Mathematical Analysis of a Structure–Preserving Approximation of the Bidimensional Vorticity Equation

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Abstract. *We show the consistency and the convergence of a spectral approximation of the bidimensional vorticity equation, proposed by V. Zeitlin in [13] and studied numerically by I. Szunyogh, B. Kadar, and D. Dévényi in [12], whose main feature is that it preserves the Hamiltonian structure of the vorticity equation.*

Résumé. *On démontre la consistance et la convergence d'une approximation spectrale de l'équation du tourbillon bidimensionnelle périodique, proposée par V. Zeitlin dans [13] et étudiée numériquement par I. Szunyogh, B. Kadar et D. Dévényi dans [12]; sa caractéristique principale est de préserver la structure hamiltonienne de l'équation du tourbillon.*

1 Introduction

The aim of this article is to study a particular form of spectral approximation of the bidimensional Euler equation, proposed by V. Zeitlin in [13]. Before explaining the reason for that particular approximation, let us recall a few very basic facts concerning bidimensional, inviscid, incompressible flows, from which is derived the bidimensional vorticity equation.

Let us consider an inviscid, incompressible fluid, evolving on the bidimensional plane. A particle of that fluid, at a point x and at a time t, has a velocity $v(t,x)$ which satisfies the incompressible Euler equation

$$(E) \quad \left\{ \begin{array}{rcll} \partial_t v + v\cdot\nabla v & = & -\nabla p & \text{in} \quad \mathbb{R}\times\mathbb{T}^2 \\ \operatorname{div} v & = & 0 & \\ v_{|t=0} & = & v_0\,. & \end{array} \right.$$

Here $\mathbf{T}^2$ stands for the bidimensional torus: throughout this paper, we shall suppose that the fluid is periodic in both directions of the plane. The first equation in (E) is the momentum equation, where p is the pressure, and the second equation represents the conservation of mass. We have written $\operatorname{div} v = \partial_1 v^1 + \partial_2 v^2$, where ∂_i is the partial derivative in the direction x_i, and where $v = (v^1, v^2)$. We suppose that v_0 is a periodic, divergence free vector field, and throughout this paper, the vector fields will be supposed also to be mean free on $\mathbf{T}^2$. Let us note that the unknowns are *a priori* v and p, but p can be obtained from v by plugging the incompressibility condition into the momentum equation: we get

$$-\Delta p = \sum_{i,j} \partial_i\partial_j (v^i v^j).$$

A fundamental quantity in fluid mechanics is the vorticity $\Omega \stackrel{\text{def}}{=} \operatorname{curl} v$. In the bidimensional case, it is identified with the scalar function $\omega \stackrel{\text{def}}{=} \partial_1 v^2 - \partial_2 v^1$, which satisfies

$$(VE) \qquad \partial_t \omega + v \cdot \nabla \omega = 0 \quad \text{in} \quad \mathbf{T}^2,$$

and v and ω are related by the well–known Biot–Savart law

$$(BS) \qquad v = \nabla^{\perp}(E * \omega), \quad \text{where} \quad E(x) = \frac{1}{2\pi} \log |x|.$$

We have noted $*$ for the convolution operator, and $\nabla^{\perp} = (-\partial_2, \partial_1)$. We shall not go into any detailed statement concerning systems (E) and (VE). We refer to [5], [8], [9], and [10] for extensive studies on those equations. Let us nevertheless recall that there is a unique global solution to (E) for smooth initial data (say, in $C^0(\mathbf{R}, C^r)$ if $v_0 \in C^r$ with $r > 1$), and the L^p norms of the vorticity are conserved for any $p \in [1, \infty]$.

In the following, the space H^s is the usual homogeneous Sobolev space of order $s \in \mathbf{R}$. We shall note in the following the norm of any function u in H^s by

$$|u|_s \stackrel{\text{def}}{=} \left(\sum_{n \in \mathbf{Z}^2} |n|^{2s} |\widehat{u}(n)|^2 \right)^{\frac{1}{2}},$$

where $\widehat{u}(n)$ denotes the discrete Fourier transform of u, which we shall also sometimes note as $\mathcal{F}u(n)$:

$$\forall\, n \in \mathbf{Z}^2, \quad \widehat{u}(n) = \mathcal{F}u(n) \stackrel{\text{def}}{=} \int_{\mathbf{T}^2} e^{-in\cdot x} u(x)\, dx,$$

where $n \cdot x$ is the usual scalar product of (n_1, n_2) by (x_1, x_2). We have noted $|n|$ for the usual Euclidean norm in $\mathbf{Z}^2$, $|n|^2 = n_1^2 + n_2^2$. Moreover, we will note $(a|b)$ the scalar product in L^2 of two functions a and b.

As shown in [12] and [13], one very important feature of the vorticity equation (VE) is its Hamiltonian structure. We shall not go into any discussion of the theory of Hamiltonian systems (see the work of V. Arnold in [1], as well as [2] and [11]). All we shall recall here is that the Hamiltonian structure of (VE) implies the conservation of an infinite number of invariants: the energy, and the volume integral of any function of ω. If one wants to study numerically the system (VE), one approach is to compute the Fourier transform of the system, and to truncate at some frequency N. However, traditional truncation strategies inevitably destroy the Hamiltonian structure of the equations (see [12] for details). In [13], V. Zeitlin proposes a strategy which has the feature of preserving the Hamiltonian structure; the main interest of such a structure–preserving scheme is that it is liable to give better results than a scheme destroying the Hamiltonian structure, as such a destruction leads to a loss in the Physics underlying the equation. Let us note that the strategy proposed by V. Zeitlin, and studied in this article, is also presented in the book by V. Arnold and B.A. Khesin (see [2], page 61). The truncation strategy proposed by V. Zeitlin is based on the Fourier expansion of the equation (VE), which can be written as follows:

$$(FVE) \qquad \frac{d}{dt}\widehat{\omega}(n) = \sum_{k \in \mathbf{Z}^2} \frac{n \times k}{|k|^2} \widehat{\omega}(k) \widehat{\omega}(n-k), \quad \forall\, n \in \mathbf{Z}^2.$$

We have noted $n \times k = n_1 k_2 - n_2 k_1$. One traditional way to approximate (FVE) is to consider the ordinary differential equation

$$(FVE_N) \quad \left\{ \begin{array}{rcl} \dfrac{d}{dt}\widehat{\omega}_N(n) & = & \displaystyle\sum_{|k|\leq N} \mathbf{1}_{|n|\leq N} \frac{n \times k}{|k|^2} \widehat{\omega}(k)\widehat{\omega}(n-k), \quad \widehat{\omega}_N(n) = 0 \quad \text{if } |n| > N, \\ \widehat{\omega}_{N|t=0}(n) & = & \mathbf{1}_{|n|\leq N} \displaystyle\int_{\mathbf{T}^2} e^{-in\cdot x} \omega_0(x)\, dx\,. \end{array} \right.$$

We have noted $\mathbf{1}_{|n|\leq N}$ for characteristic function of the ball centered at the origin, of radius N. We refer to [10] for the proof of the well–posedness of (FVE_N), and for the proof of the convergence of the solutions of (FVE_N) to those of (FVE) when N goes to infinity. It is shown in [12] that (FVE_N) does not preserve the Hamiltonian structure, and the same goes for the *aliased* version of that approximation, defined in the following way:

$$\frac{d}{dt}\widehat{\omega}_N(n) = -\sum_{|k|\leq N} \widehat{v}_N\left((n-k)[2N+1]\right) \cdot k\, \widehat{\omega}_N(k),$$

where we have defined $a[M] = \dfrac{|a|[M]}{|a|}a$, where $|a|[M]$ is the smallest integer in $[-N, N]$ equal to $|a|$ mod M. Now we have all the elements to introduce the *sine–bracket truncation*, proposed by V. Zeitlin in [13]:

$$(SB_N) \quad \left\{ \begin{array}{rcl} \dfrac{d}{dt}\widehat{\omega}_N(n) & = & \displaystyle\sum_{|k|\leq N} \mathbf{1}_{|n|\leq N} \frac{2N+1}{2\pi|k|^2} \sin\left(\frac{2\pi n \times k}{2N+1}\right) \widehat{\omega}_N(k)\widehat{\omega}_N\left((n-k)[2N+1]\right) \\ \widehat{\omega}_{N|t=0}(n) & = & \mathbf{1}_{|n|\leq N}\, \widehat{\omega}_0(n). \end{array} \right.$$

The following result is proved in [12] and [13].

Proposition 1.1 *The sine–bracket truncation preserves the Hamiltonian structure of the bidimensional vorticity equation. In particular, it preserves the entropy, defined, for all functions ω, by $H(\omega) \stackrel{\text{def}}{=} |\omega|_0$.*

Remark. Concerning the terminology used to define that truncation, the "sine" part is clear, and the term "bracket" is due to the fact that (SB_N) can be written in a bracket form, due to its Hamiltonian structure (see [12], [13]).

We shall note, throughout this paper,

$$\mathcal{F}T_N(a,b)(n) \stackrel{\text{def}}{=} \sum_{|k|\leq N} \mathbf{1}_{|n|\leq N} \frac{2N+1}{2\pi|k|^2} \sin\left(\frac{2\pi n \times k}{2N+1}\right) \widehat{a}(k)\widehat{b}\left((n-k)[2N+1]\right),$$

$$\mathcal{F}T(a,b)(n) \stackrel{\text{def}}{=} \sum_{k\in\mathbf{Z}^2} \frac{n \times k}{|k|^2} \widehat{a}(k)\widehat{b}(n-k).$$

In particular, if ω_N (resp. ω) is a solution of the sine–bracket truncation (SB_N) (resp. of the bidimensional vorticity equation (VE)), then

$$\frac{d\omega_N}{dt} = T_N(\omega_N, \omega_N), \quad \text{and} \quad \frac{d\omega}{dt} = T(\omega, \omega).$$

This paper is devoted to the mathematical analysis of the truncation (SB_N). We shall not study the numerical aspect of the truncation; we refer to [12] for the analysis of numerical experiments carried out on (SB_N), and for the comparison with traditional schemes. In particular, we shall be concerned with the consistency, and the convergence of the scheme (the stability is a byproduct of the Hamiltonian structure, as seen in Proposition 1.1).

Theorem 1 (consistency) *There exists a constant C such that the following property is satisfied. Let ω be a smooth function, and let $\gamma < 4$ and $\eta > 0$ be two real numbers. Then*

$$|T_N(\omega,\omega) - T(\omega,\omega)|_0 \le C \min\left(|\omega|_1, N|\omega|_0\right) \left(N^{2-\gamma}\,|\omega|_\gamma + N^{-\eta}\,|\omega|_{\eta+2}\right).$$

Remark. That theorem indicates that the sine–bracket truncation is consistent with the vorticity equation (VE), with a rate of at most N^{-2}. The following theorem shows that rate is sharp, and in particular, that the sine–bracket truncation T_N is not spectrally accurate with respect to the operator T.

Theorem 2 *There exists a smooth function ω, a constant $C(\omega)$ and an integer $N(\omega)$, such that for all integers $N \ge N(\omega)$, we have*

$$|T_N(\omega,\omega) - T(\omega,\omega)|_0 \ge C(\omega) N^{-2}.$$

Theorem 3 (convergence) *Let ω_0 and $\omega_{N,0}$ be smooth functions, and let ω_N (resp. ω) be the solution of (SB_N) (resp. (VE)) with initial data $\omega_{N,0}$ (resp. ω_0). Then for all $\gamma < 4$, we have*

$$|\omega_N(t) - \omega(t)|_0 \le C\left(|\omega_{N,0} - \omega_0|_0 + N^{3-\gamma}\|\omega_0\|_{L^2}\|\omega\|_{L^1([0,t],H^\gamma)}\right) e^{C(\|\nabla\omega\|_{L^1([0,t],L^\infty)} + t\|\omega_0\|_{L^2})},$$

where C is a constant independent of N. In particular, the sine–bracket truncation (SB_N) converges to (VE) when N goes to infinity.

Remarks. It should be noted that $\|\nabla\omega\|_{L^1([0,t],L^\infty)}$ increases, *a priori*, as a double exponential of t. So the estimate given in the theorem is poor for large times. Moreover, one can notice that L^2 norms of the vorticity appear, which are constant in time contrary to higher order Sobolev norms. In the proof of that theorem, special care is taken in order to let such L^2 norms appear as often as possible, sometimes at the cost of higher powers of N. Finally, the L^∞ norm of $\nabla\omega$ which appears in Theorem 3 is due to the use of a smoothened cut–off operator (see Section 3), enabling one to use L^p space–norms for $p \neq 2$.

Throughout this paper, we will note all "universal constants" by the same letter C.

2 Consistency of the sine–bracket truncation

In this section, we are going to prove Theorems 1 and 2 stated in the introduction.

Proof of Theorem 1. Let us compute the ℓ^2 norm in n of

$$I_N(\omega,\omega)(n) \stackrel{\text{def}}{=} \mathcal{F}T_N(\omega,\omega)(n) - \mathcal{F}T(\omega,\omega)(n).$$

We have $I_N(\omega,\omega)(n)=\sum_{i=1}^{5} I_N^i(\omega,\omega)(n)$, where

$$I_N^1(a,b)(n)=\sum_{k\in\mathbf{Z}^2}\mathbf{1}_{|k|\le N}\mathbf{1}_{|n-k|>N}\mathbf{1}_{|n|\le N}\frac{2N+1}{2\pi|k|^2}\sin\frac{2\pi n\times k}{2N+1}\widehat{a}(k)\widehat{b}((n-k)[2N+1]), \tag{2.1}$$

$$I_N^2(a,b)(n)=-\sum_{k\in\mathbf{Z}^2}\mathbf{1}_{|k|>N}\mathbf{1}_{|n-k|\le N}\mathbf{1}_{|n|\le N}\frac{n\times k}{|k|^2}\widehat{a}(k)\widehat{b}(n-k), \tag{2.2}$$

$$I_N^3(a,b)(n)=-\sum_{k\in\mathbf{Z}^2}\mathbf{1}_{|n-k|>N}\mathbf{1}_{|n|\le N}\frac{n\times k}{|k|^2}\widehat{a}(k)\widehat{b}(n-k), \tag{2.3}$$

$$I_N^4(a,b)(n)=-\sum_{k\in\mathbf{Z}^2}\mathbf{1}_{|n|>N}\frac{n\times k}{|k|^2}\widehat{a}(k)\widehat{b}(n-k), \tag{2.4}$$

and finally where

$$I_N^5(a,b)(n)=\sum_{k\in\mathbf{Z}^2}\mathbf{1}_{|k|\le N}\mathbf{1}_{|n-k|\le N}\mathbf{1}_{|n|\le N}\frac{2N+1}{2\pi|k|^2}\left(\sin\frac{2\pi n\times k}{2N+1}-\frac{2\pi n\times k}{2N+1}\right)\widehat{a}(k)\widehat{b}(n-k). \tag{2.5}$$

Lemma 2.1 *If a and b are smooth functions, then for all $\alpha>1$ and $\beta>-1$, we have*

$$\|I_N^1(a,b)\|_{\ell^2}\le CN^{2-\alpha}|a|_{-2}|b|_\alpha+CN^{-\beta}|a|_\beta|b|_0.$$

Proof. Let us decompose $I_N^1(a,b)$ according to the relative size of $|k|$ and $N/2$. We have

$$I_N^1(a,b)=I_N^{1,1}(a,b)+I_N^{1,2}(a,b),$$

where

$$I_N^{1,1}(a,b)(n)=\sum_{\frac{N}{2}\le|k|\le N}\mathbf{1}_{|n|\le N}\mathbf{1}_{|n-k|>N}\frac{2N+1}{2\pi|k|^2}\sin\left(\frac{2\pi n\times k}{2N+1}\right)\widehat{a}(k)\widehat{b}((n-k)[2N+1]),$$

$$\text{and}\quad I_N^{1,2}(a,b)(n)=\sum_{|k|\le\frac{N}{2}}\mathbf{1}_{|n|\le N}\mathbf{1}_{|n-k|>N}\frac{2N+1}{2\pi|k|^2}\sin\left(\frac{2\pi n\times k}{2N+1}\right)\widehat{a}(k)\widehat{b}((n-k)[2N+1]).$$

We have

$$\|I_N^{1,1}(a,b)\|_{\ell^2}\le CN\left\|\sum_{|k|\ge\frac{N}{2}}|k|^{-2}|\widehat{a}(k)|\,\mathbf{1}_{N<|n-k|\le 2N+1}|\widehat{b}((n-k)[2N+1])|\right\|_{\ell^2_n}.$$

But if $N<|n-k|\le 2N+1$, then the application $n-k\mapsto(n-k)[2N+1]$ is one–to–one, hence using Young's inequality, we get

$$\|I_N^{1,1}(a,b)\|_{\ell^2}\le CN\left\|\mathbf{1}_{\frac{N}{2}\le|k|}|k|^{-2}|\widehat{a}(k)|\right\|_{\ell^1_k}\left\|\widehat{b}(m)\right\|_{\ell^2_m}.$$

We have noted ℓ^p_m the ℓ^p norm with respect to the variable m.

Now the Cauchy–Schwarz inequality implies that

$$\begin{aligned} \|I_N^{1,1}(a,b)\|_{\ell^2} &\le CN\left\| \mathbf{1}_{\frac{N}{2}\le |k|}|k|^{-2-\beta}|k|^{\beta}|\widehat{a}(k)| \right\|_{\ell^1_k} |b|_0 \\ &\le CN\left\| \mathbf{1}_{\frac{N}{2}\le |k|}|k|^{-2-\beta} \right\|_{\ell^2_k} |a|_{\beta}|b|_0 \\ &\le CN^{-\beta}|a|_{\beta}|b|_0, \end{aligned}$$

for all $\beta > -1$. So one part of the lemma is proved. We now are left with the estimate of the term $\|I_N^{1,2}(a,b)\|_{\ell^2}$. We have

$$\|I_N^{1,2}(a,b)\|_{\ell^2} \le \left\| \sum_{|k|\le \frac{N}{2}} \mathbf{1}_{|n-k|>N}\mathbf{1}_{|n|\le N}\frac{2N+1}{2\pi|k|^2}|\widehat{a}(k)|\,|\widehat{b}\left((n-k)[2N+1]\right)|\, \right\|_{\ell^2_n}.$$

But it is easy to see that if $|k| \le \dfrac{N}{2}$ and $|n-k| > N$, then $\dfrac{N}{2} \le |(n-k)[2N+1]| \le N$. So we can write, using Young's inequality, followed by Cauchy–Schwarz,

$$\begin{aligned} \|I_N^{1,2}(a,b)\|_{\ell^2} &\le CN\left\|\mathbf{1}_{N\le|m|\le 2N+1}\mathbf{1}_{\frac{N}{2}\le|m[2N+1]|\le N}|\widehat{b}(m[2N+1])|\,\right\|_{\ell^1_m}|a|_{-2} \\ &\le CN^{2-\alpha}|b|_{\alpha}|a|_{-2}, \end{aligned}$$

for all $\alpha > 1$. So the lemma is proved. .

Lemma 2.2 *If a and b are smooth functions, then for all $\beta' > 0$, we have*

$$\|I_N^2(a,b)\|_{\ell^2} \le CN^{-\beta'}|a|_{\beta'}\min(|b|_1, N|b|_0).$$

Proof. We have, again using the Cauchy–Schwarz inequality,

$$\begin{aligned} \|I_N^2(a,b)\|_{\ell^2} &\le \left\| \sum_{|k|>N} \mathbf{1}_{|n-k|\le N}\mathbf{1}_{|n|\le N}\frac{|n\times k|}{|k|^2}|\widehat{a}(k)|\,|\widehat{b}(n-k)|\, \right\|_{\ell^2_n} \\ &\le C\left\|\mathbf{1}_{|k|>N}|k|^{-1}|\widehat{a}(k)|\,\right\|_{\ell^1_k}\min(|b|_1, N|b|_0) \\ &\le CN^{-\beta'}|a|_{\beta'}\min(|b|_1, N|b|_0), \end{aligned}$$

for all $\beta' > 0$. The lemma is proved. .

Lemma 2.3 *If a and b are smooth functions, then for all $\delta > 2$, we have*

$$\|I_N^3(a,b)\|_{\ell^2} \le CN^{2-\delta}|a|_{-1}|b|_{\delta}.$$

Proof. We can write

$$\|I_N^3(a,b)\|_{\ell^2} \le \left\| \sum_{k\in\mathbf{Z}^2} \mathbf{1}_{|n-k|>N}\mathbf{1}_{|n|\le N}\frac{|n\times k|}{|k|^2}|\widehat{a}(k)|\,|\widehat{b}(n-k)|\right\|_{\ell^2_n}.$$

It follows that for all $\delta > 2$,

$$\begin{aligned}\|I_N^3(a,b)\|_{\ell^2} &\leq C\Big\| \,|k|^{-1}|\widehat{a}(k)|\,\Big\|_{\ell_k^2} \Big\|\mathbf{1}_{|m|>N}|m||\widehat{b}(m)|\,\Big\|_{\ell_m^1} \\ &\leq C|a|_{-1}\Big\|\mathbf{1}_{|m|>N}|m|^{1-\delta}\Big\|_{\ell_m^2}|b|_\delta,\end{aligned}$$

which gives the result. $\blacksquare$

Lemma 2.4 *If a and b are smooth functions, then for all $\delta > 2$ and $\beta' > 0$, we have*

$$\|I_N^4(a,b)\|_{\ell^2} \leq CN^{-\beta'}|a|_{\beta'}\min(|b|_1, N|b|_0) + CN^{2-\delta}|a|_{-1}|b|_\delta.$$

Proof. Since $|n| > N$, then necessarily, we have either $|k| > \dfrac{N}{2}$, or $|n-k| > \dfrac{N}{2}$. It follows that

$$\begin{aligned}\|I_N^4(a,b)\|_{\ell^2} &\leq \Big\|\sum_{\frac{N}{2}\leq|k|}\mathbf{1}_{|n-k|\leq N/2}\mathbf{1}_{|n|\geq N}\frac{|n\times k|}{|k|^2}|\widehat{a}(k)|\,|\widehat{b}(n-k)|\,\Big\|_{\ell_n^2} \\ &+ \Big\|\sum_{|n-k|>\frac{N}{2}}\mathbf{1}_{|k|\leq N}\mathbf{1}_{|n|\leq N}\frac{|n\times k|}{|k|^2}|\widehat{a}(k)|\,|\widehat{b}(n-k)|\,\Big\|_{\ell_n^2}.\end{aligned}$$

Then a Young inequality yields

$$\begin{aligned}\|I_N^4(a,b)\|_{\ell^2} &\leq C\Big\|\mathbf{1}_{|k|\geq N/2}|k|^{-1}|\widehat{a}(k)|\Big\|_{\ell_k^1}\min(|b|_1, N|b|_0) \\ &+ C\Big\|\mathbf{1}_{\frac{N}{2}\leq|m|}|m||\widehat{b}(m)|\Big\|_{\ell_m^1}\Big\||k|^{-1}|\widehat{a}(k)|\Big\|_{\ell_k^2},\end{aligned}$$

and the result follows by a Cauchy–Schwarz inequality. $\blacksquare$

Lemma 2.5 *If a and b are smooth functions, then for all $\gamma < 4$, we have*

$$\|I_N^5(a,b)\|_{\ell^2} \leq CN^{2-\gamma}|b|_\gamma\min(|a|_1, N|a|_0).$$

Proof. For all $x \in \mathbf{R}$, we have $|\sin x - x| \leq C|x|^3$, hence

$$\|I_N^5(a,b)\|_{\ell^2} \leq C\Big\|\sum_k \mathbf{1}_{|k|\leq N}\mathbf{1}_{|n-k|\leq N}\mathbf{1}_{|n|\leq N}\frac{|n-k|^3|k|}{(2N+1)^2}|\widehat{a}(k)|\,|\widehat{b}(n-k)|\,\Big\|_{\ell_n^2}.$$

Hence we get, by similar computations as above,

$$\begin{aligned}\|I_N^5(a,b)\|_{\ell^2} &\leq C\Big\|\mathbf{1}_{|m|\leq N}|m|^3|\widehat{b}(m)|\Big\|_{\ell_m^1}\min(N^{-1}|a|_0, N^{-2}|a|_1) \\ &\leq CN^{2-\gamma}|b|_\gamma\min(|a|_1, N|a|_0).\end{aligned}$$

for all $\gamma < 4$. The result follows. $\blacksquare$

Putting together Lemmas 2.1 to 2.5 , we get the theorem. ■

Proof of Theorem 2. It is clear from Lemmas 2.1 to and 2.4 that the functions I_N^1 to I_N^4 defined in (2.1) to (2.4) have spectral accuracy. Hence it is enough to prove the estimate of Theorem 2 for I_N^5 defined in (2.5). We are going to construct a function satisfying Theorem 2 in the following way: let us consider the function ω defined by

$$\omega(x_1, x_2) = \cos x_1 + \cos 2x_2. \tag{2.6}$$

Then we obviously have $\widehat{\omega}(1,0) = \widehat{\omega}(0,2) = 1$, and $\widehat{\omega}(k) = 0$ for all other k in $\mathbf{Z}^2$. Let us compute $I_N^5(\omega,\omega)$ in that case. An immediate computation shows that in $I_N^5(\omega,\omega)(n)$, only the value $n = (1,2)$ gives a non zero result, and, for $N \geq 2$,

$$I_N^5(\omega,\omega)(1,2) = \frac{2N+1}{2\pi}\left(\frac{4\pi}{2N+1} - \sin\frac{4\pi}{2N+1}\right) - \frac{2N+1}{8\pi}\left(\frac{4\pi}{2N+1} - \sin\frac{4\pi}{2N+1}\right).$$

Hence we get

$$\begin{aligned} |I_N^5(\omega,\omega)(1,2)| &\geq CN\left(\frac{4\pi}{2N+1} - \sin\frac{4\pi}{2N+1}\right) \\ &\geq CN^{-2} + O(N^{-4}), \end{aligned}$$

using the fact that $x - \sin x = \frac{1}{3!}x^3 + O(x^5)$. So Theorem 2 is proved. ■

3 Convergence of the sine–bracket truncation

The aim of this section is to prove Theorem 3 stated in the introduction. We have

$$\frac{1}{2}\frac{d}{dt}|\omega_N(t) - \omega(t)|_0^2 = \mathrm{Re}\,(\omega_N - \omega \mid T_N(\omega_N,\omega_N) - T(\omega,\omega)),$$

But the structure of the truncation implies that Re $(T_N(a,b)|b) = 0$ for any a and b, hence the quantity $T_N(\omega_N,\omega_N)$ above can be replaced by $T_N(\omega_N,\omega)$. Moreover, we have

$$T_N(\omega_N,\omega) = T_N(\omega_N - \omega,\omega) + T_N(\omega,\omega),$$

so we get finally

$$\frac{1}{2}\frac{d}{dt}|\omega_N - \omega|_0^2 = \mathrm{Re}\,(T_N(\omega,\omega) - T(\omega,\omega) \mid \omega_N - \omega) + \mathrm{Re}\,(T_N(\omega_N - \omega,\omega) \mid \omega_N - \omega).$$

Then Theorem 1 enables us to infer that for all $\eta > 0$ and all $\gamma < 4$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\omega_N - \omega|_0^2 &\leq C|\omega_N - \omega|_0\|\omega\|_{L^2}(N^{3-\gamma}|\omega|_\gamma + N^{1-\eta}|\omega|_{\eta+2}) \\ &+ |T_N(\omega_N - \omega \mid \omega)|_0|\omega_N - \omega|_0. \end{aligned} \tag{3.1}$$

Lemma 3.1 *If $a \in L^2(\mathbf{T}^2)$ and if b is a smooth function, then for all $N \in \mathbf{N}$, we have for all $0 < \varepsilon < 1$,*

$$|T_N(a,b)|_0 \leq C|a|_0(\|\nabla b\|_{L^\infty} + |b|_0 + N^{-\varepsilon}|b|_{\varepsilon+3}).$$

Let us suppose temporarily that this result is proved, and let us go back to estimate (3.1). Then Lemma 3.1 implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\omega_N-\omega|_0^2 &\leq C\|\omega\|_{L^2}|\omega_N-\omega|_0(N^{3-\gamma}|\omega|_\gamma+N^{1-\eta}|\omega|_{\eta+2})\\ &+ C(\|\nabla\omega\|_{L^\infty}+\|\omega\|_{L^2}+N^{-\varepsilon}|\omega|_{\varepsilon+3})\,|\omega_N-\omega|_0^2,\end{aligned}$$

hence, by Gronwall's lemma, we get for all $t\geq 0$, for all $N\in\mathbf{N}$, for all $\gamma<4$, $\eta>0$ and $0<\varepsilon<1$,

$$\begin{aligned}|\omega_N(t)-\omega(t)|_0 &\leq |\omega_{N,0}-\omega_0|_0\exp\left(C\int_0^t\left(\|\nabla\omega\|_{L^\infty}+\|\omega\|_{L^2}+N^{-\varepsilon}|\omega|_{\varepsilon+3}\right)(\tau)\,d\tau\right)\\ &+C\,N^{3-\gamma}\|\omega_0\|_{L^2}\int_0^t|\omega|_\gamma(\tau)\,d\tau\ \exp\left(C\int_0^t\left(\|\nabla\omega\|_{L^\infty}+\|\omega\|_{L^2}+N^{-\varepsilon}|\omega|_{\varepsilon+3}\right)(\tau)\,d\tau\right)\\ &+C\,N^{1-\eta}\|\omega_0\|_{L^2}\int_0^t|\omega|_{\eta+2}(\tau)\,d\tau\ \exp\left(C\int_0^t\left(\|\nabla\omega\|_{L^\infty}+\|\omega\|_{L^2}+N^{-\varepsilon}|\omega|_{\varepsilon+3}\right)(\tau)\,d\tau\right),\quad(3.2)\end{aligned}$$

where we have used the fact that the L^2 norm of ω is constant. So the theorem is proved. .

Proof of Lemma 3.1. We have

$$|T_N(a,b)|_0\leq\|I_N^1(a,b)\|_{\ell^2}+\|I_N^5(a,b)\|_{\ell^2}+\|I_N^6(a,b)\|_{\ell^2},$$

with the notation of the previous section (see (2.5)), and where we have defined

$$I_N^6(a,b)(n)\stackrel{\text{def}}{=}\sum_{k\in\mathbf{Z}^2}\mathbf{1}_{|k|\leq N}\mathbf{1}_{|n-k|\leq N}\mathbf{1}_{|n|\leq N}\frac{n\times k}{|k|^2}\widehat{a}(k)\widehat{b}(n-k).\qquad(3.3)$$

But Lemmas 2.1 and 2.5 imply that

$$\|I_N^1(a,b)\|_{\ell^2}+\|I_N^5(a,b)\|_{\ell^2}\leq CN^{3-\gamma}|a|_0|b|_\gamma+C|a|_0|b|_0,$$

for all $2<\gamma<4$. So now we are left with the estimate of $\|I_N^6(a,b)\|_{\ell^2}$. A direct estimate here will lead nowhere: one sees easily that it yields

$$\|I_N^6(a,b)\|_{\ell^2}\leq C|a|_{-1}\left\|\mathbf{1}_{|m|\leq N}|m||\widehat{b}(m)|\right\|_{\ell^1_m},$$

which if we only consider L^2-based Sobolev norms for b, is unbounded as N increases, and that is not wanted since that estimate appears in an exponential after Gronwall's lemma (see equation (3.2) above). To circumvent this difficulty, we are going to replace the ℓ^1_m estimate on $\mathbf{1}_{|m|\leq N}|m||\widehat{b}(m)|$ above by an L^∞ estimate on ∇b, which will not cost any power of N. The way to do so is to use an elementary form of J.-M. Bony's paradifferential calculus (see [3]), by replacing the cut–off $\mathbf{1}_{|m|\leq N}$ by a smooth truncation defined as follows: let us consider a smooth, compactly supported, radial function φ, with values in $[0,1]$, such that

$$\varphi(x)=1\quad\text{if}\quad x\in[0,1],\qquad\varphi(x)=0\quad\text{if}\quad|x|\geq 2\,.$$

Then the tame truncation operator S_N, for all $N\in\mathbf{N}$, is defined by

$$S_Nu\stackrel{\text{def}}{=}\mathcal{F}^{-1}\left(\varphi\left(\frac{|\cdot|}{N}\right)\widehat{u}(\cdot)\right),$$

and it is easy to see that if $u \in \mathcal{D}'$, then $S_N u$ converges to u in $\mathcal{D}'$, when N goes to infinity. Moreover, we have the following crucial estimate:

$$\|S_N u\|_{L^p} \leq C\|u\|_{L^p},$$

for any $p \in [1,\infty]$, where C does not depend on N. Let us note that such an estimate is false in the case of the cut–off, except of course when $p = 2$. Now we write $I_N^6 = I_N^{6,1} + I_N^{6,2}$, where

$$I_N^{6,1}(a,b)(n) = \sum_{k\in\mathbf{Z}^2} \mathbf{1}_{|k|\leq N}\mathbf{1}_{|n|\leq N}\frac{n\times k}{|k|^2}\widehat{a}(k)\varphi\left(\frac{|n-k|}{N}\right)\widehat{b}(n-k)$$

and

$$I_N^{6,2}(a,b)(n) = \sum_{k\in\mathbf{Z}^2} \mathbf{1}_{|k|\leq N}\mathbf{1}_{|n|\leq N}\frac{n\times k}{|k|^2}\widehat{a}(k)\,(1-\varphi)\left(\frac{|n-k|}{N}\right)\widehat{b}(n-k).$$

We have

$$\|I_N^{6,1}(a,b)\|_{\ell^2} \leq \sum_{i\neq j}\left\|\sum_{k\in\mathbf{Z}^2} \mathbf{1}_{|k|\leq N}\mathbf{1}_{|n|\leq N}\frac{(n_i-k_i)k_j}{|k|^2}\widehat{a}(k)\varphi\left(\frac{|n-k|}{N}\right)\widehat{b}(n-k)\right\|_{\ell^2_n},$$

hence by Plancherel's theorem,

$$\begin{aligned}
\|I_N^{6,1}(a,b)\|_{\ell^2} &\leq \sum_{i\neq j}\left\|\mathcal{F}^{-1}\left(\mathbf{1}_{|k|\leq N}\frac{k_j}{|k|^2}\widehat{a}(k)\right)S_N(\partial_i b)\right\|_{L^2}\\
&\leq \sum_{i\neq j}\left\|S_N(\partial_i b)\right\|_{L^\infty}\left\|\mathbf{1}_{|k|\leq N}\frac{k_j}{|k|^2}\widehat{a}(k)\right\|_{\ell^2_k}\\
&\leq C\|\nabla b\|_{L^\infty}|a|_{-1}.
\end{aligned}$$

Finally the last term is estimated as follows:

$$\begin{aligned}
\|I_N^{6,2}(a,b)\|_{\ell^2} &\leq \left\|\sum_{k\in\mathbf{Z}^2} \mathbf{1}_{|k|\leq N}\mathbf{1}_{|n-k|\geq N}\frac{|n-k|}{|k|}|\widehat{a}(k)|\,|\widehat{b}(n-k)|\right\|_{\ell^2}\\
&\leq C|a|_{-1}\left\|\mathbf{1}_{|m|\geq N}|m|^{1-\varepsilon'}\right\|_{\ell^1_m}|b|_{\varepsilon'}\\
&\leq CN^{2-\varepsilon'}|a|_{-1}|b|_{\varepsilon'},
\end{aligned}$$

as soon as $\varepsilon' > 2$. The lemma follows. .

Concluding remark

This paper shows that the sine–bracket truncation does converge towards the Euler equation, but that the consistency is not satisfactory, contrary to a traditional spectral truncation approximation, which has spectral accuracy. However it is important from a practical point of view to conserve as much invariants as possible in the approximation; hence if one desires a better balance between accuracy and conservation of a large number of invariants, one is led to looking for a new type of approximation.

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