Applications of Schochet's Methods to Parabolic Equations

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Abstract. *Methods used by S. Schochet in [34] enable one to find a lower bound for the life span of solutions of hyperbolic PDEs with a small parameter. We prove a similar theorem for such equations where a diffusion term has been added, with the minimal assumption on the Sobolev regularity of the initial data ($H^{\frac{d}{2}-1}$ in the d–dimensional torus). When the data is smooth, and under a "small divisor" assumption on the perturbation, the first term of an asymptotic expansion of the solution is computed. Those results are then applied to prove global existence theorems, for arbitrary initial data, in the case of the primitive system of the quasigeostrophic equations, followed by the rotating fluid equations. We finally prove a more precise existence theorem for the latter, using anisotropic Sobolev and Besov spaces.*

Résumé. *Des méthodes utilisées par S. Schochet dans [34] permettent d'obtenir une borne inférieure pour le temps d'existence des solutions d'équations aux dérivées partielles hyperboliques avec petit paramètre. On démontre un théorème similaire pour de telles équations, où un terme de diffusion a été ajouté, avec l'hypothèse minimale de régularité Sobolev sur la donnée initiale ($H^{\frac{d}{2}-1}$ dans le tore de dimension d). Quand la donnée est régulière, et sous une hypothèse de type "petit diviseurs" sur la perturbation, le premier terme d'un développement asymptotique de la solution est calculé. Ces résultats sont ensuite utilisés pour démontrer des théorèmes d'existence globale, pour des données quelconques, dans le cas du système primitif des équations quasigéostrophiques, puis pour les équations des fluides tournants. On démontre enfin un théorème d'existence plus précis pour ce dernier système, en utilisant des espaces de Sobolev et de Besov anisotropes.*

1 Introduction and statement of the results

This study is centered around the following system:

$$(\mathcal{S}^\varepsilon) \quad \begin{cases} \partial_t v^\varepsilon + q(v^\varepsilon, v^\varepsilon) + a_2(D)v^\varepsilon + \dfrac{L(v^\varepsilon)}{\varepsilon} & = & f \quad \text{in} \quad \mathcal{T}^d \\ v^\varepsilon_{|t=0} & = & v_0 \end{cases}$$

where ε is a small, strictly positive parameter. Throughout this article, $\mathcal{T}^d$ will be a periodic box in $\mathbf{R}^d$, where $d \geq 2$ is the dimension. Each side of $\mathcal{T}^d$, in the direction x_i, $1 \leq i \leq d$, has a size equal to $2\pi a_i$, where the a_i are strictly positive real numbers. The vector field v^ε has d' components, and is defined on $\mathbf{R}^+ \times \mathcal{T}^d$. The operator L, which shall be studied in some detail in the next section, is a skew–symmetric operator, which can be written as

$$L(v) = \mathcal{F}^{-1} \sum_{a=1}^{d'} i\omega^a(n) \left(\mathcal{F}v(n), e^a(n)\right) e^a(n), \tag{1.1}$$

where $\mathcal{F}v(n)$ represents the discrete Fourier transform of v, $\omega^a(n)$ are real numbers, and where $(e^a(n))_{1 \leq a \leq d'}$ is a set of orthonormal vectors for all n. To simplify, we will suppose in the following that L is of order 0, *i.e.* , that there exists a constant C such that

$$\forall a, \, \forall n, \quad \omega^a(n) \neq 0 \Rightarrow |\omega^a(n)|^{-1} \leq C. \tag{1.2}$$

We will indicate how to modify the theorems and their proofs when there is no such assumption. The operator $a_2(D)$ is second order elliptic: for all u, $(a_2(D)u)_i = a_{ij}(D)u^j$, where $a_{ij}(D)$ is a Fourier multiplicator of order 2, and where u^j is the j–th component of u, with $j \in [1, .., d']$. We have used the Einstein convention on the summation over repeated indexes: here and throughout the paper, we have omitted the sum on j. The notation (u, v) represents the usual scalar product, which we shall also sometimes write $u \cdot v$. Furthermore, the bilinear form q is defined by

$$q(u, v) = A_{i,j}(D)(u^i v^j), \tag{1.3}$$

where $A_{i,j}(D)$ is a symmetric Fourier multiplicator of order 1. Finally f is an exterior forcing term, on which we shall make some regularity assumptions as we go along.

We shall see in section 2 that $(\mathcal{S}^\varepsilon)$ has a limit system, in a sense which will be made clear later, when ε goes to 0. This system can be written in the form

$$(\mathcal{S}_0) \quad \begin{cases} \partial_t u + \mathcal{Q}(u, u) + \mathcal{A}_2(D)u & = & \bar{f} \\ u_{|t=0} & = & v_0 \end{cases}$$

where $\mathcal{Q}$, $\bar{f}$, and $\mathcal{A}_2(D)$ are defined by $\mathcal{Q}(u, v) = \lim\limits_{\varepsilon \to 0} \mathcal{L}(-\frac{t}{\varepsilon}) q(\mathcal{L}(\frac{t}{\varepsilon})u, \mathcal{L}(\frac{t}{\varepsilon})v)$, $\bar{f} = \lim\limits_{\varepsilon \to 0} \mathcal{L}(-\frac{t}{\varepsilon})f$, and $\mathcal{A}_2(D) = \lim\limits_{\varepsilon \to 0} \mathcal{L}(-\frac{t}{\varepsilon}) a_2(D) \mathcal{L}(\frac{t}{\varepsilon})$ where $\mathcal{L}$ is the semi–group generated by L: we will come back to that in section 2.2.

Remark. We will see in section 2.2 that necessarily $\mathcal{A}_2(D)$ is elliptic.

Systems of that type, with or without the $a_2(D)$ term, have been studied by a number of authors (among others, we can refer to [20], [23], [24], [25], or [32] for instance). It is in particular well known that the system $(\mathcal{S}^\varepsilon)$ has solutions locally in time, as stated by the following theorem.

Theorem 0 *If $v_0 \in H^s(\mathcal{T}^d)$, with $s \geq \frac{d}{2} - 1$, if f is in $C^0(\mathbf{R}^+, H^{s-2}) \cap L^2(\mathbf{R}^+, H^{s-1})$, then there exists a time $T > 0$, independent of ε, and a unique strong solution v^ε of $(\mathcal{S}^\varepsilon)$ in the space $C^0([0,T], H^s(\mathcal{T}^d)) \cap L^2([0,T], H^{s+1}(\mathcal{T}^d))$.*

We are interested here in finding a lower bound for the life span of v^ε, using the life span of the solutions of the limit system $(\mathcal{S}_0)$. We will prove the following theorem.

Theorem 1 *If $v_0 \in H^s(\mathcal{T}^d)$, with $s \geq \frac{d}{2} - 1$, if f is in $C^0(\mathbf{R}^+, H^{s-2}) \cap L^2(\mathbf{R}^+, H^{s-1})$, with $\partial_t f \in L^2(\mathbf{R}^+, H^{s-3})$, and if u, solution of the system $(\mathcal{S}_0)$, satisfies*

$$u \in C^0([0,T], H^s(\mathcal{T}^d)) \cap L^2([0,T], H^{s+1}(\mathcal{T}^d)),$$

then for ε small enough, the associate solution v^ε of $(\mathcal{S}^\varepsilon)$ is also defined on $[0,T]$, and

$$v^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})u = o(1) \quad in \quad C^0([0,T], H^s(\mathcal{T}^d)) \cap L^2([0,T], H^{s+1}(\mathcal{T}^d)).$$

To prove that theorem, we shall use the method introduced by S. Schochet in the case of symmetric hyperbolic systems, with no $a_2(D)$ term. The result he obtains (Theorem 2.3 in [34]) is similar to the theorem above, except for the fact that v^ε is only approximated by $\mathcal{L}(\frac{t}{\varepsilon})u$ in $C^0([0,T], H^{s-1})$ for an initial data in H^s with $s \geq \frac{d}{2} + 3$. Thus $a_2(D)$ will enable us to gain some regularity, since in our case v^ε remains in $C^0([0,T], H^s)$; we also obtain here the minimal assumption on the Sobolev regularity of the initial data.

However it must be noted that Theorem 1 gives no information on the rate of convergence of $v^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})u$ to zero. Such a rate of convergence, in $O(\varepsilon)$, is obtained by A. Babin, A. Mahalov and B. Nicolaenko in [2], in special cases we will mention more precisely later in this introduction. In [2], that is strongly linked to the presence of "small divisor" estimates on L, which are defined below.

Definition 1.1 (Small divisor estimates) *We will say that L satisfies a small divisor estimate of order $p \geq 2$ and of degree α if there exists a constant $C_{p,\alpha}$ such that for all $(k_1, .., k_p)$ in $\mathbf{Z}^{pd}$, for all $(b_1, .., b_{p+1})$ in $\{1, .., d'\}^{p+1}$, we have*

$$\sum_{i=1}^{p} \omega^{b_i}(k_i) \neq \omega^{b_{p+1}}(\sum_{i=1}^{p} k_i) \quad \Rightarrow \quad |\sum_{i=1}^{p} \omega^{b_i}(k_i) - \omega^{b_{p+1}}(\sum_{i=1}^{p} k_i)|^{-1} \leq C_{p,\alpha} \prod_{i=1}^{p} (1 + |k_i|)^\alpha,$$

where as previously the $\omega^{b_i}(k_i)$ are the eigenvalues of $\mathcal{F}L(k_i)$. Moreover, L satisfies a small divisor estimate of order 1 and of degree α_1 if there exists a constant C_1 such that for all k and all (b_1, b_2) satisfying $\omega^{b_1}(k) \neq \omega^{b_2}(k)$, we have $\left|\omega^{b_1}(k) - \omega^{b_2}(k)\right|^{-1} \leq C_1 (1 + |k|)^{\alpha_1}$.

Remark. Such an estimate at order p can be understood as a measure of the $(p+1)$–wave interactions created by the perturbation. If one has some precise control over those interactions, then one can also control the speed of convergence of $v^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})u$ to zero, if one also assumes enough regularity on the initial data.

Theorem 2 *Suppose L satisfies first and second order small divisor estimates of degree α_1 and α_2 respectively, with $\alpha_i \geq 0$, and let $s > \frac{d}{2}$. We define $\alpha_{12} = \max(1+\alpha_1, \alpha_2)$. Suppose that*

$$\forall\, j \in \{0,1\}, \quad \partial_t^j f \in C^0([0,T], H^{s-2-2j+\alpha_{12}}) \cap L^2([0,T], H^{s-1-2j+\alpha_{12}}),$$

and that $\partial_t^2 f \in L^2([0,T], H^{s-5+\alpha_{12}})$. Then if $v_0 \in H^{s+\alpha_{12}}$ and if u, solution of $(\mathcal{S}_0)$, satisfies

$$u \in C^0([0,T], H^{s+\alpha_{12}}(\mathcal{T}^d)) \cap L^2([0,T], H^{s+1+\alpha_{12}}(\mathcal{T}^d)),$$

then for ε small enough, the solution v^ε of $(\mathcal{S}^\varepsilon)$ is such that $u^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})v^\varepsilon$ satisfies

$$u^\varepsilon = u + \varepsilon h - \varepsilon \tilde{R}^\varepsilon_{osc}(u) + o(\varepsilon) \quad in \quad C^0([0,T], H^{\frac{d}{2}-1}(\mathcal{T}^d)) \cap L^2([0,T], H^{\frac{d}{2}}(\mathcal{T}^d)),$$

where h is the solution in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$ of the linearized limit system

$$(\mathcal{S}_1) \qquad \left\{ \begin{array}{rcl} \partial_t h + 2\mathcal{Q}(u,h) + \mathcal{A}_2(D)h &=& R(u) \\ h_{|t=0} &=& \tilde{R}^\varepsilon_{osc}(u)_{|t=0}. \end{array} \right.$$

Both functions $\tilde{R}^\varepsilon_{osc}(u)$ and $R(u)$ can be computed explicitly, as explained below.

Remarks. If $[a_2(D), L] = 0$, then we will see that $u \in C^0([0,T], H^{s+\alpha_2}) \cap L^2([0,T], H^{s+1+\alpha_2})$ is a sufficient condition to have the result. Moreover, we will see that the assumption that L is of order zero, written as (1.2), is not strictly necessary: one can suppose that L is of order r, *i.e.*, that $\omega^a(n) \sim (1+|n|)^r$, if one also supposes more regularity on f.

Finally, we will compute explicitly the functions $R(u)$ and $\tilde{R}^\varepsilon_{osc}(u)$. We will show that

$$\begin{aligned} \tilde{R}^\varepsilon_{osc}(u) \;&=\; -\mathcal{F}^{-1} \sum_{\omega^a(n)\neq 0} \frac{i\, e^{i\frac{t}{\varepsilon}\omega^a(n)}}{\omega^a(n)} f^a(n) \;-\; \mathcal{F}^{-1} \sum_{\omega_n^{a,b}\neq 0} \frac{i\, e^{i\frac{t}{\varepsilon}\omega_n^{a,b}}}{\omega_n^{a,b}} \Big(a_2 u^b(n)\Big)^a (n) \\ &+\; \mathcal{F}^{-1} \sum_{\omega_{k,n}^{a,b,c}\neq 0} \frac{i\, e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}}}{\omega_{k,n}^{a,b,c}} \Big(A_{j,j'} u^{a,j}(k) u^{b,j'}(n-k)\Big)^c (n), \end{aligned} \tag{1.4}$$

where we have written

$$\omega_n^{a,b} = \omega^a(n) - \omega^b(n), \quad \omega_{k,n}^{a,b,c} = \omega^a(k) + \omega^b(n-k) - \omega^c(n),$$

$$\omega_{k',k,n}^{a,b,c,d} = \omega^a(k') + \omega^b(k-k') + \omega^c(n-k) - \omega^d(n).$$

Moreover, we have noted

$$u^a(k) = (\mathcal{F}u(k), e^a(k))\, e^a(k).$$

The function $R(u)$ is more complicated to write (its expression is given page 26). It consists in a sum of linear, quadratic and cubic terms in u, depending also on f; that function $R(u)$ is due to the fact that in a term such as $\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc}(u), u)$, which appears when one considers the equation satisfied by the function $\varepsilon^{-1}(u^\varepsilon - u) + \tilde{R}^\varepsilon_{osc}(u)$, the oscillation phases brought by $\tilde{R}^\varepsilon_{osc}(u)$ and $\mathcal{Q}^\varepsilon$ can interfere and cancel out. That leads to the formation of additional "slow waves", at the first order of the expansion of u^ε, and that accounts for the function h, solution of the system $(\mathcal{S}_1)$ given in the theorem. The other part of the first order term in the expansion of u^ε is made of a "fast" term, which is the oscillating function $\tilde{R}^\varepsilon_{osc}(u)$.

Let us note that in [18], a similar result as Theorem 2 is proved in the hyperbolic case, *i.e.*, when there is no elliptic $a_2(D)$ term.

In [26], H.-O. Kreiss, J. Lorenz, and M.J. Naughton compute an asymptotic expansion of the type of that given in Theorem 2, only under an assumption on the initial data of the "bounded derivative principle" type: in the case of the compressible Navier–Stokes equations, if the solution has two time derivatives bounded independently of ε at $t = 0$, then the solution v^ε satisfies $v^\varepsilon = V + \varepsilon^2(V_1 + v_1) + O(\varepsilon^4)$, where V is the solution of a limiting equation, V_1 is the solution of the linearized incompressible equation, and v_1 is highly oscillatory in time. The proof does not require any filtering of the equation, due to the strong assumptions made on the initial data.

In [7], G. Browning and H.-O. Kreiss also show how to prepare the initial data so that the solutions and their derivatives can be bounded independently of ε.

Finally, let us note that it is proved in Appendix C that a large class of perturbations L satisfy, for almost all periodic boxes, a second order small divisor estimate at a certain degree. In particular, that will hold in the two examples we will study in detail in this article, namely the rotating fluids and the primitive equations, both presented below in this section. Let us also remark that in those cases, precise small divisor estimates are computed in [3], and we will also see in section 5 that in some special cases the small divisor estimates are trivial.

The proofs of Theorems 1 and 2 are the objects of sections 3 and 4 respectively. They both use a general result about equations where the right hand side oscillates fast with time, Lemma 3.1 proved in Appendix A. Those highly oscillating functions are the object of the coming definition.

Definition 1.2 ((p,σ)-oscillating functions) *Let $T > 0$, $p \geq 1$ and $\sigma \in \mathbf{R}$ be fixed. We write $\vec{k_q} = (k_1, .., k_q)$, where $k_i \in \mathbf{Z}^d$, and we will call $|\vec{k_q}| = \max_{1 \leq i \leq q} |k_i|$. Then $R^\varepsilon_{osc}(t)$ will be said to be (p,σ)–oscillating if there exist real–valued functions $\beta_1, .., \beta_p$, r_0 and $(f_i)_{i \geq 0}$ such that R^ε_{osc} can be written in the following way, where $\mathbf{1}_X$ is the characteristic function of X:*

$$R^\varepsilon_{osc}(t) = \sum_{q \in \{1,..,p\}} R^\varepsilon_{q,osc}(t), \quad \textit{where} \quad \mathcal{F}R^\varepsilon_{1,osc}(t,n) = \mathbf{1}_{\{n \,|\, \beta_1(n) \neq 0\}} e^{-i\frac{t}{\varepsilon}\beta_1(n)} f_0(t,n),$$

$$\textit{and} \quad \forall q \geq 2, \quad \mathcal{F}R^\varepsilon_{q,osc}(t,n) = \sum_{\vec{k_q} \in K^n_q} e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})} r_0(n,\vec{k_q}) f_1(t,k_1)..f_q(t,k_q), \tag{1.5}$$

with $K_q^n = \left\{ \vec{k_q} \in \mathbf{Z}^{dq} \mid \sum_{i=1}^{q} k_i = n \text{ and } \beta_q(n, \vec{k_q}) \neq 0 \right\}$, *and where* r_0 *and* $(f_i)_{i \geq 0}$ *satisfy*

$$|r_0(n, \vec{k_q})| \leq C\,(1+|n|)^{\rho_0}\,(1+|k_1|)^{\rho_1} \,..\, (1+|k_q|)^{\rho_q}\,, \quad \textit{with} \quad \rho_i \geq 0 \;\; \textit{and} \;\; \rho_0 + \sigma + \frac{d}{2} > 0\,;$$

$$\mathcal{F}^{-1} f_0(t,\cdot) \quad \textit{is an element of} \quad C^0([0,T], H^\sigma) \cap L^2([0,T], H^{\sigma+1})\,;$$

$$\mathcal{F}^{-1} \partial_t f_0(t,\cdot) \quad \textit{is an element of} \quad L^2([0,T], H^{\sigma-1})\,;$$

$$\forall\, i \geq 1, \quad \mathcal{F}^{-1} f_i(t,\cdot) \quad \textit{is an element of} \quad C^0([0,T], H^{\sigma_i+\rho_i}) \cap L^2([0,T], H^{\sigma_i+1+\rho_i}),$$

where the $(\sigma_i)_{1 \leq i \leq q}$ *are such that the product* $\prod_{i \geq 1} \mathcal{F}^{-1}(1+|\cdot|)^{\rho_i} f_i(t,\cdot)$ *is an element of the space* $C^0([0,T], H^{\sigma+\rho_0}) \cap L^2([0,T], H^{\sigma+1+\rho_0})$*; and finally we suppose that there exists* $(\bar{\sigma}_i)_{i \geq 1}$ *such that* $\bar{\sigma}_i + \sum_{j \neq i} \sigma_j > 0$*, and* $\mathcal{F}^{-1} \partial_t f_i(t,\cdot)$ *is an element of* $L^2([0,T], H^{\bar{\sigma}_i+\rho_i})$.

Remarks. It is obvious that a (p,σ)–oscillating function is in particular bounded in the space $C^0([0,T], H^\sigma) \cap L^2([0,T], H^{\sigma+1})$. It can be noted that if R^ε_{osc} is a (p,σ)–oscillating function, then its first derivative in time, $\partial_t R^\varepsilon_{osc}$, is not bounded in ε, whereas $\varepsilon\, \partial_t R^\varepsilon_{osc}$ is. Finally let us note that we will see that under the assumptions of Theorem 2, the function $\tilde{R}^\varepsilon_{osc}(u)$ defined in (1.4) is a $(2, \frac{d}{2} - 1)$–oscillating function (see Lemma 4.1).

Those theorems will then be applied to the study of two particular systems which can be written in the form $(\mathcal{S}^\varepsilon)$, in three dimensions of space: first, in section 5, we shall consider the case of the primitive equations of the quasigeostrophic system. The unknown is a four component vector field $V^\varepsilon = (v^\varepsilon, T^\varepsilon)$, where v^ε represents the velocity, and is divergence free, and the scalar field T^ε is known as the potential temperature. According to the study of J.-Y. Chemin in [10], one can write

$$(PE^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t V^\varepsilon + v^\varepsilon \cdot \nabla V^\varepsilon - \mathcal{D} V^\varepsilon + \varepsilon^{-1} L V^\varepsilon & = & \varepsilon^{-1}(-\nabla \Phi^\varepsilon, 0) \quad \text{in} \quad \mathcal{T}^3 \\ V^\varepsilon & = & (v^\varepsilon, T^\varepsilon) \\ \operatorname{div} v^\varepsilon & = & 0 \quad \text{in} \quad \mathcal{T}^3 \\ V^\varepsilon_{|t=0} & = & V_0, \end{array} \right.$$

with $L = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}$. The operator $\mathcal{D}$ is defined by $\mathcal{D} V^\varepsilon = (\nu \Delta v^\varepsilon, \nu' \Delta T^\varepsilon)$, where ν and ν' are two strictly positive real numbers, representing the viscosities along v^ε and T^ε. Finally the parameter F^{-1} is a strictly positive constant representing the ratio of the Rossby and Froude numbers (see [14]).

The physical origin of such a system is discussed in [19] and [33] for instance. For the derivation of such a model, one can refer to the work of J.-L. Lions, R. Temam and S. Wang in [29]–[31]; in [6], T. Beale and A. Bourgeois study and validate such a system, when there are no viscous terms ($\nu = \nu' = 0$). The limit of that inviscid case is also studied by D. Iftimie in [21], in the whole space. The primitive equations are finally also studied by A. Babin, A. Mahalov, B. Nicolaenko, and Y. Zhou in [2], [3], [4] and [5]; one aspect of their study is the respective influence of the rotation and the "stratification" in the equations, depending essentially on the size of the parameter F. This will not be considered here, as F will be supposed to be a constant throughout our study.

Let us note that this system can be diagonalized, as in [3], [10] or [14] for instance, by setting $V^\varepsilon = V^\varepsilon_{QG} + V^\varepsilon_{osc}$, where V^ε_{QG} is defined by

$$V^\varepsilon_{QG} = (-\partial_2\Delta^{-1}\Omega^\varepsilon, \partial_1\Delta^{-1}\Omega^\varepsilon, 0, -F\partial_3\Delta^{-1}\Omega^\varepsilon), \tag{1.6}$$

and $\Omega^\varepsilon = \partial_1 v^{\varepsilon,2} - \partial_2 v^{\varepsilon,1} - F^{-1}\partial_3 T^\varepsilon$.

We will see in section 5 that the limit system $(\mathcal{S}_0)$ corresponding to the system (PE_ε) is

$$(QG) \quad \left\{ \begin{array}{rcl} \partial_t\Omega + v_{QG}\cdot\nabla\Omega + a_{QG}(D)\Omega & = & f_{pot} \\ \partial_t U_{osc} + 2\mathcal{Q}(V_{QG}, U_{osc}) + a_{osc}(D)U_{osc} & = & 0 \\ V_{|t=0} & = & V_0 \end{array} \right.$$

where $a_{QG}(D)$ and $a_{osc}(D)$ are second order elliptic operators, and where f_{pot} is an exterior force, and is obtained from f in the same way as the potential vorticity Ω^ε is obtained from V^ε above. We have written U for the limit of the filtered function $U^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})V^\varepsilon$. In [14], and in the inviscid case, P. Embid and A. Majda show that the quasigeostrophic equations (QG) on Ω are a valid limiting equation in the weak topology, for arbitrary initial data, of the primitive equations; they in particular study the resonances due to the penalization, and we will come back to those computations in section 5. Using that special diagonal form (QG) of the limit system, as well as both Theorems 1 and 2, we will prove the following result.

Theorem 3 *For almost all (a_1, a_2, a_3), the following result is true. Let V_0 be in $H^s(\mathcal{T}^3)$, with $s \geq 1$, and suppose that $f \in L^\infty(\mathbf{R}^+, H^{s-2}) \cap L^2(\mathbf{R}^+, H^{s-1})$, with $\partial_t f \in L^2(\mathbf{R}^+, H^{s-3})$. Then for ε small enough, (PE^ε) is globally well posed in $C^0(\mathbf{R}^+, H^s) \cap L^2(\mathbf{R}^+, H^{s+1})$. Furthermore, if U is the solution of (QG), then V^ε satisfies*

$$V^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})U = o(1) \quad in \quad C^0(\mathbf{R}^+, H^s) \cap L^2(\mathbf{R}^+, H^{s+1}).$$

Finally, for almost all $\mathcal{T}^3$, there exists $\sigma \geq \frac{3}{2}$ such that if $U_0 \in H^\sigma(\mathcal{T}^3)$, and if f is smooth enough, then the first order expansion of Theorem 2 holds.

Let us note that in [10], and in the case of the entire space $\mathbf{R}^3$, J.-Y. Chemin proves a global existence theorem, if the oscillating part of the initial data satisfies

$$\|V_{0,osc}\|_{H^{-1}} \leq \frac{C\nu^4}{\|V_0\|^3_{H^1}} \exp\left(-\frac{\|V_0\|_{H^1}\|V_0\|_{L^2}}{C\nu^2}\right).$$

The proof in [10] lies in a diagonalization along the quasigeostrophic and oscillating parts of the velocity and the temperature fields, without using S. Schochet's theorem. The use of the latter theorem enables us not to require any kind of smallness condition on the data.

In the second example we shall consider, the vector fields are tridimensional and divergence free, the bilinear term corresponds to the Navier–Stokes equations, $q(u,v) = P(u\cdot\nabla v)$ where P

is the L^2–orthogonal projector onto the space of non divergent vector fields, and $a_2(D)$ represents the diffusion operator $-\nu\Delta$, where the viscosity ν is a strictly positive real number. Then one chooses the operator L to be the Coriolis force $v \times B$, where B is a vertical vector of norm 1 in $\mathcal{T}^3$: $B = {}^t(0,0,1)$. That leads to the rotating fluid equations:

$$(RF^\varepsilon) \quad \begin{cases} \partial_t v^\varepsilon + P(v^\varepsilon \cdot \nabla v^\varepsilon) - \nu\Delta v^\varepsilon + P\dfrac{v^\varepsilon \times B}{\varepsilon} & = & f \quad \text{in} \quad \mathcal{T}^3 \\ \operatorname{div} v^\varepsilon & = & 0 \quad \text{in} \quad \mathcal{T}^3 \\ v^\varepsilon_{|t=0} & = & v_0 \,. \end{cases}$$

That system will be studied in section 6. We will show that the limit system $(\mathcal{S}_0)$ corresponding to (RF^ε) is composed of two systems: one is a bidimensional Navier–Stokes equation

$$(NS2D) \quad \begin{cases} \partial_t \bar{u} - \nu\Delta_2 \bar{u} + \bar{u} \cdot \nabla_2 \bar{u} & = & (-\nabla_2 \bar{p}, 0) + \bar{f} \\ \partial_3 \bar{u} & = & 0 \\ \nabla_2 \cdot \bar{u} & = & 0 \\ \bar{u}_{|t=0} & = & \bar{v}_0 \end{cases}$$

where $\bar{u}$ is the bidimensional part of the limit u, that is to say its vertical average, and where we have noted $\Delta_2 = \partial_1^2 + \partial_2^2$ and $\nabla_2 = (\partial_1, \partial_2)$. The other equation concerns the remainder u_{osc}, defined by $u_{osc} = u - \bar{u}$, and is a linear equation:

$$(L_{osc}) \quad \begin{cases} \partial_t u_{osc} - \nu\Delta u_{osc} + 2\mathcal{Q}(\bar{u}, u_{osc}) & = & 0 \\ \operatorname{div} u_{osc} & = & 0 \\ u_{osc|t=0} & = & v_{0,osc} \,. \end{cases}$$

As in the case of the primitive equations presented above, we will prove, using Theorems 1 and 2, the following result.

Theorem 4 *When a_1 and a_2 are fixed, then for almost all a_3, the following result is true: if $v_0 \in H^s(\mathcal{T}^3)$, with $s \geq \frac{1}{2}$, and if the force f is in $L^\infty(\mathbf{R}^+, H^{s-2}) \cap L^2(\mathbf{R}^+, H^{s-1})$, with $\partial_t f \in L^2(\mathbf{R}^+, H^{s-3})$, then for $\varepsilon > 0$ small enough, the system (RF^ε) associated with the initial data v_0 is globally well posed in $C^0(\mathbf{R}^+, H^s(\mathcal{T}^3)) \cap L^2(\mathbf{R}^+, H^{s+1}(\mathcal{T}^3))$.*

Moreover, if v^ε is the solution of (RF^ε) and u is the solution of the limit system $(NS2D, L_{osc})$, then $v^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})u = o(1)$ in $C^0(\mathbf{R}^+, H^s(\mathcal{T}^3)) \cap L^2(\mathbf{R}^+, H^{s+1}(\mathcal{T}^3))$.

Finally, for almost all $\mathcal{T}^3$, there exists $\alpha \geq \frac{3}{2}$ such that if $v_0 \in H^\alpha(\mathcal{T}^3)$, and if f is smooth enough, then the first order expansion of Theorem 2 holds.

Remark. The fact that Theorems 3 and 4 hold for almost all periodic boxes is due to two different and independent reasons. The first one is that the limit system diagonalizes into (QG) and $(NS2D, L_{osc})$ respectively, for almost all values of (a_1, a_2, a_3). That diagonalization is crucial to get the global existence result, and the associate values of the a_i define non resonant domains. The second reason is that the perturbation satisfies a small divisor estimate at some degree for almost all values of the a_i; that is proved in Appendix C.

Let us note that some stability results concerning (RF^ε) are proved in [16] and [17], and that in [12], T. Colin and P. Fabrie prove a global existence result for $v^\varepsilon_{0,osc}$ small in H^1 (that is to say for well prepared initial data, which is also the setting in [16] and [17]). Moreover, in [2] and [3], A. Babin, A. Mahalov and B. Nicolaenko prove a similar theorem of global existence in a non resonant domain (Theorem 8.2), with an additional non resonance condition on a_2, which does not appear here. Our proof will use part of their computations, namely for the study of the resonant set of the operator L. However, once the diagonalized limit system is obtained, the use of the methods of S. Schochet will enable us, *via* Theorems 1 and 2, to find directly the result with no more computations, and in particular with no additional assumption on the a_i other than the one on a_3. Let us note that that fact is used in [3], in Theorem 6.3: it enables the authors to avoid the restriction on a_2, but only for smooth initial data, and the higher order terms are not computed.

The proof of Theorem 4 shows that the vertical variable plays an important part in the problem; therefore we shall try to see, in section 7, if that fact cannot be used in the study of those equations, the aim of that study being to decrease the smoothness assumptions made on the initial data, as in [21] and [22] in the case of the Navier–Stokes equations. That will be made possible by the use of anisotropic Sobolev and Besov spaces, studied by D. Iftimie in [21] and [22].

Without going into any precise definition at this stage (that will be done in section 7), let us just say that in the statement of the two following theorems, the space $H^{s,s'}$ (resp. the space $HB^{s,s'}$) is to be understood as the space of functions that are H^s in the horizontal variables, and $H^{s'}$ (resp. Besov $B^{s'}_{2,1}$) in the vertical variables.

Theorem 5 *If a_1 and a_2 are fixed, then for almost all a_3, the following result is true. Let v_0 have a vertical average $\bar{v}_0$ in $L^2(\mathcal{T}^2)$, with $v_{0,osc} \in H^{\delta,\frac{1}{2}-\delta}$, where δ is a real number in the interval $]0,1[$, and let the vertical average of the exterior force, $\bar{f}$ be an element of $L^2(\mathbf{R}^+, H^{-1})$, with $f_{osc} \in L^2(\mathbf{R}^+, H^{2\delta-1,\frac{1}{2}-\delta})$, and $\partial_t f_{osc} \in L^2(\mathbf{R}^+, H^{2\delta-\frac{7}{2},\frac{1}{2}-\delta})$, where $f_{osc} = f - \bar{f}$.*

Then for ε sufficiently small, (RF^ε) is globally well posed, and the solution v^ε satisfies

$$v^\varepsilon - \bar{u} - \mathcal{L}(\frac{t}{\varepsilon})u_{osc} = o(1) \quad \text{in} \quad L^\infty(\mathbf{R}^+, H^{\delta,\frac{1}{2}-\delta}) \cap L^4(\mathbf{R}^+, H^{\frac{1}{2}+\frac{\delta}{2},\frac{1}{2}-\frac{\delta}{2}}),$$

where $\bar{u}$ is the solution in $L^\infty(\mathbf{R}^+, L^2) \cap L^2(\mathbf{R}^+, H^1)$ of $(NS2D)$, and u_{osc} is the solution in $L^\infty(\mathbf{R}^+, H^{\delta,\frac{1}{2}-\delta}) \cap L^4(\mathbf{R}^+, H^{\frac{1}{2}+\frac{\delta}{2},\frac{1}{2}-\frac{\delta}{2}})$ of (L_{osc}).

The interesting aspect of that result is that $\bar{v}_0$ is taken in the energy space $L^2(\mathcal{T}^2)$ instead of $H^{\frac{1}{2}}$ as previously. Actually, one can even suppose the "horizontal smoothness" of $v_{0,osc}$ as well, to be only in L^2, as stated in the following theorem.

Theorem 6 *If a_1 and a_2 are fixed, then for almost all a_3, the following result is true. Let v_0 have a vertical average $\bar{v}_0$ in $L^2(\mathcal{T}^2)$, with $v_{0,osc} \in HB^{0,\frac{1}{2}}$, and let $\bar{f} \in L^2(\mathbf{R}^+, H^{-1})$, with $f_{osc} \in L^2(\mathbf{R}^+, HB^{-1,\frac{1}{2}})$, and $\partial_t f_{osc} \in L^2(\mathbf{R}^+, HB^{-\frac{7}{2},\frac{1}{2}})$. Then for ε sufficiently small, (RF^ε) is globally well posed, and*

$$v^\varepsilon - \bar{u} - \mathcal{L}(\frac{t}{\varepsilon})u_{osc} = o(1) \quad \text{in} \quad L^\infty(\mathbf{R}^+, HB^{0,\frac{1}{2}}) \cap L^4(\mathbf{R}^+, HB^{\frac{1}{2},\frac{1}{2}}).$$

Contents

2 Notations and preliminary results

2.1 Elements of Littlewood–Paley theory

Throughout this article, we shall consider vector fields with a mean equal to zero over the periodic box $\mathcal{T}^d$, and that are divergence free. In all the definitions and properties we shall state in this section, we have defined

$$|n|^2 = \sum_i \frac{n_i^2}{a_i^2}, \quad \text{and similarly} \quad n \cdot x = \sum_i \frac{n_i x_i}{a_i^2}.$$

We will work in homogeneous Sobolev spaces, defined by the following norm.

Definition 2.1 *Let s be a real number and let a be a distribution defined on $\mathcal{T}^d$. Then a is an element of the homogeneous Sobolev space H^s if the following norm is finite:*

$$|a|_s \stackrel{def}{=} \| \, |k|^s |\mathcal{F}a(k)| \, \|_{\ell^2(\mathbf{Z}^d)} \, ,$$

where $\mathcal{F}a$ represents the discrete Fourier transform of a.

In the following we will no longer specify that the spaces are homogeneous. The rules of product in those spaces are the following.

Lemma 2.1 *Let a and b be two distributions, and s and t be two real numbers such that $s,t < \frac{d}{2}$ and $s+t>0$. If $a \in H^s(\mathcal{T}^d)$ and $b \in H^t(\mathcal{T}^d)$, then $ab \in H^{s+t-\frac{d}{2}}(\mathcal{T}^d)$, and*

$$|ab|_{s+t-\frac{d}{2}} \leq C_{s,t}|a|_s|b|_t,$$

where $C_{s,t}$ is a constant depending only on s and t.

Moreover, for all $s > \frac{d}{2}$, H^s is an algebra.

In the rest of this study, we shall use the generic notation C for all "universal" constants.

To end this section, let us present a few elements of the Littlewood–Paley theory on the torus, refering for instance to the study of D. Iftimie in [21]: we consider a radial, positive function φ in C^∞, supported in the ring centered at the origin, of small radius $3/4$ and large radius $8/3$. That function is at the basis of a dyadic partition of unity corresponding to a decomposition of the frequency space into dyadic blocks.

Definition 2.2 *Let a be a distribution defined on $\mathcal{T}^d$. The operators Δ_q of frequency truncation are defined by*

$$\forall q \in \mathbf{N}, \quad \Delta_q a = \sum_{n \in \mathbf{Z}^d} a_n e^{in\cdot x} \varphi\left(\frac{|n|}{2^q}\right),$$

where the a_n are the Fourier coefficients of a: $a(x) = \sum_{n\in\mathbf{Z}^d} a_n e^{in\cdot x}$.

Let us notice that there exists an integer N_0 such that

$$|p-q| \geq N_0 \Rightarrow supp\, \varphi(2^{-p}\cdot) \cap supp\, \varphi(2^{-q}\cdot) = \emptyset.$$

That quasi–orthogonality property will come in useful in the computations that follow. In particular it implies that for all a, $|p-q| \geq N_0 \Rightarrow \Delta_p\Delta_q a = 0$.

The main interest of that decomposition will essentially lie for us in the following lemmas (the proofs of which, as well as a general presentation of the Littlewood–Paley theory for periodic functions, can be found in [21]).

Lemma 2.2 *For any function a, the following equivalence is true:*

$$|a|_s^2 \sim \sum_{q\in\mathbf{N}} 2^{2qs}|\Delta_q a|_0^2,$$

which can also be written $2^{qs}|\Delta_q a|_0 = c_q|a|_s$, with $c_q \in \ell^2(\mathbf{N})$, of norm equal to 1.

The local aspect of the Δ_q will also be used through the following lemma.

Lemma 2.3 *For any function a, one has the equivalence $|\nabla\Delta_q a|_0 \sim 2^q|\Delta_q a|_0$, with constants of equivalence independent of q and of a.*

In [11], one can find the proof of the following result.

Lemma 2.4 *For all couples of functions (a,b) such that a is divergence free, if s and t are two real numbers such that $s,t < \frac{d}{2}+1$ and $s+t>0$, then*

$$|(\Delta_q(a\cdot\nabla b)|\Delta_q b)| \leq Cc_q 2^{-q(s+t-\frac{d}{2}-1)}|\Delta_q b|_0|a|_s|b|_t,$$

where c_q is a sequence of $\ell^2(\mathbf{N})$, of norm equal to 1.

From now on, the notation $(a|b)$ represents the scalar product in L^2.

Finally in the course of the proofs we shall use the following very simple lemma, the proof of which can be found in [21]:

Lemma 2.5 *For all functions a, and all couples of real numbers (p,q) such that $1\leq p\leq q$, the following inequality is true:*

$$\| \, \|a_n(t)\|_{\ell^p} \, \|_{L^q_t} \leq \| \, \|a_n(t)\|_{L^q_t} \, \|_{\ell^p}.$$

2.2 The system $(\mathcal{S}^\varepsilon)$

In this section we are going to study the system $(\mathcal{S}^\varepsilon)$ presented in the introduction:

$$(\mathcal{S}^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t v^\varepsilon + q(v^\varepsilon,v^\varepsilon) + a_2(D)v^\varepsilon + \dfrac{L(v^\varepsilon)}{\varepsilon} & = & f \quad \text{in} \quad \mathcal{T}^d \\ v^\varepsilon_{|t=0} & = & v_0 \, . \end{array} \right.$$

We shall work here on the general bilinear form defined in (1.3), and will mention at the end of the proof of Theorem 1 how to adapt the study to the case of the Navier–Stokes equations, where the vector fields are divergence free. Let us note that in that case, $q(u,v)=P(u\cdot\nabla v)$, where P is the L^2–orthogonal projector on the divergence free vector fields.

Before proving Theorems 1 and 2, let us concentrate a while on the system $(\mathcal{S}^\varepsilon)$, and in particular on the penalization operator L. It is a skew–symmetric linear operator, called an oscillating perturbation (concerning this aspect of the study one can refer to the work of E. Grenier in [20]), meaning that for all $s\in\mathbf{R}$ and all $v_0\in H^s$, the problem

$$\left\{ \begin{array}{rcl} \partial_t v + L(v) & = & 0 \quad \text{in} \quad \mathcal{T}^d \\ v_{|t=0} & = & v_0 \end{array} \right.$$

has a unique global solution $v(t)=\mathcal{L}(t)v_0$, with

$$\forall t\in\mathbf{R}, \quad |\mathcal{L}(t)v_0|_s = |{}^t\mathcal{L}(t)v_0|_s = |v_0|_s.$$

According to (1.1), $\mathcal{L}$ is defined by

$$\mathcal{L}(t)v = \mathcal{F}^{-1} \sum_{a=1}^{d'} e^{-i\omega^a(n)t} \left(\mathcal{F}v(n), e^a(n)\right) e^a(n).$$

As shown in [20], the fact that L is skew–symmetric leads to the following property, which is essential to get existence results for $(\mathcal{S}^\varepsilon)$, and in particular Theorem 0 stated in the introduction, as it implies that the perturbation term does not appear in the energy estimates:

$$\forall v, \quad \int_{\mathcal{T}^d} L(v) \cdot v = 0.$$

So the classical proofs of existence of solutions to the unperturbated equations hold in our case (see for instance [9], [15], [27], [28], [36]), and that accounts for Theorem 0.

In the case when the vector fields considered are divergence free, which will be in particular the case for the rotating fluids and for the quasigeostrophic system, $\mathcal{L}$ is the semi–group associated with PL, where P is the L^2–orthogonal projection on the divergence free vector fields.

Let us finally note that the system $(\mathcal{S}^\varepsilon)$ can be made more general, for example by adding space variables other than those concerned by the penalization — that is useful in geometric optics for example; such a system is studied in [34]. One can also rather consider f^ε or v_0^ε, converging to f and v_0. We will study here the simpler system $(\mathcal{S}^\varepsilon)$, and the results obtained remain true in those other cases.

As stated in Theorem 0, for an initial data in $H^{\frac{d}{2}-1}$, the system $(\mathcal{S}^\varepsilon)$ has a strong solution v^ε, uniformly bounded in $C^0([0,T], H^{\frac{d}{2}-1})$, and its life span T_0 is bounded from below independently of ε. However the fact that $\partial_t v^\varepsilon$ is *a priori* not bounded in ε prevents one from taking directly the limit of the equation. A method — see [3], [20], [34] — is to use the semi–group associated with the perturbation : the filtered vector field defined by $u^\varepsilon = \mathcal{L}(-\dfrac{t}{\varepsilon})v^\varepsilon$ satisfies the equation

$$(\tilde{\mathcal{S}}^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t u^\varepsilon + \mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) + \mathcal{A}_2^\varepsilon(D)u^\varepsilon & = & \mathcal{L}(-\frac{t}{\varepsilon})f \quad \text{in} \quad \mathcal{T}^d \\ u^\varepsilon_{|t=0} & = & v_0 \end{array} \right.$$

where we have written

$$\mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) = \mathcal{L}(-\frac{t}{\varepsilon})q\left(\mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon, \mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon\right), \text{ and } \quad \mathcal{A}_2^\varepsilon(D)u^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})a_2(D)\mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon. \tag{2.1}$$

Then $\partial_t v^\varepsilon$ is bounded in some H^σ (see [20]), and by application of Ascoli's theorem one finds the following convergence result.

Proposition 2.1 *For an initial data $v_0 \in H^{\frac{d}{2}-1}(\mathcal{T}^d)$, the solution u^ε of $(\tilde{\mathcal{S}}^\varepsilon)$ converges strongly in $C^0([0,T], H^{s'})$ for all $s' < \frac{d}{2} - 1$, to a function u, which satisfies the limit system $(\mathcal{S}_0)$.*

The system $(\mathcal{S}_0)$ was presented above; let us recall that

$$(\mathcal{S}_0) \quad \begin{cases} \partial_t u + \mathcal{Q}(u,u) + \mathcal{A}_2(D)u &= \bar{f} \\ u_{|t=0} &= v_0, \end{cases}$$

where $\bar{f}$, $\mathcal{Q}(u,u)$ and $\mathcal{A}_2(D)u$ are the limits in $\mathcal{D}'$ respectively of $\mathcal{L}(-\frac{t}{\varepsilon})f$, $\mathcal{Q}^\varepsilon(u^\varepsilon,u^\varepsilon)$, and of $\mathcal{A}_2^\varepsilon(D)u^\varepsilon$.

Remark. As the operator $\mathcal{L}$ preserves the H^σ norms, $\mathcal{Q}(u,u)$ and $\mathcal{A}_2(D)$ are in fact also the limits of $\mathcal{Q}^\varepsilon(u,u)$ and of $\mathcal{A}_2^\varepsilon(D)u$.

The operators $\mathcal{Q}$ and $\mathcal{A}_2(D)$ satisfy the following fundamental property:

Lemma 2.6 *The operators $\mathcal{Q}$ and $\mathcal{A}_2(D)$, defined by*

$$\mathcal{Q}(u,v) = \lim_{\varepsilon\to 0} \mathcal{L}(-\frac{t}{\varepsilon})q\left(\mathcal{L}(\frac{t}{\varepsilon})u, \mathcal{L}(\frac{t}{\varepsilon})v\right), \quad \mathcal{A}_2(D)u = \lim_{\varepsilon\to 0} \mathcal{L}(-\frac{t}{\varepsilon})a_2(D)\mathcal{L}(\frac{t}{\varepsilon})u \quad \textit{in} \quad \mathcal{D}',$$

do not depend on the time variable.

Proof. The proof of this lemma is written in [20] in the special case of two dimensions, and can easily be adapted to our general case. Let us first study the quadratic term. We have

$$\begin{aligned} & \lim_{\varepsilon\to 0} \mathcal{L}(-\frac{t}{\varepsilon})q\left(\mathcal{L}(\frac{t}{\varepsilon})u, \mathcal{L}(\frac{t}{\varepsilon})v\right) \\ = \;& \lim_{\varepsilon\to 0} \mathcal{F}^{-1} \sum_{a,b,c} e^{-i\frac{t}{\varepsilon}\omega^{a,b,c}_{k,n}} \left(A_{j,j'}(n)u^{a,j}(k)v^{b,j'}(n-k), e^c(n)\right) e^c(n), \end{aligned}$$

with the notations presented in the introduction. But then it is just a matter of using the non stationary phase theorem (see [1] for instance), and one finds

$$\lim_{\varepsilon\to 0} \mathcal{Q}^\varepsilon(u,v) = \mathcal{F}^{-1} \sum_{\omega^{a,b,c}_{k,n}=0} \left(A_{j,j'}(n)u^{a,j}(k)v^{b,j'}(n-k), e^c(n)\right) e^c(n),$$

which proves Lemma 2.6 for the bilinear term.

For $\mathcal{A}_2(D)$ the same type of argument holds: we have

$$\lim_{\varepsilon\to 0} \mathcal{A}_2^\varepsilon(D)u = \lim_{\varepsilon\to 0} \mathcal{F}^{-1} \sum_{a,b} e^{i\frac{t}{\varepsilon}\omega^{a,b}_n} \left(a_2(n)u^b(n), e^a(n)\right) e^a(n),$$

where $\omega^{a,b}_n = \omega^a(n) - \omega^b(n)$, which leads directly to

$$\mathcal{A}_2(D)u = \mathcal{F}^{-1} \sum_{\omega^{a,b}_n=0} \left(a_2(n)u^b(n), e^a(n)\right) e^a(n), \tag{2.2}$$

and the lemma is proved. $\square$

Finally, as mentionned in the introduction, one can prove the following result.

Lemma 2.7 *For all t, for all ε, the operator $\mathcal{A}_2^\varepsilon(D) = \mathcal{L}(-\frac{t}{\varepsilon})a_2(D)\mathcal{L}(\frac{t}{\varepsilon})$ is elliptic, with the same constant of ellipticity as $a_2(D)$. The same holds for $\mathcal{A}_2(D)$.*

Proof. It is trivial, if we notice that as $\mathcal{L}$ is unitary, we have for all vector fields u,

$$
\begin{aligned}
(\mathcal{A}_2^\varepsilon(D)u \mid u) &= \left(\mathcal{L}(-\frac{t}{\varepsilon})a_2(D)\mathcal{L}(\frac{t}{\varepsilon})u \mid u\right) \\
&= \left(a_2(D)\mathcal{L}(\frac{t}{\varepsilon})u \mid \mathcal{L}(\frac{t}{\varepsilon})u\right),
\end{aligned}
$$

so the ellipticity of $a_2(D)$ implies that of $\mathcal{A}_2^\varepsilon(D)$ for all ε, with the same constant as $a_2(D)$, and therefore also the ellipticity of $\mathcal{A}_2(D)$. $\square$

This last remark puts an end to the general study of system $(\mathcal{S}^\varepsilon)$. We are now going to prove the various theorems stated in the introduction, using results proved in Appendix A; it is there that the methods of S. Schochet in [34] will be used.

3 Proof of Theorem 1

3.1 Introduction

The object of this section is to prove Theorem 1, stated in the introduction. So let us first recall the various equations on which we are going to work here: we start with the equation

$$
(\mathcal{S}^\varepsilon) \quad \left\{ \begin{aligned} \partial_t v^\varepsilon + q(v^\varepsilon, v^\varepsilon) + a_2(D)v^\varepsilon + \frac{L(v^\varepsilon)}{\varepsilon} &= f \quad \text{in} \quad \mathcal{T}^d \\ v^\varepsilon_{|t=0} &= v_0 \end{aligned} \right.
$$

and the limit system

$$
(\mathcal{S}_0) \quad \left\{ \begin{aligned} \partial_t u + \mathcal{Q}(u,u) + \mathcal{A}_2(D)u &= \bar{f} \\ u_{|t=0} &= v_0 \end{aligned} \right.
$$

where $\mathcal{Q}$, $\mathcal{A}_2(D)$, and $\bar{f}$ are defined by

$$
\mathcal{Q}(u,v) = \lim_{\varepsilon \to 0} \mathcal{L}(-\frac{t}{\varepsilon})q(\mathcal{L}(\frac{t}{\varepsilon})u, \mathcal{L}(\frac{t}{\varepsilon})v) \quad \text{in} \quad \mathcal{D}',
$$

$$
\mathcal{A}_2(D)u = \lim_{\varepsilon \to 0} \mathcal{L}(-\frac{t}{\varepsilon})a_2(D)\mathcal{L}(\frac{t}{\varepsilon})u \quad \text{and} \quad \bar{f} = \lim_{\varepsilon \to 0} \mathcal{L}(-\frac{t}{\varepsilon})f, \quad \text{in} \quad \mathcal{D}'.
$$

We suppose that $u \in C^0([0,T],H^s) \cap L^2([0,T],H^{s+1})$, with $s \geq \frac{d}{2} - 1$ and we want to show that

$$
v^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})u = o(1) \quad \text{in} \quad C^0([0,T],H^s) \cap L^2([0,T],H^{s+1}).
$$

Since the operator $\mathcal{L}$ preserves the H^σ norms, one can just as well work on the filtered solution u^ε rather than on v^ε. That filtered solution u^ε, defined by $u^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})v^\varepsilon$, satisfies

$$(\tilde{\mathcal{S}}^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t u^\varepsilon + \mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) + \mathcal{A}_2^\varepsilon(D)u^\varepsilon & = & \mathcal{L}(-\frac{t}{\varepsilon})f \quad \text{in} \quad \mathcal{T}^d \\ u^\varepsilon_{|t=0} & = & v_0 \end{array} \right.$$

where $\mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) = \mathcal{L}(-\frac{t}{\varepsilon})q\left(\mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon, \mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon\right)$, $\mathcal{A}_2^\varepsilon(D) = \mathcal{L}(-\frac{t}{\varepsilon})a_2(D)\mathcal{L}(\frac{t}{\varepsilon})$, and q was defined in (1.3).

Let us therefore define $w^\varepsilon = u^\varepsilon - u$. From $(\tilde{\mathcal{S}}^\varepsilon)$ and $(\mathcal{S}_0)$, we get the following equation for w^ε :

$$\partial_t w^\varepsilon + \mathcal{Q}^\varepsilon(w^\varepsilon, w^\varepsilon + 2u) + \mathcal{A}_2^\varepsilon(D)w^\varepsilon = g^\varepsilon - \mathcal{Q}^\varepsilon(u,u) + \mathcal{Q}(u,u) + \mathcal{A}_2(D)u - \mathcal{A}_2^\varepsilon(D)u, \quad (3.1)$$

with an initial data equal to zero since $u^\varepsilon_{|t=0} = u_{|t=0} = v_0$. We have defined $g^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})f - \bar{f}$.

We want to show that w^ε is defined on $[0,T]$, and satisfies

$$w^\varepsilon = o(1) \quad \text{in} \quad C^0([0,T], H^s) \cap L^2([0,T], H^{s+1}).$$

The right-hand side of equation (3.1) satisfied by w^ε is an oscillating term; we are going to prove Theorem 1 using the following general lemma, the proof of which is left to Appendix A.

Lemma 3.1 *Let $T > 0$ and $\sigma \geq \frac{d}{2} - 1$ be two real numbers, let b^ε be a family of functions, bounded in the space $C^0([0,T], H^\sigma) \cap L^2([0,T], H^{\sigma+1})$, and let a_0^ε be a function going to zero with ε in H^σ. Let $\mathcal{Q}^\varepsilon$ and $\mathcal{A}_2^\varepsilon$ be respectively a quadratic form and a second order elliptic operator of the type* (2.1)*, and finally let R^ε_{osc} be a $(p, \sigma - 2)$-oscillating function, with $p \geq 1$, and let F^ε be a function going to zero with ε in the space $L^2([0,T], H^{\sigma-1})$.*

Then the function a^ε, solution of

$$(\mathcal{E}^\varepsilon_{osc}) \quad \left\{ \begin{array}{rcl} \partial_t a^\varepsilon + \mathcal{Q}^\varepsilon(a^\varepsilon, a^\varepsilon + b^\varepsilon) + \mathcal{A}_2^\varepsilon(D)a^\varepsilon & = & R^\varepsilon_{osc} + F^\varepsilon \quad in \quad \mathcal{T}^d \\ a^\varepsilon_{|t=0} & = & a_0^\varepsilon \, , \end{array} \right.$$

is an $o(1)$ in the space $C^0([0,T], H^\sigma) \cap L^2([0,T], H^{\sigma+1})$.

3.2 Proof of Theorem 1

All that has to be done in order to use Lemma 3.1 is to check that (3.1) can be written in the form $(\mathcal{E}^\varepsilon_{osc})$. The initial data is zero, and u has the smoothness required for b^ε in Lemma 3.1, so we just have to check that the right-hand side is a $(p, s-2)$-oscillating function, for some $p \geq 1$.

Proposition 3.1 *Under the assumptions of Theorem 1, the right-hand side of equation* (3.1) *is a $(2, s-2)$-oscillating function.*

Proof. First of all, the forcing term g^ε is defined by

$$g^\varepsilon = \mathcal{F}^{-1} \sum_{\omega^a(n)\neq 0} e^{i\frac{t}{\varepsilon}\omega^a(n)} \left(\mathcal{F}f(t,n), e^a(n)\right) e^a(n), \tag{3.2}$$

where let us recall that $\omega^a(n)$ are the eigenvalues of L, and $e^a(n)$ are the associate eigenvectors.

But $f \in C^0([0,T], H^{s-2}) \cap L^2([0,T], H^{s-1})$, and $\partial_t f \in L^2([0,T], H^{s-3})$, so the function g^ε is a $(1, s-2)$-oscillating term, where we have taken $f_0(t,n) = f(t,n)$ with the notation of Definition 1.2.

We then have a linear term, noted $\mathcal{D}^\varepsilon u$, which is defined by

$$\begin{aligned} \mathcal{D}^\varepsilon u &= \mathcal{A}_2^\varepsilon(D)u - \mathcal{A}_2(D)u \\ &= \mathcal{F}^{-1} \sum_{\omega_n^{a,b}\neq 0} e^{i\frac{t}{\varepsilon}\omega_n^{a,b}} \Big(a_2(n)u^b(n), e^a(n)\Big)\, e^a(n), \end{aligned} \tag{3.3}$$

where $\omega_n^{a,b} = \omega^a(n) - \omega^b(n)$, $u^b(n) = \Big(\mathcal{F}u(n)|e^b(n)\Big)\, e^b(n)$, and where let us recall that $a_2(D)$ is a second order elliptic operator. Hence we can take

$$f_0(t,n) = a_2(n)u^b(n),$$

so $\mathcal{F}^{-1}f_0(t,\cdot)$ is an element of the space $C^0([0,T], H^{s-2}) \cap L^2([0,T], H^{s-1})$, and $\mathcal{F}^{-1}\partial_t f_0(t,\cdot)$ is an element of $L^2([0,T], H^{s-3})$, since $\partial_t u$ has the smoothness of $a_2(D)u$. So $\mathcal{D}^\varepsilon u$ is a $(1, s-2)$-oscillating term.

Finally we have the quadratic term

$$\begin{aligned} F^\varepsilon(u,v) &= \mathcal{Q}^\varepsilon(u,v) - \mathcal{Q}(u,v) \\ &= \mathcal{F}^{-1} \sum_{\omega_{k,n}^{a,b,c}\neq 0} e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} \Big(A_{j,j'}(n)u^{a,j}(k)v^{b,j'}(n-k), e^c(n)\Big)\, e^c(n), \end{aligned} \tag{3.4}$$

where let us recall that $\omega_{k,n}^{a,b,c} = \omega^a(k) + \omega^b(n-k) - \omega^c(n)$.

Then we choose

$$r_0(n,k,m) = A_{j,j'}(n),$$

$$f_1(t,k) = u^{a,j}(t,k), \qquad f_2(t,m) = u^{b,j'}(t,m).$$

We have, with the notation of Definition 1.2, $\rho_0 = 1$ and $\rho_1 = \rho_2 = 0$. Then let us suppose that $s > \frac{d}{2}$; then H^s is an algebra, so $\prod_i \mathcal{F}^{-1}f_i(t,\cdot)$ is in particular an element of the space $C^0([0,T], H^{s-1})$. If $s < \frac{d}{2}$, we have the same result, using the rules of product recalled in Lemma 2.1 associated with the fact that $2s - \frac{d}{2} \geq s-1$. Finally if $s = \frac{d}{2}$, then let η be a real number in $]0, \frac{1}{2}[$. We have $u \in C^0([0,T], H^{\frac{d}{2}-\eta})$ and $2(\frac{d}{2}-\eta) - \frac{d}{2} \geq \frac{d}{2} - 1$, so finally we have proved that, under the assumptions of Theorem 1,

$$\prod_i \mathcal{F}^{-1}f_i(t,\cdot) \quad \text{is an element of} \quad C^0([0,T], H^{s-1}).$$

Similarly, we find

$$\prod_i \mathcal{F}^{-1} f_i(t,\cdot) \quad \text{is an element of} \quad L^2([0,T],H^s).$$

So since moreover $\mathcal{F}^{-1}\partial_t f_i(t,\cdot)$ is an element of $L^2([0,T],H^{s-1})$, we conclude that $F^\varepsilon(u,u)$ is a $(2,s-2)$–oscillating term.

Proposition 3.1 is proved. $\square$

Conclusion. We have just seen that (3.1) is an equation of the the type $(\mathcal{E}^\varepsilon_{osc})$, so all we have to do now is to use Lemma 3.1, which gives directly

$$w^\varepsilon = o(1) \quad \text{in} \quad C^0([0,T],H^s) \cap L^2([0,T],H^{s+1}),$$

and Theorem 1 is proved. $\square$

3.3 The case of the Navier–Stokes equations

In the case when the bilinear term is $q(u,v) = P(u\cdot\nabla v)$ with divergence free vector fields on $\mathcal{T}^d$, the previous result is identical : it is just a matter of replacing in the preceding proof the operator L by PL, where P is the L^2–orthogonal projection operator onto the space of divergence free vector fields.

Then the constraint on the divergence implies that PL is written along $d'-1$ vectors:

$$PLv = \mathcal{F}^{-1} \sum_{a=1}^{d'-1} i\omega^a(n)\,(\mathcal{F}v(n), e^a(n))\, e^a(n).$$

The remarkable property of those operators is that they commute with the derivation operators. Thus nothing is to be modified in the previous proof to get the expected result : the solutions are defined in $C^0([0,T],H^s)\cap L^2([0,T],H^{s+1})$, and $u^\varepsilon - u = o(1)$ in that same space.

3.4 The case when T is infinite

In the two applications we will study, we wish to use Theorem 1 to obtain the global existence of systems (RF^ε) and (PE^ε), for $\varepsilon > 0$ small enough. However we must beware that in the proof of Theorem 1, we have explicitly used the fact that the life time T of the limit system was finite (that appears in the proof of Lemma 3.1 in Appendix A, in the study of the high frequencies: the cut–off frequency N depends on T, and therefore so does ε, and nothing enables one to say for certain that N can be chosen independently of T). Hence even if we prove that $(\mathcal{S}_0)$ is defined for all time $T>0$, Theorem 1 only leads to the existence of $\varepsilon_T > 0$ such that $(\mathcal{S}^\varepsilon)$ is defined on $[0,T]$ for all $\varepsilon < \varepsilon_T$. But there again ε_T is *a priori* not uniform with respect to T.

However in the particular case of the rotating fluid equations, as well as for the primitive system, such a result will be sufficient to obtain the global existence for systems (RF^ε) and (PE^ε), for $\varepsilon > 0$ small enough. Indeed, for both these equations, it is classical that if the solution exists for a sufficiently long time T_0, depending on the initial data, then it exists in fact globally. Hence in both cases, one just has to find ε_0 small enough so that equation (RF^ε) (resp. (PE^ε)) is defined on $[0, T_0]$, using Theorem 1. Then for all $\varepsilon \leq \varepsilon_0$, (RF^ε) (resp. (PE^ε)) is globally well–posed. In other words in both those cases, Theorem 1 can be extended to the case when $T = \infty$. The same holds for Theorem 2, which is proved in the coming section.

4 Proof of Theorem 2

4.1 Introduction

This section is devoted to the proof of Theorem 2: under a certain number of small divisor assumptions on the operator L, we shall prove that

$$u^\varepsilon - u = O(\varepsilon) \quad \text{in} \quad C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}}),$$

and analyse precisely what that $O(\varepsilon)$ term is made of.

4.2 A first order expansion

From Theorem 1, we know that

$$u^\varepsilon - u = o(1) \quad \text{in} \quad C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}}). \tag{4.1}$$

To prove that we have in fact, under some conditions, $u^\varepsilon - u = O(\varepsilon)$, we are going to study as in section 3.2, the equation satisfied by $w^\varepsilon = u^\varepsilon - u$:

$$\partial_t w^\varepsilon + \mathcal{A}_2^\varepsilon(D) w^\varepsilon + \mathcal{Q}^\varepsilon(w^\varepsilon, w^\varepsilon + 2u) = g^\varepsilon - \mathcal{D}^\varepsilon u - F^\varepsilon(u,u), \tag{4.2}$$

where g^ε, $\mathcal{D}^\varepsilon u$, and $F^\varepsilon(u,u)$ are defined in (3.2), (3.3), and (3.4) respectively.

Proposition 4.1 *Let j_0 be the integer part of $\frac{1}{2}(s + \alpha_{12} - \frac{d}{2})$. Suppose that $v_0 \in H^{s+\alpha_{12}}(\mathcal{T}^d)$, and that for all $0 \leq j \leq j_0$, $\partial_t^j f \in C^0([0,T], H^{s-2j-2+\alpha_{12}}) \cap L^2([0,T], H^{s-2j-1+\alpha_{12}})$; then for all $0 \leq j \leq j_0$, we have*

$$\partial_t^j u \in C^0([0,T], H^{s-2j+\alpha_{12}}) \cap L^2([0,T], H^{s-2j+1+\alpha_{12}}).$$

Proof. A simple recursive argument gives the result: it is of course true when $j = 0$, and let us suppose that for all $0 \leq k \leq j-1$, we have

$$\partial_t^k u \in C^0([0,T], H^{s-2k+\alpha_{12}}) \cap L^2([0,T], H^{s-2k+1+\alpha_{12}}).$$

Then differentiating $j-1$ times the system $(\mathcal{S}_0)$ yields

$$\partial_t^j u + \partial_t^{j-1} \mathcal{Q}(u,u) + \mathcal{A}_2(D) \partial_t^{j-1} u = \partial_t^{j-1} \bar{f}.$$

Since, as recalled in Lemma 2.1, H^τ is an algebra if $\tau > \frac{d}{2}$, the recurrence assumption implies that $\partial_t^{j-1}\mathcal{Q}(u,u)$ is in $C^0([0,T],H^{s-2j+1+\alpha_{12}})\cap L^2([0,T],H^{s-2j+2+\alpha_{12}})$. Furthermore, we have $\mathcal{A}_2(D)\partial_t^{j-1}u \in C^0([0,T],H^{s-2j+\alpha_{12}}) \cap L^2([0,T],H^{s-2j+1+\alpha_{12}})$; the proposition follows. □

In order to prove that w^ε is an $O(\varepsilon)$ in $C^0([0,T],H^{\frac{d}{2}-1}) \cap L^2([0,T],H^{\frac{d}{2}})$, we shall use Lemma 3.1, as in the proof of Theorem 1. But first let us note that we have seen in section 3.2, Proposition 3.1, that the right-hand side of (4.2) is oscillating in time. The method we shall follow is that used by S. Schochet in [34]: it is inspired by the following simple ordinary differential equation, which modelizes (4.2). Consider

$$\partial_t\varphi - F(\varphi) = \cos\frac{t}{\varepsilon},$$

and define the function

$$\psi = \varphi - \varepsilon\sin\frac{t}{\varepsilon}.$$

Then ψ and φ are of the same size, up to an ε, and ψ satisfies the equation

$$\partial_t\psi = F\left(\psi + \varepsilon\sin\frac{t}{\varepsilon}\right),$$

which is an equation where, up to an ε, the oscillating part has disappeared. By analogy with that model problem, let us define

$$\psi^\varepsilon = w^\varepsilon + \varepsilon\tilde{R}^\varepsilon_{osc}(u), \tag{4.3}$$

where $\tilde{R}^\varepsilon_{osc}(u)$ is defined in (1.4): we have

$$\tilde{R}^\varepsilon_{osc}(u) \stackrel{def}{=} \tilde{g}^\varepsilon + \tilde{\mathcal{D}}^\varepsilon u + \tilde{F}^\varepsilon(u,u), \tag{4.4}$$

with

$$\tilde{g}^\varepsilon(t) \stackrel{def}{=} \mathcal{F}^{-1}\sum_{\omega^a(n)\neq 0}\frac{-i\,e^{i\frac{t}{\varepsilon}\omega^a(n)}}{\omega^a(n)}\left(\mathcal{F}f(t,n),e^a(n)\right)e^a(n), \tag{4.5}$$

$$\tilde{\mathcal{D}}^\varepsilon u \stackrel{def}{=} \mathcal{F}^{-1}\sum_{\omega_n^{a,b}\neq 0}\frac{-i\,e^{i\frac{t}{\varepsilon}\omega_n^{a,b}}}{\omega_n^{a,b}}\Big(a_2(n)u^b(n),e^a(n)\Big)e^a(n), \tag{4.6}$$

$$\tilde{F}^\varepsilon(u,u) \stackrel{def}{=} \mathcal{F}^{-1}\sum_{\omega_{k,n}^{a,b,c}\neq 0}\frac{i\,e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}}}{\omega_{k,n}^{a,b,c}}\Big(A_{j,j'}(n)u^{a,j}(k)u^{b,j'}(n-k),e^c(n)\Big)e^c(n). \tag{4.7}$$

The notation is the same as in the introduction, and as in section 2.2.

Lemma 4.1 *Under the hypotheses of Theorem 2, $\tilde{R}^\varepsilon_{osc}(u)$ is a $(2,\frac{d}{2}-1)$-oscillating function.*

Proof. The forcing term $\tilde{g}^\varepsilon(t)$ is defined in (4.5). The assumptions of Theorem 2 imply in particular that

$$f \in C^0([0,T],H^{\frac{d}{2}-1}) \cap L^2([0,T],H^{\frac{d}{2}})$$

as well as $\partial_t f \in L^2([0,T], H^{\frac{d}{2}-2})$; since moreover L is an operator of order zero, (1.2) implies the result for the first term in $\tilde{R}^\varepsilon_{osc}(u)$. Let us note that, as remarked in the introduction of this article, if instead of assuming (1.2), we suppose that L is of order r, then supposing that $f \in C^0([0,T], H^{s-2+\alpha_{12}+r}) \cap L^2([0,T], H^{s-1+\alpha_{12}+r})$ and $\partial_t f \in L^2([0,T], H^{s-3+\alpha_{12}+r})$ gives the same result.

For the linear term $\tilde{\mathcal{D}}^\varepsilon u$ defined in (4.6), we can take

$$f_0(t,n) = (\omega_n^{a,b})^{-1} a_2(n) u^b(n),$$

which is an element of the space $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}-2})$ since we have assumed that u is an element of $C^0([0,T], H^{s+1+\alpha_1}) \cap L^2([0,T], H^{s+2+\alpha_1})$, with $s > \frac{d}{2}$. Then as $\partial_t u$ is in $C^0([0,T], H^{s-1+\alpha_1}) \cap L^2([0,T], H^{s+\alpha_1})$ according to Proposition 4.1, the result follows for $\tilde{\mathcal{D}}^\varepsilon u$.

Finally for $\tilde{F}^\varepsilon(u,u)$, we can choose

$$r_0(n,k,m) = (\omega_{k,n}^{a,b,c})^{-1} A_{j,j'}(n), \quad f_1(t,k) = u^{a,j}(k), \quad \text{and} \quad f_2(t,m) = u^{b,j'}(m),$$

$$\text{with} \quad \rho_0 = 1 \quad \text{and} \quad \rho_1 = \rho_2 = \alpha_2.$$

The result follows immediately, since u is in $C^0([0,T], H^{s+\alpha_2}) \cap L^2([0,T], H^{s+1+\alpha_2})$ with $s > \frac{d}{2}$, and $\partial_t u$ is an element of $C^0([0,T], H^{s-2+\alpha_2}) \cap L^2([0,T], H^{s-1+\alpha_2})$ by Proposition 4.1.

So Lemma 4.1 is proved. □

Remark. Up to now, we have written that $u^\varepsilon - u = \psi^\varepsilon - \varepsilon \tilde{R}^\varepsilon_{osc}$. So one of the terms of the expansion given in Theorem 2 is found: the "fast part" $\varepsilon \tilde{R}^\varepsilon_{osc}$. Now we are going to prove that there exists a function h, independent of ε, such that $\psi^\varepsilon = \varepsilon h + o(\varepsilon)$. That will prove the theorem.

So let us define the function $\psi_1^\varepsilon = \varepsilon^{-1}\psi^\varepsilon$. It is easy to see, according to the definition of ψ^ε given in (4.3), that ψ_1^ε satisfies the following equation:

$$\partial_t \psi_1^\varepsilon + \mathcal{A}_2^\varepsilon(D)\psi_1^\varepsilon + \mathcal{Q}^\varepsilon(\psi_1^\varepsilon, \psi^\varepsilon + 2u - 2\varepsilon \tilde{R}^\varepsilon_{osc}) = \tilde{R}^{\varepsilon,t}_{osc} + \mathcal{A}_2^\varepsilon(D)\tilde{R}^\varepsilon_{osc} + \mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc}, 2u - \varepsilon \tilde{R}^\varepsilon_{osc}), \quad (4.8)$$

where we have defined $\tilde{R}^{\varepsilon,t}_{osc}$ by $\tilde{R}^{\varepsilon,t}_{osc} = \tilde{g}^{\varepsilon,t} + \tilde{\mathcal{D}}^{\varepsilon,t} + \tilde{F}^{\varepsilon,t}$, and, with the notation of the introduction,

$$\tilde{g}^{\varepsilon,t} \stackrel{def}{=} \mathcal{F}^{-1} \sum_{\omega^a(n)\neq 0} \frac{-i\, e^{i\frac{t}{\varepsilon}\omega^a(n)}}{\omega^a(n)} \partial_t f^a(t,n),$$

$$\tilde{\mathcal{D}}^{\varepsilon,t} u \stackrel{def}{=} \mathcal{F}^{-1} \sum_{\omega_n^{a,b}\neq 0} \frac{-i\, e^{i\frac{t}{\varepsilon}\omega_n^{a,b}}}{\omega_n^{a,b}} \Big(a_2(n)\partial_t u^b(t,n), e^a(n) \Big)\, e^a(n),$$

$$\tilde{F}^{\varepsilon,t}(u,u) \stackrel{def}{=} \mathcal{F}^{-1} \sum_{\omega_{k,n}^{a,b,c}\neq 0} \frac{i\, e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}}}{\omega_{k,n}^{a,b,c}} \partial_t \Big(A_{j,j'}(n) u^{a,j}(t,k) u^{b,j'}(t,n-k), e^c(n) \Big)\, e^c(n).$$

Lemma 4.2 *Under the assumptions of Theorem 2, $\tilde{R}^{\varepsilon,t}_{osc}$ is a $(2, \frac{d}{2}-3)$-oscillating function.*

Proof. The proof is very similar to that leading to Lemma 4.1. The fact that $\partial_t f$ is in $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$ and $\partial_t^2 f \in L^2([0,T], H^{\frac{d}{2}-4})$ gives the result for $\tilde{g}^{\varepsilon,t}$. As in the case of Lemma 4.1, if one does not suppose the condition (1.2), the result still holds if $\partial_t f$ and $\partial_t^2 f$ are smooth enough.

Then Proposition 4.1 implies that $\partial_t u \in C^0([0,T], H^{s-1+\alpha_1}) \cap L^2([0,T], H^{s+\alpha_1})$, which, with the first order small divisor estimate satisfied by L, as well as the smoothness of $\partial_t^2 u$, leads to the result for $\tilde{\mathcal{D}}^{\varepsilon,t} u$.

Finally for $\tilde{F}^{\varepsilon,t}(u,u)$, we can take

$$r_0(n,k,m) = (\omega_{k,n}^{a,b,c})^{-1} A_{j,j'}(n),$$

as well as

$$f_1(t,k) = \partial_t u^{a,j}(k) \quad \text{and} \quad f_2(t,m) = u^{b,j'}(m).$$

Then, with the notation of Definition 1.2, we have $\sigma_1 = s-2$ and $\sigma_2 = s$, with $s > \frac{d}{2}$, as well as $\bar{\sigma}_i = s-4+\alpha_2$, $\rho_0 = 1$, and $\rho_1 = \rho_2 = \alpha_2$, which gives the result: without loss of generality, we can suppose that $s < \frac{d}{2}+2$; then one considers η such that $s - \frac{\eta}{2} > \frac{d}{2}$, and $s - \eta < \frac{d}{2}$. Then the rules of product of Lemma 2.1 and the fact that $s-\eta+s-2-\frac{d}{2} \geq \frac{d}{2}-2$ gives the result. $\square$

Proposition 4.2 *Under the assumptions of Theorem* 2, *we have*

$$\psi^\varepsilon = O(\varepsilon) \quad in \quad C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}}).$$

Proof. Let us prove that ψ_1^ε, solution of (4.8), is a $O(1)$ in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$.

That result is simply due to the fact that the right-hand side of equation (4.8), which we can call G^ε, is bounded in $L^2([0,T], H^{\frac{d}{2}-2})$: for the terms $\mathcal{A}_2^\varepsilon(D)\tilde{R}_{osc}^\varepsilon$ and $\tilde{R}_{osc}^{\varepsilon,t}$, that is an obvious consequence of Lemmas 4.1 and 4.2, and for the term $\mathcal{Q}^\varepsilon(\tilde{R}_{osc}^\varepsilon, 2u - \varepsilon\tilde{R}_{osc}^\varepsilon)$, the rules of product of Lemma 2.1 give the result. Moreover, w^ε is an $o(1)$ in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$, according to (4.1), so ψ^ε defined in (4.3) also, and finally $\psi^\varepsilon + 2u - 2\varepsilon\tilde{R}_{osc}^\varepsilon$ is bounded in the space $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$. Calling that last term c^ε, it follows that ψ_1^ε satisfies an equation of the type

$$\partial_t \psi_1^\varepsilon + \mathcal{A}_2^\varepsilon(D)\psi_1^\varepsilon + 2\mathcal{Q}^\varepsilon(\psi_1^\varepsilon, c^\varepsilon) = G^\varepsilon,$$

where c^ε is bounded in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$, and where G^ε is bounded in the space $L^2([0,T], H^{\frac{d}{2}-2})$. Then a classical estimate (see [9] or [21] for instance, or Appendix B) implies that ψ_1^ε is bounded in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$. The proposition follows. $\square$

Proposition 4.3 *Under the assumptions of Theorem* 2, *ψ_1^ε defined in* (4.8) *satisfies an equation that can be written as*

$$\partial_t \psi_1^\varepsilon + \mathcal{A}_2^\varepsilon(D)\psi_1^\varepsilon + 2\mathcal{Q}^\varepsilon(\psi_1^\varepsilon, u) = R_{osc}^{1,\varepsilon}(u) + \varepsilon R^{2,\varepsilon}(u) + R(u), \tag{4.9}$$

where $R_{osc}^{1,\varepsilon}$ is a $(3, \frac{d}{2}-3)$-oscillating term, where $R^{2,\varepsilon}$ is bounded in the space $L^2([0,T], H^{\frac{d}{2}-2})$, and where $R(u)$ is a function independent of ε, in $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$.

Proof. Let us analyse the right-hand side of the equation (4.8), satisfied by ψ_1^ε. The first term, $\tilde{R}_{osc}^{\varepsilon,t}$, can be written as $R_{osc}^{1,\varepsilon}(u)$ of the proposition, according to Lemma 4.2.

Moreover, the rules of product recalled in Lemma 2.1, associated with Lemma 4.1 and Proposition 4.2 imply that

$$\mathcal{Q}^\varepsilon(\psi_1^\varepsilon, \psi^\varepsilon - 2\varepsilon\tilde{R}_{osc}^\varepsilon) + \varepsilon\mathcal{Q}^\varepsilon(\tilde{R}_{osc}^\varepsilon, \tilde{R}_{osc}^\varepsilon) = \varepsilon R^{2,\varepsilon}(u),$$

where $R^{2,\varepsilon}(u)$ is bounded in $L^2([0,T], H^{\frac{d}{2}-2})$.

Now we are left with two terms, $\mathcal{A}_2^\xi(D)\tilde{R}_{osc}^\varepsilon$ and $\mathcal{Q}^\varepsilon(\tilde{R}_{osc}^\varepsilon, u)$.

For $\mathcal{A}_2^\xi(D)\tilde{R}_{osc}^\varepsilon(u)$, the study is somewhat complicated by the fact that the oscillations due to $\mathcal{A}_2^\xi(D)$ can interfere with those of $\tilde{R}_{osc}^\varepsilon(u)$: we have

$$\mathcal{A}_2^\xi(D)\tilde{g}^\varepsilon = \mathcal{F}^{-1}\sum_{a,b} e^{i\frac{t}{\varepsilon}\omega_n^{a,b}}\left(a_2(n)\tilde{g}^{\varepsilon,b}(n), e^a(n)\right)e^a(n),$$

$$\text{with}\quad \tilde{g}^\varepsilon = \mathcal{F}^{-1}\sum_{\omega^b(n)\neq 0}\frac{-i\, e^{i\frac{t}{\varepsilon}\omega^b(n)}}{\omega^b(n)}\left(\mathcal{F}f(t,n), e^b(n)\right)e^b(n),$$

so the oscillation phase in $\mathcal{A}_2^\xi(D)\tilde{g}^\varepsilon$ is $\omega^a(n) - \omega^b(n) + \omega^b(n) = \omega^a(n)$. So the terms, in the sum defining $\mathcal{A}_2^\xi(D)\tilde{g}^\varepsilon$ above, such that $\omega^a(n) = 0$, do not depend on ε and can be written as $R(u)$, whereas if $\omega^a(n) \neq 0$, then we have a purely oscillating term which goes into $R_{osc}^{1,\varepsilon}(u)$: it is clearly a $(1, \frac{d}{2}-3)$-oscillating term, as

$$f \in C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}}) \quad\text{and}\quad \partial_t f \in L^2([0,T], H^{\frac{d}{2}-2}).$$

Similarly, the restriction of $\mathcal{A}_2^\xi(D)\tilde{g}^\varepsilon$ to $\omega^a(n) = 0$ is in $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$.

For the term $\mathcal{A}_2^\xi(D)\tilde{\mathcal{D}}^\varepsilon u$ we find the phase $\omega^a(n) - \omega^b(n) + \omega^b(n) - \omega^{b'}(n) = \omega_n^{a,b'}$, since

$$\tilde{\mathcal{D}}^\varepsilon u = \mathcal{F}^{-1}\sum_{\omega_n^{b,b'}\neq 0}\frac{-i\, e^{-i\frac{t}{\varepsilon}\omega_n^{b,b'}}}{\omega_n^{b,b'}}\left(a_2(n)u^b(n), e^a(n)\right)e^a(n),$$

and the phase of $\mathcal{A}_2^\xi(D)\tilde{F}^\varepsilon(u,u)$ is $\omega^a(n) - \omega^b(n) + \omega^b(n) - \omega^{a'}(k) - \omega^{b'}(n-k) = -\omega_{k,n}^{a',b',a}$, since $\tilde{F}^\varepsilon(u,u)$ is equal to

$$\mathcal{F}^{-1}\sum_{\omega_{k,n}^{a',b',b}\neq 0}\frac{i}{\omega_{k,n}^{a',b',b}}e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a',b',b}}\left(A_{\ell,\ell'}(n)u^{a',\ell}(k)u^{b',\ell'}(n-k), e^b(n)\right)e^b(n).$$

So in both those cases, one can also separate the function into $R(u)$, independent of ε and an element of $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$, and $R_{osc}^{1,\varepsilon}(u)$: in a similar way to the previous cases we have encountered, it is easy to see that the oscillating part of those functions, $\left(\mathcal{A}_2^\xi(D)\tilde{\mathcal{D}}^\varepsilon u\right)_{osc}$ and $\left(\mathcal{A}_2^\xi(D)\tilde{F}^\varepsilon(u,u)\right)_{osc}$, are respectively $(1, \frac{d}{2}-3)$ and $(2, \frac{d}{2}-3)$-oscillating. One can indeed take, in the case of $\mathcal{A}_2^\xi(D)\tilde{\mathcal{D}}^\varepsilon u$,

$$f_0(t,n) = a_2^2(n)(\omega_n^{b,b'})^{-1}u^b(n),$$

and in the case of $\mathcal{A}_2^\varepsilon(D)\tilde{F}^\varepsilon(u,u)$,

$$r_0(n,k,m) = a_2(n)A_{\ell,\ell'}(n)(\omega_{k,n}^{a',b',b})^{-1},$$

$$f_1(t,k) = u^{a',\ell}(k) \quad \text{and} \quad f_2(t,m) = u^{b',\ell'}(m),$$

with $\rho_0 = 3$ and $\rho_1 = \rho_2 = \alpha_2$. Then $u \in C^0([0,T], H^{s+1+\alpha_1}) \cap L^2([0,T], H^{s+2+\alpha_1})$ as well as $u \in C^0([0,T], H^{s+\alpha_2}) \cap L^2([0,T], H^{s+1+\alpha_2})$ give the expected result in both cases, respectively.

Finally we are left with one last term, $\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc}, u)$.

The study for that term is similar to that of the term $\mathcal{A}_2^\varepsilon(D)\tilde{R}^\varepsilon_{osc}(u)$, as the function $\mathcal{Q}^\varepsilon$ introduces oscillations that can cancel out those of $\tilde{R}^\varepsilon_{osc}$. We have indeed

$$\mathcal{Q}^\varepsilon(\tilde{g}^\varepsilon, u) = \mathcal{F}^{-1} \sum_{k,a,b,c} e^{i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} \left(A_{j,j'}(n)\tilde{g}^{\varepsilon,a,j}(k)u^{b,j'}(n-k), e^c(n)\right) e^c(n),$$

$$\text{with} \quad \tilde{g}^\varepsilon = \mathcal{F}^{-1} \sum_{\omega^a(k)\neq 0} \frac{-i\, e^{i\frac{t}{\varepsilon}\omega^a(k)}}{\omega^a(k)} \left(\mathcal{F}f(t,k), e^a(k)\right) e^a(k),$$

so the function $\mathcal{Q}^\varepsilon(\tilde{g}^\varepsilon, u)$ can be split in two, depending if $\omega^b(n-k) = \omega^c(n)$ or not.

The oscillating part, $\mathcal{Q}^\varepsilon(\tilde{g}^\varepsilon, u)_{osc}$, is clearly $(2, \frac{d}{2}-3)$-oscillating, considering the assumptions made on f. Similarly, in the case when we have $\omega^b(n-k) = \omega^c(n)$, the corresponding function is in $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$, so it can be written as $R(u)$.

In the same way, $\mathcal{Q}^\varepsilon(\tilde{\mathcal{D}}^\varepsilon, u)$ can be cut into two parts, since we have

$$\mathcal{Q}^\varepsilon(\tilde{\mathcal{D}}^\varepsilon u, u) = \mathcal{F}^{-1} \sum_{k,a,b,c} e^{i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} \left(A_{j,j'}(n)\tilde{\mathcal{D}}^{\varepsilon,a,j}u(k)\; u^{b,j'}(n-k), e^c(n)\right) e^c(n),$$

$$\text{with} \quad \tilde{\mathcal{D}}^\varepsilon u \stackrel{def}{=} \mathcal{F}^{-1} \sum_{\omega_k^{a,b}\neq 0} \frac{-i\, e^{i\frac{t}{\varepsilon}\omega_k^{a,b}}}{\omega_k^{a,b}} \left(a_2(k)u^b(k), e^a(k)\right) e^a(k).$$

For the same reasons as previously, the function $\mathcal{Q}^\varepsilon(\tilde{\mathcal{D}}^\varepsilon u, u)_{osc}$ is $(2, \frac{d}{2}-3)$-oscillating: we can take

$$r_0(n,k,m) = A_{j,j'}(n)a_2(k)(\omega_k^{a,b})^{-1},$$

$$f_1(t,k) = u^b(k) \quad \text{and} \quad f_2(t,m) = u^{b,j'}(m),$$

which gives directly the result, since $u \in C^0([0,T], H^{s+1+\alpha_1}) \cap L^2([0,T], H^{s+1+\alpha_1})$, and using moreover Proposition 4.1.

Similarly, the restriction of $\mathcal{Q}^\varepsilon(\tilde{\mathcal{D}}^\varepsilon u, u)$ to the case when $\omega^{b'}(k) + \omega^b(n-k) = \omega^c(n)$ is an element of $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$, hence can be written as $R(u)$.

Finally the same goes for $\mathcal{Q}^\varepsilon(\tilde{F}^\varepsilon, u)$: we have

$$\mathcal{Q}^\varepsilon(\tilde{F}^\varepsilon, u) = \mathcal{F}^{-1} \sum_{k,a,b,c} e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} \left(A_{j,j'}(n)\tilde{F}^{\varepsilon,a,j}(k)u^{b,j'}(n-k), e^c(n)\right) e^c(n),$$

where $\tilde F^{\varepsilon,a,j}(u,u)(k)$ is equal to

$$\mathcal{F}^{-1}\sum_{\omega^{a',b',a}_{k',k}\neq 0}\frac{i}{\omega^{a',b',a}_{k',k}}e^{-i\frac{t}{\varepsilon}\omega^{a',b',a}_{k',k}}\Big(A_{\ell,\ell'}(k)u^{a',\ell}(k')u^{b',\ell'}(k-k'),e^a(k)\Big)\,e^{a,j}(k).$$

So in $\mathcal{Q}^\varepsilon(\tilde F^\varepsilon,u)$ the phase is $\omega^{a,b,c}_{k,n}+\omega^{a',b',a}_{k',k}=\omega^b(n-k)-\omega^c(n)+\omega^{a'}(k')+\omega^{b'}(k-k')$. The function restricted to the case when the phase vanishes, is in $C^0([0,T],H^{\frac d2-3})\cap L^2([0,T],H^{\frac d2-2})$, so can be written as $R(u)$. When the phase is not zero, which corresponds to $\mathcal{Q}^\varepsilon(\tilde F^\varepsilon,u)_{osc}$, it is easy to see that we have a $(3,\frac d2-3)$-oscillating function, taking

$$r_0(k',m',m)=A_{j,j'}(n)A_{\ell,\ell'}(k)(\omega^{a',b',a}_{k',k})^{-1},$$

as well as

$$f_1(t,k')=u^{a',\ell}(k'),\quad f_2(t,m')=u^{b',\ell'}(m'),\quad\text{and}\quad f_3(t,m)=u^{b,j'}(m),$$

with $\rho_0=1$ and $\rho_1=\rho_2=1+\alpha_2$. Hence Proposition 4.3 is proved. $\Box$

Remarks. In all the results proved here, it is easy to see that if $[L,a_2(D)]=0$, then it is enough to suppose that $u\in C^0([0,T],H^{s+\alpha_2})\cap L^2([0,T],H^{s+1+\alpha_2})$ to have the same results. Let us note that the computations above imply that

$$\begin{aligned}
\mathcal{F}\,R(u)(n)=&\sum_{\substack{\omega^{a',b',b}_{k,n}\neq 0\\ \omega^{a',b',a}_{k,n}=0}}\frac{i}{\omega^{a',b',b}_{k,n}}\Big(a_2\Big(A_{\ell,\ell'}u^{a',\ell}(k)u^{b',\ell'}(n-k)\Big)^b(n)\Big)^a(n)\\
&-\sum_{\substack{\omega^a(k)\neq 0\\ \omega^b(n-k)=\omega^c(n)}}\frac{i}{\omega^a(k)}\Big(A_{j,j'}f^{a,j}(k)u^{b,j'}(n-k)\Big)^c(n)\\
&-\sum_{\substack{\omega^{a,b'}_n=0\\ \omega^{b,b'}_n\neq 0}}\frac{i}{\omega^{b,b'}_n}\Big(a_2\Big(a_2u^{b'}(n)\Big)^b(n)\Big)^a(n)-\sum_{\substack{\omega^b(n)\neq 0\\ \omega^a(n)=0}}\frac{i}{\omega^b(n)}\Big(a_2f^b(n)\Big)^a(n)\\
&-\sum_{\substack{\omega^{a,b}_k\neq 0,\\ \omega^{b',b,c}_{k,n}=0}}\frac{i}{\omega^{a,b}_k}\Big(A_{j,j'}\Big(a_2u^b(k)\Big)^{a,j}(k)\;u^{b,j'}(n-k)\Big)^c(n)\\
&+\sum_{\substack{\omega^{a',b',a}_{k',k}\neq 0\\ \omega^{a',b',b,c}_{k',k,n}=0}}\frac{i}{\omega^{a',b',a}_{k',k}}\Big(A_{j,j'}\Big(A_{\ell,\ell'}u^{a',\ell}(k')u^{b',\ell'}(k-k')\Big)^{a,j}(k)u^{b,j'}(n-k)\Big)^c(n),
\end{aligned}$$

with the usual notation, and where $\omega^{a,b,c,d}_{k',k,n}=\omega^a(k')+\omega^b(k-k')+\omega^c(n-k)-\omega^d(n)$.

Now let us define the function h, solution of the following equation:

$$(\mathcal{S}_1)\left\{\begin{array}{rcl}\partial_t h+\mathcal{A}_2(D)h+2\mathcal{Q}(u,h)&=&R(u)\quad\text{in}\quad\mathcal{T}^d\\ h_{|t=0}&=&\tilde R^\varepsilon_{osc}(u)_{|t=0},\end{array}\right.$$

and let us introduce the function $w_1^\varepsilon = \psi_1^\varepsilon - h$.

The definition of $\tilde{R}^\varepsilon_{osc}(u)$ in (4.4)–(4.7) shows that $h_{|t=0}$ does not depend on ε.

Proposition 4.4 *Under the assumptions of Theorem 2, we have*

$$h \in C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}}), \quad \textit{and}$$

$$w_1^\varepsilon = o(1) \quad \textit{in} \quad C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}}).$$

Moreover, if $[L, a_2(D)] = 0$, *then* $u \in C^0([0,T], H^{s+\alpha_2}) \cap L^2([0,T], H^{s+1+\alpha_2})$ *is enough to obtain the same result.*

Remark. Let us suppose that Proposition 4.4 has been proved. Then we have

$$\begin{aligned} u^\varepsilon - u &= \psi^\varepsilon - \varepsilon\, \tilde{R}^\varepsilon_{osc}(u) \\ &= \varepsilon\, \psi_1^\varepsilon - \varepsilon\, \tilde{R}^\varepsilon_{osc}(u) \\ &= \varepsilon\, h - \varepsilon\, \tilde{R}^\varepsilon_{osc}(u) + \varepsilon\, w_1^\varepsilon, \end{aligned}$$

with $w_1^\varepsilon = o(1)$ in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$, so the first order expansion of u^ε is found, and Theorem 2 is proved.

Proof of Proposition 4.4. We have just proved, with Proposition 4.3, that $R(u)$ is an element of $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$. So classical results on parabolic equations imply that h is in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$ (those results are recalled in Appendix B).

So now let us study the function $w_1^\varepsilon = \psi_1^\varepsilon - h$, which satisfies the following equation:

$$\partial_t w_1^\varepsilon + \mathcal{A}_2^\varepsilon(D) w_1^\varepsilon + \mathcal{Q}^\varepsilon(w_1^\varepsilon, u) = R^{1,\varepsilon}_{osc}(u) + \varepsilon R^{2,\varepsilon} - \tilde{\mathcal{D}}^\varepsilon(h) + 2\mathcal{Q}(h,u) - 2\mathcal{Q}^\varepsilon(h,u). \tag{4.10}$$

Lemma 4.3 *Under the assumptions of Theorem 2, the right-hand side of* (4.10) *is of the type* $R^\varepsilon_{osc} + F^\varepsilon$, *where* R^ε_{osc} *is a* $(3, \frac{d}{2} - 3)$*-oscillating function, and* F^ε *goes to zero with* ε *in* $L^2([0,T], H^{\frac{d}{2}-2})$. *Moreover, if* $[a_2(D), L] = 0$, *then* $u \in C^0([0,T], H^{s+\alpha_2}) \cap L^2([0,T], H^{s+1+\alpha_2})$ *is a sufficient condition.*

Proof of Lemma 4.3. Let us start by proving that $R^{1,\varepsilon}_{osc}(u) - \tilde{\mathcal{D}}^\varepsilon(h) + 2\mathcal{Q}(h,u) - 2\mathcal{Q}^\varepsilon(h,u)$ is a $(3, \frac{d}{2} - 3)$-oscillating function.

We have already proved the result for the function $R^{1,\varepsilon}_{osc}(u)$, with Proposition 4.3. Hence we are left with $2\mathcal{Q}(h,u) - 2\mathcal{Q}^\varepsilon(h,u) - \tilde{\mathcal{D}}^\varepsilon(h)$. We have

$$\tilde{\mathcal{D}}^\varepsilon(h) = -\mathcal{F}^{-1} \sum_{\omega^a(n) - \omega^b(n) \neq 0} e^{i\frac{t}{\varepsilon}(\omega^a(n) - \omega^b(n))} \Big(a_2(n) h^b(n), e^a(n) \Big)\, e^a(n).$$

But since h is an element of the space $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$, according to Proposition 4.4, then $a_2(D)h$ is in $C^0([0,T], H^{\frac{d}{2}-3}) \cap L^2([0,T], H^{\frac{d}{2}-2})$. Besides, $\partial_t h$ has the smoothness of $a_2(D)h$, so $\partial_t a_2(D)h$ has the smoothness of $a_2^2(D)h$, hence $\partial_t a_2(D)h$ is an element of $L^2([0,T], H^{\frac{d}{2}-4})$. So $\tilde{\mathcal{D}}^\varepsilon(h)$ is a $(1, \frac{d}{2}-3)$-oscillating function, and hence has the right form for Lemma 4.3.

Moreover, $\mathcal{Q}^\varepsilon(h,u) - \mathcal{Q}(h,u)$ is equal to

$$\mathcal{F}^{-1} \sum_{\omega_{k,n}^{a,b,c} \neq 0} e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} \left(A_{j,j'}(n) h^{a,j}(k) u^{b,j'}(n-k), e^c(n) \right) e^c(n),$$

which is clearly a $(2, \frac{d}{2}-3)$-oscillating function.

So we have proved that $R_{osc}^{1,\varepsilon} + 2\mathcal{Q}(h,u) - 2\mathcal{Q}^\varepsilon(h,u) - \tilde{\mathcal{D}}^\varepsilon(h)$ is a $(3, \frac{d}{2}-3)$-oscillating function.

Moreover, if $[L, a_2] = 0$, then those results hold with $\sigma = s + \alpha_2$. So Lemma 4.3 is proved. $\square$

End of the proof of Proposition 4.4. To conclude, we just have to remember that the initial data satisfies $w^\varepsilon_{1|t=0} = 0$. So Lemma 3.1 implies directly that w_1^ε is an $o(1)$ in $C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}})$, and Proposition 4.4 is proved. $\square$

So as remarked after the statement of Proposition 4.4, we have computed the first order expansion of u^ε:

$$u^\varepsilon - u = \varepsilon h - \varepsilon \tilde{R}^\varepsilon_{osc}(u) + o(\varepsilon) \quad \text{in} \quad C^0([0,T], H^{\frac{d}{2}-1}) \cap L^2([0,T], H^{\frac{d}{2}}).$$

So Theorem 2 is proved. $\square$

Remark. Were it not for the fear of over–complicated computations, it would be easy to extend the previous proofs to a higher order expansion: after the study we have undertaken, it seems clear that under more regularity assumptions on the initial data, as well as with more small divisor estimates on the perturbation, u^ε has an asymptotic expansion at any order.

5 Application to the primitive equations

5.1 Presentation of the equations

Let us consider in this section the primitive equations of the quasigeostrophic system:

$$(PE^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t V^\varepsilon + v^\varepsilon \cdot \nabla V^\varepsilon - \mathcal{D} V^\varepsilon + \varepsilon^{-1} L V^\varepsilon & = & \varepsilon^{-1}(-\nabla \Phi^\varepsilon, 0) + f \quad \text{in} \quad \mathcal{T}^3 \\ \operatorname{div} v^\varepsilon & = & 0 \quad \text{in} \quad \mathcal{T}^3 \\ V^\varepsilon_{|t=0} & = & V_0 \end{array} \right.$$

$$\text{with} \quad L = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} \nu\Delta & 0 & 0 & 0 \\ 0 & \nu\Delta & 0 & 0 \\ 0 & 0 & \nu\Delta & 0 \\ 0 & 0 & 0 & \nu'\Delta \end{pmatrix},$$

where let us recall that $V^\varepsilon = (v^\varepsilon, T^\varepsilon)$.

We will sometimes find it more convenient to write $v = (V^1, V^2, V^3)$ and $T = V^4$.

The goal of this section is to prove Theorem 3, stating the global existence of the solutions of (PE^ε) when ε is small enough. Since we want to use Theorems 1 and 2 proved in the previous sections, we are going to compute the limit of the system $(\widetilde{PE}^\varepsilon)$, found by filtering (PE^ε) with the operator $\mathcal{L}\left(-\frac{t}{\varepsilon}\right)$. Here, since the velocity field v^ε is non divergent, $\mathcal{L}$ is the semi–group associated with the operator PL, where PV^ε projects v^ε onto the divergence free vector fields without changing the temperature T^ε; we will come back to its explicit form in the next section.

Let us define $U^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})V^\varepsilon$. The vector field U^ε satisfies the following equation

$$(\widetilde{PE}^\varepsilon) \quad \begin{cases} \partial_t U^\varepsilon + \mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) - \mathcal{L}(-\dfrac{t}{\varepsilon})\mathcal{D}\mathcal{L}(\dfrac{t}{\varepsilon})U^\varepsilon &= \tilde{f}^\varepsilon \quad \text{in} \quad \mathcal{T}^3 \\ \operatorname{div} v^\varepsilon &= 0 \quad \text{in} \quad \mathcal{T}^3 \\ U^\varepsilon_{|t=0} &= V_0 \end{cases}$$

$$\text{where} \quad \mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon) = \mathcal{L}(-\frac{t}{\varepsilon})\left(\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon \cdot \nabla \mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon\right), \qquad \tilde{f}^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})f.$$

5.2 The limit system

Finding the limit of $(\widetilde{PE}^\varepsilon)$ comes down to finding the limit of the quadratic term $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ as well as the linear term $\mathcal{L}(-\dfrac{t}{\varepsilon})\mathcal{D}\mathcal{L}(\dfrac{t}{\varepsilon})U^\varepsilon$: calling those limits respectively $\mathcal{Q}(U,U)$ and $\tilde{\mathcal{D}}U$, and writing $\tilde{f}$ for the limit of $\tilde{f}^\varepsilon$, the limit equation is $\partial_t U - \tilde{\mathcal{D}}U + \mathcal{Q}(U,U) = \tilde{f}$.

Those limits are obtained by studying the eigenvalues of PL in the Fourier space. So we first are going to compute those eigenvalues, which will enable us not only to find $\mathcal{Q}(U,U)$ and $\tilde{\mathcal{D}}U$ but also will give us a means to diagonalize the limit system, following also the method of J.-Y. Chemin in [10], in a way which will make it very easy to check that it is well posed.

5.2.1 The operator PL

Let us consider the following equation:

$$\partial_t V + PLV = 0,$$

where as stated above, P is the orthogonal projection of v onto the divergence free vector fields, without changing T. In Fourier coordinates, P has the following form:

$$P_n = \mathrm{Id} \, - \frac{1}{|n|^2}\begin{pmatrix} n_1^2/a_1^2 & n_1n_2/a_1a_2 & n_1n_3/a_1a_3 & 0 \\ n_2n_1/a_2a_1 & n_2^2/a_2^2 & n_2n_3/a_2a_3 & 0 \\ n_3n_1/a_3a_1 & n_3n_2/a_3a_2 & n_3^2/a_3^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $|n|^2 = \sum n_j^2/a_j^2$, and Id is the identity matrix.

One then computes the eigenvalues of $P_n L$: in [14], P. Embid and A. Majda find the eigenvalues 0 (double) and $\pm i \dfrac{(n_3^2/a_3^2 + F^{-2}|n'|^2)^{\frac{1}{2}}}{|n|}$. Hence when $F = a_3^{-1}$, the eigenvalues of $P_n L$ are 0, i and $-i$, and the associate eigenvectors, where one only considers those whose first three components are "divergence free" are, in that particular case,

$$e^0(n) = \frac{1}{|n|}\begin{pmatrix} -n_2 \\ n_1 \\ 0 \\ -n_3 \end{pmatrix} \; ; \quad e^i(n) = \frac{1}{|n|\sqrt{2}}\begin{pmatrix} n_3 \\ -in_3 \\ -n_1 + in_2 \\ -in_1 - n_2 \end{pmatrix} \; ; \quad e^{-i}(n) = \frac{1}{|n|\sqrt{2}}\begin{pmatrix} n_3 \\ in_3 \\ -n_1 - in_2 \\ in_1 - n_2 \end{pmatrix}.$$

In the following, we shall call $e^a(n)$ the eigenvector of $P_n L$ associated with the eigenvalue a. Let us note that in the case $F = a_3^{-1}$, the eigenvalues of $P_n L$ do not depend on n. So in that very special situation, the small divisor estimates of Definition 1.1 are trivial.

Before using those results to find the limit of $(\widetilde{PE}^\varepsilon)$, let us introduce, in the same way as J.-Y. Chemin in [10], the following diagonalization: we start by defining the "potential vorticity" by

$$\Omega^\varepsilon = \partial_1 U^{\varepsilon,2} - \partial_2 U^{\varepsilon,1} - F^{-1}\partial_3 U^{\varepsilon,4}.$$

The diagonalization is then obtained by writing $U^\varepsilon = U^\varepsilon_{QG} + U^\varepsilon_{osc}$, with

$$U^\varepsilon_{QG} = (\nabla_2^\perp \Delta^{-1}\Omega^\varepsilon, 0, -\partial_3 F \Delta^{-1}\Omega^\varepsilon),$$

where $\nabla_2^\perp = (-\partial_2, \partial_1)$. The field U^ε_{QG} is known as the quasigeostrophic part of U^ε, whereas the field U^ε_{osc} is the oscillating part. One can then immediately notice that e^0 defined above is non oscillating, that is to say that $e^0 = e^0_{QG}$, and the eigenvectors associated with $\pm i$ have a quasigeostrophic component equal to zero, since $e^{\pm i} = e^{\pm i}_{osc}$.

Let us also note that the equation satisfied by Ω will simply be the projection of the equation on U, onto $|n|e^0(n)$, in the Fourier space, whereas for U_{osc} it corresponds to the projection onto $e^{\pm i}(n)$.

Finally, as U^ε_{QG} is defined along the vectors in the kernel of $P_n L$, then U^ε_{QG} is in fact equal to V^ε_{QG}, and the same holds for U_{QG} and V_{QG}, which are equal.

By analogy with the notation above, we shall note in the following

$$f_{pot} = \partial_1 f^2 - \partial_2 f^1 - F^{-1}\partial_3 f^4,$$

and $f = f_{QG} + f_{osc}$, where

$$f_{QG} = (\nabla_2^\perp \Delta^{-1} f_{pot}, 0, -\partial_3 F \Delta^{-1} f_{pot}). \tag{5.1}$$

Let us note that for all t, we have $\mathcal{L}(t) f_{QG} = f_{QG}$.

5.2.2 The limit

To make things simpler, we will write the computations here in the case when $F = a_3^{-1}$. The case $F \neq a_3^{-1}$ is easily obtained from that particular case, at the price of longer computations, written for the most part in [14]; we will refer to them as we go along.

Proposition 5.1 *The limit of* $(\widetilde{PE}^{\varepsilon})$ *can be separated into two equations : the first one is*

$$(\widetilde{D_{QG}}) \quad \left\{ \begin{array}{rcl} \partial_t \Omega + v_{QG} \cdot \nabla \Omega + a_{QG}(D)\Omega & = & f_{pot} \\ U_{QG|t=0} & = & V_{0,QG} \, , \end{array} \right.$$

where $a_{QG}(D)$ *is a second order elliptic operator, and in the case when* $F = a_3^{-1}$, *that equation can be written* $\partial_t \Omega + v_{QG} \cdot \nabla \Omega - \nu \Delta_2 \Omega - \nu' \partial_3^2 \Omega = f_{pot}$, *where* $\Delta_2 = \partial_1^2 + \partial_2^2$.

The second equation is linear in U_{osc}, *for almost all* (a_1, a_2, a_3)*:*

$$(\widetilde{D_{osc}}) \quad \left\{ \begin{array}{rcl} \partial_t U_{osc} + 2\mathcal{Q}(V_{QG}, U_{osc}) + a_{osc}(D) U_{osc} & = & 0 \\ U_{osc|t=0} & = & V_{0,osc} \, , \end{array} \right.$$

where $a_{osc}(D)$ *is a second order elliptic operator, and in the case when* $F = a_3^{-1}$, *that equation can be written* $\partial_t U_{osc} + 2\mathcal{Q}(V_{QG}, U_{osc}) - \dfrac{\nu + \nu'}{2} \Delta_2 U_{osc} - \nu \partial_3^2 U_{osc} = 0.$

Proof. Let us start by noticing that the weak limit of $\mathcal{L}(-\frac{t}{\varepsilon}) f_{osc}$ is zero. Then the proof of Proposition 5.1 is made of two parts: in the first part, we shall investigate the limit of the quadratic term (Lemmas 5.1 and 5.3), and then we will study the limit of the linear term $\mathcal{L}(-\frac{t}{\varepsilon}) \mathcal{D} \mathcal{L}(\frac{t}{\varepsilon}) U^{\varepsilon}$.

- The quadratic term:

Lemma 5.1 *The limit of* $(\mathcal{F}\mathcal{Q}^{\varepsilon}(U^{\varepsilon}, U^{\varepsilon}), |n| e^0(n))$ *is equal to* $v_{QG} \cdot \nabla \Omega$.

Proof. Let us prove the result in the case when $F = a_3^{-1}$; the case $F \neq a_3^{-1}$ is treated exactly in the same way, but the computations are somewhat more complicated. Let us note that in [14], P. Embid and A. Majda state that in the general case, the limit of that quadratic term is $v_{QG} \cdot \nabla \Omega$, as given in the lemma.

The method to find the limit of such a term is standard, and shall be used continuously in the following. As in the proof of Lemma 2.6, it all comes down to studying, by application of the non stationary phase theorem, the resonances of the operator PL, defined by the equation

$$\omega^a(k) + \omega^b(m) = \omega^c(n), \qquad \text{with} \quad k + m = n,$$

where each of the terms in the equation is an eigenvalue of PL in the Fourier space, which we have computed above. Hence in the case when $F = a_3^{-1}$, the computations are going to be remarkably easy, due to the fact that the eigenvalues of $P_n L$ do not depend on the Fourier variable n.

Lemma 5.2 *When $F = a_3^{-1}$, the triplets $(\pm i, \mp i, 0)$ do not contribute to the resonances.*

Let us suppose for a moment that this lemma has been proved. To find the limit of the quadratic term, one considers the possibilities of resonances, which correspond here to the case where the eigenvalues $(\omega^a(k), \omega^b(m), \omega^c(n))$ are elements of $\{(0,0,0),\ (\pm i, \mp i, 0)\}$: this is due to the fact that Ω^ε does not oscillate, so is defined along the vectors $e^0(n)$. But Lemma 5.2 indicates that $(\pm i, \mp i, 0)$ does not contribute to the resonances, so it follows that one is left with $(0,0,0)$ only, which corresponds exactly to the quadratic term in $(\widetilde{D_{QG}})$, and to Lemma 5.1. □

Proof of Lemma 5.2. Let us compute the following expression, where we have written the sum of the limits of $\mathcal{Q}^\varepsilon$ restricted respectively to $(i, -i, 0)$ and $(-i, i, 0)$:

$$\begin{aligned} &\sum_{k+m=n} \left(U(k), e^i(k)\right) \left(U(m), e^{-i}(m)\right) e^{i,j}(k) m_j \left(e^{-i}(m), e^0(n)\right) e^0(n) \\ +\ &\sum_{k+m=n} \left(U(k), e^{-i}(k)\right) \left(U(m), e^{i}(m)\right) e^{-i,j}(k) m_j \left(e^{i}(m), e^0(n)\right) e^0(n). \end{aligned} \tag{5.2}$$

A simple computation, shows that, at a fixed value of n, and with $k+m=n$,

$$\sum_j e^{i,j}(k) m_j \left(e^{-i}(m), e^0(n)\right) = -i\mathcal{R}(k,m),$$

where $\mathcal{R}(k,m)$ is real and symmetric in k and m:

$$2|k||m||n|\ \mathcal{R}(k,m) = (m_1k_3 - k_1m_3)^2 + (m_2k_3 - k_2m_3)^2.$$

But the term $(U(k), e^i(k))\ (U(m), e^{-i}(m))$ can be written as $\mathcal{C}(k,m) + i\mathcal{D}(k,m)$, with $\mathcal{C}(k,m)$ and $\mathcal{D}(k,m)$ real, and furthermore

$$\mathcal{C}(m,k) + i\mathcal{D}(m,k) = \mathcal{C}(k,m) - i\mathcal{D}(k,m).$$

Using the symmetry of the domain of summation, still with n fixed, one can therefore write

$$\begin{aligned} \sum_{k+m=n} i\mathcal{R}(k,m)(\mathcal{C}(k,m) + i\mathcal{D}(k,m)) &= \sum_{k+m=n} i\mathcal{R}(k,m)(\mathcal{C}(k,m) - i\mathcal{D}(k,m)) \\ &= \sum_{k+m=n} i\mathcal{R}(k,m)\mathcal{C}(k,m), \end{aligned}$$

which means that this sum is in fact a purely imaginary number. And since the second sum in (5.2) is its complex conjugate, the lemma follows. □

Lemma 5.3 *For almost all (a_1, a_2, a_3), the limit of $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)_{osc}$ is $2\mathcal{Q}(V_{QG}, U_{osc})$.*

Proof. Like above, we shall first consider the case when $F = a_3^{-1}$. Let us write the bilinear term $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)$ in the form

$$\mathcal{Q}^{\varepsilon,\ell}(U^\varepsilon, U^\varepsilon) = \mathcal{F}^{-1} \sum_{k+m=n} \sum_{(a,b,c)\in\{-i,0,i\}^3} e^{-\frac{t}{\varepsilon}(a+b-c)} U^{\varepsilon,a,j}(k) m_j U^{\varepsilon,b,c,\ell}(m,n), \tag{5.3}$$

with the following notations:

$$U^a(k) = (\mathcal{F}U(k), e^a(k))\, e^a(k) \quad \text{and} \quad U^{b,c}(m,n) = \Big(U^b(m), e^c(n)\Big)\, e^c(n). \tag{5.4}$$

Since by definition U^ε_{osc} oscillates, then necessarily e^c oscillates, which means that $c \in \{-i; i\}$. But then after taking the limit in ε, the couple (a,b) can only be equal to $(\pm i, 0)$ or $(0, \pm i)$, which implies that between e^a and e^b, one is equal to e^0, and in particular does not oscillate, whereas the other is equal to $e^{\pm i}$, and oscillates. It follows that the limit of $\mathcal{Q}^\varepsilon(U^\varepsilon, U^\varepsilon)_{osc}$ can be written as $2\mathcal{Q}(V_{QG}, U_{osc})$, and Lemma 5.3 is proved in this particular case, when $F = a_3^{-1}$, in $\mathcal{T}^3$.

In the case when $F \neq a_3^{-1}$, the resonances are obviously not so easy to find. However a simple argument leads to Lemma 5.3: the equation $\omega^a(k) + \omega^b(m) = \omega^c(k+m)$, for $\omega^a(k)$, $\omega^b(m)$ and $\omega^c(k+m)$ non zero, can be considered as the equation in (a_1, a_2, a_3) of a surface of $\mathbf{R}^3$, with coefficients depending on a countable number of variables (that equation is non trivial, since when $a_3 = F^{-1}$, there is no solution). Therefore the set of solutions is of measure zero in $\mathbf{R}^3$. So out of a set of measure 0, that identity never occurs, for any value of k and m. Hence the only possible quadratic terms in the limit of the equation on U^ε_{osc} have V_{QG} in the product, which proves the lemma. $\square$

- The linear term:

To find the limit system, we now are left with one last term: $\mathcal{L}(-\frac{t}{\varepsilon})\mathcal{D}\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon$. As we have seen in Lemma 2.7, the limit of the operator $\mathcal{L}(-\frac{t}{\varepsilon})\mathcal{D}\mathcal{L}(\frac{t}{\varepsilon})$ is elliptic. Let us compute that limit against the vectors $e^a(n)$. We have, considering (2.2),

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathcal{F}\mathcal{L}(-\frac{t}{\varepsilon})\mathcal{D}\mathcal{L}(\frac{t}{\varepsilon})\mathcal{F}^{-1}e^a(n) &= (a_2(n)e^a(n), e^a(n))\, e^a(n) \\
&= \sum_i a_i(n)|e^{a,i}(n)|^2 e^a(n),
\end{aligned}$$

where we have written $\mathcal{D} = \mathrm{diag}(a_i)_{1\leq i\leq 4}$. So in particular we have

$$\lim_{\varepsilon\to 0}\left(\mathcal{F}\mathcal{L}(-\frac{t}{\varepsilon})\mathcal{D}\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon, |n|e^0(n)\right) = a_{QG}(D)\Omega, \qquad \lim_{\varepsilon\to 0}\mathcal{F}\mathcal{L}(-\frac{t}{\varepsilon})\mathcal{D}\mathcal{L}(\frac{t}{\varepsilon})U^\varepsilon_{osc} = a_{osc}(D)U_{osc},$$

where $a_{QG}(D)$ and $a_{osc}(D)$ are elliptic. So using the results found above for the quadratic terms, the case where F is arbitrary in Proposition 5.1, is proved.

In the case when $F = a_3^{-1}$, very simple computations enable us to find precisely the expression of $a_{QG}(D)$ and $a_{osc}(D)$, since we have computed the value of the eigenvectors $e^a(n)$ in that simpler case. It is thus trivial to see that

$$\begin{aligned}
\sum_j a_j(n)|e^{0,j}(n)|^2 &= -\nu\,(n_1^2+n_2^2) - \nu'\, n_3^2, \\
\text{and} \quad \sum_j a_j(n)|e^{\pm i,j}(n)|^2 &= -\nu\, n_3^2 - \frac{\nu}{2}(n_1^2+n_2^2) - \frac{\nu'}{2}(n_1^2+n_2^2) \\
&= -\nu\, n_3^2 - \frac{\nu+\nu'}{2}(n_1^2+n_2^2).
\end{aligned}$$

That ends the proof of the Proposition 5.1. $\square$

So finally one has found the following result: for the quasigeostrophic part of U^ε, after taking the limit in ε, one is only left with bilinear terms in V_{QG}, and the limit equation is the following:

$$(\widetilde{D_{QG}}) \quad \left\{ \begin{array}{rcl} \partial_t \Omega + v_{QG} \cdot \nabla \Omega + a_{QG}(D)\Omega & = & f_{pot} \\ V_{QG|t=0} & = & V_{0,QG} \end{array} \right.$$

and for the oscillating part of U^ε, for almost all (a_1, a_2, a_3), the limit system in U_{osc} is linear in U_{osc}, and of the form

$$(\widetilde{D_{osc}}) \quad \left\{ \begin{array}{rcl} \partial_t U_{osc} + 2\mathcal{Q}(V_{QG}, U_{osc}) + a_{osc}(D) U_{osc} & = & 0 \\ U_{osc|t=0} & = & V_{0,osc}\,. \end{array} \right.$$

Remark. The values of (a_1, a_2, a_3) leading to $(\widetilde{D_{osc}})$ define the non–resonant domains for the primitive equations.

5.3 Global existence for the limit system

We shall in this part prove that the limit system obtained above is globally well–posed in the space $C^0(\mathbf{R}^+, H^s) \cap L^2(\mathbf{R}^+, H^{s+1})$ for an initial data in H^s, with $s \geq 1$. Let us recall that in (5.1), we have defined f_{QG} by

$$f_{QG} = (\nabla_2^\perp \Delta^{-1} f_{pot}, 0, -\partial_3 F \Delta^{-1} f_{pot}).$$

Proposition 5.2 *If $V_0 \in H^s(\mathcal{T}^3)$, with $s \geq 1$, and if $f_{QG} \in L^2(\mathbf{R}^+, H^{s-1})$, then*

$$V_{QG} \in C^0(\mathbf{R}^+, H^s) \cap L^2(\mathbf{R}^+, H^{s+1}).$$

Proof. The proof of that proposition is based on energy estimates in H^s. For technical reasons, we are going to consider different possible values of s. Everything lies on the result when $s = 1$, and that result is due to the particular form of the equation, as since v_{QG} is divergence free, a direct energy estimate in L^2 on the equation satisfied by Ω leads to

$$\|\Omega\|^2_{C^0(\mathbf{R}^+, L^2)} + c\|\Omega\|^2_{L^2(\mathbf{R}^+, H^1)} \leq |V_0|_1^2 + C\|f_{pot}\|^2_{L^2(\mathbf{R}^+, H^{-1})}, \tag{5.5}$$

where c and C are two "universal" constants.

Then writing $V_{QG,q} = \Delta_q V_{QG}$, we have, using Lemma 2.4,

$$\begin{array}{rcl} \dfrac{1}{2}\dfrac{d}{dt}|V_{QG,q}(t)|_0^2 + c2^{2q}|V_{QG,q}(t)|_0^2 & \leq & Cc_q(t)2^{-\frac{3q}{2}}|V_{QG}(t)|_2^2 |V_{QG,q}(t)|_0 \\ & + & Cc_q(t)2^{-q(s-1)}|f_{QG}(t)|_{s-1}|V_{QG,q}(t)|_0, \end{array}$$

where $c_q(t)$ is a sequence of $\ell^2(\mathbf{N})$, of norm 1.

Then by Gronwall's lemma, that implies that

$$|V_{QG,q}(t)|_0 \leq |V_{QG,q,0}|_0 e^{-c2^{2q}t} + C\int_0^t c_q(\tau)2^{-\frac{3q}{2}}|V_{QG}(\tau)|_2^2 e^{-c2^{2q}(t-\tau)}d\tau$$

$$+C\int_0^t c_q(\tau)2^{-q(s-1)}|f_{QG}(\tau)|_{s-1}e^{-c2^{2q}(t-\tau)}d\tau.$$

Then multiplying by $2^{q(s+1)}$, taking the L^2 norm followed by the ℓ^2 norm in q and using Young's inequality yields, if $1 \leq s \leq \frac{3}{2}$,

$$\|V_{QG}\|_{L^2([0,T],H^{s+1})} \leq |V_{QG,0}|_s + C\|V_{QG}\|^2_{L^2([0,T],H^2)} + C\|f_{QG}\|_{L^2([0,T],H^{s-1})}.$$

Similarly, multiplying by 2^{qs}, taking the L^∞ norm followed by the ℓ^2 norm in q, and using Lemma 2.5 gives, if $1 \leq s \leq \frac{3}{2}$,

$$\|V_{QG}\|_{L^\infty([0,T],H^s)} \leq |V_{QG,0}|_s + C\|V_{QG}\|^2_{L^2([0,T],H^2)} + C\|f_{QG}\|_{L^2([0,T],H^{s-1})}.$$

So using (5.5), the result is proved when $1 \leq s \leq \frac{3}{2}$, and we have

$$\forall s \in [1,\frac{3}{2}], \quad \text{if } V_0 \in H^s, \quad \text{then} \quad V_{QG} \in C^0([0,T],H^s) \cap L^2([0,T],H^{s+1}). \tag{5.6}$$

Then let us suppose that $\frac{3}{2} \leq s < \frac{5}{2}$; we can again use Lemma 2.4, which yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|V_{QG,q}(t)|_0^2 + c2^{2q}|V_{QG,q}(t)|_0^2 &\leq Cc_q(t)2^{-q(2s-\frac{5}{2})}|V_{QG}(t)|_s^2|V_{QG,q}(t)|_0 \\ &+ Cc_q(t)2^{-q(s-1)}|f_{QG}(t)|_{s-1}|V_{QG,q}(t)|_0. \end{aligned}\tag{5.7}$$

Then similar computations to the ones written above imply finally that

$$\|V_{QG}\|_{L^2([0,T],H^{s+1})} \leq |V_{QG,0}|_s + C\|V_{QG}\|^2_{L^4([0,T],H^s)} + C\|f_{QG}\|_{L^2([0,T],H^{s-1})},$$

$$\|V_{QG}\|_{C^0([0,T],H^s)} \leq |V_{QG,0}|_s + C\|V_{QG}\|^2_{L^4([0,T],H^s)} + C\|f_{QG}\|_{L^2([0,T],H^{s-1})}.$$

So using (5.6), we have

$$\forall s \in [1,2], \quad \text{if } V_0 \in H^s, \quad \text{then} \quad V_{QG} \in C^0([0,T],H^s) \cap L^2([0,T],H^{s+1}). \tag{5.8}$$

Finally let us suppose that $s \geq 2$. Then we have

$$\frac{1}{2}\frac{d}{dt}|V_{QG}(t)|_s^2 + c|V_{QG}(t)|_{s+1}^2 \leq C|V_{QG}(t)|_s^2|V_{QG}(t)|_{s+1} + C|f_{QG}(t)|_{s-1}|V_{QG}(t)|_{s+1},$$

hence by a Gronwall estimate we get

$$|V_{QG}(t)|_s^2 + \frac{c}{2}\int_0^t |V_{QG}(\tau)|_{s+1}^2\,d\tau \leq (|V_{QG,0}|_s^2 + C\|f_{QG}\|^2_{L^2([0,T],H^{s-1})})\exp\left(C\int_0^t |V_{QG}(\tau)|_s^2\,d\tau\right).$$

So we can conclude by recurrence: if $V_{QG} \in L^2(\mathbf{R}^+,H^s)$, then the estimate above implies that $V_{QG} \in C^0([0,T],H^s) \cap L^2(\mathbf{R}^+,H^{s+1})$. So Proposition 5.2 is proved. $\square$

Now we can turn to the oscillating part U_{osc}.

Proposition 5.3 *If $V_0 \in H^s(\mathcal{T}^3)$, with $s \geq 1$, then the solution U_{osc} of the system $(\widetilde{D_{osc}})$ satisfies*

$$U_{osc} \in C^0(\mathbf{R}^+, H^s) \cap L^2(\mathbf{R}^+, H^{s+1}).$$

Proof. This proposition lies on the fact that the equation satisfied by U_{osc} is linear.

We can use Lemma B.1, proved in Appendix B: taking $A = U_{osc}$, $B = V_{QG}$, $\tilde{q} = 0$ as well as $W = 0$, we get, for all $s \geq 1$,

$$\|U_{osc}\|_{L^2(\mathbf{R}^+,H^{s+1})} \leq C|U_{osc,0}|_s \exp C\|V_{QG}\|^2_{L^2(\mathbf{R}^+,H^{s+1})},$$

$$\|U_{osc}\|_{L^\infty(\mathbf{R}^+,H^s)} \leq |U_{osc,0}|_s + C\|U_{osc}\|_{L^2(\mathbf{R}^+,H^{s+1})}\|V_{QG}\|_{L^2(\mathbf{R}^+,H^{s+1})}.$$

The proposition is proved. □

5.4 Proof of Theorem 3

The first part of the theorem concerns the case when $V_0 \in H^s(\mathcal{T}^3)$, with $s \geq 1$. One can apply Theorem 1 to our system: we have found the limit system, for almost all (a_1, a_2, a_3), and proved in Propositions 5.2 and 5.3 that it is well posed in $C^0(\mathbf{R}^+, H^s) \cap L^2(\mathbf{R}^+, H^{s+1})$. Then using section 3.4 and Theorem 1, we get directly the first part of Theorem 3:

$$V^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})U = o(1) \quad \text{in} \quad C^0(\mathbf{R}^+, H^s) \cap L^2(\mathbf{R}^+, H^{s+1}).$$

In order to obtain the second part of the theorem, *i.e.* the first order expansion of the solution, the application of Theorem 2 requires a second order small divisor estimate on the perturbation. In the case when $F = a_3^{-1}$, we have seen that the small divisor estimates are trivial, since the eigenvalues of $P_n L$ do not depend on n. So the theorem is proved in that particular case. In the general case, we use the result proved in Appendix C (Proposition C.1), stating that such an estimate can be achieved for almost all values of the a_i's, at a certain degree. Then provided the initial data and the exterior force are smooth enough, we have the theorem. □

Let us note that two independent restrictions on the a_i's have occurred: the first one in order to get a "diagonalized" limit system, and the second one in order to have small divisor estimates for the perturbation.

6 Application to the rotating fluid equations

6.1 Introduction

In this part, we propose to use Theorems 1 and 2 in order to prove Theorem 4 concerning the rotating fluid equations. The system of equations governing the motion of rotating fluids is the following, as presented in the introduction, and where p^ε is the pressure:

$$(RF^\varepsilon)\quad \begin{cases} \partial_t v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon - \nu\Delta v^\varepsilon + \dfrac{v^\varepsilon\times B}{\varepsilon} &= -\nabla p^\varepsilon + f \quad \text{in} \quad \mathcal{T}^3 \\ \operatorname{div} v^\varepsilon &= 0 \quad \text{in} \quad \mathcal{T}^3 \\ v^\varepsilon_{t=0} &= v_0\,. \end{cases}$$

Then Theorem 4 can be proved using Theorem 1: one only has to prove that the limit system of the equations above is globally well–posed. This system will turn out being particularly simple if, prior to taking the limit in ε, one diagonalizes the vector field u^ε, obtained as above, as the filtered solution $\mathcal{L}(-\frac{t}{\varepsilon})v^\varepsilon$: this diagonalization consists, as in [2] or [20], in a separation of u^ε into a bidimensional field and a remainder, called the oscillating part, in the following way (following the notations of [10]):

$$u^\varepsilon = \bar u^\varepsilon + u^\varepsilon_{osc}, \quad \text{with} \quad \bar u^\varepsilon = \int_{x_3} u^\varepsilon dx_3.$$

So $\bar u^\varepsilon$ is the vertical mean of u^ε; once the system has been diagonalized (which is the purpose of section 6.3), one can take the limit in ε, by the means of a study of the resonances of the operator PL, and following the computations of A. Babin, A. Mahalov and B. Nicolaenko in [3]. Finally in the last section we will show that those limit equations are globally well–posed.

6.2 The operator PL

The expression of $\mathcal{L}$ is found by solving the equation

$$\partial_t v + PLv = 0,$$

where the matrix L is defined by $L = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and in Fourier variables, the operator P is

$$P_n = \mathrm{Id} - \frac{1}{|n|^2}\begin{pmatrix} n_1^2/a_1^2 & n_1n_2/a_1a_2 & n_1n_3/a_1a_3 \\ n_2n_1/a_2a_1 & n_2^2/a_2^2 & n_2n_3/a_2a_3 \\ n_3n_1/a_3a_1 & n_3n_2/a_3a_2 & n_3^2/a_3^2 \end{pmatrix},$$

where $|n|^2 = \sum n_j^2/a_j^2$, and Id is the identity matrix.

A simple computation shows that the eigenvalues of P_nL are 0, $in_3/a_3|n|$, and $-in_3/a_3|n|$, and the two last eigenvalues have associate eigenvectors which are "divergence free".

One can then show (see [3]) that $\mathcal{L}$ can be written as

$$2\mathcal{F}\mathcal{L}(t) = e^{i\frac{n_3}{a_3|n|}t}P_n^+ + e^{-i\frac{n_3}{a_3|n|}t}P_n^-, \tag{6.1}$$

where we have written $P_n^\pm = \mathrm{Id} \pm \dfrac{i}{|n|}\begin{pmatrix} 0 & -n_3/a_3 & n_2/a_2 \\ n_3/a_3 & 0 & -n_1/a_1 \\ -n_2/a_2 & n_1/a_1 & 0 \end{pmatrix}.$

The kernel of PL is therefore made of the vector fields that do not depend on the third variable. That suggests considering more precisely that type of vector, and that is why we shall diagonalize the system (RF^ε), as announced in the introduction, by integrating u^ε along the vertical variable.

6.3 Diagonalization

According to section 2.2, if one defines $u^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})v^\varepsilon$, then u^ε satisfies the following system:

$$(\widetilde{RF}^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t u^\varepsilon + \mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) - \nu \Delta u^\varepsilon & = & \tilde{f}^\varepsilon \\ \mathrm{div}\, u^\varepsilon & = & 0 \\ u^\varepsilon_{|t=0} & = & v_0 \end{array} \right.$$

where $\mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) = \mathcal{L}(-\frac{t}{\varepsilon})\left(\mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon \cdot \nabla \mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon\right)$ and $\tilde{f}^\varepsilon = \mathcal{L}(-\frac{t}{\varepsilon})f^\varepsilon$.

Let us now define, as suggested in the introduction, $u^\varepsilon = \bar{u}^\varepsilon + u^\varepsilon_{osc}$. The integration of $(\widetilde{RF}^\varepsilon)$ along the vertical variable leads to

$$(\overline{RF}^\varepsilon) \quad \left\{ \begin{array}{rcl} \partial_t \bar{u}^\varepsilon - \nu \Delta_2 \bar{u}^\varepsilon + \bar{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) & = & \bar{f}^\varepsilon \\ \partial_3 \bar{u}^\varepsilon & = & 0 \\ \nabla_2 \cdot \bar{u}^\varepsilon & = & 0 \\ \bar{u}^\varepsilon_{|t=0} & = & \bar{v}_0, \end{array} \right.$$

and the remainder u^ε_{osc} satisfies

$$(RF^\varepsilon_{osc}) \quad \left\{ \begin{array}{rcl} \partial_t u^\varepsilon_{osc} - \nu \Delta u^\varepsilon_{osc} + Q^\varepsilon_{osc}(u^\varepsilon, u^\varepsilon) & = & \tilde{f}^\varepsilon_{osc} \\ \mathrm{div}\, u^\varepsilon_{osc} & = & 0 \\ u^\varepsilon_{osc,|t=0} & = & v_{0,osc}\ , \end{array} \right.$$

where for any function h, $\bar{h}$ represents its vertical average, and h_{osc} the remainder, defined by $h_{osc} = h - \bar{h}$. We have written $\Delta_2 = \partial_1^2 + \partial_2^2$, and $\nabla_2 = (\partial_1, \partial_2)$.

Let us note that in Fourier variables, those two equations are simply the projections of $(\widetilde{RF}^\varepsilon)$ onto $\{n \mid n_3 = 0\}$ for $(\overline{RF}^\varepsilon)$, and onto $\{n \mid n_3 \neq 0\}$ for (RF^ε_{osc}).

6.4 The limit

6.4.1 The quadratic terms

We shall consider the limit $\varepsilon \to 0$ of the equations $(\overline{RF}^\varepsilon)$ and (RF^ε_{osc}) obtained above. It is therefore a matter of taking the limit in ε of the quadratic terms, which, according to the non stationary phase theorem, amounts to studying the resonances of the operator $\mathcal{L}$: the limit of the quadratic form $\mathcal{Q}^\varepsilon$ can be written

$$\lim_{\varepsilon \to 0} \mathcal{Q}^\varepsilon(u,u) = \mathcal{F}^{-1} \sum_{(k,m,n) \in\ K} \sum_{a,b,c} e^{-i\frac{t}{\varepsilon}(\omega^a(k) + \omega^b(m) - \omega^c(n))} u^{a,j}(k) m_j u^{b,c}(m,n),$$

with the same notations as in (5.4). In particular the $\omega^a(n)$ are the eigenvalues of $P_n L$, defined by $\omega^a(n) = \pm \frac{n_3}{a_3|n|}$; we have noted K the set of resonances of PL,

$$K = \left\{ (k,m,n) \mid \frac{k_3}{a_3|k|} \pm \frac{m_3}{a_3|m|} \pm \frac{n_3}{a_3|n|} = 0 \quad \text{and} \quad k+m=n \right\}. \tag{6.2}$$

In the following, we shall call $\mathcal{K}$ the set of triplets (k,m,n) actually present in the summation after the limit has been taken, that is to say whose contribution to the limit is non zero; one has of course $\mathcal{K} \subset K$. The computations that follow, concerning the quadratic terms, can be found in [3]. We recall them here for the convenience of the reader.

Adopting the notations of [3], the set K can be written as $K = K_1 \cup K_2 \cup K_3 \cup K_4$, with

$$\begin{aligned}
K_1 &= \left\{ (k,m,n) \mid \frac{k_3}{|k|} + \frac{m_3}{|m|} + \frac{n_3}{|n|} = 0 \quad \text{and} \quad k+m=n \right\} \\
K_2 &= \left\{ (k,m,n) \mid \frac{k_3}{|k|} - \frac{m_3}{|m|} + \frac{n_3}{|n|} = 0 \quad \text{and} \quad k+m=n \right\} \\
K_3 &= \left\{ (k,m,n) \mid \frac{k_3}{|k|} - \frac{m_3}{|m|} - \frac{n_3}{|n|} = 0 \quad \text{and} \quad k+m=n \right\} \\
K_4 &= \left\{ (k,m,n) \mid \frac{k_3}{|k|} + \frac{m_3}{|m|} - \frac{n_3}{|n|} = 0 \quad \text{and} \quad k+m=n \right\}.
\end{aligned}$$

Before starting the study of these sets in both the cases we are concerned with, which correspond to the two equations $(\overline{RF}^\varepsilon)$ and (RF^ε_{osc}), let us study them more closely, following [3].

Lemma 6.1 *The following identity is true:*

$$K \cap \{(k,m,n) \mid k_3 m_3 n_3 = 0\} \quad = \quad K_{14} \cup K_{24} \cup K_{34} \cup K_{2D}, \tag{6.3}$$

where $K_{2D} = \{k_3 = m_3 = n_3 = 0\}$*, and* $K_{ij} = K_i \cap K_j - K_{2D}$*.*

Proof. As in [3], elementary computations lead to this result: let us first notice that the sets K_{ij} can be written as

$$\begin{aligned}
K_{14} &= \left\{ (k,m,n) \mid \frac{k_3}{|k|} + \frac{m_3}{|m|} = 0, \quad k+m=n, \quad \text{and} \quad n_3 = 0 \right\} - K_{2D} \\
K_{24} &= \left\{ (k,m,n) \mid \frac{m_3}{|m|} - \frac{n_3}{|n|} = 0, \quad k+m=n, \quad \text{and} \quad k_3 = 0 \right\} - K_{2D} \\
K_{34} &= \left\{ (k,m,n) \mid \frac{n_3}{|n|} - \frac{k_3}{|k|} = 0, \quad k+m=n, \quad \text{and} \quad m_3 = 0 \right\} - K_{2D} \\
K_{2D} &= \{(k,m,n) \mid k_3 = m_3 = n_3 = 0\}.
\end{aligned}$$

So let $(k,m,n) \in K \cap \{(k,m,n) \mid k_3 m_3 n_3 = 0, \text{ and } k+m=n\}$. Then there exist two real numbers $\varepsilon_1, \varepsilon_2$ such that $|\varepsilon_i| = 1$ and

$$\frac{k_3}{|k|} + \varepsilon_1 \frac{m_3}{|m|} + \varepsilon_2 \frac{n_3}{|n|} = 0.$$

If one of the three integers k_3, m_3 or n_3 is equal to 0, then

– either both the remaining integers are non zero, and we are in the case of K_{j4} ;

– or one of the remaining integers is equal to zero, and the fact that $k_3 + m_3 = n_3$ implies that the last integer is also zero, which corresponds to the space K_{2D}.

So Lemma 6.1 is proved. $\Box$

One also has, as in [3], the following lemma, which will come in very useful in the study of the resonances.

Lemma 6.2 *The set K_{14} does not contribute to the resonances:*

$$\lim_{\varepsilon\to 0} \mathcal{F}^{-1} \sum_{(k,m,n)\in K_{14}} \sum_{a,b,c} e^{-i\frac{t}{\varepsilon}(\omega^a(k)+\omega^b(m)-\omega^c(n))} u^{a,j}(k) m_j u^{b,c}(m,n) = 0.$$

Proof. It consists in a straightforward computation. Indeed, according to (6.1), one has the following expression for $\mathcal{L}$, where the exponentials have been written as sums of *cos* and *sin*:

$$\mathcal{F}\mathcal{L}(-\frac{t}{\varepsilon})u(k) = c(k)u_k - \frac{1}{|k|}s(k)k \times u_k,$$

where $u_k = \mathcal{F}u(k)$, and we have defined

$$c(k) = \cos(\frac{k_3}{a_3|k|}\frac{t}{\varepsilon}) \quad \text{and} \quad s(k) = \sin(\frac{k_3}{a_3|k|}\frac{t}{\varepsilon}).$$

It can be noticed, as in [2], that $(v \cdot \nabla)v = -v \times curl\, v + \nabla\frac{|v|^2}{2}$. Since the projection P of a gradient is equal to zero, only $v \times curl\, v$ will have a part to play here.

We are interested in the study of the bilinear term $\mathcal{Q}^\varepsilon$, whose definition is

$$\mathcal{Q}^\varepsilon(u^\varepsilon, u^\varepsilon) = \mathcal{L}(-\frac{t}{\varepsilon})\left(\mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon \cdot \nabla\mathcal{L}(\frac{t}{\varepsilon})u^\varepsilon\right).$$

If we restrict the summation to K_{14}, hence in particular to the case $n_3 = 0$, we come up with the following terms:

$$\begin{array}{ll} -c(k)c(m)u_k \times (m \times u_m) & -|m|c(k)s(m)u_k \times u_m \\ \dfrac{|m|}{|k|}s(k)s(m)(k \times u_k) \times u_m & \dfrac{1}{|k|}s(k)c(m)(k \times u_k) \times (m \times u_m) \end{array}$$

once we have noticed that $m \times (m \times u_m) = -|m|^2 u_m$.

The left hand side column is made of the even terms in t, and the right hand side of the odd terms. Let us consider for example the case of the even terms: one has the relation $\frac{k_3}{|k|} + \frac{m_3}{|m|} = 0$, which leads to $|k| = |m|$, since $k_3 + m_3 = 0$, and thus in particular $c(k) = c(m)$ and $s(k) = -s(m)$.

So if n is set to a fixed value, the even terms lead to two contributions:

$$-c^2(m)u_k \times (m \times u_m) \quad \text{and} \quad - s^2(m)(k \times u_k) \times u_m,$$

and after summation over k and m, these two indexes can be interchanged by symmetry, which gives finally

$$-c(2m)u_k \times (m \times u_m),$$

which leads to a non–zero contribution. So one finds finally an element of the type $\cos(\alpha \frac{t}{\varepsilon})$, with $\alpha \neq 0$, thus the triplets of K_{14} are not present in the limit of $\mathcal{Q}^\varepsilon$, and the lemma is proved. $\Box$

6.4.2 The limit of $(\overline{RF}^\varepsilon)$

The goal of this section is to prove the following proposition.

Proposition 6.1 *The limit of $(\overline{RF}^\varepsilon)$ is a bidimensional Navier–Stokes equation*

$$(NS2D) \quad \begin{cases} \partial_t \bar{u} - \nu \Delta_2 \bar{u} + \bar{u} \cdot \nabla_2 \bar{u} &= (-\nabla_2 \bar{p}, 0) + \bar{f} \\ \partial_3 \bar{u} &= 0 \\ \nabla_2 \cdot \bar{u} &= 0 \\ \bar{u}_{|t=0} &= \bar{v}_0 \,. \end{cases}$$

Proof. For the forcing term, we just have to notice that $\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\bar{f} = \bar{f}$. So now the proof of this proposition consists in the study of the limit of $\bar{\mathcal{Q}}^\varepsilon(u^\varepsilon, u^\varepsilon)$. Here we are in the case when $n_3 = 0$, so the resonant set satisfies

$$\mathcal{K} \quad \subset \quad K \cap \{(k, m, n) \,|\, n_3 = 0 \quad \text{and} \quad k + m = n\}.$$

According to Lemma 6.1, one therefore has $\mathcal{K} \subset K_{14} \cup K_{24} \cup K_{34} \cup K_{2D}$, and since K_{14} does not contribute to the resonances (that was proved in Lemma 6.2), one has actually

$$\mathcal{K} \quad \subset \quad K_{24} \cup K_{34} \cup K_{2D}.$$

Let us suppose that $(k, m, n) \in \mathcal{K} \cap K_{24}$: then by definition of K_{24}, one has $\frac{m_3}{|m|} - \frac{n_3}{|n|} = 0$ and $k_3 = 0$, which implies, since $n_3 = 0$, that $n_3 = k_3 = m_3 = 0$, which is not possible, since $K_{24} \cap K_{2D} = \emptyset$. The computation is identical in the case of K_{34}, once k and m have been swapped.

Therefore we have proved that $\mathcal{K} \subset K_{2D}$, which means that the only terms of $\bar{\mathcal{Q}}^\varepsilon(u^\varepsilon, u^\varepsilon)$ that do not lead to a limit equal to zero are bilinear in $\bar{u}^\varepsilon$, that is to say more precisely that

$$\lim_{\varepsilon \to 0} \bar{\mathcal{Q}}^\varepsilon(u^\varepsilon, u^\varepsilon) = \lim_{\varepsilon \to 0} \bar{\mathcal{Q}}^\varepsilon(\bar{u}^\varepsilon, \bar{u}^\varepsilon).$$

But for any vector field v that does not depend on the third variable, one clearly has $\mathcal{L}v = v$. Therefore the limit of $\bar{\mathcal{Q}}^\varepsilon(\bar{u}^\varepsilon, \bar{u}^\varepsilon)$ in $\mathcal{D}'$ is simply $P(\bar{u} \cdot \nabla \bar{u})$, which is the result we wanted. $\Box$

6.4.3 The limit of (RF^{ε}_{osc})

Let us define $u_{osc} = u - \bar{u}$, where $\bar{u}$ is the solution of the bidimensional equation $(NS2D)$, and u is the limit of u^{ε}. Let us prove the following proposition, due to [3].

Proposition 6.2 *For almost all values of a_3, the limit equation of (RF^{ε}_{osc}) is a linear equation on u_{osc}:*

$$(L_{osc}) \quad \left\{ \begin{array}{rcl} \partial_t u_{osc} - \nu \Delta u_{osc} + 2\mathcal{Q}(\bar{u}, u_{osc}) & = & 0 \\ div\, u_{osc} & = & 0 \\ u_{osc,|t=0} & = & v_{0,osc}\,. \end{array} \right.$$

Remark. Proposition 6.2 is the crucial step which will enable us to find an infinite existence time for the limit system. The admissible values of a_3 are said to be non resonant, and Lemma 6.3 below highlights the role played by the vertical variable a_3 in the following definition of a non resonant domain.

Definition 6.1 *A set of triplets (a_1, a_2, a_3) is said to be non resonant if the following condition is satisfied:*

$$(NR) \qquad K \quad \subset \quad \{(k,m,n) \in \mathbf{Z}^9 \mid k_3 m_3 n_3 = 0 \quad and \quad k + m = n\},$$

where K is the resonant set defined in (6.2).

Lemma 6.3 *For any couple of real numbers (a_1, a_2), there exists a set of measure zero of $\mathbf{R}$ outside of which for every value of a_3, the condition (NR) is satisfied.*

Remark. Let us note that a non resonant domain depends therefore on a_1 and a_2, but that its complementary in $\mathbf{R}$ is of measure zero for every (a_1, a_2).

Let us suppose temporarily that we have proved that lemma.

Proof of Proposition 6.2. It follows immediately from Lemma 6.3: we have seen previously that (RF^{ε}_{osc}) is obtained by projection, in Fourier variables, onto the set $\{n \in \mathbf{Z}^3 \mid n_3 \neq 0\}$. Therefore we have n_3 non zero in (RF^{ε}_{osc}). Consequently, for a_3 non resonant with a_1 and a_2 set to fixed values, the only possible resonances — and thus the only terms of non zero limit in the quadratic form $\mathcal{Q}^{\varepsilon}_{osc}$ — correspond to $k_3 m_3 = 0$, which means that in $\mathcal{Q}^{\varepsilon}_{osc}(u^{\varepsilon}, u^{\varepsilon})$, one of the two terms in the product does not depend on the third variable. Let us note, by the way, that one and only one of the two terms does not depend on the third variable, for if $m_3 = k_3 = 0$, then $n_3 = 0$, which contradicts the assumption made. Hence Proposition 6.2 is proved, once we have noticed that $\mathcal{L}\left(-\frac{t}{\varepsilon}\right) f^{\varepsilon}_{osc}$ has for limit zero in $\mathcal{D}'$. $\square$

Proof of Lemma 6.3. As in [2], let us define, for a parameter $\ell \in \{1, .., 4\}$, the function $\lambda^{\ell}(a,b,c) = a \pm b \pm c$. We are going to prove that, if a_1 and a_1 are set arbitrarily, then for almost all a_3, the product

$$\prod_{\ell=1}^{4} \lambda^{\ell}\left(\frac{k_3}{a_3|k|}, \frac{m_3}{a_3|m|}, \frac{n_3}{a_3|n|}\right)$$

is never zero, for any value of k,m,n, where one still has $n = k + m$ and $|n|^2 = \sum n_i^2/a_i^2$.

Let us define the quantity $X = a_3^{-1}$. Following [2], one can notice that the product above can be written as a polynomial in X, since if one defines

$$a = k_3|m||n|, \quad b = m_3|n||k|, \quad c = n_3|k||m|,$$

then $a_3^4|k|^4\,|m|^4\,|n|^4 \prod_{\ell=1}^{4} \lambda^\ell(a,b,c) = (a^2+b^2-c^2)^2 - 4a^2b^2$, and

$$\begin{array}{rcl} a^2 & = & k_3^2(|m'|^2 + X^2 m_3^2)(|n'|^2 + X^2 n_3^2) \\ b^2 & = & m_3^2(|n'|^2 + X^2 n_3^2)(|k'|^2 + X^2 k_3^2) \\ c^2 & = & n_3^2(|k'|^2 + X^2 k_3^2)(|m'|^2 + X^2 m_3^2), \end{array}$$

where $|n'|^2 = n_1^2/a_1^2 + n_2^2/a_2^2$.

Here and in the following, it is supposed that $k_3 m_3 n_3 \neq 0$. Then if we set k, m, and n to fixed arbitrary values, we are in fact considering a polynomial in X of order 4 and of dominant coefficient $k_3 m_3 n_3$ which is not zero, so which has at most 4 solutions. Then as k, m, and n vary in a countable set, there is a countable number of X, hence of a_3^{-1}, such that $\prod_{\ell=1}^{4} \lambda^\ell(\frac{k_3}{a_3|k|}, \frac{m_3}{a_3|m|}, \frac{n_3}{a_3|n|}) = 0$. This means that for all (a_1, a_2), the set of a_3 such that for all (k,m,n), the product above is zero, is of measure zero in $\mathbf{R}$. That is exactly Lemma 6.3. $\square$

6.5 Global existence for the limit system

In the previous section, we have found the limit system of the diagonalized form of the rotating fluid equations: it is

$$(NS2D) \quad \left\{ \begin{array}{rcl} \partial_t \bar{u} - \nu\Delta_2 \bar{u} + \bar{u}\cdot\nabla_2 \bar{u} & = & (-\nabla_2 \bar{p}, 0) + \bar{f} \\ \partial_3 \bar{u} & = & 0 \\ \nabla_2 \cdot \bar{u} & = & 0 \\ \bar{u}_{|t=0} & = & \bar{v}_0 \end{array} \right.$$

for the bidimensional part, and

$$(L_{osc}) \quad \left\{ \begin{array}{rcl} \partial_t u_{osc} - \nu\Delta u_{osc} + 2\mathcal{Q}(\bar{u}, u_{osc}) & = & 0 \\ \operatorname{div} u_{osc} & = & 0 \\ u_{osc,0} & = & v_{0,osc} \end{array} \right.$$

for the remainder, for almost all a_3.

To end the proof of Theorem 4 , all we have to do is check that those equations are globally well posed. That is given through the following proposition.

Proposition 6.3 *If $v_0 \in H^s$, with $s \geq \frac{1}{2}$, then the solution of the limit system $(NS2D, L_{osc})$ is defined in $C^0(\mathbf{R}^+, H^s(\mathcal{T}^3)) \cap L^2(\mathbf{R}^+, H^{s+1}(\mathcal{T}^3))$.*

Proof. The first equation is a bidimensional Navier–Stokes equation, well–known to be globally well posed (see [9], [13], [27]); the fact that the vector fields we are working with here have three components rather than two does not change anything to the classical proofs for the traditional bidimensional case. So the result is proved for $(NS2D)$.

One now just has to look at the second equation: the absence of quadratic terms in u_{osc} enables us to find the result, by a straightforward energy estimate in H^s. However, one can note that in [3] (Theorem 5.3), it is proved that for all real numbers s,

$$(\mathcal{Q}(\bar{u}, u_{osc}) \mid (P\Delta)^s u_{osc}) = 0.$$

So one has the following simpler estimate:

$$\frac{1}{2}\frac{d}{dt}|u_{osc}|_s^2 + \nu|u_{osc}|_{s+1}^2 = 0,$$

and therefore we find, for all $t \geq 0$,

$$\|u_{osc}\|_{L^\infty([0,t],H^s)}^2 + 2\nu\|u_{osc}\|_{L^2([0,t],H^{s+1})}^2 = |v_{0,osc}|_s^2. \tag{6.4}$$

Hence Proposition 6.3 is proved. □

Now one can use Theorem 1, along with section 3.4, to get the global existence for (RF^ε).

Let us note finally, as at the end of the proof of Theorem 3, that the first order expansion is found for a smooth enough initial data, for almost all $\mathcal{T}^3$: it is proved in Appendix C that a second order small divisor estimate for the perturbation is obtained for almost all $\mathcal{T}^3$, at a certain degree. So using Theorem 2, Theorem 4 is proved. □

6.6 A remark on the well prepared case

Let us make a few comments on the case when the initial data is well prepared, *i.e.* , in the case when $v_0 = \bar{v}_0$ and $f = \bar{f}$. In that case, estimate (6.4) implies that for all t, we have $u(t) = \bar{u}(t)$, which means that the limit system reduces to the bidimensional Navier–Stokes equation $(NS2D)$.

Moreover, it is easy to see that in those conditions, $\tilde{R}^\varepsilon_{osc}(u) = \tilde{R}^\varepsilon_{osc}(\bar{u}) = 0$, with the notations of Theorem 2; so in particular $\tilde{R}^\varepsilon_{osc}(u)_{|t=0} = 0$. And since similarly $R(u) = 0$, we have $h = 0$, still with the notations of Theorem 2.

So we get $u^\varepsilon = \bar{u} + o(\varepsilon)$: we have recovered part of a result proved in [17] which states that if the initial data of the tridimensional Navier–Stokes equation (resp. of (RF^ε)) is bidimensional, then the solution is also bidimensional, and is the solution of $(NS2D)$.

7 An anisotropic study of the rotating fluid equations

7.1 Introduction

Some of the results obtained in section 6, in the proof of Theorem 4 stating the global existence of the rotating fluid equations, show clearly that the vertical variable plays an important role: for instance it turned out to be easy to prove that the limit system of (RF^ε) was globally well posed, once we computed the vertical average of the equation.

However one could wish to see this special dependence appear more explicitly in the proof, and it can be expected that a more precise study of the role of the different variables, in the particular case of the rotating fluids, may lead to better results. With that in view, we are going to use, in this section, regularity spaces of the Sobolev and Besov type, where one considers different regularities along the horizontal and the vertical variables.

In the first part, we will define the Sobolev-type spaces $H^{s,s'}$, as well as the Besov-type $HB^{s,s'}$; all of the definitions and the properties we will state here come from the study of D. Iftimie in [21] and [22], which we will constantly be referring to.

We will then check that the limit system of (RF^ε) is globally well posed in spaces of that type, and the last part will consist in writing again part of the proof of Theorem 1 in those new spaces.

That study will lead to the proof of Theorems 5 and 6 stated in the introduction.

7.2 Anisotropic Sobolev and Besov spaces

7.2.1 An anisotropic Littlewood–Paley decomposition

Before defining the spaces we want to work with, let us give the definition of the following anisotropic Littlewood–Paley decomposition, as it is presented by D. Iftimie in [21].

Definition 7.1 *Let $\chi \in C^\infty(\mathbf{R})$ be a positive function, compactly supported in $[-1,1]$, and equal to 1 near $[0,\frac{1}{2}]$. For all positive integers q and q', we define the following horizontal and vertical truncation operators:*

$$S_q^h u = \sum_{n\in\mathbf{Z}^3} u_n e^{in\cdot x}\chi\left(\frac{|n'|}{2^q}\right); \quad \Delta_q^h = S_q^h - S_{q-1}^h; \quad \Delta_0^h = S_0^h\;;$$

$$S_{q'}^v u = \sum_{n\in\mathbf{Z}^3} u_n e^{in\cdot x}\chi\left(\frac{|n_3|}{2^{q'}}\right); \quad \Delta_{q'}^v = S_{q'}^v - S_{q'-1}^v; \quad \Delta_0^v = S_0^v.$$

Definition 7.2 *We define $L^{p,q} = \left\{ u \mid \|u\|_{L^{p,q}} \stackrel{def}{=} \left\| \|u(x')\|_{L^q_{x_3}} \right\|_{L^p_{x'}} < \infty \right\}$, and in a similar way for sequences, we define the spaces $\ell^{p,q}$.*

7.2.2 Anisotropic Sobolev and Besov spaces

We are going to write here results stated and proved by D. Iftimie in [21], regarding the following anisotropic spaces:

Definition 7.3 *The space $H^{s,s'}$ is defined by the following norm, for all $u \in \mathcal{D}'(\mathcal{T}^3)$:*

$$|u|_{s,s'} = \|(1+|n'|_{\mathcal{T}})^s(1+|n_3|_{\mathcal{T}})^{s'}|u_n|\|_{\ell^2},$$

where we have written for all n, $n' = (n_1, n_2)$.

As we are working with $2\pi a_j$–periodic functions, the norms are defined by

$$|n_i|_{\mathcal{T}} = \left|\frac{n_i}{a_i}\right| \quad \text{and similarly} \quad n \cdot x = \sum_i \frac{n_i x_i}{a_i^2}.$$

Remark. The space $H^{s,s'}$ can be understood as the space of H^s functions in the horizontal variables, and $H^{s'}$ in the vertical variable.

We shall now define in a similar way the anisotropic Besov spaces. First let us recall that the spaces $B^s_{p,q}(\mathcal{T}^d)$, for $q < \infty$, are defined as the completion of $\mathcal{S}$ for the following norm:

$$\|u\|_{B^s_{p,q}} \stackrel{def}{=} \left(\sum_{k \in \mathbf{N}} 2^{kqs}\|\Delta_k u\|^q_{L^p}\right)^{\frac{1}{q}},$$

and where Δ_k is the standard Littlewood–Paley decomposition defined in section 2.1.

We can then define an anisotropic version of those spaces, still following [21]:

Definition 7.4 *The space $HB^{s,s'}$ is defined by the following norm, for all $u \in \mathcal{D}'(\mathcal{T}^3)$:*

$$|u|_{HB^{s,s'}} = \|2^{qs}2^{q's'}|\Delta^h_q \Delta^v_{q'} u|_0\|_{\ell^{2,1}}.$$

Remark. We shall more particularly work with the space $HB^{0,\frac{1}{2}}$, which can be understood as the space of L^2 functions in the horizontal variables, and $B^{\frac{1}{2}}_{2,1}$ in the vertical variable, the advantage of the latter space being that it is an algebra, contrary to $H^{\frac{1}{2}}(\mathcal{T})$. It is easy to see that $HB^{s,\frac{1}{2}} \subset C(\mathcal{T}_{x_3}, H^s(\mathcal{T}^2))$. Let us also note that $HB^{s,s'}$ is strictly imbedded in $H^{s,s'}$.

To finish this introductory section, let us state the various properties satisfied by those spaces, which shall be used in the course of our study. The proofs are all written in [21]; in the following lemma, $X^{s,s'}$ is indifferently $H^{s,s'}$ or $HB^{s,s'}$.

Lemma 7.1 *If $u \in H^{s,s'}$, then $|u|_{s,s'} \sim \|2^{qs+q's'}|\Delta_q^h \Delta_{q'}^v u|_0\|_{\ell^{2,2}}$. If $u \in \mathcal{D}'$ has mean zero on $\mathcal{T}^3$, then*

$$|\nabla u|_{X^{s,s'}} \sim |u|_{X^{s+1,s'}} + |u|_{X^{s,s'+1}} \sim \sup_{\alpha\in[0,1]} |u|_{X^{s+\alpha,s'+1-\alpha}}.$$

Let s, s', t, t' be four real numbers, let $\alpha \in [0,1]$, and consider a function u in $X^{s,s'} \cap X^{t,t'}$. Then we have $u \in X^{\alpha s+(1-\alpha)t, \alpha s'+(1-\alpha)t'}$, and

$$|u|_{X^{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}} \leq C|u|^{\alpha}_{X^{s,s'}}|u|^{1-\alpha}_{X^{t,t'}}.$$

If $u \in X^{s,s'}$ and $v \in X^{t,t'}$, with $s < 1$, $t < 1$, $s+t > 0$, $s' < \frac{1}{2}$, $t' < \frac{1}{2}$, and $s'+t' > 0$, then we have $uv \in X^{s+t-1,s'+t'-\frac{1}{2}}$, and

$$|uv|_{X^{s+t-1,s'+t'-\frac{1}{2}}} \leq C|u|_{X^{s,s'}}|v|_{X^{t,t'}}.$$

If $u \in H^s(\mathcal{T}^2)$ and $v \in X^{t,t'}$, with $s < 1$, $t < 1$, and $s+t > 0$, then $uv \in X^{s+t-1,t'}$, and

$$|uv|_{X^{s+t-1,t'}} \leq C|u|_s|v|_{X^{t,t'}}.$$

7.3 The study of the limit system

The limit system corresponding to (RF^ε) was computed in section 6. Before we prove that the filtered solution u^ε converges to the solution u of the limit system in the anisotropic spaces, let us check that the limit system is globally well posed. Defining $u = \bar{u} + u_{osc}$, we have, in a non–resonant domain, the following limit system:

$$(NS2D) \quad \begin{cases} \partial_t \bar{u} - \nu\Delta_2 \bar{u} + \bar{u}\cdot\nabla_2 \bar{u} &= (-\nabla_2 \bar{p}, 0) + \bar{f} \\ \partial_3 \bar{u} &= 0 \\ \nabla_2 \cdot \bar{u} &= 0 \\ \bar{u}_{|t=0} &= \bar{v}_0 \end{cases}$$

$$(L_{osc}) \quad \begin{cases} \partial_t u_{osc} - \nu\Delta u_{osc} + 2\mathcal{Q}(\bar{u}, u_{osc}) &= 0 \\ \nabla\cdot u_{osc} &= 0 \\ u_{osc,|t=0} &= v_{0,osc}\,. \end{cases}$$

The hypotheses of Theorems 5 and 6 assume that $\bar{v}_0 \in L^2(\mathcal{T}^2)$, and that $\bar{f}$ and f_{osc} are elements of $L^2(\mathbf{R}^+, H^{-1}(\mathcal{T}^2))$ and $L^2(\mathbf{R}^+, H^{2\delta-1,\frac{1}{2}-2\delta})$ or $L^2(\mathbf{R}^+, HB^{-1,\frac{1}{2}})$ respectively. Then classical results concerning the bidimensional Navier–Stokes equations (see [9], [13], [27], [36]) lead to the existence and uniqueness of a solution $\bar{u}$ of $(NS2D)$, and

$$\bar{u} \in C^0(\mathbf{R}^+, L^2(\mathcal{T}^2)) \cap L^2(\mathbf{R}^+, H^1(\mathcal{T}^2)).$$

We are now going to study the system (L_{osc}), first in the case of the $H^{s,s'}$ spaces, and then for the $HB^{s,s'}$ spaces. From now on, δ shall be a real number in the interval $]0,1[$.

Proposition 7.1 *If $v_{0,osc}$ satisfies $v_{0,osc} \in H^{\delta,\frac{1}{2}-\delta}$, then the system (L_{osc}) is globally well posed in the space $C_b^0(\mathbf{R}^+, H^{\delta,\frac{1}{2}-\delta}) \cap L^4(\mathbf{R}^+, H^{\frac{1}{2}+\frac{\delta}{2},\frac{1}{2}-\frac{\delta}{2}})$.*

Proposition 7.2 *If $v_{0,osc}$ satisfies $v_{0,osc} \in HB^{0,\frac{1}{2}}$, then the system (L_{osc}) is globally well posed in the space $C_b^0(\mathbf{R}^+, HB^{0,\frac{1}{2}}) \cap L^4(\mathbf{R}^+, HB^{\frac{1}{2},\frac{1}{2}})$.*

Proof of Proposition 7.1. We are going to study the equation

$$\partial_t u_{osc} - \nu \Delta u_{osc} + 2\mathcal{Q}(\bar{u}, u_{osc}) = 0, \tag{7.1}$$

with $\bar{u} \in C^0(\mathbf{R}^+, L^2(\mathcal{T}^2)) \cap L^2(\mathbf{R}^+, H^1(\mathcal{T}^2))$. Let us start by noticing, as in section 6, that the term $\mathcal{Q}(\bar{u}, u_{osc})$ does not appear in the energy estimates. That is due to Theorem 5.3 of [3], which states in particular that

$$\sum_n \mathcal{F}\mathcal{Q}(\bar{u}, u_{osc})(n) |n'|^s \mathcal{F}u_{osc}(n) = 0, \quad \text{for all } s \in \mathbf{R}, \tag{7.2}$$

as well as

$$\Big(\mathcal{Q}(\bar{u}, u_{osc}) \mid \partial_3^\ell u_{osc}\Big) = 0, \quad \text{for all } \ell. \tag{7.3}$$

It is moreover easy to see, from the proof of Theorem 5.3 in [3], that one also has

$$\sum_{n,q,q'} \mathcal{F}\mathcal{Q}(\bar{u}, u_{osc})(n) \chi\left(\frac{|n'|}{2^q}\right) \chi\left(\frac{|n_3|}{2^{q'}}\right) \mathcal{F}u_{osc}(n) = 0, \tag{7.4}$$

i.e. that $|n'|^s$ in (7.2) can be replaced by any function of $|n'|$, and similarly for $|n_3|^\ell$ in the case of (7.3). Those remarkable properties are due to the fact that very few frequencies are left in the convolution product, when $\mathcal{Q}(\bar{u}, u_{osc})$ is written in Fourier variables. Namely, using the notation presented page 39, we have

$$\mathcal{K} \cap \{(k, m, n) \mid k_3 = 0 \quad \text{and} \quad n_3 \neq 0\} \quad = \quad K_{24}\,.$$

So if k is the Fourier variable associated with $\bar{u}$, and m is the Fourier variable associated with u_{osc}, with $k + m = n$, then one can exchange m and n in the convolution, which gives the result exactly in the same way as in [3] for the case of (7.2) and (7.3); so we will not go into the details of the computation here.

That property will make the study considerably easier than in the general case (see [21] for instance).

Let us follow computations in [21]. If we apply the operators $\Delta_q^h \Delta_{q'}^v$ to equation (7.1), we get, writing $c_{q,q'}$ for a sequence of norm 1 in $\ell^{2,2}$,

$$\frac{1}{2}\frac{d}{dt}|\Delta_q^h \Delta_{q'}^v u_{osc}|_0^2 + \nu |\Delta_q^h \Delta_{q'}^v \nabla u_{osc}|_0^2 \leq 2\left(\Delta_q^h \Delta_{q'}^v \mathcal{Q}(\bar{u}, u_{osc}) \mid \Delta_q^h \Delta_{q'}^v u_{osc}\right).$$

Then after multiplication by $2^{2q\delta + 2q'(\frac{1}{2}-\delta)}$ and summation, we immediately get, using (7.4),

$$\frac{d}{dt}|u_{osc}|^2_{\delta,\frac{1}{2}-\delta} + \nu |\nabla u_{osc}|^2_{\delta,\frac{1}{2}-\delta} \leq 0,$$

using Lemma 7.1 which implies in particular that $|u_{osc}|_{1,\frac{1}{2}} \leq C|\nabla u_{osc}|_{\delta,\frac{1}{2}-\delta}$. This leads to

$$|u_{osc}(t)|^2_{\delta,\frac{1}{2}-\delta} + \nu \int_0^t |\nabla u_{osc}(\tau)|^2_{\delta,\frac{1}{2}-\delta}\, d\tau \leq |u_{osc,0}|^2_{\delta,\frac{1}{2}-\delta}.$$

Then by an interpolation using Lemma 7.1, we get the expected result. $\square$

Proof of Proposition 7.2. As in [21], let us first of all note that

$$c(2^{2q} + 2^{2q'})|\Delta^h_q \Delta^v_{q'} u_{osc}|^2_0 \leq |\Delta^h_q \Delta^v_{q'} \nabla u_{osc}|^2_0.$$

By Lemma 7.1, and if we call $c_{q,q'}$ the sequences of norm 1 in $\ell^{2,1}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\Delta^h_q \Delta^v_{q'} u_{osc}|^2_0 + c\nu(2^{2q} + 2^{2q'})|\Delta^h_q \Delta^v_{q'} u_{osc}|^2_0 \quad &\leq \quad C2^{\frac{q}{2}-\frac{q'}{2}} c_{q,q'}|\bar{u}|_{\frac{1}{2}} |\nabla u_{osc}|_{HB^{0,\frac{1}{2}}} |\Delta^h_q \Delta^v_{q'} u_{osc}|_0 \\ &+ \quad C2^{\frac{q}{2}-\frac{q'}{2}} c_{q,q'}|\bar{u}|_1 |u_{osc}|_{HB^{\frac{1}{2},\frac{1}{2}}} |\Delta^h_q \Delta^v_{q'} u_{osc}|_0.\end{aligned}$$

A variant of Gronwall's Lemma leads to the following estimate, after multiplication by $2^{\frac{q'}{2}}$ and summation on q':

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{q'} 2^{\frac{q'}{2}}|\Delta^h_q \Delta^v_{q'} u_{osc}|_0 \quad &+ \quad c\nu \sum_{q'} 2^{\frac{q'}{2}}(2^{2q} + 2^{2q'})|\Delta^h_q \Delta^v_{q'} u_{osc}|_0 \\ &\leq \quad C2^{\frac{q}{2}} b_q |\bar{u}|_{\frac{1}{2}} |\nabla u_{osc}|_{HB^{0,\frac{1}{2}}} + C2^{\frac{q}{2}} b_q |\bar{u}|_1 |u_{osc}|_{HB^{\frac{1}{2},\frac{1}{2}}},\end{aligned}$$

where b_q is a sequence of norm 1 in ℓ^2.

Let us multiply this estimate by $\sum_{q'} 2^{\frac{q'}{2}}|\Delta^h_q \Delta^v_{q'} u_{osc}|_0$, and take the sum on q. We find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_q (\sum_{q'} 2^{\frac{q'}{2}}|\Delta^h_q \Delta^v_{q'} u_{osc}|_0)^2 \\ + \quad & c\nu \sum_q \left(\sum_{q'} 2^{\frac{q'}{2}}(2^{2q} + 2^{2q'})|\Delta^h_q \Delta^v_{q'} u_{osc}|_0 \right) \left(\sum_{q'} 2^{\frac{q'}{2}}|\Delta^h_q \Delta^v_{q'} u_{osc}|_0 \right) \\ \leq \quad & C(|\bar{u}|_{\frac{1}{2}} |\nabla u_{osc}|_{HB^{0,\frac{1}{2}}} + |\bar{u}|_1 |u_{osc}|_{HB^{\frac{1}{2},\frac{1}{2}}}) \sum_q 2^{\frac{q}{2}} b_q \sum_{q'} 2^{\frac{q'}{2}}|\Delta^h_q \Delta^v_{q'} u_{osc}|_0.\end{aligned}$$

But Cauchy–Schwartz implies that

$$C \sum_{q',q''} 2^{\frac{q'}{2}}(2^{2q} + 2^{2q'})|\Delta^h_q \Delta^v_{q'} u_{osc}|_0 2^{\frac{q''}{2}}|\Delta^h_q \Delta^v_{q''} u_{osc}|_0 \geq \left(\sum_{q'} 2^{\frac{q'}{2}}(2^q + 2^{q'})|\Delta^h_q \Delta^v_{q'} u_{osc}|_0 \right)^2,$$

and similarly

$$\sum_q 2^{\frac{q}{2}} b_q \sum_{q'} 2^{\frac{q'}{2}}|\Delta^h_q \Delta^v_{q'} u_{osc}|_0 \leq \left(\sum_q (\sum_{q'} 2^{\frac{q+q'}{2}}|\Delta^h_q \Delta^v_{q'} u_{osc}|_0)^2 \right)^{\frac{1}{2}}.$$

Finally, after interpolating between $HB^{0,\frac{1}{2}}$ and $HB^{1,\frac{1}{2}}$, and using Lemma 7.1, we get

$$\frac{1}{2}\frac{d}{dt}|u_{osc}|^2_{HB^{0,\frac{1}{2}}} + \frac{\nu}{C}|\nabla u_{osc}|^2_{HB^{0,\frac{1}{2}}} \leq \frac{C}{\nu}\left(|\bar{u}|_1^2 + \frac{C}{\nu}|\bar{u}|^4_{\frac{1}{2}}\right)|u_{osc}|^2_{HB^{0,\frac{1}{2}}}.$$

Then as above, the proposition is proved. $\square$

Remark. It can be noticed that the initial data $v_{0,osc}$ is of vertical mean equal to zero, that $\bar{u}$ does not depend on the third variable, and that (L_{osc}) is linear, hence u_{osc} remains of vertical mean zero for all time. But the space $\left\{w \in HB^{0,\frac{1}{2}} \mid \int_{x_3} w\, dx_3 = 0\right\}$ is invariant for the usual scaling of the Navier–Stokes equations as well as for the vertical scaling $x_3 \mapsto \lambda x_3$. Hence the estimates, when $v_{0,osc}$ in $HB^{0,\frac{1}{2}}$, have the same scale–invariance, and in particular are invariant for the vertical scaling.

7.4 The limit of $(\widetilde{RF}^{\varepsilon})$

In this section we are going to follow the same method as the one leading to Theorem 1: we want to prove that u^{ε}, solution of $(\widetilde{RF}^{\varepsilon})$, converges strongly to u, solution of $(NS2D, L_{osc})$. In the isotropic case, that result was found by studying the equation satisfied by the vector field w^{ε} defined by $w^{\varepsilon} = u^{\varepsilon} - u$:

$$\partial_t w^{\varepsilon} - \nu\Delta w^{\varepsilon} + \mathcal{Q}^{\varepsilon}(w^{\varepsilon}, w^{\varepsilon}) + 2\mathcal{Q}^{\varepsilon}(w^{\varepsilon}, u) = \mathcal{Q}(u,u) - \mathcal{Q}^{\varepsilon}(u,u) + g^{\varepsilon}.$$

But the right–hand side can be written a little differently if we notice that since $\bar{u}$ does not depend on the third variable, then $\mathcal{Q}^{\varepsilon}(\bar{u},\bar{u}) = P(\bar{u}\cdot\nabla\bar{u})$, which means that $\mathcal{Q}^{\varepsilon}(\bar{u},\bar{u}) = \mathcal{Q}(\bar{u},\bar{u})$. Therefore we have

$$\mathcal{Q}(u,u) - \mathcal{Q}^{\varepsilon}(u,u) = \mathcal{Q}(u_{osc},u_{osc}) - \mathcal{Q}^{\varepsilon}(u_{osc},u_{osc}) + 2\mathcal{Q}(u_{osc},\bar{u}) - 2\mathcal{Q}^{\varepsilon}(u_{osc},\bar{u}).$$

Moreover, in a non resonant domain we have $\mathcal{Q}(u_{osc},u_{osc}) = 0$ and $\mathcal{Q}^{\varepsilon}(u_{osc},u_{osc})$ converges weakly to 0. Similarly, we can notice that as $\bar{f} = \mathcal{L}(-\frac{t}{\varepsilon})\bar{f}$, we have $g^{\varepsilon} = \mathcal{L}(-\frac{t}{\varepsilon})f_{osc}$, where we have defined $f_{osc} = f - \bar{f}$. Hence the equation on w^{ε} is in fact

$$\partial_t w^{\varepsilon} - \nu\Delta w^{\varepsilon} + \mathcal{Q}^{\varepsilon}(w^{\varepsilon},w^{\varepsilon}) + 2\mathcal{Q}^{\varepsilon}(w^{\varepsilon},u) = 2\mathcal{Q}(u_{osc},\bar{u}) - 2\mathcal{Q}^{\varepsilon}(u_{osc},\bar{u}) - \mathcal{Q}^{\varepsilon}(u_{osc},u_{osc}) + g^{\varepsilon} \quad (7.5)$$

where $\mathcal{Q}(u_{osc},\bar{u}) - \mathcal{Q}^{\varepsilon}(u_{osc},\bar{u})$, $\mathcal{Q}^{\varepsilon}(u_{osc},u_{osc})$ and g^{ε} converge weakly to 0 in $\mathcal{D}'$.

We can now follow the method used to prove Lemma 3.1, in Appendix A: a change of function brings the problem down to the study of a similar equation, where the right–hand side converges strongly to 0. Let us therefore read again the proof of Theorem 1, in section 3, and see what modifications have to be brought in order to treat the anisotropic case, first when the initial data is in $H^{\delta,\frac{1}{2}-\delta}$ (in section 7.4.1), and then when the data is in $HB^{0,\frac{1}{2}}$ (in section 7.4.2).

7.4.1 The case when the initial data is in $H^{\delta,\frac{1}{2}-\delta}$

Following the method leading to Lemma 3.1, proved in Appendix A, let us carry out the following change of function:

$$\psi_N^\varepsilon = w^\varepsilon + \varepsilon \tilde{F}_N^\varepsilon(u,u) + \varepsilon \tilde{g}_N^\varepsilon, \tag{7.6}$$

where the terms $\tilde{F}_N^\varepsilon(u,u)$ and $\tilde{g}_N^\varepsilon$ correspond to the right–hand side of (7.5), where only the low frequencies in u are considered, and where we have divided each term in the sum by the quantity $\omega^a(k)+\omega^b(n-k)-\omega^c(n)$ and $\omega^a(k)$ respectively — where let us recall that we have $\omega^a(k) = \pm \dfrac{k_3}{|k|}$. We get

$$\tilde{F}_N^\varepsilon(u,u) = \mathcal{F}^{-1} \sum_{\substack{k+m=n \\ |k|,|m|\leq N}} \sum_{\omega_{k,n}^{a,b,c}\neq 0} \frac{i}{\omega_{k,n}^{a,b,c}} e^{-i\frac{t}{\varepsilon}\omega_{k,n}^{a,b,c}} u^{a,j}(k) m_j u^{b,c}(m,n),$$

where $\omega_{k,n}^{a,b,c} = \omega^a(k)+\omega^b(n-k)-\omega^c(n)$, and similarly

$$\tilde{g}_N^\varepsilon = \mathcal{F}^{-1} \sum_{|k|\leq N, k_3\neq 0} \frac{i}{\omega^a(k)} e^{\omega^a(k)\frac{t}{\varepsilon}} \left(\mathcal{F}f(k), e^a(k)\right) e^a(k).$$

The fact that only low frequencies are involved in the change of function, implies that from now on, we can work on ψ_N^ε rather than on w^ε, since $\tilde{g}_N^\varepsilon$ and $\tilde{F}_N^\varepsilon$ are as smooth as needed, provided one pays with enough powers of N. In the following, we shall note

$$F^{\varepsilon,N} = F^\varepsilon - F_N^\varepsilon \quad \text{and} \quad g^{\varepsilon,N} = g^\varepsilon - g_N^\varepsilon,$$

where as earlier in the article, for all functions a, we have written $a_N = \mathcal{F}^{-1}(\mathbf{1}_{\{|\cdot|\leq N\}}\mathcal{F}a(\cdot))$.

Before going into the heart of the study, let us notice that one need not prove the existence of ψ_N^ε globally in time, but just for a time T, large enough so that the global existence of (RF^ε) follows by classical arguments (as seen in section 3.4, it is well known that it is enough to control the Navier–Stokes equations for a small time, depending on the size of the initial data, to conclude to the global existence of the solutions). Hence in the following we are going to work on a time span $[0,T]$, where T is arbitrary, chosen large enough.

Proposition 7.3 *The function ψ_N^ε defined in* (7.6) *satisfies an equation of the type*

$$\partial_t \psi_N^\varepsilon - \nu\Delta\psi_N^\varepsilon + \mathcal{Q}^\varepsilon(\psi_N^\varepsilon,\psi_N^\varepsilon) + \mathcal{Q}^\varepsilon(\psi_N^\varepsilon, b+\bar{b}) = R_{1,N}^\varepsilon + \bar{R}_2^{\varepsilon,N} + \nabla\cdot R_3^{\varepsilon,N} + R_4^{\varepsilon,N},$$

where b and $\bar{b}$ depend on u and satisfy

$$b \in C^0([0,T], H^{\delta,\frac{1}{2}-\delta}), \qquad \nabla b \in L^2([0,T], H^{\delta,\frac{1}{2}-\delta}),$$
$$\bar{b} \in C^0([0,T], L^2(\mathcal{T}^2)) \cap L^2([0,T], H^1(\mathcal{T}^2)),$$

where $R_{1,N}^\varepsilon$ goes to 0 with ε, with N fixed, in $L^2([0,T], H^{\delta-1,\frac{1}{2}-\delta})$, and finally where $\bar{R}_2^{\varepsilon,N}$, $R_3^{\varepsilon,N}$, and $R_4^{\varepsilon,N}$ go to 0 when N goes to infinity, uniformly in ε, respectively in the spaces $L^2([0,T], H^{\delta-1,\frac{1}{2}-\delta})$, $L^2([0,T], H^{\delta,\frac{1}{2}-\delta})$, and $L^2([0,T], H^{2\delta-1,\frac{1}{2}-2\delta})$.

Proof. The equation satisfied by ψ_N^ε can be written as

$$\partial_t \psi_N^\varepsilon - \nu\Delta\psi_N^\varepsilon + \mathcal{Q}(\psi_N^\varepsilon,\psi_N^\varepsilon) + 2\mathcal{Q}^\varepsilon(\psi_N^\varepsilon, u - \varepsilon\tilde h_N^\varepsilon) = \varepsilon\tilde F_N^\varepsilon(\partial_t u, u) + 2\varepsilon\mathcal{Q}^\varepsilon(\tilde h_N^\varepsilon, u) + \varepsilon\tilde g_N^{\varepsilon,t}$$
$$-F^{\varepsilon,N}(\bar u, u_{osc}) - F^{\varepsilon,N}(u_{osc},u_{osc}) - g^{\varepsilon,N} - \varepsilon\nu\Delta\tilde h_N^\varepsilon(u,u) - \varepsilon^2\mathcal{Q}^\varepsilon(\tilde h_N^\varepsilon,\tilde h_N^\varepsilon), \tag{7.7}$$

keeping in mind that $F^{\varepsilon,N}(\bar u,\bar u)=0$. We have written $\tilde h_N^\varepsilon = \tilde F_N^\varepsilon + \tilde g_N^\varepsilon$. Then if

$$u - \varepsilon\tilde h_N^\varepsilon(u) = \left(\bar u - \varepsilon\tilde F_{N,s}^\varepsilon(\bar u, u_{osc})\right) + \left(u_{osc} - \varepsilon\tilde F_N^\varepsilon(u_{osc},u_{osc}) - \varepsilon\tilde g_N^\varepsilon\right),$$

remembering that $g_N^\varepsilon = \left(\mathcal{L}(-\frac{t}{\varepsilon})f_{osc}\right)_N$, the results obtained in section 7.3 imply that we can write $u-\varepsilon\tilde h_N^\varepsilon(u) = \bar b + b$, where b and $\bar b$ have the required smoothness. Indeed, in $\tilde F_N^\varepsilon(u,u)$, u only appears with low frequencies, hence $\tilde F_N^\varepsilon(u,u)$ is as smooth as wanted, at the cost of enough powers of N. The same holds for $\tilde g_N^\varepsilon$. Thus provided ε is chosen small enough, when N is fixed, one can write

$$\|b\|_{C^0([0,T],H^{\delta,\frac12-\delta})} \le 2\|u_{osc}\|_{C^0([0,T],H^{\delta,\frac12-\delta})},$$
$$\|\nabla b\|_{L^2([0,T],H^{\delta,\frac12-\delta})} \le 2\|\nabla u_{osc}\|_{L^2([0,T],H^{\delta,\frac12-\delta})},$$
$$\|\bar b\|_{C^0([0,T],L^2(\mathcal{T}^2))\cap L^2([0,T],H^1(\mathcal{T}^2))} \le 2\|\bar u\|_{C^0([0,T],L^2(\mathcal{T}^2))\cap L^2([0,T],H^1(\mathcal{T}^2))}.$$

- The low frequencies in the right–hand side of (7.7):

The terms in (7.7) where $\tilde F_N^\varepsilon$ alone appears, *i.e.* $\tilde F_N^\varepsilon(\partial_t u, u)$, $\Delta\tilde F_N^\varepsilon(u,u)$, and $\mathcal{Q}^\varepsilon(\tilde F_N^\varepsilon,\tilde F_N^\varepsilon)$, are clearly as smooth as wanted, for the reasons stated above. As to the remaining low frequency term, $\mathcal{Q}^\varepsilon(\tilde F_N^\varepsilon, u)$, it can be cut into $\mathcal{Q}^\varepsilon(\tilde F_N^\varepsilon,\bar u)$ and $\mathcal{Q}^\varepsilon(\tilde F_N^\varepsilon,u_{osc})$, and these two terms have respectively the smoothness of $\nabla\cdot(\tilde F_N^\varepsilon\otimes\bar u)$ and $\nabla\cdot(\tilde F_N^\varepsilon\otimes u_{osc})$.

Hence using Lemma 7.1, $\mathcal{Q}^\varepsilon(\tilde F_N^\varepsilon,\bar u)$ and $\mathcal{Q}^\varepsilon(\tilde F_N^\varepsilon,u_{osc})$ are bounded by a constant c_N in the space $L^2([0,T],H^{\delta-1,\frac12-\delta})$.

Finally the same holds for the low frequency terms where $\tilde g_N^\varepsilon$ appears (let us recall that we have assumed that $\partial_t f_{osc}\in L^2(\mathbf{R}^+,H^{s,s'})$ for some s and s'), therefore the term $R_{1,N}^\varepsilon$ of Proposition 7.3 has been found.

- The high frequencies in the right–hand side of (7.7):

We are now left with three terms, $F^{\varepsilon,N}(\bar u,u_{osc})$, $F^{\varepsilon,N}(u_{osc},u_{osc})$, and $\tilde g^{\varepsilon,N}$, which are going to be dealt with, using the same compactness arguments as in Lemma A.4, and based on Lebesgue's and Dini's theorems: if

$$u^N = \mathcal{F}^{-1}\left(\mathbf{1}_{|n|\ge N}\mathcal{F}u(n)\right),$$

then $\bar u^N$ and u_{osc}^N go to zero, when N goes to infinity, respectively in $C^0([0,T],L^2(\mathcal{T}^2))\cap L^2([0,T],H^1(\mathcal{T}^2))$ and in the space $C^0([0,T],H^{\delta,\frac12-\delta})\cap L^4([0,T],H^{\frac12+\frac\delta2,\frac12-\frac\delta2})$, and ∇u_{osc}^N goes to zero in $L^2([0,T],H^{\delta,\frac12-\delta})$.

Then Lemma 7.1 implies that $\nabla\cdot(\bar{u}^N\otimes u_{osc})$ and $\nabla\cdot(\bar{u}\otimes u_{osc}^N)$ go to zero in $L^2([0,T],H^{\delta-1,\frac{1}{2}-\delta})$, so $F^{\varepsilon,N}(\bar{u},u_{osc})$ can be written as the term $\bar{R}_2^{\varepsilon,N}$, and similarly $u_{osc}^N\otimes u_{osc}$ goes to zero in the space $L^2([0,T],H^{\delta,\frac{1}{2}-\delta})$: that accounts for the term $R_3^{\varepsilon,N}$. The same is true for the forcing term $g^{\varepsilon,N}$, in $L^2([0,T],H^{2\delta-1,\frac{1}{2}-2\delta})$, hence the terms $\bar{R}_2^{\varepsilon,N}$, $R_3^{\varepsilon,N}$, and $R_4^{\varepsilon,N}$ are obtained, and Proposition 7.3 is proved. □

- End of the proof of Theorem 5:

In the computations that follow, in order to use simpler notations, we shall define $W_1=R_3^{\varepsilon,N}$, $W_2=R_{1,N}^\varepsilon+\bar{R}_2^{\varepsilon,N}$, and $W_3=R_4^{\varepsilon,N}$. Then ψ_N^ε satisfies

$$\partial_t\psi_N^\varepsilon-\nu\Delta\psi_N^\varepsilon+\mathcal{Q}^\varepsilon(\psi_N^\varepsilon,\psi_N^\varepsilon)+\mathcal{Q}^\varepsilon(\psi_N^\varepsilon,b+\bar{b})=\nabla\cdot W_1+W_2+W_3,$$

with

$$b\in C^0([0,T],H^{\delta,\frac{1}{2}-\delta}),\quad \nabla b\in L^2([0,T],H^{\delta,\frac{1}{2}-\delta}),$$
$$\bar{b}\in C^0([0,T],L^2)\cap L^2([0,T],H^1(\mathcal{T}^2)),$$

as well as with

$$W_1\in L^2([0,T],H^{\delta,\frac{1}{2}-\delta}),\quad W_2\in L^2([0,T],H^{\delta-1,\frac{1}{2}-\delta}),$$
$$\text{and}\quad W_3\in L^2([0,T],H^{2\delta-1,\frac{1}{2}-2\delta}).$$

Let us also note that the initial data, $\psi^\varepsilon_{N|t=0}=\varepsilon\tilde{h}_N^\varepsilon(u_{|t=0},u_{|t=0})$ goes to zero with ε in $H^{\delta,\frac{1}{2}-\delta}$, when N is fixed, since it is once again a low frequency term.

We shall now write energy estimates for this equation, writing $\psi^\varepsilon_{N,q,q'}=\Delta_q^h\Delta_{q'}^v\psi_N^\varepsilon$, and using the rules of product stated in Lemma 7.1:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi^\varepsilon_{N,q,q'}|_0^2+\nu|\nabla\psi^\varepsilon_{N,q,q'}|_0^2 &\leq C2^{-q\delta-q'(\frac{1}{2}-\delta)}c_{q,q'}|\nabla\psi^\varepsilon_{N,q,q'}|_0|W_1|_{\delta,\frac{1}{2}-\delta}\\
&+ C2^{-q(\delta-1)-q'(\frac{1}{2}-\delta)}c_{q,q'}|\psi^\varepsilon_{N,q,q'}|_0(|W_2|_{\delta-1,\frac{1}{2}-\delta}+|\bar{b}|_1|\psi_N^\varepsilon|_{\delta,\frac{1}{2}-\delta})\\
&+ C2^{-q(2\delta-1)-q'(\frac{1}{2}-2\delta)}c_{q,q'}|\psi^\varepsilon_{N,q,q'}|_0|W_3|_{2\delta-1,\frac{1}{2}-2\delta}\\
&+ C2^{-q\delta-q'(\frac{1}{2}-\delta)}c_{q,q'}|\nabla\psi^\varepsilon_{N,q,q'}|_0|\psi_N^\varepsilon|_{\frac{1}{2}+\frac{\delta}{2},\frac{1}{2}-\frac{\delta}{2}}|b|_{\frac{1}{2}+\frac{\delta}{2},\frac{1}{2}-\frac{\delta}{2}}\\
&+ C2^{-q\delta-q'(\frac{1}{2}-\delta)}c_{q,q'}|\nabla\psi^\varepsilon_{N,q,q'}|_0|\psi_N^\varepsilon|^2_{\frac{1}{2}+\frac{\delta}{2},\frac{1}{2}-\frac{\delta}{2}}\\
&+ C2^{-q(\delta-\frac{1}{2})-q'(\frac{1}{2}-\delta)}c_{q,q'}|\psi^\varepsilon_{N,q,q'}|_0|\bar{b}|_{\frac{1}{2}}|\nabla\psi_N^\varepsilon|_{\delta,\frac{1}{2}-\delta},\end{aligned}$$

where we have noted $c_{q,q'}$ for the sequences in $\ell^{2,2}$.

Let us then multiply by $2^{2q\delta+2q'(\frac{1}{2}-\delta)}$, to find, after summation,

$$\begin{aligned}\frac{d}{dt}|\psi_N^\varepsilon|^2_{\delta,\frac{1}{2}-\delta}+\frac{3\nu}{2}|\nabla\psi_N^\varepsilon|^2_{\delta,\frac{1}{2}-\delta} &\leq C|\psi_N^\varepsilon|_{\delta,\frac{1}{2}-\delta}|\nabla\psi_N^\varepsilon|^2_{\delta,\frac{1}{2}-\delta}\\
&+ \frac{C}{\nu}(|b|^2_{\frac{1}{2}+\frac{\delta}{2},\frac{1}{2}-\frac{\delta}{2}}+\nu|\bar{b}|_1+|\bar{b}|^2_{\frac{1}{2}})|\psi_N^\varepsilon|_{\delta,\frac{1}{2}-\delta}|\nabla\psi_N^\varepsilon|_{\delta,\frac{1}{2}-\delta}\\
&+ C(|W_1|_{\delta,\frac{1}{2}-\delta}+|W_2|_{\delta-1,\frac{1}{2}-\delta}+|W_3|_{2\delta-1,\frac{1}{2}-2\delta})|\nabla\psi_N^\varepsilon|_{\delta,\frac{1}{2}-\delta},\end{aligned}$$

and using the fact that

$$
\begin{aligned}
|\psi_N^\varepsilon|^2_{\frac12+\frac\delta2,\frac12-\frac\delta2} &\le |\psi_N^\varepsilon|_{\delta,\frac12-\delta}|\psi_N^\varepsilon|_{1,\frac12}\\
&\le |\psi_N^\varepsilon|_{\delta,\frac12-\delta}|\nabla\psi_N^\varepsilon|_{\delta,\frac12-\delta},
\end{aligned}
$$

$$
\begin{aligned}
|\psi_N^\varepsilon|^2_{\frac12+\delta,\frac12-\delta} &\le |\psi_N^\varepsilon|_{\delta,\frac12-\delta}|\psi_N^\varepsilon|_{1+\delta,\frac12-\delta}\\
&\le |\psi_N^\varepsilon|_{\delta,\frac12-\delta}|\nabla\psi_N^\varepsilon|_{\delta,\frac12-\delta}\,.
\end{aligned}
$$

The method is then classical: we define

$$
g_\varepsilon(t)=|\psi_N^\varepsilon(t)|_{\delta,\frac12-\delta}\exp\left(-\lambda(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))\right),
$$

where we have defined $B^4(t)=\|\,|b(\tau)|_{\frac12+\frac\delta2,\frac12-\frac\delta2}\,\|^4_{L^4([0,t])}$, $\bar B^2(t)=\|\,|\bar b(\tau)|_1\,\|^2_{L^2([0,t])}$, and where $\bar B^4(t)=\|\,|\bar b(\tau)|_{\frac12}\,\|^4_{L^4([0,t])}$. Finally λ is a real number we will define later.

Then

$$
\begin{aligned}
\frac{d}{dt}g_\varepsilon^2(t) &+ \frac{3\nu}{2}|\nabla\psi_N^\varepsilon|^2_{\delta,\frac12-\delta}e^{-2\lambda(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))}\le -2\lambda(|b|^4_{\frac12+\frac\delta2,\frac12-\frac\delta2}+\nu|\bar b|_1^2+|\bar b|^4_{\frac12})g_\varepsilon^2(t)\\
&+ Cg_\varepsilon(t)e^{-\lambda(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))}|\nabla\psi_N^\varepsilon|^2_{\delta,\frac12-\delta}\\
&+ \frac{C}{\nu}(|b|^2_{\frac12+\frac\delta2,\frac12-\frac\delta2}+\nu|\bar b|_1+|\bar b|^2_{\frac12})|\nabla\psi_N^\varepsilon|_{\delta,\frac12-\delta}g_\varepsilon(t)e^{-\lambda(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))}\\
&+ C(|W_1|_{\delta,\frac12-\delta}+|W_2|_{\delta-1,\frac12-\delta}+|W_3|_{2\delta-1,\frac12-2\delta})|\nabla\psi_N^\varepsilon|_{\delta,\frac12-\delta}e^{-2\lambda(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))},
\end{aligned}
$$

which leads to the following estimate, where we have taken $\lambda=\dfrac{C}{2\nu^2}$,

$$
\begin{aligned}
\frac{d}{dt}g_\varepsilon^2(t) &+ \nu|\nabla\psi_N^\varepsilon|^2_{\delta,\frac12-\delta}e^{-\frac{C}{\nu^2}(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))}\\
&\le Cg_\varepsilon(t)e^{-\frac{C}{2\nu^2}(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))}|\nabla\psi_N^\varepsilon|^2_{\delta,\frac12-\delta}\\
&+ \frac{C}{\nu}(|W_1|^2_{\delta,\frac12-\delta}+|W_2|^2_{\delta-1,\frac12-\delta}+|W_3|^2_{2\delta-1,\frac12-2\delta})e^{-\frac{C}{\nu^2}(B^4(t)+\nu\bar B^2(t)+\bar B^4(t))}.
\end{aligned}
$$

Let η be an arbitrary real number, chosen such that

$$
\eta\le\frac{\nu^{\frac12}}{6C}\exp\left(-\frac{C}{2\nu^2}\int_0^T\left(|b(\tau)|^4_{\frac12+\frac\delta2,\frac12-\frac\delta2}+\nu|\bar b(\tau)|_1^2+|\bar b(\tau)|^4_{\frac12}\right)\,d\tau\right).
$$

Then let N be chosen large enough, and then ε be small enough such that $|\psi_{N,0}^\varepsilon|^2\le\nu\eta^2$ and that $\dfrac{C}{\nu}\displaystyle\int_0^T\left(|W_1(t)|^2_{\delta,\frac12-\delta}+|W_2(t)|^2_{\delta-1,\frac12-\delta}+|W_3|^2_{2\delta-1,\frac12-2\delta}\right)\,dt\le\nu\eta^2$. Finally let T_ε^* be the maximal time such that for all $t\le T_\varepsilon^*$ we have $g_\varepsilon(t)\le 3\eta\nu^{\frac12}$. Let us integrate the estimate above on $[0,t]$, with $t<T_\varepsilon^*$: writing

$$
X^2(t)=\int_0^t|\nabla\psi_N^\varepsilon(\tau)|^2_{\delta,\frac12-\delta}e^{-\frac{C}{\nu^2}(B^4(\tau)+\nu\bar B^2(\tau)+\bar B^4(\tau))}\,d\tau, \tag{7.8}
$$

we get

$$g_\varepsilon^2(t) + \nu X^2(t) \leq 3C\nu^{\frac{1}{2}} X^2(t)\eta e^{\frac{C}{2\nu^2}(B^4(t)+\nu \bar{B}^2(t)+\bar{B}^4(t))} + 2\eta^2\nu,$$

which leads to $X^2(t) \leq 4\eta^2$ for $t < T_\varepsilon^*$. Then $g_\varepsilon^2(t) \leq 4\nu\eta^2$, and in a standard way we deduce that $T_\varepsilon^* = T$. Hence, using the same argument as in section 3.4, Theorem 5 is proved. □

7.4.2 The case when the initial data is in $HB^{0,\frac{1}{2}}$

Proposition 7.4 *The function ψ_N^ε defined in* (7.6) *satisfies an equation of the type*

$$\partial_t \psi_N^\varepsilon - \nu\Delta\psi_N^\varepsilon + \mathcal{Q}^\varepsilon(\psi_N^\varepsilon, \psi_N^\varepsilon) + \mathcal{Q}^\varepsilon(\psi_N^\varepsilon, b + \bar{b}) = R_{1,N}^\varepsilon + R_2^{\varepsilon,N} + R_3^{\varepsilon,N},$$

where b and $\bar{b}$ depend on u and satisfy

$$b \in C^0([0,T], HB^{0,\frac{1}{2}}), \quad \nabla b \in L^2([0,T], HB^{0,\frac{1}{2}}),$$

and where

$$\bar{b} \in C^0([0,T], L^2(\mathcal{T}^2)) \cap L^2([0,T], H^1(\mathcal{T}^2)),$$

where $R_{1,N}^\varepsilon$ goes to zero in the space $L^{\frac{4}{3}}([0,T], HB^{-\frac{1}{2},\frac{1}{2}})$ when N is fixed and ε goes to zero, and where $R_2^{\varepsilon,N}$ and $R_3^{\varepsilon,N}$ go to 0 in $L^{\frac{4}{3}}([0,T], HB^{-\frac{1}{2},\frac{1}{2}})$ and $L^2([0,T], HB^{-1,\frac{1}{2}})$ respectively, when N goes to infinity, uniformly in ε.

Proof. We are not going to go too deep into the details of the proof here, as the computations leading to Proposition 7.4 are very similar to those of the previous section, proving Proposition 7.3.

The smoothness required on b and $\bar{b}$ is that given by Proposition 7.2, and the study of the low frequencies in equation (7.7) is a direct application of Lemma 7.1, using large enough powers of N to obtain the expected smoothness for the term $R_{1,N}^\varepsilon$.

For the high frequency term $R_2^{\varepsilon,N}$, one uses the same compactness results as above, along with the following estimates:

$$\begin{aligned}
\|\nabla\cdot(u_{osc}\otimes\bar{u})\|_{L^{\frac{4}{3}}([0,T],HB^{-\frac{1}{2},\frac{1}{2}})} &\leq \|u_{osc}\|_{L^4([0,T],HB^{\frac{1}{2},\frac{1}{2}})}\|\bar{u}\|_{L^2([0,T],H^1(\mathcal{T}^2))} \\
&+ \|\nabla u_{osc}\|_{L^2([0,T],HB^{0,\frac{1}{2}})}\|\bar{u}\|_{L^4([0,T],H^{\frac{1}{2}}(\mathcal{T}^2))},
\end{aligned}$$

$$\|\nabla\cdot(u_{osc}\otimes u_{osc})\|_{L^{\frac{4}{3}}([0,T],HB^{-\frac{1}{2},\frac{1}{2}})} \leq \|u_{osc}\|_{L^4([0,T],HB^{\frac{1}{2},\frac{1}{2}})}\|\nabla u_{osc}\|_{L^2([0,T],HB^{0,\frac{1}{2}})}.$$

Finally the high frequency term $R_3^{\varepsilon,N}$ is due to the exterior force, which as we have seen in the previous part is of the form $\left(\mathcal{L}(-\frac{t}{\varepsilon})f_{osc}\right)^N$, and hence goes to zero when N goes to infinity, in the space $L^2([0,T], HB^{-1,\frac{1}{2}})$.

That proves the proposition. □

We can now prove Theorem 6, following computations in [21].

By Lemma 7.1, and if we call $c_{q,q'}$ the sequences of norm 1 in $\ell^{2,1}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi^\varepsilon_{N,q,q'}|_0^2 \quad &+ \quad \nu|\nabla\psi^\varepsilon_{N,q,q'}|_0^2 \leq \left|\left(\Delta^h_q\Delta^v_{q'}(\mathcal{Q}^\varepsilon(\psi^\varepsilon_N,\psi^\varepsilon_N)+\mathcal{Q}^\varepsilon(\psi^\varepsilon_N,b+\bar{b}))\mid\psi^\varepsilon_{N,q,q'}\right)\right| \\ &+ \quad C2^{\frac{q-q'}{2}}|W_1|_{HB^{-\frac{1}{2},\frac{1}{2}}}c_{q,q'}|\psi^\varepsilon_{N,q,q'}|_0+C2^{q-\frac{q'}{2}}|W_2|_{HB^{-1,\frac{1}{2}}}c_{q,q'}|\psi^\varepsilon_{N,q,q'}|_0,\end{aligned}$$

where $W_1=R^\varepsilon_{1,N}+R^{\varepsilon,N}_2$ and $W_2=R^{\varepsilon,N}_3$. Let us define, as in [21],

$$F_{q,q'}=\left|\frac{\left(\Delta^h_q\Delta^v_{q'}\mathcal{Q}^\varepsilon(\psi^\varepsilon_N,\psi^\varepsilon_N)\mid\psi^\varepsilon_{N,q,q'}\right)}{|\psi^\varepsilon_{N,q,q'}|_0}\right|,\quad F_{q,q'}=0\ \text{ if }\ |\psi^\varepsilon_{N,q,q'}|_0=0,$$

$$G_{q,q'}=\left|\frac{\left(\Delta^h_q\Delta^v_{q'}\mathcal{Q}^\varepsilon(\psi^\varepsilon_N,b)\mid\psi^\varepsilon_{N,q,q'}\right)}{|\psi^\varepsilon_{N,q,q'}|_0}\right|,\quad G_{q,q'}=0\ \text{ if }\ |\psi^\varepsilon_{N,q,q'}|_0=0,$$

$$H_{q,q'}=\left|\frac{\left(\Delta^h_q\Delta^v_{q'}\mathcal{Q}^\varepsilon(\psi^\varepsilon_N,\bar{b})\mid\psi^\varepsilon_{N,q,q'}\right)}{|\psi^\varepsilon_{N,q,q'}|_0}\right|,\quad H_{q,q'}=0\ \text{ if }\ |\psi^\varepsilon_{N,q,q'}|_0=0.$$

We find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi^\varepsilon_{N,q,q'}|_0^2+c\nu(2^{2q}+2^{2q'})|\psi^\varepsilon_{N,q,q'}|_0^2 &\leq C2^{\frac{q-q'}{2}}|W_1|_{HB^{-\frac{1}{2},\frac{1}{2}}}c_{q,q'}|\psi^\varepsilon_{N,q,q'}|_0 \\ +C2^{q-\frac{q'}{2}}|W_2|_{HB^{-1,\frac{1}{2}}}c_{q,q'}|\psi^\varepsilon_{N,q,q'}|_0 &+(F_{q,q'}+G_{q,q'}+H_{q,q'})|\psi^\varepsilon_{N,q,q'}|_0.\end{aligned}$$

Then as in the proof of Proposition 7.2, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi^\varepsilon_N|^2_{HB^{0,\frac{1}{2}}}+c\nu|\nabla\psi^\varepsilon_N|^2_{HB^{0,\frac{1}{2}}} &\leq C|W_1|_{HB^{-\frac{1}{2},\frac{1}{2}}}|\psi^\varepsilon_N|_{HB^{\frac{1}{2},\frac{1}{2}}}+\frac{C}{\nu}|W_2|^2_{HB^{-1,\frac{1}{2}}} \\ &+\sum_q\left(\sum_{q'}2^{\frac{q'}{2}}(F_{q,q'}+G_{q,q'}+H_{q,q'})\right)\left(\sum_{q'}2^{\frac{q'}{2}}|\psi^\varepsilon_{N,q,q'}|_0\right).\end{aligned}$$

But the rules of product in Lemma 7.1 imply that

$$\|2^{\frac{q'-q}{2}}F_{q,q'}\|_{\ell^{2,1}}\leq C|\psi^\varepsilon_N|_{HB^{\frac{1}{2},\frac{1}{2}}}|\nabla\psi^\varepsilon_N|_{HB^{0,\frac{1}{2}}},$$

$$\|2^{\frac{q'-q}{2}}G_{q,q'}\|_{\ell^{2,1}}\leq C(|\psi^\varepsilon_N|_{HB^{\frac{1}{2},\frac{1}{2}}}|\nabla b|_{HB^{0,\frac{1}{2}}}+|b|_{HB^{\frac{1}{2},\frac{1}{2}}}|\nabla\psi^\varepsilon_N|_{HB^{0,\frac{1}{2}}}),$$

$$\|2^{\frac{q'-q}{2}}H_{q,q'}\|_{\ell^{2,1}}\leq C(|\psi^\varepsilon_N|_{HB^{\frac{1}{2},\frac{1}{2}}}|\nabla\bar{b}|_0+|\bar{b}|_{\frac{1}{2}}|\nabla\psi^\varepsilon_N|_{HB^{0,\frac{1}{2}}}).$$

Therefore we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi^\varepsilon_N|^2_{HB^{0,\frac{1}{2}}}+c\nu|\nabla\psi^\varepsilon_N|^2_{HB^{0,\frac{1}{2}}} \quad &\leq \quad C|W_1|_{HB^{-\frac{1}{2},\frac{1}{2}}}|\psi^\varepsilon_N|_{HB^{\frac{1}{2},\frac{1}{2}}}+C|\psi^\varepsilon_N|_{HB^{0,\frac{1}{2}}}|\nabla\psi^\varepsilon_N|^2_{HB^{0,\frac{1}{2}}} \\ &+ \quad \frac{C}{\nu}(|\nabla b|^2_{HB^{0,\frac{1}{2}}}+|\bar{b}|_1^2)|\psi^\varepsilon_N|^2_{HB^{0,\frac{1}{2}}}+\frac{C}{\nu}|W_2|^2_{HB^{-1,\frac{1}{2}}} \\ &+ \quad \frac{C}{\nu}(|b|^2_{HB^{\frac{1}{2},\frac{1}{2}}}+|\bar{b}|^2_{\frac{1}{2}})|\psi^\varepsilon_N|_{HB^{0,\frac{1}{2}}}|\nabla\psi^\varepsilon_N|_{HB^{0,\frac{1}{2}}}.\end{aligned}$$

Let us as previously define

$$g_\varepsilon(t) = |\psi_N^\varepsilon(t)|_{HB^{0,\frac{1}{2}}} \exp\left(-\lambda B^2(t)\right),$$

with $B^2(t) = \displaystyle\int_0^t (|\nabla b(\tau)|^2_{HB^{0,\frac{1}{2}}} + |\bar{b}(\tau)|_1^2)\, d\tau.$

Then $g_\varepsilon(t)$ satisfies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}g_\varepsilon^2 &+ c\nu|\nabla\psi_N^\varepsilon|^2_{HB^{0,\frac{1}{2}}} e^{-2\lambda B^2(t)} \leq C|W_1|_{HB^{-\frac{1}{2},\frac{1}{2}}}|\psi_N^\varepsilon|_{HB^{\frac{1}{2},\frac{1}{2}}} e^{-2\lambda B^2(t)} \\
&- 2\lambda(|\nabla b|^2_{HB^{0,\frac{1}{2}}} + |\bar{b}|_1^2)g_\varepsilon^2 + C|\nabla\psi_N^\varepsilon|^2_{HB^{0,\frac{1}{2}}} e^{-\lambda B^2(t)} g_\varepsilon \\
&+ \frac{C}{\nu}(|\nabla b|^2_{HB^{0,\frac{1}{2}}} + |\bar{b}|_1^2)g_\varepsilon^2 + \frac{C}{\nu}(|b|^2_{HB^{\frac{1}{2},\frac{1}{2}}} + |\bar{b}|^2_{\frac{1}{2}})|\nabla\psi_N^\varepsilon|_{HB^{0,\frac{1}{2}}} e^{-\lambda B^2(t)} g_\varepsilon \\
&+ \frac{C}{\nu}|W_2|^2_{HB^{-1,\frac{1}{2}}} e^{-2\lambda B^2(t)}.
\end{aligned}$$

But

$$\begin{aligned}
\frac{C}{\nu}(|b|^2_{HB^{\frac{1}{2},\frac{1}{2}}} + |\bar{b}|^2_{\frac{1}{2}})|\nabla\psi_N^\varepsilon|_{HB^{0,\frac{1}{2}}} e^{-\lambda B^2(t)} g_\varepsilon(t) &\leq \frac{c\nu}{2}|\nabla\psi_N^\varepsilon|^2_{HB^{0,\frac{1}{2}}} e^{-2\lambda B^2(t)} \\
&+ \frac{C}{\nu^2}(|b|^2_{HB^{0,\frac{1}{2}}}|\nabla b|^2_{HB^{0,\frac{1}{2}}} + |\bar{b}|_1^2|\bar{b}|_0^2)g_\varepsilon^2(t),
\end{aligned}$$

so let us define $2\lambda = \dfrac{C}{\nu} + \dfrac{C}{\nu^2}(\|b\|^2_{C^0(\mathbf{R}^+,HB^{0,\frac{1}{2}})} + \|\bar{b}\|^2_{C^0(\mathbf{R}^+,L^2)})$; integrating the previous estimate we find

$$\begin{aligned}
g_\varepsilon^2(t) &+ c\nu\int_0^t |\nabla\psi_N^\varepsilon(\tau)|^2_{HB^{0,\frac{1}{2}}} e^{-2\lambda B^2(\tau)} d\tau \leq \frac{C}{\nu}\int_0^t |W_2(\tau)|^2_{HB^{-1,\frac{1}{2}}} e^{-2\lambda B^2(\tau)}\, d\tau \\
&+ C\int_0^t |W_1(\tau)|_{HB^{-\frac{1}{2},\frac{1}{2}}}|\psi_N^\varepsilon(\tau)|_{HB^{\frac{1}{2},\frac{1}{2}}} e^{-2\lambda B^2(\tau)}\, d\tau \\
&+ |\psi_{N,0}^\varepsilon|^2_{HB^{0,\frac{1}{2}}} + C\int_0^t |\nabla\psi_N^\varepsilon(\tau)|^2_{HB^{0,\frac{1}{2}}} e^{-\lambda B^2(\tau)} g_\varepsilon(\tau) d\tau.
\end{aligned}$$

But

$$\begin{aligned}
\|\psi_N^\varepsilon e^{-2\lambda B^2(\tau)}\|_{L^4([0,t],HB^{\frac{1}{2},\frac{1}{2}})} &\leq \|\psi_N^\varepsilon e^{-\lambda B^2(\tau)}\|^{\frac{1}{2}}_{C^0([0,t],HB^{0,\frac{1}{2}})}\|\nabla\psi_N^\varepsilon e^{-\lambda B^2(\tau)}\|^{\frac{1}{2}}_{L^2([0,t],HB^{0,\frac{1}{2}})} \\
&\leq \|g_\varepsilon\|^{\frac{1}{2}}_{C^0([0,t])} X^{\frac{1}{2}}(t),
\end{aligned}$$

with the notations used in the previous section, in (7.8).

The inequality $\|fg\|_{L^1} \leq \|f\|_{L^{\frac{4}{3}}}\|g\|_{L^4}$, associated with a similar method to the one used in the case of the $H^{s,s'}$ spaces leads to

$$\begin{aligned}
g_\varepsilon^2(t) + c\nu X^2(t) &\leq \frac{C}{\nu}\left(\|W_1\|^2_{L^{\frac{4}{3}}([0,T],HB^{-\frac{1}{2},\frac{1}{2}})} + \|W_2\|^2_{L^2([0,T],HB^{-1,\frac{1}{2}})}\right) \\
&+ C\nu\|g_\varepsilon\|_{C^0([0,t])}X(t) + |\psi_{N,0}^\varepsilon|^2_{HB^{0,\frac{1}{2}}}.
\end{aligned}$$

Then the same method as the one used in the case of the $H^{s,s'}$ spaces leads to the theorem. $\square$

A Proof of Lemma 3.1

The aim of this section is to prove Lemma 3.1. That will consist in the study of equations of the type

$$(\mathcal{E}^\varepsilon_{osc}) \qquad \begin{cases} \partial_t a^\varepsilon + \mathcal{Q}^\varepsilon(a^\varepsilon, a^\varepsilon + b^\varepsilon) + \mathcal{A}^\varepsilon_2(D)a^\varepsilon & = & R^\varepsilon_{osc} + F^\varepsilon \quad \text{in} \quad \mathcal{T}^d \\ a^\varepsilon_{|t=0} & = & a^\varepsilon_0 \,, \end{cases}$$

where a^ε_0 goes to zero with ε in H^σ, with $\sigma \geq \frac{d}{2} - 1$, and where b^ε is a family of functions, bounded in the space $C^0([0,T], H^\sigma) \cap L^2([0,T], H^{\sigma+1})$. The operators $\mathcal{Q}^\varepsilon$ and $\mathcal{A}^\varepsilon_2(D)$ are defined in (2.1).

The right-hand side is made of two terms: F^ε goes to zero with ε in $L^2([0,T], H^{\sigma-1})$, and R^ε_{osc} is a $(p, \sigma - 2)$-oscillating term, in the sense of Definition 1.2. The method used here to deal with that type of rapidly oscillating forcing term is founded in the ideas introduced by S. Schochet in [34]; we will refer to that study as we go along.

The difficulty here is due to the fact that R^ε_{osc} goes weakly to zero with ε, and not strongly (that is simply due to the non stationary phase theorem, as we have seen in section 2.2). The idea in [34] is to decompose R^ε_{osc} into two parts, cutting at an arbitrary frequency N. We shall show that the high frequency terms converge strongly to zero when N goes to infinity, whereas the low frequency terms can be eliminated, up to an ε, by a change of function.

The proof will therefore be structured in the following way :

- The term R^ε_{osc} is cut into two parts, one with the low frequencies and the other with the high frequencies, with an arbitrary cut–off parameter N;
- The low frequency part is dealt with by considering a function ψ^ε_N of same H^s norm as a^ε up to an ε, which satisfies an equation similar to that satisfied by a^ε, but where the low frequencies have disappeared, up to an ε;
- We consider the forcing term of the equation on ψ^ε_N, which can be divided into two categories : one is made of the high frequency terms, which we show converge to 0 uniformly in ε, when the cut–off parameter N goes to infinity, and the other consists in the low frequency terms which appeared with the change of function, and that are $O(\varepsilon)$ and depend on N;
- We write the energy estimates satisfied by ψ^ε_N, using the results found in the preceding steps, as well as Lemma B.1 stated and proved in Appendix B.

So let us write equation $(\mathcal{E}^\varepsilon_{osc})$ in the form

$$\partial_t a^\varepsilon + \mathcal{Q}^\varepsilon(a^\varepsilon, a^\varepsilon + b^\varepsilon) + \mathcal{A}^\varepsilon_2(D)a^\varepsilon = R^\varepsilon_{osc,N} + R^{\varepsilon,N}_{osc} + F^\varepsilon,$$

where as planned we have separated R^ε_{osc} into a high frequency part $R^{\varepsilon,N}_{osc}$, and a low frequency part $R^\varepsilon_{osc,N}$. Writing, for any space X, $\mathbf{1}_X$ for its characteristic function, we have defined

$$R^\varepsilon_{osc,N} = \sum_{q \in \{1,..,p\}} R^\varepsilon_{q,osc,N},$$

where, with the notation of Definition 1.2,

$$\begin{array}{rcl} R^{\varepsilon}_{1,osc,N} & = & \mathcal{F}^{-1}\left(\mathbf{1}_{\{|n|\leq N\}} R^{\varepsilon}_{1,osc}\right) \\ \forall q \geq 2, \quad R^{\varepsilon}_{q,osc,N} & = & \mathcal{F}^{-1}\left(\mathbf{1}_{\{|n|,|\vec{k_q}|\leq N\}} R^{\varepsilon}_{q,osc}\right) \end{array}$$

$$\text{and where} \quad R^{\varepsilon,N}_{osc} \stackrel{def}{=} R^{\varepsilon}_{osc} - R^{\varepsilon}_{osc,N}.$$

In a similar way to the model problem $\partial_t \varphi - F(\varphi) = \cos\frac{t}{\varepsilon}$ discussed in section 4, let us carry out the following change of function:

$$\psi^{\varepsilon}_N = a^{\varepsilon} + \varepsilon \tilde{R}^{\varepsilon}_{osc,N}, \tag{A.1}$$

where $\tilde{R}^{\varepsilon}_{osc,N}$ is defined in the following way, with the notation of Definition 1.2:

$$\tilde{R}^{\varepsilon}_{osc,N} = \sum_{q\in\{1,..,p\}} \tilde{R}^{\varepsilon}_{q,osc,N},$$

$$\text{with} \quad \mathcal{F}\tilde{R}^{\varepsilon}_{1,osc,N}(n) \stackrel{def}{=} \sum_{|n|\leq N} \mathbf{1}_{\beta_1(n)\neq 0} \frac{i\, e^{-i\frac{t}{\varepsilon}\beta_1(n)}}{\beta_1(n)} f_0(t,n),$$

$$\forall\, q \geq 2, \quad \mathcal{F}\tilde{R}^{\varepsilon}_{q,osc,N}(n) \stackrel{def}{=} \sum_{|n|,|\vec{k_q}|\leq N} \sum_{\vec{k_q}\in K^n_q} \frac{i\, e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})}}{\beta_q(n,\vec{k_q})} r_0(n,\vec{k_q}) f_1(t,k_1)..f_q(t,k_q).$$

Lemma A.1 *There exists a constant $C_N(T)$ such that*

$$\|\tilde{R}^{\varepsilon}_{osc,N}\|_{C^0([0,T],H^{\sigma})\cap L^2([0,T],H^{\sigma+1})} \leq C_N(T).$$

Proof. Let us first of all note that each of the $(\beta_p)^{-1}$ is bounded from above by a constant depending on N, as each β_p takes a finite number of non zero values. Then since in each of the quantities estimated here, one sums on low frequencies only, they are as smooth as needed provided one pays with enough powers of N. □

That lemma implies that Lemma 3.1 is proved if one shows that

$$\psi^{\varepsilon}_N = o(1) \quad \text{in} \quad C^0([0,T],H^{\sigma}) \cap L^2([0,T],H^{\sigma+1}).$$

Lemma A.2 *The function ψ^{ε}_N satisfies an equation of the following type :*

$$\left\{ \begin{array}{rcl} \partial_t \psi^{\varepsilon}_N + \mathcal{A}^{\varepsilon}_2(D)\psi^{\varepsilon}_N + \mathcal{Q}^{\varepsilon}(\psi^{\varepsilon}_N, \psi^{\varepsilon}_N + b^{\varepsilon} - \varepsilon \tilde{R}^{\varepsilon}_{osc,N}) & = & \varepsilon R^{1,\varepsilon}_{osc,N} + R^{2,\varepsilon,N}_{osc} + F^{\varepsilon} \\ \psi^{\varepsilon}_{N|t=0} & = & \psi^{\varepsilon}_{N,0}\,, \end{array} \right. \tag{A.2}$$

where $\psi^{\varepsilon}_{N,0}$ goes to 0 with ε in H^{σ}, $R^{1,\varepsilon}_{osc,N}$ is bounded uniformly in ε, in $L^2([0,T],H^{\sigma-1})$, by a constant depending on N, and $R^{2,\varepsilon,N}_{osc}$ goes to 0 in $L^2([0,T],H^{\sigma-1})$ when N goes to infinity, uniformly in ε. The function F^{ε} goes to zero in $L^2([0,T],H^{\sigma-1})$.

Proof. Let us first note that $\psi^\varepsilon_{N|t=0} = a^\varepsilon_0 + \varepsilon \tilde{R}^\varepsilon_{osc,N|t=0}$, so the assumption on a^ε_0, as well as Lemma A.1, imply that $\psi^\varepsilon_{N|t=0}$ goes to zero with ε in H^σ, when N is fixed.

Furthermore, the change of function introduced in (A.1) implies that ψ^ε_N satisfies the following equation.

$$\begin{aligned}\partial_t \psi^\varepsilon_N + \mathcal{A}^\varepsilon_2(D)\psi^\varepsilon_N + \mathcal{Q}^\varepsilon(\psi^\varepsilon_N, \psi^\varepsilon_N + b^\varepsilon - 2\varepsilon\, \tilde{R}^\varepsilon_{osc,N}) = R^\varepsilon_0 + \varepsilon\, \tilde{R}^{\varepsilon,t}_{osc,N} + \varepsilon\, \mathcal{A}^\varepsilon_2(D)\tilde{R}^\varepsilon_{osc,N} \\ + R^{\varepsilon,N}_{osc} + \varepsilon\, \mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc,N}, b^\varepsilon - \varepsilon\, \tilde{R}^\varepsilon_{osc,N}) + F^\varepsilon,\end{aligned}$$

where $\tilde{R}^{\varepsilon,t}_{osc,N}$ is defined by $\tilde{R}^{\varepsilon,t}_{osc,N} = \sum\limits_{q\in\{1,..,p\}} \tilde{R}^{\varepsilon,t}_{q,osc,N}$, with

$$\tilde{R}^{\varepsilon,t}_{1,osc,N} \stackrel{def}{=} \mathcal{F}^{-1}\left(\mathbf{1}_{\{n\,|\,|n|\le N,\,\beta_1(n)\neq 0\}} \frac{i}{\beta_1(n)} e^{-i\frac{t}{\varepsilon}\beta_1(n)} \partial_t f_0(t,n)\right),$$

$$\text{and} \quad \forall q \ge 2, \quad \tilde{R}^{\varepsilon,t}_{q,osc,N} \stackrel{def}{=} \mathcal{F}^{-1} \sum_{j\in\{2,..q\}} \sum_{\substack{\vec{k_q}\in K^n_q \\ |n|,|\vec{k_q}|\le N}} \frac{i\, e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})}}{\beta_q(n,\vec{k_q})} \partial_t f_j(t,k_j) \prod_{i\neq j} f_i(t,k_i).$$

- The low frequencies :

Lemma A.3 *The terms $\tilde{R}^{\varepsilon,t}_{osc,N}$, $\mathcal{A}^\varepsilon_2(D)\tilde{R}^\varepsilon_{osc,N}$, and $\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc,N}, b^\varepsilon - \varepsilon\tilde{R}^\varepsilon_{osc,N})$, are bounded, uniformly in ε, in $L^2([0,T], H^{\sigma-1})$, with an upper bound $C_N(T)$ which depends on N.*

Proof. For the first two terms, the result is simply due to the fact that all the functions considered are in low frequencies, hence are as smooth as wanted provided $C_N(T)$ is large enough. Let us recall that we have supposed in Definition 1.2 that each of the $\mathcal{F}^{-1}\partial_t f_j(t,\cdot)$ is in $L^2([0,T], H^{\bar{\sigma}_j})$ for some $\bar{\sigma}_j$.

For $\mathcal{Q}^\varepsilon(\tilde{R}^\varepsilon_{osc,N}, b^\varepsilon - \varepsilon\tilde{R}^\varepsilon_{osc,N})$, the rules of product given in Lemma 2.1 give the result since b^ε and $\tilde{R}^\varepsilon_{osc,N}$ are in $C^0([0,T], H^\sigma) \cap L^2([0,T], H^{\sigma+1})$.

So Lemma A.3 is proved, and the term $R^{1,\varepsilon}_{osc,N}$ of (A.2) is found. □

- The high frequencies :

Lemma A.4 *The high frequency term $R^{\varepsilon,N}_{osc}$ goes to zero when N goes to infinity, in the space $C^0([0,T], H^{\sigma-2}) \cap L^2([0,T], H^{\sigma-1})$, uniformly in ε.*

Proof. Let us study each of the terms composing $R^{\varepsilon,N}_{osc}$:

– the first term, $R^{\varepsilon,N}_{1,osc}$, goes to zero in $L^2([0,T], H^{\sigma-1})$, uniformly in ε, according to Lebesgue's monotone convergence theorem.

Moreover, $\|R_{1,osc}^{\varepsilon,N}\|_{C^0([0,t],H^{\sigma-2})}$ goes to zero uniformly in t, since it is the remainder of a series of continuous functions defined on a compact set $[0,T]$, so Dini's theorem applies.

– the other terms: defining $\mathcal{K}^{q,N} = \{(m_1,..,m_{q+1}) \mid \exists i, |m_i| \geq N\}$, we have

$$R_{q,osc}^{\varepsilon,N} = \mathcal{F}^{-1} \sum_{\substack{\vec{k_q}\in K_q^n \\ (n,\vec{k_q})\in\mathcal{K}^{q,N}}} e^{-i\frac{t}{\varepsilon}\beta_q(n,\vec{k_q})} r_0(n,\vec{k_q}) f_1(t,k_1)..f_q(t,k_q),$$

so they are dealt with exactly in the same way, using the fact that f_j^N goes to zero in the space $C^0([0,T],H^{\sigma+\rho_j})\cap L^2([0,T],H^{\sigma+1+\rho_j})$, with the notation of Definition 1.2, for the same reason as above, and that the other f_i's are in $C^0([0,T],H^{\sigma+\rho_i})$. Then the result follows from the classical rules of product in Sobolev spaces. So Lemma A.4 is proved, and the term $R_{osc}^{2,\varepsilon,N}$ in (A.2) is found. □

Let us note that in this study of the high frequencies, nothing has been said about the smallness of ε : for the term $R_{osc}^{\varepsilon,N}$ to be arbitrarily small, the cut–off parameter must be chosen big enough, uniformly in ε. Only in the study of the low frequencies does the smallness of ε have a fundamental role to play.

We have proved Lemma A.2. □

- Energy estimates and end of the proof :

According to Lemma A.2, the equation on ψ_N^ε can be written in the following way:

$$\partial_t\psi_N^\varepsilon - \mathcal{A}_2^\varepsilon(D)\psi_N^\varepsilon + \mathcal{Q}^\varepsilon(\psi_N^\varepsilon, \psi_N^\varepsilon + b^\varepsilon - 2\varepsilon\tilde{R}_{osc,N}^\varepsilon) = \varepsilon R_{osc,N}^{1,\varepsilon} + R_{osc}^{2,\varepsilon,N} + F^\varepsilon,$$

$$\begin{aligned} \text{with} \quad \|R_{osc,N}^{1,\varepsilon}\|_{L^2([0,t],H^{\sigma-1})} &\leq C_N(t), \\ \|R_2^{\varepsilon,N}\|_{L^2([0,t],H^{\sigma-1})} &\leq c_N(t), \\ \|F^\varepsilon\|_{L^2([0,t],H^{\sigma-1})} &\leq c_\varepsilon(t), \end{aligned}$$

where $C_N(t)$ and $c_N(t)$ are constants depending on N and $t \leq T$, equal to zero when $t = 0$, and $c_N(t)$ goes to 0 when N goes to infinity, uniformly in t. Neither $C_N(t)$ nor $c_N(t)$ depend on ε, if one assumes $\varepsilon < 1$ for example. The constant $c_\varepsilon(t)$ goes to zero with ε, and is equal to zero when $t = 0$. Furthermore the initial data satisfies

$$\psi^\varepsilon_{|t=0} = a_0^\varepsilon + \varepsilon\left(\tilde{R}_{osc,N}^\varepsilon\right)_{|t=0},$$

and is therefore bounded in H^σ, by $c_{0,\varepsilon} + \varepsilon\, c_{0,N}$, where $c_{0,N}$ is a constant depending on $|b_0^\varepsilon|_0$ and on N.

Then it is just a a matter of using Lemma B.1, with $A = \psi_N^\varepsilon, B = b^\varepsilon - 2\varepsilon\tilde{R}_{osc,N}^\varepsilon$, and finally with $W = \varepsilon R_N^{1,\varepsilon} + R_{osc}^{2,\varepsilon,N} + F^\varepsilon$.

Let us start by supposing that $\sigma > \frac{d}{2} - 1$. Then if we define $X(t) = \|\psi_N^\varepsilon\|_{L^2([0,t],H^{\sigma+1})}$ and $Y(t) = \|\psi_N^\varepsilon\|_{L^\infty([0,t],H^\sigma)}$, we have the following estimates on $X(t)$ and $Y(t)$:

$$e^{-C\mathcal{B}_2(t)}X(t) \leq C(c_{0,\varepsilon} + \varepsilon c_{0,N}) + C\left(\varepsilon C_N(t) + c_N(t) + c_\varepsilon(t)\right) + CX^2(t),$$

where $\mathcal{B}_2(t)$ is an upper bound for $\|b^\varepsilon - 2\varepsilon \tilde{R}^\varepsilon_{osc,N}\|_{L^2([0,t],H^{\sigma+1})}$, and

$$Y(t) \leq C(c_{0,\varepsilon} + \varepsilon c_{0,N}) + C\left(\varepsilon C_N(t) + c_N(t) + c_\varepsilon(t)\right) + CX^2(t) + C\mathcal{B}_2(t)X(t).$$

Let $\eta > 0$ be an arbitrarily small real number, such that $\eta \leq \dfrac{1}{12C}\exp\left(-2C\mathcal{B}_2^2(T)\right)$, and let $T_\varepsilon^* > 0$ be the maximal time such that for all $t \leq T_\varepsilon^*$, we have $X(t) \leq \eta \exp\left(C\mathcal{B}_2^2(t)\right)$. Finally let T_0 be an arbitrary time. We suppose that $T_0, T_\varepsilon^* \leq T$.

Then for all $t \leq \min(T_\varepsilon^*, T_0)$, we have

$$X(t) \leq C\exp\left(C\mathcal{B}_2^2(t)\right)\left(c_{0,\varepsilon} + \varepsilon c_{0,N} + \varepsilon C_N(t) + c_N(t) + c_\varepsilon(t)\right) + C\eta^2 \exp\left(3C\mathcal{B}_2^2(t)\right).$$

Then one choses N large enough so that

$$c_N(T) \leq \frac{\eta}{12C},$$

followed by ε small enough so that

$$\varepsilon \leq \frac{\eta}{12C}(c_{0,N} + C_N(T))^{-1} \quad \text{and} \quad c_{0,\varepsilon} + c_\varepsilon(T) \leq \frac{\eta}{12C}.$$

That leads to $X(t) \leq \dfrac{\eta}{2}e^{C\mathcal{B}_2^2(t)}$, therefore $T_\varepsilon^* \geq T_0$, so $X(t)$ is defined for all $t \in [0,T]$ and we have $X(t) = o(1)$. Then we have $Y(t) \leq O(\eta)$, and Lemma 3.1 is proved, in the case when $\sigma > \frac{d}{2} - 1$.

In the case when $\sigma = \frac{d}{2} - 1$, the same method as above works: one starts by estimating the quantity $\|\psi_N^\varepsilon\|_{L^4([0,t],H^{\sigma+\frac{1}{2}})}$, using (B.3), which leads to

$$\|\psi_N^\varepsilon\|_{L^4([0,t],H^{\sigma+\frac{1}{2}})} = o(1) \quad \forall t \in [0,T],$$

for N large enough and ε small enough. Then for $\|\psi_N^\varepsilon\|_{L^2([0,t],H^{\sigma+1})}$ and $\|\psi_N^\varepsilon\|_{L^\infty([0,t],H^\sigma)}$, the estimates (B.4) and (B.5) of Lemma B.1 give the result, since we have

$$\begin{aligned}\|\psi_N^\varepsilon\|_{L^2([0,t],H^{\sigma+1})} &\leq C(c_{0,\varepsilon} + \varepsilon c_{0,N}) + C\left(\varepsilon C_N(t) + c_N(t) + c_\varepsilon(t)\right) + C\|\psi_N^\varepsilon\|^2_{L^4([0,t],H^{\sigma+\frac{1}{2}})} \\ &+ C\mathcal{B}_4(t)\|\psi_N^\varepsilon\|_{L^4([0,t],H^{\sigma+\frac{1}{2}})},\end{aligned}$$

where $\mathcal{B}_4(t)$ is an upper bound for $\|b^\varepsilon - 2\varepsilon \tilde{R}^\varepsilon_{osc,N}\|_{L^4([0,t],H^{\sigma+\frac{1}{2}})}$, and the same type of estimate goes for $\|\psi_N^\varepsilon\|_{L^\infty([0,t],H^\sigma)}$.

So the lemma is proved. $\square$

B Some estimates on evolution equations

The aim of this section is to prove some estimates on the solutions of equations of the type

$$\partial_t A + a_2(D)A + q(A,B) + \tilde{q}(A,A) = f$$

which we have used in the preceding pages, and that we had assumed to be true.

The operator $a_2(D)$ is second order elliptic, and the functions q and $\tilde{q}$ are of the form

$$q(A,B) = a_{i,j}(D)(A^i B^j), \qquad \tilde{q}(A,B) = \tilde{a}_{i,j}(D)(A^i B^j),$$

where $a_{i,j}(D)$ and $\tilde{a}_{i,j}(D)$ are symmetric Fourier multiplicators of order 1.

Remark. As $\mathcal{L}$ is a unitary operator, the results proved below hold when q (or $\tilde{q}$) and $a_2(D)$ are replaced respectively by $\mathcal{Q}^\varepsilon$ and $\mathcal{A}_2^\varepsilon(D)$, as defined in (2.1): in the following, the constants that shall appear in the estimates only depend on the ellipticity constant of a_2, as well as the continuity constants of q and $\tilde{q}$.

Lemma B.1 *Let $\sigma \geq \frac{d}{2} - 1$ and $T > 0$ be two real numbers. Let A_0 and B be two vector fields, defined on $\mathcal{T}^d$, such that $B \in L^4([0,T], H^{\sigma+\frac{1}{2}}) \cap L^2([0,T], H^{\sigma+1})$, and A_0 is in H^σ.*

Finally let W be an element of $L^2([0,T], H^{\sigma-1})$, and let A be a solution of the following equation :

$$\left\{ \begin{array}{rcl} \partial_t A + a_2(D)A + q(A,B) + \tilde{q}(A,A) & = & W \quad in \quad \mathcal{T}^d \\ A_{|t=0} & = & A_0\,. \end{array} \right. \tag{B.1}$$

Then there exists constants C and $\tilde{C}$ such that the following properties are satisfied.

Suppose that $\sigma > \frac{d}{2} - 1$. Then if $X_2(t) = \|A\|_{L^2([0,t],H^{\sigma+1})}$, then

$$X_2(t) e^{-C\mathcal{B}_2^2(t)} \leq C|A_0|_\sigma + C\mathcal{W}(t) + \tilde{C} X_2^2(t), \tag{B.2}$$

and $X_\infty(t) = \|A\|_{L^\infty([0,t],H^\sigma)}$ satisfies

$$X_\infty(t) \leq |A_0|_\sigma + C\mathcal{W}(t) + C X_2(t) \left(\tilde{C} X_2(t) + B_2(t)\right).$$

We have written $\mathcal{B}_2 = \|B\|_{L^2([0,t],H^{\sigma+1})}$ and $\mathcal{W}(t) = \|W\|_{L^2([0,t],H^{\sigma-1})}$.

If $\sigma = \frac{d}{2} - 1$, then if $X_4(t) = \|A\|_{L^4([0,t],H^{\sigma+\frac{1}{2}})}$ and $\mathcal{B}_4(t) = \|B\|_{L^4([0,t],H^{\sigma+\frac{1}{2}})}$, then

$$X_4(t) e^{-C\mathcal{B}_4^4(t)} \leq C|A_0|_\sigma + C\mathcal{W}(t) + \tilde{C} X_4^2(t), \tag{B.3}$$

$$X_2(t) \leq |A_0|_\sigma + C\mathcal{W}(t) + C X_4(t)(\tilde{C} X_4(t) + \mathcal{B}_4(t)), \tag{B.4}$$

$$X_\infty(t) \leq |A_0|_\sigma + C\mathcal{W}(t) + C X_4(t) \left(\tilde{C} X_4(t) + \mathcal{B}_4(t)\right). \tag{B.5}$$

Remark. In the linear case, when $\tilde{q} = 0$, the results above hold with $\tilde{C} = 0$.

Proof of Lemma B.1. Let us apply the truncation operators Δ_q to equation (B.1). If we write $A_q = \Delta_q A$, then we get

$$\partial_t A_q + a_2(D)\Delta A_q + \Delta_q \tilde{q}(A, A) + \Delta_q q(A, B) = \Delta_q W.$$

Let us suppose that $s > \frac{d}{2} - 1$.

We take the scalar product of this equation with A_q, and integrate the result over $\mathcal{T}^d$. The results we recalled in section 2.1 give, since $\sigma > \frac{d}{2} - 1$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|A_q(t)|_0^2 + c2^{2q}|A_q(t)|_0^2 &\leq Cc_q(t)2^{-q\sigma}|A_q(t)|_0|A(t)|_{\sigma+1}(|B(t)|_{\sigma+1} + \tilde{C}|A(t)|_{\sigma+1}) \\ &+ Cc_q(t)2^{-q(\sigma-1)}|W(t)|_{\sigma-1}|A_q(t)|_0, \end{aligned} \tag{B.6}$$

where $\sum_q c_q^2(t) = 1$.

The idea now is to conjugate the function A by the means of an exponential, enabling one to have some control over the linear term $q(A, B)$ of equation (B.1).

Let us thus define $g_q(t) = |A_q(t)|_0 \exp\left(-\lambda \int_0^t |B(s)|_{\sigma+1}^2 \, ds\right)$, where λ is a strictly positive real number, the precise value of which we shall define later.

Then

$$\frac{d}{dt}g_q^2(t) = -2\lambda|B(t)|_{\sigma+1}^2 g_q^2(t) + e^{-2\lambda\int_0^t |B(s)|_{\sigma+1}^2\,ds}\frac{d}{dt}|A_q(t)|_0^2,$$

and defining $\mathcal{B}_2^2(t) = \displaystyle\int_0^t |B(s)|_{\sigma+1}^2 \, ds$, we get

$$\begin{aligned} \frac{d}{dt}g_q^2(t) + c2^{2q}g_q^2(t) &\leq -2\lambda|B(t)|_{\sigma+1}^2 g_q^2(t) \\ &+ Cc_q(t)2^{-q\sigma}e^{-\lambda\mathcal{B}_2^2(t)}g_q(t)|A(t)|_{\sigma+1}\left(|B(t)|_{\sigma+1} + \tilde{C}|A(t)|_{\sigma+1}\right) \\ &+ Cc_q(t)2^{-q(\sigma-1)}|W(t)|_{\sigma-1}e^{-\lambda\mathcal{B}_2^2(t)}g_q(t). \end{aligned}$$

Then Gronwall's lemma leads to

$$\begin{aligned} g_q(t) &\leq g_{q|t=0}e^{-c2^{2q}t-2\lambda\mathcal{B}_2^2(t)} \\ &+ \tilde{C}\int_0^t c_q(\tau)2^{-q\sigma}e^{-\lambda\mathcal{B}_2^2(\tau)-c2^{2q}(t-\tau)-2\lambda\int_\tau^t |B(s)|_{\sigma+1}^2\,ds}|A(\tau)|_{\sigma+1}^2 \, d\tau \\ &+ C\int_0^t c_q(\tau)2^{-q\sigma}e^{-\lambda\mathcal{B}_2^2(\tau)-c2^{2q}(t-\tau)-2\lambda\int_\tau^t |B(s)|_{\sigma+1}^2\,ds}|A(\tau)|_{\sigma+1}|B(\tau)|_{\sigma+1} \, d\tau \\ &+ C\int_0^t c_q(\tau)2^{-q(\sigma-1)}e^{-\lambda\mathcal{B}_2^2(\tau)-c2^{2q}(t-\tau)-2\lambda\int_\tau^t |B(s)|_{\sigma+1}^2\,ds}|W(\tau)|_{\sigma-1} \, d\tau. \end{aligned}$$

Let us multiply this estimate by $2^{q(\sigma+1)}$; we can use Young's inequality

$$\|f \star g\|_{L^2} \leq \|f\|_{L^2}\|g\|_{L^1}$$

and take the ℓ^2 norm in q, which brings, using Lemma 2.5 and where $\mathcal{W}(t) = \|W\|_{L^2([0,t],H^{\sigma-1})}$,

$$\begin{aligned}\|e^{-\lambda\mathcal{B}_2^2(\tau)}|A|_{\sigma+1}\|_{L^2([0,t])} &\leq \tilde{C}X_2^2(t) + CX_2(t)e^{-2\lambda\mathcal{B}_2^2(t)}\|e^{\lambda\mathcal{B}_2^2(\tau)}|B(\tau)|_{\sigma+1}\|_{L^2([0,t])}\\ &+ C|A_0|_\sigma + C\mathcal{W}(t). \qquad \text{(B.7)}\end{aligned}$$

But one can notice that

$$\begin{aligned}\|e^{\lambda\mathcal{B}_2^2(\tau)}|B(\tau)|_{\sigma+1}\|^2_{L^2([0,t])} &= \int_0^t |B(\tau)|^2_{\sigma+1} e^{2\lambda\int_0^\tau |B(s)|^2_{\sigma+1}\,ds}\,d\tau\\ &= \frac{1}{2\lambda}\int_0^t \frac{d}{d\tau} e^{2\lambda\int_0^\tau |B(s)|^2_{\sigma+1}\,ds}\,d\tau\\ &\leq \frac{1}{2\lambda}e^{2\lambda\mathcal{B}_2^2(t)},\end{aligned}$$

therefore if λ is chosen such that $2\dfrac{C}{(2\lambda)^{\frac{1}{2}}} = 1$, the estimate (B.7) becomes

$$e^{-4C^2\mathcal{B}_2^2(t)}X_2(t) \leq 2C|A_0|_\sigma + 2\tilde{C}X_2^2(t) + 2C\mathcal{W}(t).$$

Estimate (B.2) is therefore proved. To get the fact that A is bounded in time, we can write the same computations as above up to the point (B.6), and then Gronwall's lemma leads to

$$\begin{aligned}|A_q(t)|_0 &\leq C2^{-q(\sigma-1)}\int_0^t c_q(\tau)|W(\tau)|_{\sigma-1}e^{-c2^{2q}(t-\tau)}d\tau + |A_{q,0}|_\sigma e^{-c2^{2q}t}\\ &+ C2^{-q\sigma}\int_0^t c_q(\tau)|A(\tau)|_{\sigma+1}(\tilde{C}|A(\tau)|_{\sigma+1} + |B(\tau)|_{\sigma+1})e^{-C2^{2q}(t-\tau)}d\tau.\end{aligned}$$

One then just has to multiply this estimate by $2^{q\sigma}$ and take the L^∞ norm in time, with

$$\begin{aligned}\|f \star g\|_{L^\infty} &\leq \|f\|_{L^2}\|g\|_{L^2}\\ &\leq \|f\|_{L^\infty}\|g\|_{L^1}.\end{aligned}$$

Taking the ℓ^2 norm in q and using Lemma 2.5 leads to the expected result. Hence Lemma B.1 is proved, in the case when $\sigma > \frac{d}{2} - 1$.

In the case when $\sigma = \frac{d}{2} - 1$, this method does not work as it stands, as the estimate (B.6) is no longer correct. In that case, one writes rather, since $\sigma + \frac{1}{2} < \frac{d}{2}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A_q(t)|_0^2 + c2^{2q}|A_q(t)|_0^2 &\leq Cc_q(t)2^{-q(\sigma-1)}|A_q(t)|_0|A(t)|_{\sigma+\frac{1}{2}}(|B(t)|_{\sigma+\frac{1}{2}} + \tilde{C}|A(t)|_{\sigma+\frac{1}{2}})\\ &+ Cc_q(t)2^{-q(\sigma-1)}|W(t)|_{\sigma-1}|A_q(t)|_0. \qquad \text{(B.8)}\end{aligned}$$

Then the computations are very similar to the ones above: we define

$$h_q(t) = |A_q(t)|_0 \exp\left(-\mu\int_0^t |B(s)|^4_{\sigma+\frac{1}{2}}\,ds\right),$$

where μ is defined by

$$4\frac{C}{(4\mu)^{\frac{1}{4}}} = 1.$$

Then we get, writing $\mathcal{B}_4^4(t) = \displaystyle\int_0^t |B(s)|_{\sigma+\frac{1}{2}}^4 \, ds$,

$$
\begin{array}{rcl}
h_q(t) & \leq & h_{q|t=0} e^{-c2^{2q}t - 2\mu \mathcal{B}_4^4(t)} \\
& + & \tilde{C} \displaystyle\int_0^t c_q(\tau) 2^{-q(\sigma-1)} e^{-\mu \mathcal{B}_4^4(\tau) - c2^{2q}(t-\tau) - 2\mu \int_\tau^t |B(s)|_{\sigma+\frac{1}{2}}^4 \, ds} |A(\tau)|_{\sigma+\frac{1}{2}}^2 \, d\tau \\
& + & C \displaystyle\int_0^t c_q(\tau) 2^{-q(\sigma-1)} e^{-\mu \mathcal{B}_4^4(\tau) - c2^{2q}(t-\tau) - 2\mu \int_\tau^t |B(s)|_{\sigma+\frac{1}{2}}^2 \, ds} |A(\tau)|_{\sigma+\frac{1}{2}} |B(\tau)|_{\sigma+\frac{1}{2}} \, d\tau \\
& + & C \displaystyle\int_0^t c_q(\tau) 2^{-q(\sigma-1)} e^{-\mu \mathcal{B}_4^4(\tau) - c2^{2q}(t-\tau) - 2\mu \int_\tau^t |B(s)|_{\sigma+\frac{1}{2}}^2 \, ds} |W(\tau)|_{\sigma-1} \, d\tau.
\end{array}
$$

Then using Young's inequality

$$
\|f * g\|_{L^4} \leq \|f\|_{L^4} \|g\|_{L^1},
$$

we finally get the following estimate:

$$
e^{-4C^4 \mathcal{B}_4^4(t)} X_4(t) \leq 2C|A_0|_\sigma + 2\tilde{C} X_4^2(t) + 2C\mathcal{W}(t).
$$

Then the estimates on X_∞ and X_2 are found exactly as the estimate on X_∞ in the previous case, when $\sigma > \frac{d}{2} - 1$, by a Gronwall lemma in equation (B.8), which gives

$$
\begin{array}{rcl}
|A_q(t)|_0 & \leq & C 2^{-q(\sigma-1)} \displaystyle\int_0^t c_q(\tau) |W(\tau)|_{\sigma-1} e^{-c2^{2q}(t-\tau)} d\tau + |A_{q,0}|_\sigma e^{-c2^{2q}t} \\
& + & C 2^{-q(\sigma-1)} \displaystyle\int_0^t c_q(\tau) |A(\tau)|_{\sigma+\frac{1}{2}} (\tilde{C}|A(\tau)|_{\sigma+\frac{1}{2}} + |B(\tau)|_{\sigma+\frac{1}{2}}) e^{-C2^{2q}(t-\tau)} d\tau,
\end{array}
$$

followed by Young's inequalities recalled above. The lemma is proved. □

C The small divisor estimates

In this section we are going to study the small divisor estimates which have been used in Theorem 2. Let us recall that in Theorem 2, in order to get the first term of an asymptotic expansion of the filtered solution u^ε, we have supposed that L satisfied a second order small divisor estimate in $\mathcal{T}^d$. It is hence natural to wonder whether such an assumption is reasonable, other than in very special cases such as in the primitive equations when $F = a_3^{-1}$ where the eigenvalues of $\mathcal{F}P_nL$ do not depend on n.

We are going to suppose that the eigenvalues of $\mathcal{F}L$ (or $\mathcal{F}P_nL$ in the case when there are divergence free conditions), called $\omega^a(n)$ with $a \in \{1, .., d'\}$, satisfy a relation of the type

$$\exists\, a_j, \quad \forall (k,m) \in \mathbf{Z}^{2d}, \quad \left|\omega^a(k) + \omega^b(m) - \omega^c(k+m)\right| = \frac{\sum_{i=1}^{3} \varepsilon_i P_{k,m,i}^{\frac{1}{2}}(a_j^{-1})}{P_{k,m,0}^{\frac{1}{2}}(a_j^{-1})}, \tag{C.1}$$

where $\varepsilon_i^2 = 1$, $(P_{k,m,i}(x))_{0\leq i\leq 3}$ are polynomials in x of degree $p_0 > 0$, with coefficients that are polynomials in $(k, m, |k|, |m|)$ and depend on $(a_k)_{k\neq j}$; let us recall that the a_k are the sizes of the sides of the periodic box $\mathcal{T}^d$. The coefficients also depend on (a, b, c).

In the following we will write $\omega^a(k) + \omega^b(m) - \omega^c(k+m) = \omega_{k,k+m}^{a,b,c}$.

Remark. The computations written in sections 5 and 6, show that such a relation is for instance satisfied in the case of the rotating fluid equations, or for the primitive system: let us recall that in the case of the primitive equations and the rotating fluids respectively, we have seen that

$$\omega^a(k) = \frac{(k_3^2/a_3^2 + F^{-2}|k'|^2)^{\frac{1}{2}}}{|k|}, \quad \text{and} \quad \omega^a(k) = \frac{k_3/a_3}{|k|},$$

so a reduction to the same denominator, in $|\omega^a(k) + \omega^b(m) - \omega^c(k+m)|$, gives the form (C.1).

Proposition C.1 *If the eigenvalues of $\mathcal{F}L$ satisfy a relation of the type (C.1), then the set*

$$\mathcal{A} = \left\{ a_j \in \mathbf{R}_+^* \mid \forall (A,\alpha) \in (\mathbf{R}_+^*)^2, \exists (k,m,a,b,c),\ |\omega_{k,k+m}^{a,b,c}|^{-1} \geq A\,(1+|k|)^\alpha\,(1+|m|)^\alpha \right\}$$

is of Lebesgue measure zero in $\mathbf{R}_+^*$.

Remark. A stated in the introduction, that proposition implies that the second order assumption made on L in Theorem 2 is true for almost all periodic boxes $\mathcal{T}^d$.

We will only consider second order estimates here. In the case of different order estimates, the computations below must be adapted, namely by supposing that a relation of the type (C.1) is satisfied at the same order.

Proof of the proposition. The first steps of the proof consist in a series of transformations enabling one to bring the problem down to the case when $|\omega_{k,k+m}^{a,b,c}|$ is a polynomial in a_j^{-1}, with coefficients depending on (k, m).

From now on we shall define $x = a_j^{-1}$, and suppose that $x \leq K$, where K is an arbitrary integer. Only at the very end will we let K go to infinity.

We will note, for any measurable function X, $\mu(X)$ for its Lebesgue measure.

Then let us define

$$\mathcal{A}_K = \left\{ x \in [0, K] \mid \forall (A, \alpha) \in (\mathbf{R}_+^*)^2, \exists (k, m, a, b, c),\ |\omega_{k,k+m}^{a,b,c}|^{-1} \geq A\,(1 + |k|)^\alpha\,(1 + |m|)^\alpha \right\}.$$

Of course, here and in the following, we restrict ourselves to the case when

$$|\omega_{k,k+m}^{a,b,c}| \neq 0.$$

The measure of $\mathcal{A}_K$ is zero if the same goes for that of the space $\tilde{\mathcal{A}}_K$ defined by

$$\left\{ x \in [0, K] \mid \forall (A, \alpha) \in (\mathbf{R}_+^*)^2, \exists (k, m),\ |\sum_{i=1}^{3} \varepsilon_i P_{k,m,i}^{\frac{1}{2}}(x)| \leq A^{-1}\,(1 + |k|)^{-\alpha}\,(1 + |m|)^{-\alpha} \right\}.$$

Let us suppose for instance that $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = -1$. Then an immediate computation, using $a - b = (a^2 - b^2)(a + b)^{-1}$, leads to the fact that

$$|\sum_{i=1}^{3} \varepsilon_i P_{k,m,i}^{\frac{1}{2}}(x)| = |\sum_{i=1}^{3} \varepsilon_i P_{k,m,i}(x) - P_{k,m,4}(x)|\, |Q_{k,m}(x)|^{-1},$$

where $P_{k,m,4}(x)$ is a polynomial in x, and $Q_{k,m}(x)$ is a strictly positive function of x, and both are polynomial in $(k, m, |k|, |m|)$. From now on we will call generically $P_{k,m}(x)$ the polynomial in the numerator of the expression above, and define q_0 as its degree.

Then we have reduced the problem to showing that the set

$$\tilde{\mathcal{A}}'_K = \left\{ x \in [0, K] \mid \forall (A, \alpha) \in (\mathbf{R}_+^*)^2, \exists (k, m),\ |P_{k,m}(x)| \leq A^{-1}\,(1 + |k|)^{-\alpha}\,(1 + |m|)^{-\alpha} \right\}$$

is of Lebesgue measure zero in $\mathbf{R}$. Now we define a sequence in $\mathbf{R}_+^* \times \mathbf{R}_+^*$, called $(A_n, \alpha_n)_{n \in \mathbf{N}}$, such that A_n and α_n go to infinity with n, and $\sum_n A_n^{-\frac{1}{q_0}} < \infty$, where let us recall that q_0 is the degree of $P_{k,m}$ defined above. Then the measure of the set $\tilde{\mathcal{A}}'_K$ will be smaller than that of the intersection for $n \in \mathbf{N}$ of the sets

$$\mathcal{A}_{K,n} = \left\{ x \in [0, K] \mid \exists (k, m),\ |P_{k,m}(x)| \leq A_n^{-1}\,(1 + |k|)^{-\alpha_n}\,(1 + |m|)^{-\alpha_n} \right\}.$$

Now let us number the couples (k, m) of $\mathbf{Z}^d \times \mathbf{Z}^d$, using an index $j \in \mathbf{N}$. The numbering is such that there exists a countable sequence of embedded subsets N_j whose reunion is the lattice $\mathbf{Z}^d \times \mathbf{Z}^d$, where each N_j has a cardinal $|N_j| \sim (1 + |k_j|)^d (1 + |m_j|)^d$, and where we have $(k_j, m_j) \in N_j$. In other words, we number the couples (k, m) increaslingly according to their size.

Finally let us define, fixing a couple of integers (k_j, m_j) in N_j,

$$\mathcal{A}_{K,n}^j = \left\{ x \in [0, K] \mid |P_{k_j,m_j}(x)| \leq A_n^{-1}\,(1 + |k_j|)^{-\alpha_n}\,(1 + |m_j|)^{-\alpha_n} \right\}.$$

The measure of $\mathcal{A}^j_{K,n}$ is smaller than that of

$$\tilde{\mathcal{A}}^j_{K,n} = \left\{ x \in [0,K] \mid \exists (k,m) \in N_j,\ |P_{k,m}(x)| \leq A_n^{-1} \left(1+|k_j|\right)^{-\alpha_n} \left(1+|m_j|\right)^{-\alpha_n} \right\}.$$

But in [3] is proved the following result:

Lemma C.1 *Let $\mathcal{P}_{k,m}$ be a polynomial of degree p_0, with coefficients that depend on discrete parameters k and m. Let $\eta > 0$ and $\delta \in]0,1[$ be fixed numbers. Then the measure of the set $\mathcal{S}_j$ defined by*

$$\mathcal{S}_j = \left\{ x \in \mathbf{R}^*_+ \mid \exists (k,m) \in N_j,\ |\mathcal{P}_{k,m}(x)| \leq \delta^{p_0} |N_j|^{-p_0(1+\eta)} \right\}$$

satisfies $\mu\left(\mathcal{S}_j\right) \leq 2p_0 \delta |N_j|^{-\eta}$.

That lemma applied here leads to

$$\begin{aligned} \mu\left(\mathcal{A}^j_{K,n}\right) &\leq \mu\left(\tilde{\mathcal{A}}^j_{K,n}\right) \\ &\leq 2q_0 A_n^{-\frac{1}{q_0}} |N_j|^{-\frac{\alpha_n \eta}{q_0(1+\eta)}} \\ &\leq 2q_0 A_n^{-\frac{1}{q_0}} \left(1+|k_j|\right)^{-\frac{\alpha_n \eta d}{q_0(1+\eta)}} \left(1+|m_j|\right)^{-\frac{\alpha_n \eta d}{q_0(1+\eta)}}. \end{aligned}$$

Summing over (k_j, m_j) yields $\mu\left(\mathcal{A}_{K,n}\right) \leq C_{d,q_0} q_0 A_n^{-\frac{1}{q_0}}$, for α_n large enough in front of η and q_0, where C_{d,q_0} is a constant depending only on d and q_0.

We have assumed that $\sum_n A_n^{-\frac{1}{q_0}} < \infty$, so we get finally $\sum_n \mu\left(\mathcal{A}_{K,n}\right) < \infty$. Then it is just a matter of using Borel–Cantelli's Lemma (see [35] for instance): we find that $\mu\left(\limsup_n \mathcal{A}_{K,n}\right) = 0$, which means in other words that almost all $x \in [0,K]$ belong to a finite number only of spaces $\mathcal{A}_{K,n}$. Hence the measure of $\mathcal{A}_K$ is zero, so we can then consider the countable union of the $\mathcal{A}_K$, which leads to the fact that $\mathcal{A}$ is of measure zero, so the proposition is proved. □

Remark. In [3], A. Babin, A. Mahalov, and B. Nicolaenko compute, in the particular cases we have studied in this paper, a precise second order small divisor estimate: they find $\alpha_2 = 4 + \varepsilon_0$ (see Theorem 3.2 of [3]), but that result only holds when a_3 remains outside a set of (small) strictly positive measure. That is made precise in Remark 3.2 of [3].

References

[1] S. Alinhac, P. Gérard: Opérateurs pseudodifférentiels et théorème de Nash–Moser, *Savoirs actuels, Interéditions*, 1991.

[2] A. Babin, A. Mahalov, and B. Nicolaenko: Global Splitting, Integrability and Regularity of 3D Euler and Navier–Stokes Equations for Uniformly Rotating Fluids, *European Journal of Mechanics*, **15**, 3, pages 291-300, 1996.

[3] A. Babin, A. Mahalov et B. Nicolaenko: Global Splitting, Integrability and Regularity of 3D Euler and Navier–Stokes Equations for Uniformly Rotating Fluids, *European Journal of Mechanics*, **15**, 3, pages 291-300, 1996.

[4] A. Babin, A. Mahalov, and B. Nicolaenko: Resonances qnd Regularity for Boussinesq Equations, *Russian Journal of Mathematical Physics*, **4**, 4, pages 417-428, 1996.

[5] A. Babin, A. Mahalov, B. Nicolaenko, and Y. Zhou: On the Asymptotic Regimes and the Strongly Stratified Limit of Rotating Boussinesq Equations, *Journal of Theoretical and Computational Fluid Dynamics,* **9**, pages 223-251, 1997.

[6] T. Beale and A. Bourgeois: Validity of the Quasigeostrophic Model for Large–Scale Flow in the Atmosphere and the Ocean, *SIAM Journal of Mathematical Analysis*, **25**, pages 1023-1068, 1994.

[7] G. Browning and H.-O. Kreiss: Problems with Different Time Scales for Nonlinear Partial Differential Equations, *SIAM Journal of Applied Mathematics*, **42**, 4, pages 704-718, 1982.

[8] J.-Y. Chemin: Fluides parfaits incompressibles, *Astérisque*, **230**, 1995.

[9] J.-Y. Chemin: About the Navier–Stokes System, *Preprint, Laboratoire d'Analyse Numérique, Paris 6*, 1996.

[10] J.-Y. Chemin: À propos d'un problème de pénalisation de type antisymétrique, *Journal de Mathématiques Pures et Appliquées*, **76**, pages 739–755, 1997.

[11] J.-Y. Chemin and N. Lerner: Flot de champs de vecteurs non Lipschitziens et équations de Navier–Stokes, *Journal of Differential Equations*, **121**, pages 314-328, 1992.

[12] T. Colin and P. Fabrie: Rotating Fluid at High Rossby Number Driven by a Surface Stress: Existence and Convergence, *Advances in Differential Equations*, **2**, pages 715-751, 1997.

[13] P. Constantin and C. Foias: Navier–Stokes Equations, *Chicago University Press*, 1988.

[14] P. Embid and A. Majda: Averaging Over Fast Gravity Waves For Geophysical Flows with Arbitrary Potential Vorticity, *Communications in Partial Differential Equations*, **21**, pages 619-658, 1996.

[15] H. Fujita, T. Kato: On the Navier–Stokes Initial Value Problem I, *Archive for Rational Mechanics and Analysis*, **16**, pages 269–315, 1964.

[16] I. Gallagher: Un résultat de stabilité pour les équations des fluides tournants, *Notes aux Comptes-Rendus de l'Académie des Sciences de Paris*, **324**, Série I, pages 183-186, 1997.

[17] I. Gallagher: The Tridimensional Navier–Stokes Equations with Almost Bidimensional Data: Stability, Uniqueness and Life Span, *International Mathematics Research Notices*, **18**, pages 919-935, 1997.

[18] I. Gallagher: Asymptotics of the Solutions of Hyperbolic Equations With a Skew–Symmetric Perturbation, to appear in *Journal of Differential Equations,* and *Preprint of the Laboratoire d'Analyse Numérique, Paris 6*, 1997.

[19] H.P. Greenspan: The Theory of Rotating Fluids, *Cambridge University Press*, 1968.

[20] E. Grenier: Oscillatory Perturbations of the Navier–Stokes Equations, *Journal de Mathématiques Pures et Appliquées*, **76**, pages 477-498, 1997.

[21] D. Iftimie: La résolution des équations de Navier–Stokes dans des domaines minces et la limite quasigéostrophique, Thèse de l'Université Paris 6, 1997.

[22] D. Iftimie: The resolution of the Navier–Stokes equations in anisotropic spaces, to appear in *Revista Matematica Ibero–Americana*, 1998.

[23] J.-L. Joly, G. Métivier and J. Rauch: Generic Rigorous Asymptotic Expansions for Weakly Nonlinear Multidimensional Oscillatory Waves, *Duke Mathematical Journal*, **70**, pages 373-404, 1993.

[24] J.-L. Joly, G. Métivier and J. Rauch: Coherent and Focusing Multidimensional Nonlinear Geometric Optics, *Annales Scientifiques de l'ENS Paris*, **28**, pages 51-113, 1995.

[25] S. Klainerman and A. Majda: Singular Limits of Quasilinear Hyperbolic Systems with Large Parameters, and the Incompressible Limit of Compressible Fluids, *Communications on Pure and Applied Mathematics*, **34**, pages 481-524, 1981.

[26] H.-O. Kreiss, J. Lorenz, and M.J. Naughton: Convergence of the Solutions of the Compressible to the Solutions of the Incompressible Navier–Stokes Equations, *Advances in Pure and Applied Mathematics*, **12**, pages 187-214, 1991.

[27] J. Leray: Sur le mouvement d'un liquide visqueux remplissant l'espace, *Acta Math.*, **63**, pages 193–248, 1934.

[28] J.-L. Lions: Quelques méthodes de résolution de problèmes aux limites non linéaires, Dunod, Paris, 1969.

[29] J.-L. Lions, R. Temam, and S. Wang: New Formulations of the Primitive Equations of the Atmosphere and Applications, *Nonlinearity*, **5**, pages 237-288, 1992.

[30] J.-L. Lions, R. Temam, and S. Wang: The Equations of Large–Scale Ocean, *Nonlinearity*, **5**, pages 1007-1053, 1992.

[31] J.-L. Lions, R. Temam, and S. Wang: Geostrophic Asymptotics of the Primitive Equations of the Atmosphere, *Topological Methods in Non Linear Analysis*, **4**, pages 1-35, 1994.

[32] A. Majda: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, *Applied Mathematical Sciences*, **53**, *Springer Verlag*, New York, 1984.

[33] J. Pedlosky: Geophysical Fluid Dynamics, 2nd Edition, *Springer Verlag*, New York, 1987.

[34] S. Schochet: Fast Singular Limits of Hyperbolic PDEs, *Journal of Differential Equations*, **114**, pages 476-512, 1994.

[35] L. Schwartz: Analyse III, Calcul Intégral, Hermann Editions, 1993.

[36] R. Temam: Navier–Stokes Equations, Theory and Numerical Analysis, North Holland, Amsterdam, 1984.