)}, \psi^{(K+1)})\,, \tag{2.23}$$

and $p_h^{(K+1)}$ is the linear concentrating wave associated with the data $(\varphi^{(K+1)}, \psi^{(K+1)}, h, x_h^{(K+1)}, t_h^{(K+1)})$.

Then lemma III.6 applied to $w_h^{(K)}$ and $p_h^{(K+1)}$ implies that

$$E_0(w_h^{(K)}) = E_0(p_h^{(K+1)}) + E_0(w_h^{(K+1)}) + o(1)\,, \quad h \to 0\,, \tag{2.24}$$

where $w_h^{(K+1)} = w_h^{(K)} - p_h^{(K+1)}$; hence to conclude the proof of the proposition, all we need is to prove the orthogonality of $(x_h^{(K+1)}, t_h^{(K+1)})$ and $(x_h^{(j)}, t_h^{(j)})$ for $j \leq K$, as well as the fact that for any $T > 0$,

$$\overline{\lim_{h\to 0}}\, ||w_h^{(K)}||_{L^\infty([-T,T],L^6(\Omega))} \xrightarrow[K\to\infty]{} 0\,. \tag{2.25}$$

The latter fact is due to lemma III.6 which yields, with conservation of energy,

$$E_0(v_h) = \sum_{k=1}^{K} \left(||\nabla\varphi^{(k)}||_{L^2}^2 + ||\psi^{(k)}||_{L^2}^2\right) + E_0(w_h^{(K)}) + o(1)\,, \quad h \to 0\,. \tag{2.26}$$

In turn that implies that

$$E_0(v_h) = \sum_{k=1}^{\infty} \left(||\nabla\varphi^{(k)}||_{L^2}^2 + ||\psi^{(k)}||_{L^2}^2\right) + E_0(w_h) + o(1)\,, \quad h \to 0\,, \tag{2.27}$$

where we replaced K by a sequence $K_h \xrightarrow[h\to\infty]{} \infty$.

So in particular, the series of general term $||\nabla\varphi^{(k)}||_{L^2}^2 + ||\psi^{(k)}||_{L^2}^2$ converges. But by definition of $(\varphi^{(k)}, \psi^{(k)})$ in (2.22), we have

$$\delta(\underline{w}^{(k)}) \leq 2\left(||\nabla\varphi^{(k+1)}||_{L^2}^2 + ||\psi^{(k+1)}||_{L^2}^2\right). \tag{2.28}$$

That implies that

$$\lim_{K\to\infty} \delta(\underline{w}^{(K)}) = 0\,, \tag{2.29}$$

and lemma III.5 yields the result : we have

$$\overline{\lim_{h\to 0}}\, ||w_h^{(K)}||_{L^\infty([-T,T],L^6(\Omega))} \xrightarrow[K\to\infty]{} 0\,. \tag{2.30}$$

Finally we are left with the proof of the orthogonality of the concentration cores $x_h^{(k)}$ and times $t_h^{(k)}$. In the following, considering two families $(z_h^{(j)})_{h>0}$ and $(z_h^{(k)})_{h>0}$ of elements of $\mathbb{R}^d$, $d \geq 1$, we shall write $z_h^{(j)} \perp_h z_h^{(k)}$ if $\frac{|z_h^{(j)} - z_h^{(k)}|}{h} \xrightarrow[h\to 0]{} \infty$, and symmetrically, we will write $z_h^{(j)} \not\perp_h z_h^{(k)}$ if $\frac{|z_h^{(j)} - z_h^{(k)}|}{h} \xrightarrow[h\to 0]{} \lambda \in \mathbb{R}^+$.

Let us prove the following result

Lemma III.7. *Let $\{j, j'\} \in \{1, \ldots, K\}^2$ be such that*

$$(x_h^{(j)}, t_h^{(j)}) \not\perp_h (x_h^{(K+1)}, t_h^{(K+1)}) \quad \text{and} \quad (x_h^{(j)}, t_h^{(j)}) \perp_h (x_h^{(j')}, t_h^{(j')})\,.$$

Then $\mathcal{W}_h^{(j)} w_h^{(K+1)}$ and $\mathcal{W}_h^{(j)} p_h^{(j')}$ go weakly to zero in the energy space, as h goes to zero (see (2.6) for the definition of $\mathcal{W}_h^{(j)}$).

Proof of the lemma. Let us start by considering the case of $\mathcal{W}_h^{(j)} w_h^{(K+1)}$. By assumption, there exist $\tau_h \in \mathbb{R}$ and $\xi_h \in \mathbb{R}^3$ such that

$$t_h^{(j)} = t_h^{(K+1)} + \tau_h h\,, \quad x_h^{(j)} = x_h^{(K+1)} + \xi_h h\,, \tag{2.31}$$

and $\tau_h \xrightarrow[h\to 0]{} \tau \in \mathbb{R}$, $\xi_h \xrightarrow[h\to 0]{} \xi \in \mathbb{R}^3$. By definition of $\mathcal{W}_h^{(j)} w_h^{(K+1)}$ given in (2.6), we have

$$\mathcal{W}_h^{(j)} w_h^{(K+1)}(s,y) = h^{1/2} w_h^{(K+1)}(t_h^{(j)} + hs, x_h^{(j)} + hy)\,, \tag{2.32}$$

so $\mathcal{W}_h^{(j)} w_h^{(K+1)}$ satisfies the following equation :

$$\begin{cases} \Box \mathcal{W}_h^{(j)} w_h^{(K+1)} = 0 \ \text{ in } \ \mathbb{R}_s \times \Omega_{1/h}^{(j)}, \quad \mathcal{W}_h^{(j)} w_h^{(K+1)}|_{\mathbb{R}_s \times \partial\Omega_{1/h}^{(j)}} = 0 \\ \big(\mathcal{W}_h^{(j)} w_h^{(K+1)}, \partial_s \mathcal{W}_h^{(j)} w_h^{(K+1)}\big)|_{s=0} = D_h^{(j)} w_h^{(K+1)}\,. \end{cases} \tag{2.33}$$

Now let us define

$$\tilde{w}_h^{(j,K+1)}(s,y) \stackrel{\text{def}}{=} \mathcal{W}_h^{(j)} w_h^{(K+1)}(s-\tau, y-\xi_h)\,. \tag{2.34}$$

Then (2.33) becomes

$$\begin{cases} \Box \tilde{w}_h^{(j,K+1)} = 0 \ \text{ in } \ \mathbb{R}_s \times \Omega_{1/h}^{(K+1)}, \quad \tilde{w}_h^{(j,K+1)}|_{\mathbb{R}_s \times \partial\Omega_{1/h}^{(K+1)}} = 0 \\ \big(\tilde{w}_h^{(j,K+1)}, \partial_s \tilde{w}_h^{(j,K+1)}\big)|_{s=\tau-\tau_h} = D_h^{(K+1)} w_h^{(K+1)}\,. \end{cases} \tag{2.35}$$

Now it is just a matter of using proposition II.1.3 : we know, by definition, that

$$D_h^{(K+1)} w_h^{(K+1)} \underset{h\to 0}{\rightharpoonup} (0,0) \ \text{ in } \ E(\Omega)\,, \tag{2.36}$$

which implies that

$$\mathcal{W}_h^{(j)} w_h^{(K+1)}(\cdot - \tau, \cdot - \xi_h) \rightharpoonup 0 \tag{2.37}$$

in the energy space, so the result is proved for $\mathcal{W}_h^{(j)} w_h^{(K+1)}$.

Now let us study the case of $\mathcal{W}_h^{(j)} p_h^{(j')}$: we have by definition,

$$\mathcal{W}_h^{(j)} p_h^{(j')}(s,y) = h^{1/2} p_h^{(j')}(t_h^{(j)} + hs, x_h^{(j)} + hy)\,. \tag{2.38}$$

But by definition of $p_h^{(j')}$, proposition II.3.1 implies that

$$\lim_{h\to 0} ||p_h^{(j)}(t_h^{(j)} + hs, \cdot)||_{L^6(\mathbb{R}^3)} = 0\,, \quad \forall s \in \mathbb{R}, \tag{2.39}$$

which after rescaling yields

$$\lim_{h\to 0} ||\mathcal{W}_h^{(j)} p_h^{(j')}(s,\cdot)||_{L^6(\mathbb{R}^3)} = 0\,, \quad \forall s \in \mathbb{R}, \tag{2.40}$$

which the expected result $\mathcal{W}_h^{(j)} p_h^{(j')}$, in the case of time-orthogonality.

Now let us consider the case where $t_h^{(j)} \not\perp_h t_h^{(j')}$. It is enough to prove that

$$D_h^{(j)} p_h^{(j')} \rightharpoonup (0,0)\,. \tag{2.41}$$

So let $(g^1,g^2)\in(C_0^\infty(\mathbb{R}^3))^2$. By time-equicontinuity, we can suppose to simplify that $t_h^{(j)}=t_h^{(j')}$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^3} D_h^{(j)}p_h^{(j')}(y)\cdot(g^1,g^2)(y)\,dy &= \int_{\mathbb{R}^3}\mathcal{P}_{\Omega^{(j)}_{1/h}}\varphi^{(j')}\Big(y+\frac{x_h^{(j)}-x_h^{(j')}}{h}\Big)g^1(y)\,dy\\ &\quad+\int_{\mathbb{R}^3}\mathbf{1}_{\Omega^{(j)}_{1/h}}\psi^{(j')}\Big(y+\frac{x_h^{(j)}-x_h^{(j')}}{h}\Big)g^2(y)\,dy\\ &=\int_{\mathbb{R}^3}\mathcal{P}_{\Omega^{(j')}_{1/h}}\varphi^{(j')}(z)g^1\Big(z-\frac{x_h^{(j)}-x_h^{(j')}}{h}\Big)\,dz\\ &\quad+\int_{\mathbb{R}^3}\mathbf{1}_{\Omega^{(j')}_{1/h}}(z)\psi^{(j')}(z)g^2\Big(z-\frac{x_h^{(j)}-x_h^{(j')}}{h}\Big)\,dz\,,\end{aligned}$$

and (2.41) follows from the fact that g^1 and g^2 are compactly supported. ∎

Now we are ready to prove the orthogonality result : let us define

$$j_K=\max\Big\{j\in\{1,\dots,K\}\mid(t_h^{(j)},x_h^{(j)})\not\perp_h(t_h^{(K+1)},x_h^{(K+1)})\Big\}, \tag{2.42}$$

supposing such an index exists.

Lemma III.7 implies that

$$\mathcal{W}_h^{(j_K)}w_h^{(K+1)}\rightharpoonup 0\,. \tag{2.43}$$

We may suppose that $t_h^{(j_K)}=t_h^{(K+1)}$ and $x_h^{(j_K)}=x_h^{(K+1)}$. Then

$$\mathcal{W}_h^{(j_K)}p_h^{(K+1)}(y)=\mathcal{W}_h^{(K+1)}p_h^{(K+1)}(y)\,. \tag{2.44}$$

Which implies that $\mathcal{W}_h^{(j_K)}p_h^{(K+1)}$ has a weak limit which is not zero. But by definition of $w_h^{(j_K)}$, we have

$$w_h^{(j_K)}=\sum_{k=j_K+1}^{K+1}p_h^{(k)}+w_h^{(K+1)}\,, \tag{2.45}$$

so we infer from (2.36) and (2.43) that

$$\mathcal{W}_h^{(j_K)}\Big(\sum_{k=j_K+1}^{K+1}p_h^{(k)}\Big)\underset{h\to0}{\rightharpoonup}0\,. \tag{2.46}$$

If $j_K=K$, the contradiction is obvious. If not, the definition of j_K and lemma III.7 imply that $\mathcal{W}_h^{(j_K)}(\sum_{k=j_K+1}^{K}p_h^{(k)})$ goes weakly to 0, which contradicts (2.46) and (2.44).

So proposition III.3 is proved, along with theorem 1. ∎

IV. The nonlinear equation

IV.1. Proof of theorem 2

To simplify the notation, we shall consider concentrating data $(\varphi, \psi, h, x_h, t_h)$, $h > 0$, and we shall only deal with the case $\frac{t_h}{h} \xrightarrow[h\to 0]{} +\infty$; the other cases are similar and left to the reader. Recall that the function p_h solves

$$
(1.1) \qquad \begin{cases} \Box p_h = 0 \text{ in } \mathbb{R}_t \times \Omega, \quad p_h|_{\mathbb{R}_t\times\partial\Omega} = 0 \\ (p_h, \partial_t p_h)|_{t=t_h}(x) = \Big(\dfrac{1}{h^{1/2}} \mathcal{P}_\Omega\Big(\varphi\Big(\dfrac{x-x_h}{h}\Big)\Big), \dfrac{1}{h^{3/2}} \mathbf{1}_\Omega(x)\psi\Big(\dfrac{x-x_h}{h}\Big)\Big). \end{cases}
$$

Introduce the following nonlinear wave :

$$
(1.2) \qquad \begin{cases} \Box q_h + |q_h|^4 q_h = 0 \text{ in } \mathbb{R}_t \times \Omega, \quad q_h|_{\mathbb{R}_t\times\partial\Omega} = 0 \\ (q_h, \partial_t q_h)|_{t=0} = (p_h, \partial_t p_h)|_{t=0}, \end{cases}
$$

and call P_h the function (noted $\mathcal{W}_h p_h$ in section III)

$$
(1.3) \qquad P_h(s,y) \stackrel{\text{def}}{=} h^{1/2} p_h(t_h + hs, x_h + hy),
$$

which satisfies the linear wave equation in $\Omega_{1/h} \stackrel{\text{def}}{=} \frac{\Omega - x_h}{h}$. As seen in section II.1, the domain $\Omega_{1/h}$ converges, as h goes to zero, towards a limit domain Ω_∞, and we will denote P the solution of the linear wave equation in Ω_∞ :

$$
(1.4) \qquad \begin{cases} \Box P = 0 \text{ in } \mathbb{R}_s \times \Omega_\infty, \quad P|_{\mathbb{R}_s\times\partial\Omega_\infty} = 0 \\ (P, \partial_s P)|_{s=0} = (\mathcal{P}_{\Omega_\infty}\varphi, \mathbf{1}_{\Omega_\infty}\psi). \end{cases}
$$

Then we can associate with P the following nonlinear equation :

$$
(1.5) \qquad \begin{cases} \Box Q^\lambda + |Q^\lambda|^4 Q^\lambda = 0 \text{ in } \mathbb{R}_s \times \Omega_\infty, \quad Q^\lambda|_{\mathbb{R}_s\times\partial\Omega_\infty} = 0 \\ (Q^\lambda, \partial_s Q^\lambda)|_{s=-\lambda} = (P, \partial_s P)|_{s=-\lambda}, \end{cases}
$$

where λ is a positive real number. We call $\overline{q}_h^\lambda$ the associate rescaled function

$$
(1.6) \qquad \overline{q}_h^\lambda(t,x) \stackrel{\text{def}}{=} \frac{1}{h^{1/2}} Q^\lambda\Big(\frac{t-t_h}{h}, \frac{x-x_h}{h}\Big).
$$

Finally, let us define $S_{\Omega_\infty} \stackrel{\text{def}}{=} P_{\Omega_\infty} S R_{\Omega_\infty}$, with the notation of definition I.6. Define the solution $\tilde{P}_h$ of the following equation :

$$
(1.7) \qquad \begin{cases} \Box \tilde{P}_h = 0 \text{ in } \mathbb{R}_s \times \Omega_{1/h}, \quad \tilde{P}_h|_{\mathbb{R}_s\times\partial\Omega_{1/h}} = 0 \\ (\tilde{P}_h \partial_s \tilde{P}_h)|_{s=0} = S_{\Omega_\infty}(\mathcal{P}_{\Omega_\infty}\varphi, \mathbf{1}_{\Omega_\infty}\psi), \end{cases}
$$

and set

$$
(1.8) \qquad \tilde{p}_h(t,x) \stackrel{\text{def}}{=} \frac{1}{h^{1/2}} \tilde{P}_h\Big(\frac{t-t_h}{h}, \frac{x-x_h}{h}\Big).
$$

This section is devoted to the proof of the following results :

$$
(1.9) \qquad \overline{\lim_{h\to 0}} \Big(\sup_{-T\le t\le t_h-\lambda h} E_0(p_h - q_h, t)^{1/2} + ||p_h - q_h||_{L^5([-T, t_h-\lambda h], L^{10}(\Omega))} \Big) \xrightarrow[\lambda\to\infty]{} 0;
$$

$$
(1.10) \qquad \overline{\lim_{h\to 0}} \, |||q_h - \overline{q}_h^\lambda|||_{I_h^\lambda} \xrightarrow[\lambda\to\infty]{} 0;
$$

$$(1.11)\qquad \overline{\lim_{h\to 0}}\left(\sup_{t_h+\lambda h\le t\le T} E_0(\tilde p_h - q_h, t)^{1/2} + ||\tilde p_h - q_h||_{L^5([t_h+\lambda h;T],L^{10}(\Omega))}\right) \xrightarrow[\lambda\to\infty]{} 0\,,$$

where $T>0$ is arbitrary, and where we have defined

$$(1.12)\qquad |||f|||_{I_h^\lambda} \stackrel{\text{def}}{=} \sup_{t_h-\lambda h\le t\le t_h+\lambda h} E_0(f,t)^{1/2} + ||f||_{L^5([t_h-\lambda h;t_h+\lambda h],L^{10}(\mathbb{R}^3))}\,.$$

The proof of (1.9)–(1.11) obviously yields theorem 2.

Proof of (1.9). The result is due to a straightforward adaptation of the linearization theorem of [G1] to the case of an open domain, since proposition II.3.1 implies that

$$(1.13)\qquad \lim_{h\to 0} ||p_h||_{L^\infty([-T,t_h-\lambda h],L^6(\Omega))} \xrightarrow[\lambda\to\infty]{} 0\,.$$

We omit the details, as they are contained in [G1], section 3.

Proof of (1.10). We shall separate the proof into two parts : let us start by defining Q_h^λ, the solution of

$$(1.14)\qquad \begin{cases} \Box Q_h^\lambda + |Q_h^\lambda|^4\, Q_h^\lambda = 0 \ \text{ in } \ \mathbb{R}_s\times\Omega_{1/h}\,, \quad Q_h^\lambda|_{\mathbb{R}_s\times\partial\Omega_{1/h}} = 0 \\ (Q_h^\lambda, \partial_s Q_h^\lambda)|_{s=-\lambda} = (P_h, \partial_s P_h)|_{s=-\lambda}\,, \end{cases}$$

where P_h was defined in (1.3). We shall start by proving that

$$(1.15)\qquad \overline{\lim_{h\to 0}}\, |||Q_h^\lambda - Q^\lambda|||_\lambda = 0\,, \quad \forall \lambda\in\mathbb{R},$$

where similarly to (1.12), we have defined

$$(1.16)\qquad |||f|||_\lambda \stackrel{\text{def}}{=} \sup_{s\in[-\lambda,\lambda]} E_0(f,s)^{1/2} + ||f||_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))}\,.$$

Let us start by noticing that Q_h^λ converges weakly towards Q^λ, as h goes to zero, by similar arguments to those leading to proposition II.1.3. Then conservation of energy yields the strong convergence in the energy space, so we are left, to obtain (1.15), with the proof of

$$(1.17)\qquad \overline{\lim_{h\to 0}}\, ||Q_h^\lambda - Q^\lambda||_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))} = 0\,, \quad \forall\lambda.$$

Hölder's inequality enables us to write that

$$(1.18)\qquad \begin{aligned} &||Q_h^\lambda - Q^\lambda||^5_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))} \\ &\qquad \le ||Q_h^\lambda - Q^\lambda||_{L^\infty([-\lambda,\lambda],L^6(\mathbb{R}^3))}\, ||Q_h^\lambda - Q^\lambda||^4_{L^4([-\lambda,\lambda],L^{12}(\mathbb{R}^3))}\,. \end{aligned}$$

But Q^λ and Q_h^λ both satisfy Strichartz' estimate, so (1.18) becomes, using the convergence to zero in the energy space,

$$(1.19)\qquad \begin{aligned} ||Q_h^\lambda - Q^\lambda||^5_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))} \le C\,\varepsilon_\lambda(h)\Big(&||(\varphi,\psi)||_E^2 + ||Q^\lambda||^{20}_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))} \\ &+||Q_h^\lambda||^{20}_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))}\Big)\,. \end{aligned}$$

where $\varepsilon_\lambda(h)$ satisfies

$$\varepsilon_\lambda(h) \xrightarrow[h\to 0]{} 0\,, \quad \forall \lambda \in \mathbb{R}. \tag{1.20}$$

It is easy to see, by rescaling in time, that the constant C appearing in (1.19) does not depend on λ, nor on h, provided $h\lambda$ is bounded.

So finally we obtain

$$\begin{aligned} ||Q_h^\lambda - Q^\lambda||^5_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))} \leq C\,\varepsilon_\lambda(h)\Big(&||(\varphi,\psi)||_E^2 + ||Q^\lambda||^{20}_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))} \\ &+ ||Q_h^\lambda - Q^\lambda||^{20}_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))}\Big)\,, \end{aligned} \tag{1.21}$$

and we conclude by superlinear bootstrap (see for instance [BG1], lemma 2.2) : (1.17) is proved. Now we are left with the proof of the following limit :

$$\overline{\lim_{h\to 0}}\; |||Q_h - Q_h^\lambda|||_\lambda \xrightarrow[\lambda\to\infty]{} 0\,, \tag{1.22}$$

where

$$Q_h(s,y) \stackrel{\text{def}}{=} h^{1/2} q_h(t_h + hs, x_h + hy)\,. \tag{1.23}$$

Let us define $R_h^\lambda \stackrel{\text{def}}{=} Q_h - Q_h^\lambda$, which satisfies the following equation :

$$\begin{cases} \Box R_h^\lambda + |R_h^\lambda + Q_h^\lambda|^4\,(R_h^\lambda + Q_h^\lambda) - |Q_h^\lambda|^4\,Q_h^\lambda = 0 \ \text{ in } \ \mathbb{R}_s \times \Omega_{1/h} \\ R_h^\lambda|_{\mathbb{R}_s\times\partial\Omega_{1/h}} = 0 \\ (R_h^\lambda, \partial_s R_h^\lambda)|_{s=-\lambda} = (Q_h, \partial_s Q_h)|_{s=-\lambda} - (P_h, \partial_s P_h)|_{s=-\lambda}\,. \end{cases} \tag{1.24}$$

The method of proof of (1.22) follows closely [BG2]. Equation (1.9) implies that

$$\overline{\lim_{h\to 0}}\; E_0(R_h^\lambda, -\lambda) \xrightarrow[\lambda\to+\infty]{} 0\,, \tag{1.25}$$

so by Strichartz' estimate, we have for any $T_0 > -\lambda$,

$$\begin{aligned} &||R_h^\lambda||_{L^5([-\lambda,T_0],L^{10}(\mathbb{R}^3))} \\ &\quad\leq C\Big(\varepsilon(\lambda,h) + \sum_{j=1}^{5} ||(R_h^\lambda)^j (Q_h^\lambda)^{5-j}||_{L^1([-\lambda,T_0],L^2(\mathbb{R}^3))}\Big) \\ &\quad\leq C\Big(\varepsilon(\lambda,h) + \sum_{j=1}^{5} ||R_h^\lambda||^j_{L^5([-\lambda,T_0],L^{10}(\mathbb{R}^3))}\, ||Q_h^\lambda||^{5-j}_{L^5([-\lambda,T_0],L^{10}(\mathbb{R}^3))}\Big)\,, \end{aligned} \tag{1.26}$$

where $\overline{\lim\limits_{h\to 0}}\; \varepsilon(\lambda,h) \xrightarrow[\lambda\to+\infty]{} 0.$

Now according to (1.17), we can replace Q_h^λ by Q^λ in (1.26). Moreover, it is easy to see that Q^λ converges, as λ goes to infinity, both in energy and in Strichartz norms, to Q^- solution of

$$\begin{cases} \Box Q^- + |Q^-|^4\,Q^- = 0 \ \text{ in } \ \mathbb{R}_s \times \Omega_\infty\,, \quad Q^-|_{\mathbb{R}_s\times\partial\Omega_\infty} = 0 \\ E_0(Q^- - P, s) \xrightarrow[s\to-\infty]{} 0\,. \end{cases} \tag{1.27}$$

Since Ω_∞ is either $\emptyset$, $\mathbb{R}^3$ or a half-space, we know from [BS] that Q^- is an element of $L^5(\mathbb{R}, L^{10}(\Omega_\infty))$. So we can choose λ and $|T_0|$ large enough so that $||Q^-||_{L^5([-\lambda,T_0],L^{10}(\mathbb{R}^3))}$ is small enough, and we infer by superlinear bootstrap that

$$
(1.28) \qquad \overline{\lim_{h\to 0}}\, ||R_h^\lambda||_{L^5([-\lambda,T_0],L^{10}(\mathbb{R}^3))} \xrightarrow[\lambda\to\infty]{} 0\,.
$$

The rest of the proof is a simple deformation argument and left to the reader (see for instance [BG2]).

Proof of (1.11). It is identical to (1.9), using the scattering results of [BG1]–[BG2], which extend by reflection to the case when Ω_∞ is a half-space.

Theorem 2 is proved. ■

Remark IV.1.

By proposition II.2.6, p_h and $\tilde{p}_h$ are strictly $(h)-$oscillatory functions. Moreover, $\overline{q}_h^\lambda$ is clearly an $h-$oscillatory function, so (1.9)–(1.11) imply that

$$
(1.33) \qquad q_h \ \text{ is an } \ h-\text{oscillatory function}\,.
$$

IV.2. Proof of theorem 3

To prove theorem 3, we shall follow closely the strategy used in [BG1] : we know from theorem 1 that any finite energy solution to the linear wave equation can be decomposed into concentrating waves, and theorem 2 provides the analysis of the nonlinear equation, compared to the linear equation with concentrating data. The combination of both theorems yields theorem 3. As most arguments are identical to the case of the whole space studied in [BG1], we shall merely recall most results here, and give proofs only when differences appear.

Let us recall the statement of theorem 3 : with the notation of theorem 1, we shall prove that the solution of

$$
(2.1) \qquad \begin{cases} \Box u_n + |u_n|^4 u_n = 0 \ \text{ in } \ \mathbb{R}_t \times \Omega\,, \quad u_n|_{\mathbb{R}_t\times\partial\Omega} = 0 \\ (u_n, \partial_t u_n)|_{t=0} = (v_n, \partial_t v_n)|_{t=0} \rightharpoonup (0,0) \ \text{ in } \ E(\Omega) \end{cases}
$$

can be decomposed in the following way :

$$
(2.2) \qquad u_n(t,x) = \sum_{j=1}^{\ell} q_n^{(j)}(t,x) + w_n^{(\ell)}(t,x) + r_n^{(\ell)}(t,x)\,,
$$

where $q_n^{(j)}$ is the nonlinear concentrating wave associated with $p_n^{(j)}$, and where

$$
(2.3) \qquad \overline{\lim_{n\to\infty}} \left(\sup_{-T\le t\le T} E_0(r_n^{(\ell)},t)^{1/2} + ||r_n^{(\ell)}||_{L^5([-T,T],L^{10})} \right) \xrightarrow[\ell\to\infty]{} 0\,, \quad \forall T>0\,.
$$

Let us define the function β in the following way :

$$
(2.4) \qquad \forall \omega \in \mathbb{C}, \quad \beta(\omega) \stackrel{\text{def}}{=} |\omega|^4\,\omega\,.
$$

Lemma IV.2. *Let $(q_n^{(j)})_{n\in\mathbb{N}}$, for $1 \leq j \leq \ell$, be nonlinear concentrating waves associated with $(p_n^{(j)})_{n\in\mathbb{N}}$. Then we have, for any $T > 0$,*

$$\overline{\lim_{n\to\infty}} \Big|\Big|\beta\Big(\sum_{j=1}^{\ell} q_n^{(j)}\Big) - \sum_{j=1}^{\ell} \beta(q_n^{(j)})\Big|\Big|_{L^1([-T,T],L^2)} = 0. \tag{2.5}$$

Proof of the lemma. As in [BG1], let us write, for any $T > 0$,

$$\Big|\Big|\beta\Big(\sum_{j=1}^{\ell} q_n^{(j)}\Big) - \sum_{j=1}^{\ell} \beta(q_n^{(j)})\Big|\Big|_{L^1([-T,T],L^2)} = \sum_{1\leq j_1,\ldots,j_5\leq \ell} \Big|\Big|\prod_{k=1}^{5} q_n^{(j_k)}\Big|\Big|_{L^1([-T,T],L^2)}, \tag{2.6}$$

where at least two $q_n^{(j_k)}$'s are different. Let us suppose for instance that

$$\lim_{n\to\infty} \Big|\log \frac{h_n^{(j_1)}}{h_n^{(j_2)}}\Big| = +\infty. \tag{2.7}$$

Then we write, by Hölder's inequality,

$$\Big|\Big|\prod_{k=1}^{5} q_n^{(j_k)}\Big|\Big|_{L^1([-T,T],L^2)} \leq ||q_n^{(j_1)}\, q_n^{(j_2)}||_{L^\infty([-T,T],L^3)} \Big|\Big|\prod_{k=3}^{5} q_n^{(j_k)}\Big|\Big|_{L^1([-T,T],L^6)}. \tag{2.8}$$

On the one hand, we have

$$\Big|\Big|\prod_{k=2}^{5} q_n^{(j_k)}\Big|\Big|_{L^1([-T,T],L^6)} \leq \prod_{k=3}^{5} ||q_n^{(j_k)}||_{L^3([-T,T],L^{18})}, \tag{2.9}$$

which is bounded, by Strichartz' estimate. As to the term $q_n^{(j_1)}\, q_n^{(j_2)}$, (1.33) states that $q_n^{(j_1)}$ (resp. $q_n^{(j_2)}$) is strictly $(h_n^{(j_1)})$, (resp. $(h_n^{(j_2)})$)$-$oscillatory. Hence by lemma III.2, we have

$$\overline{\lim_{n\to\infty}} ||q_n^{(j_1)}\, q_n^{(j_2)}||_{L^\infty([-T,T],L^3(\mathbb{R}^3))} = 0, \tag{2.10}$$

so the result follows in the case where two scales are orthogonal.

Otherwise, let us suppose that $h_n^{(j_1)} = \ldots = h_n^{(j_5)} = h_n$, and let us write generically, by Hölder's inequality,

$$\sum_{1\leq j_1,\ldots,j_5\leq \ell} \Big|\Big|\prod_{k=1}^{5} q_n^{(j_k)}\Big|\Big|_{L^1([-T,T],L^2)} \leq C\, ||q_n^1\, q_n^2||_{L^{5/2}([-T,T],L^5)}. \tag{2.11}$$

Writing, for any $i \in \{1,2\}$,

$$q_n^i(t,x) = \frac{1}{h_n^{1/2}}\, Q_n^i\Big(\frac{t-t_n^i}{h_n}, \frac{x-x_n^i}{h_n}\Big), \tag{2.12}$$

and defining $s = \frac{t-t_n^2}{h_n}$ and $y = \frac{x-x_n^2}{h_n}$, we have

$$\begin{aligned} &||q_n^1\, q_n^2||^{5/2}_{L^{5/2}([-T,T],L^5)} \\ &\quad = \int_{s\in I_n} \Big(\int_{\mathbb{R}^3} |Q_n^1|^5\Big(\frac{t_n^2-t_n^1}{h_n} + s,\, \frac{x_n^2-x_n^1}{h_n} + y\Big)\, |Q_n^2|^5\,(s,y)\, dy\Big)^{1/2} ds, \end{aligned} \tag{2.13}$$

where $I_n = \left[\frac{-t_n^2 - T}{h_n}, \frac{-t_n^2+T}{h_n}\right]$. Supposing, without loss of generality, that $\frac{t_n^2}{h_n} \xrightarrow[n\to\infty]{} +\infty$ or 0, let us write $I_n = \bigcup_{j=1}^{3} I_n^{j,\lambda}$, where

$$I_n^{1,\lambda} = \left[\frac{-t_n^2 - T}{h_n}, -\lambda\right] \text{ and } I_n^{2,\lambda} = [-\lambda, \lambda]. \tag{2.14}$$

Theorem 2 yields

$$\overline{\lim_{n\to\infty}} \left(\sup_{s\in I_n^{1,\lambda}} E_0(Q_n^2 - P_n^2, s) + ||Q_n^2 - P_n^2||_{L^5(I_n^{1,\lambda}, L^{10})} \right) \xrightarrow[\lambda\to\infty]{} 0, \tag{2.15}$$

and it is easy to see that (2.15) implies

$$\overline{\lim_{n\to\infty}} \left(\sup_{s\in I_n^{1,\lambda}} E_0(Q_n^2 - P^2, s) + ||Q_n^2 - P^2||_{L^5(I_n^{1,\lambda}, L^{10})} \right) \xrightarrow[\lambda\to\infty]{} 0; \tag{2.16}$$

similarly we have

$$\overline{\lim_{n\to\infty}} \left(\sup_{s\in I_n^{3,\lambda}} E_0(Q_n^2 - \tilde{P}^2, s) + ||Q_n^2 - \tilde{P}^2||_{L^5(I_n^{3,\lambda}, L^{10})} \right) \xrightarrow[\lambda\to\infty]{} 0, \tag{2.17}$$

where $\tilde{P}^2$ satisfies the linear wave equation in Ω_∞, with initial data $(S_{\Omega_\infty}(\mathcal{P}_{\Omega_\infty}\varphi, \mathbf{1}_{\Omega_\infty}\psi))$. So

$$\begin{aligned} ||Q_n^1 Q_n^2||_{L^{5/2}(I_n^{1,\lambda}\cup I_n^{3,\lambda}; L^5)} \le ||Q_n^1||^{1/2}_{L^5(I_n; L^{10})} \Big(&||Q_n^2 - P^2||_{L^5(I_n^{1,\lambda}; L^{10})} + ||P^2||_{L^5(I_n^{1,\lambda}; L^{10})} \\ &+ ||Q_n^2 - \tilde{P}^2||_{L^5(I_n^{2,\lambda}; L^{10})} + ||\tilde{P}^2||_{L^5(I_n^{3,\lambda}; L^{10})} \Big). \end{aligned}$$

Now P^2 and $\tilde{P}^2$ satisfy global Strichartz estimates, so choosing λ large enough, we can have $||P^2||_{L^5([-\infty,-\lambda], L^{10})} + ||\tilde{P}^2||_{L^5([\lambda;+\infty[, L^{10})}$ arbitrarily small. Then (2.16) and (2.17), along with Strichartz' estimate on Q_n^1, yield the result for $I_n^{1,\lambda}$ and $I_n^{3,\lambda}$.

Finally, we can write

$$\begin{aligned} ||Q_n^1 Q_n^2||^{5/2}_{L^{5/2}(I_n^{2,\lambda}; L^5)} &\le \int_{-\lambda}^{\lambda} \left(\int_{\mathbb{R}^3} |Q_n^1 - Q^{1,\lambda}|^5 \Big(\frac{t_n^2 - t_n^1}{h_n} + s, \frac{x_n^2 - x_n^1}{h_n} + y\Big) |Q_n^2|^5(s,y)\, dy \right)^{1/2} ds \\ &+ \int_{-\lambda}^{\lambda} \left(\int_{\mathbb{R}^3} |Q^{1,\lambda}|^5 \Big(\frac{t_n^2 - t_n^1}{h_n} + s; \frac{x_n^2 - x_n^1}{h_n} + y\Big) |Q_n^2|^5(s,y)\, dy \right)^{1/2} ds. \end{aligned} \tag{2.18}$$

One can approximate $Q^{1,\lambda}$ by compactly supported functions, and then (1.10) along with the orthogonality of (x_n^1, t_n^1) and (x_n^2, t_n^2), yield the result.

So lemma IV.2 is proved. ∎

The next lemma can be found in [BG] in the case $\mathbb{R}^3$. Its proof remains unchanged in the case of a domain

Lemma IV.3. *Let $T > 0$ be fixed. There exists $\delta_1 > 0$ such that if*

$$\overline{\lim_{n\to\infty}} ||v_n||_{L^5([-T,T], L^{10})} \le \delta_1, \tag{2.19}$$

then theorem 3 is proved.

So we are left with the case where the $L^5_t L^{10}_x$ norms are not small. Let us prove the following result.

Lemma IV.4. *Let $\delta > 0$ be fixed, and let $\tilde{q}_n$ be a sequence in $L^5([0,T], L^{10})$, such that*

$$\overline{\lim_{n\to\infty}} \, ||\tilde{q}_n||_{L^5([0,T],L^{10})} \leq \delta \,. \tag{2.20}$$

Then for any $\delta' > \delta$, there exists a finite decomposition of $[0,T]$,

$$[0,T] = \bigcup_{i=1}^{L} I_n^{(i)} \,,$$

where the $I_n^{(i)}$'s are closed intervals, such that the sequence

$$\Gamma_n \stackrel{\text{def}}{=} \sum_{j=1}^{\ell} q_n^{(j)} + \tilde{q}_n$$

satisfies

$$\overline{\lim_{n\to\infty}} \, ||\Gamma_n||_{L^5(I_n^{(i)},L^{10})} \leq \delta' \,.$$

Proof of the lemma. We start with the case $\ell = 1$, and separate the study according to the various limit of $\frac{t_n^{(1)}}{h_n^{(1)}}$, as in [BG1]. If $\lim\limits_{n\to\infty} \frac{t_n^{(1)}}{h_n^{(1)}} = +\infty$, then let us write

$$[0,T] = \left[0, t_n^{(1)} - \lambda h_n^{(1)}\right] \cup \left[t_n^{(1)} - \lambda h_n^{(1)}, t_n^{(1)} + \lambda h_n^{(1)}\right] \cup \left[t_n^{(1)} + \lambda h_n^{(1)}, T\right] , \tag{2.21}$$

with $\lambda > 0$ to be determined. Writing $I_n^{(1)} = \left[0, t_n^{(1)} - \lambda h_n^{(1)}\right]$, we know from section IV.1, equation (1.9), that

$$||q_n^{(1)}||_{L^5(I_n^{(1)},L^{10}(\Omega))} \leq \varepsilon(n,\lambda) + ||p_n^{(1)}||_{L^5(I_n^{(1)},L^{10}(\Omega))} \,, \tag{2.22}$$

where $\overline{\lim\limits_{n\to\infty}} \, \varepsilon(n,\lambda) \xrightarrow[\lambda\to\infty]{} 0$, which also reads, for n large enough,

$$||q_n^{(1)}||_{L^5(I_n^{(1)},L^{10}(\Omega))} \leq \varepsilon(n,\lambda) + 2||P^{(1)}||_{L^5(]-\infty,-\lambda],L^{10}(\mathbb{R}^3))} \tag{2.23}$$

with $P^{(1)}$ defined as in section IV.1, equation (1.14). So for λ large enough,

$$\overline{\lim_{n\to\infty}} \, ||q_n^{(1)}||_{L^5(I_n^{(1)},L^{10}(\Omega))} \leq \delta' - \delta \,. \tag{2.24}$$

The case of $I_n^{(3)}$ is identical, replacing everywhere $p_n^{(1)}$ by $\tilde{p}_n^{(1)}$. Finally in the case of $I_n^{(2)}$, we write, according to section IV.1,

$$||q_n^{(1)}||_{L^5(I_n^{(2)},L^{10}(\Omega))} \leq \varepsilon(n,\lambda) + 2||Q^{(1),-}||_{L^5([-\lambda,\lambda],L^{10}(\mathbb{R}^3))} \,, \tag{2.25}$$

where $Q^{(1),-}$ is defined as in section IV.1, equation (1.27). Then we just need to decompose $[-\lambda, \lambda']$ into a finite number of intervals $I_\lambda^{(i)}$ on which the $L^5(I_\lambda^{(i)}, L^{10}(\mathbb{R}^3))$ norm of $Q^{(1),-}$ is small enough, which yields the result.

The cases $\frac{t_n^{(1)}}{h_n^{(1)}} \xrightarrow[n\to\infty]{} -\infty$ or 0 are similar and left to the reader. So the case $\ell = 1$ is proved. The induction is straightforward : if Γ_n satisfies

$$\overline{\lim_{n\to\infty}} \, ||\Gamma_n||_{L^5(I_n^{(i)}, L^{10}(\mathbb{R}^3))} \leq \frac{\delta' + \delta}{2}, \tag{2.26}$$

then as above one can find two intervals on which the $L_t^5 L_x^{10}$ norm of $q_n^{(\ell+1)}$ is smaller than $\frac{\delta'-\delta}{2}$, and decomposing $[0, T]$ using the intersection of all those intervals yields the result. ■

The end of the proof of theorem 3 is similar to the case of the whole space : we refer to [BG1] for details (lemma IV.4 is slightly different to lemma 4.4 of [BG1] but yields the result in the same way : see the work of S. Keraani in [K], in the context of Schrödinger equations, for instance). ■

Appendix

Strichartz estimates and Lipschitz bounds for the nonlinear evolution group

In this appendix, we prove corollary 1 and corollary 2.

Proof of corollary 1.

Let $T > 0$, $M > 0$. We intend to prove that, for every solution of

$$\Box u + |u|^4 u = 0 \text{ in } \mathbb{R} \times \Omega\,, \quad u|_{\mathbb{R}\times\partial\Omega} = 0 \tag{A.1}$$

such that $E(u) \leq M$, the following Strichartz estimate holds

$$||u||_{L^5([0,T],L^{10}(\Omega))} \leq A(M)\,. \tag{A.2}$$

In a first step, let us assume moreover that $(u(0), \partial_t u(0))$ is supported in the closed ball B_R of radius R centered at 0, and let us prove (A.2) with a right hand side $A(M,R)$. By contradiction, there would exist a sequence (u_n) such that $E(u_n) \leq M$, $\mathrm{supp}(u_n(0), \partial_t u_n(0)) \subset B_R$, and

$$||u_n||_{L^5([0,T],L^{10}(\Omega))} \xrightarrow[n\to\infty]{} +\infty\,.$$

By theorem 3 we can decompose a subsequence of (u_n) as a sum of a fixed solution u, almost orthogonal concentrating waves $q_n^{(j)}$, and of a remainder term r_n such that, say,

$$\overline{\lim_{n\to\infty}}\, ||r_n||_{L^5([0,T],L^{10})} \leq 1\,.$$

Calculating the $L^5_T L^{10}$ norm of this subsequence as in lemma IV.2, we get a contradiction.

Now let us prove (A.2) in its general context. Let $R > 0$ such that

$$\Omega^c \subset B_{R_0} \text{ with } R_0 < R - 2T\,. \tag{A.3}$$

Let $\chi_0 \in C_0^\infty(\mathbb{R}^3)$ such that $\chi_0 = 1$ near B_{R_0}, and $\mathrm{supp}(\chi_0) \subset \overset{\circ}{B}_{R-2T}$. Moreover, let $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi = 1$ near B_R.

For every solution u of (A.1) such that $E(u) \leq M$, denote by u_{in} the solution of (A.1) defined by

$$(u_{in}, \partial_t u_{in})|_{t=0} = \chi(u, \partial_t u)|_{t=0} \tag{A.4}$$

and by u_0 the solution of

$$\Box u_0 + |u_0|^4 u_0 = 0 \text{ in } \mathbb{R}_t \times \mathbb{R}^3 \tag{A.5}$$

defined by

$$(u_0, \partial_t u_0)|_{t=0} = \chi(1-\chi_0)(u, \partial_t u)|_{t=0}\,. \tag{A.6}$$

By the first step of the proof, we have

$$E(u_{in}) \leq A_1(M)\,. \tag{A.7}$$

Moreover, by corollary 2 of [BG1],

$$E_f(u_0) \leq A_2(M)\,, \tag{A.8}$$

where E_f is the energy for the free nonlinear problem (A.5). By the principle of finite propagation speed, we have $u_{in} = u_0$ for $t \in [0,T]$, x outside B_{R-T}. Now let r be the solution of

$$\Box r + |u_0 + r|^4 (u_0 + r) - |u_0|^4 u_0 = 0 \text{ in } \mathbb{R}_t \times \mathbb{R}^3\,, \tag{A.9}$$

defined by

$$(r, \partial_t r)|_{t=0} = (1-\chi)(u, \partial_t u)|_{t=0}\,. \tag{A.10}$$

By finite propagation speed, if $t \in [0,T]$, $r(t)$ is supported outside B_{R-T}, thus r is also solution of (A.9) with u_0 replaced by u_{in}, and with the boundary condition $r|_{\mathbb{R}\times\partial\Omega} = 0$. As a consequence of the uniqueness for the Cauchy problem for (A.1), we conclude, in view of (A.4), (A.10),

$$u = u_{in} + r = u_{in} + u_0 + r - u_0\,,$$

and (A.2) follows from (A.7), (A.8) and

$$E_f(u_0 + r) \leq A_3(M) \tag{A.11}$$

by the same argument as for (A.8). ■

Proof of corollary 2.

Let us state again corollary 2 in a more precise way.

Proposition A.1. *For every $T > 0$, consider the nonlinear map*

$$\begin{aligned} F:\quad & E(\Omega) \longrightarrow C([0,T], E(\Omega)) \\ & (\varphi, \psi) \longmapsto (u, \partial_t u) \end{aligned}$$

where u is the solution of (A.1) such that

$$(u, \partial_t u)|_{t=0} = (\varphi, \psi)\,.$$

Then F is a Lipschitz map on bounded subsets of $E(\Omega)$.

Proof. The strategy is to compute the (real) tangent map of F at every point of $E(\Omega)$, and to show, using corollary 1, that this map is bounded on bounded subsets of $E(\Omega)$. This can be done through the following lemma

Lemma A.2. *Let $q_1, q_2 \in L^1([0,T], L^3(\Omega)) = L^1_T L^3$. Then, for every $(\dot\varphi, \dot\psi) \in E(\Omega)$, the equation*

$$\begin{cases} \Box v + q_1 v + q_2 \overline{v} = 0 \ \text{ in } \ \mathbb{R}_t \times \Omega\,, \quad v|_{\mathbb{R}_t\times\Omega} = 0 \\ (v, \partial_t v)|_{t=0} = (\dot\varphi, \dot\psi) \end{cases} \tag{A.12}$$

has a unique solution v such that $(v, \partial_t v) \in C([0,T], E(\Omega))$. Moreover we have the estimate

$$\sup_{0\le t\le T} ||(v,\partial_t v)(t)||_E \le e^{C[||q_1||_{L^1_T L^3}+||q_2||_{L^1_T L^3}]} \times ||(\dot\varphi,\dot\psi)||_E\,. \tag{A.13}$$

PROOF. Given $w = w(t,x)$ such that $(w, \partial_t w) \in C([0,T], E(\Omega))$, the Sobolev embedding implies $q_1 w + q_2 \overline{w} \in L^1([0,T], L^2(\Omega))$. Therefore on can define the solution $\tilde w$ to

$$\Box \tilde w + q_1 w + q_2 \overline{w} = 0 \ \text{ in } \mathbb{R}_t \times \Omega\,, \quad (\tilde w, \partial_t \tilde w)|_{t=0} = (\dot\varphi, \dot\psi) \tag{A.14}$$

with $(\tilde w, \partial_t \tilde w) \in C([0,T], E(\Omega))$.

Let $k > 0$, to be fixed further on ; we set

$$N(t,\tilde w) = ||(\tilde w, \partial_t \tilde w)(t)||^2_{E(\Omega)}\ e^{-k\int_0^t (||q_1(s)||_{L^3}+||q_2(s)||_{L^3})ds}\,.$$

Using (A.14), we have

$$\begin{aligned}\frac{d}{dt}N(t,\tilde w) &= -k\big(||q_1(t)||_{L^3}+||q_2(t)||_{L^3}\big)\,N(t,\tilde w)\\ &\quad -2\,\mathrm{Re}\,\big(q_1(t)\,w(t)+q_2(t)\,\overline{w(t)}, \partial_t\tilde w(t)\big)\,e^{-k\int_0^t(||q_1||_{L^3}+||q_2||_{L^3})ds}\\ &\le \big(||q_1(t)||_{L^3}+||q_2(t)||_{L^3}\big)\big[-kN(t,\tilde w)+CN(t,w)^{1/2}\,N(t,\tilde w)^{1/2}\big]\end{aligned} \tag{A.15}$$

by means of the Sobolev inequality. Using $C\,ab \le k\,a^2 + \frac{C^2}{4k}\,b^2$, we conclude

$$\sup_{0\le t\le T} N(t,\tilde w) \le ||(\dot\varphi,\dot\psi)||_{E(\Omega)} + \frac{C^2}{4k}\big(||q_1||_{L^1_T L^3}+||q_2||_{L^1_T L^3}\big)\sup_{0\le t\le T} N(t,w)\,. \tag{A.16}$$

Choosing $k = \frac{C^2}{2}||q||_{L^1([0,T],L^3)}$ we infer from (A.16) with $\dot\varphi = \dot\psi = 0$, that, for every $(\dot\varphi,\dot\psi)$, the affine map $w \mapsto \tilde w$ is a contraction for the norm $||w|| = \sup\limits_{0\le t\le T} N(t,w)^{1/2}$. Therefore, for every $(\dot\varphi,\dot\psi)$, a unique fixed point v exists. Coming back to (A.15) which $k = C$, we obtain (A.13). ∎

Given $(\varphi,\psi) \in E$, $(\dot\varphi,\dot\psi) \in E$, we denote by u the solution to (A.1) and by v the solution to (A.12) with $q_1 = 3|u|^4$, $q_2 = 2\,|u|^2\,\bar u^2$. Observe that

$$\frac{d}{d\varepsilon}|u+\varepsilon v|^4\,(u+\varepsilon v)|_{\varepsilon=0} = q_1 v + q_2\overline{v}\,,$$

and that $q_1, q_2 \in L^1([0,T], L^3(\Omega))$ since $u \in L^4([0,T], L^{12}(\Omega))$. We claim

Lemma A.3. $\frac{d}{d\varepsilon}F((\varphi,\psi)+\varepsilon(\dot\varphi,\dot\psi))|_{\varepsilon=0} = (v, \partial_t v)$.

PROOF. Let u_ε be the solution to (A.1) with data $(\varphi_\varepsilon,\psi_\varepsilon) = (\varphi,\psi)+\varepsilon(\dot\varphi,\dot\psi)$. Set

$$u_\varepsilon = u + \varepsilon v + \varepsilon\, r_\varepsilon\,..$$

We get, in view of (A.1),

$$\Box r_\varepsilon + q_1\,r_\varepsilon + q_2\,\overline{r}_\varepsilon = f_\varepsilon\,, \quad (r_\varepsilon, \partial_t\,r_\varepsilon)|_{t=0} = 0\,,$$

$$|f_\varepsilon| \le C\big(|u|^3|\varepsilon|\,|v+r_\varepsilon|^2 + \varepsilon^4\,|v+r_\varepsilon|^5\big)\,.$$

For every $\tau \le T$, we set

$$M_\varepsilon(\tau) = ||r_\varepsilon||_{L^5([0,T],L^{10})} + \sup_{0\le t\le\tau} ||(r_\varepsilon,\partial_t r_\varepsilon)(t)||_E\,.$$

Then, by the Strichartz estimate (A.2) and the energy inequality

$$\begin{aligned} M_\varepsilon(\tau) \le \tilde{C}\Big[&\big(||q_1||_{L^1([0,T],L^3)} + ||q_2||_{L^2([0,T],L^3)}\big)\,M_\varepsilon(\tau) \\ &+|\varepsilon|(M_\varepsilon(\tau)^2 + M_\varepsilon(\tau)^5)\Big]\,. \end{aligned} \tag{A.17}$$

By a standard bootstrap argument, we conclude from (A.17) $M_\varepsilon(\tau) = O(\varepsilon)$ for $|\tau|$ small enough. Applying the same inequalities successively on small time intervals, we finally conclude that $M_\varepsilon(T) = O(\varepsilon)$, hence goes to 0.

This yields lemma A.3. ■

Combining lemma A.2 and lemma A.3, we finally observe that

$$\Big|\Big|\frac{d}{d\varepsilon}F((\varphi,\psi)+\varepsilon(\dot\varphi,\dot\psi))|_{\varepsilon=0}\Big|\Big|_E \le e^{C||u||^4_{L^4([0,T],L^{12})}}\,||(\dot\varphi,\dot\psi)||_E\,.$$

On the other hand, if (φ,ψ) varies in a bounded subset of $E(\Omega)$, inequality (A.2) implies that $||u||_{L^4([0,T],L^{12})}$ remains bounded. We thus complete the proof by a classical argument from differential calculus on bounded convex subsets of $E(\Omega)$. ■

Bibliography

[BG1] H. Bahouri and P. Gérard : *High frequency approximation of solutions to critical nonlinear wave equations,* American Journal of Mathematics, **121**, 131–175, (1999).

[BG2] H. Bahouri and P. Gérard : *Concentration effects in critical nonlinear wave equations and scattering theory,* Geometrical Optics and Related Topics (F. Colombini and N. Lerner eds), Progress in Nonlinear Differential Equation and Applications, vol. **32**, Birkhäuser, Boston, 17–30, (1997).

[BS] H. Bahouri and J. Shatah : *Global estimate for the critical semilinear wave equation,* Annales de l'Institut Henri Poincaré, Analyse Non Linéaire, **15**, **6**, 783–789, (1998).

[Bu] N. Burq : *Contrôlabilité exacte des ondes dans des ouverts peu réguliers,* Asymptotic Analysis, **14**, 157–191, (1997).

[BuG] N. Burq and P. Gérard : *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes,* C.R. Acad. Sci. Paris, **325**, Série 1, 749–752, (1997).

[BuL] N. Burq and G. Lebeau : *Mesures de défaut de compacité, application au système de Lamé,* Preprint Ecole Polytechnique, Juin 1999, and article to appear in Annales Scientifiques de l'École Normale Supérieure.

[G1] P. Gérard : *Oscillations and concentration effects in semilinear dispersion wave equations,* Journal of Functional Analysis, **141**, 60–98, (1996).

[G2] P. Gérard : *Description du défaut de compacité de l'injection de Sobolev,* ESAIM Contrôle Optimal et Calcul des Variations, **3**, 213–233, (1998), (electronic version : http ://www.emath.fr/cocv/).

[GL] P. Gérard and E. Leichtnam : *Ergodic properties of eigenfunctions for the Dirichlet problem,* Duke Mathematical Journal, **71**, 559–607, (1993).

[GMMP] P. Gérard, P. Markowitch, N. Mauser and F. Poupaud : *Homogenization limits and Wigner transforms,* Communications on Pure and Applied Mathematics, **50**, 323–379, (1997).

[GMO] P. Gérard, Y. Meyer and F. Oru : *Inégalités de Sobolev précisées,* Séminaire EDP, Ecole Polytechnique, Décembre 1996.

[GT] D. Gilbarg and N.S. Trudinger : *Elliptic Partial Differential Equations of Second Order,* Springer Verlag, Heidelberg, (1983).

[GV] J. Ginibre and G. Velo : *Generalized Strichartz inequalities for the wave equation,* Journal of Functional Analysis, **133**, **1**, 50–68, (1995).

[Gr1] M. Grillakis : *Regularity and asymptotic behavior of the wave equation with a critical nonlinearity,* Annals of Mathematics, **132**, 485–509, (1990).

[Gr2] M. Grillakis : *Regularity for the wave equation with a critical nonlinearity,* Communications in Pure and Applied Mathematics, **45**, 749–774, (1992).

[H] L. Hörmander : *The Analysis of Linear Partial Differential Operators,* Vol. I, Springer-Verlag, New York, (1983).

[K] S. Keraani : *On the defect of compactness for the Strichartz estimates of the Schrödinger equations,* Preprint, Université de Paris-Sud, (1999).

[Le1] G. Lebeau : *Equation des ondes amorties,* in Algebraic and Geometric Methods in Mathematical Physics, A. Boutet de Monvel and V. Marchenko editors, 73–109, Kluwer Academic, (1996).

[Le2] G. Lebeau : *Personal Communication and paper in preparation.*

[L1] P.-L. Lions : *The concentration-compactness principle in the calculus of variations. The locally compact case, I,* Annales de l'Institut Henri Poincaré, **1**, 109–145, (1984).

[L2] P.-L. Lions : *The concentration-compactness principle in the calculus of variations. The limit case, II,* Revista Matematica Iberoamericana, **1**, 145–201, (1985).

[LP] P.-L. Lions and T. Paul : *Sur la mesure de Wigner,* Revista Matematica Iberoamericana, **9**, 553–618, (1993).

[M] L. Miller : *Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary,* Preprint Ecole Polytechnique, Novembre 1999, to appear in Journal de Mathématiques Pures et Appliquées.

[MS] G. Métivier and S. Schochet : *Trilinear resonant interactions of semilinear hyperbolic waves,* Duke Mathematical Journal, **95**, **2**, 241–304, (1998).

[N] K. Nakanishi : *Scattering theory for nonlinear Klein-Gordon equation with Sobolev critical power,* International Mathematics Research Notices, **1**, 31–60, (1999).

[R] J. Rauch : *The u^5-Klein-Gordon equation,* Nonlinear PDE's and applications, vol. I (H. Brézis and J.-L. Lions, eds), Pitman Research Notes Math. Series, vol. 53, Longman, Harlow, 335–364, (1981).

[ShSt] J. Shatah and M. Struwe : *Well-posedness in the energy space for semilinear wave equations with critical growth,* International Mathematics Research Notices, **7**, 303–309, (1994).

[SmSo1] H. Smith and C. Sogge : *On the critical semilinear wave equation outside convex obstacles,* Journal of the American Mathematical Society, **8**, **4**, 879–916, (1995).

[SmSo2] H. Smith and C. Sogge : *Global Strichartz estimates for nontrapping perturbation of the Laplacian,* Preprint.

[Str] W. Strauss : *Decay and asymptotics for $\Box u = F(u)$,* Journal of Functional Analysis, **2**, 409–457, (1968).

[S] M. Struwe : *Globally regular solution to the u^5 Klein-Gordon equation,* Ann. Scuola Normale Pisa, **15**, 495–513, (1988).