INVARIANT DIFFERENTIAL OPERATORS ON THE HEISENBERG GROUP AND MEIXNER-POLLACZEK POLYNOMIALS

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Abstract Consider the Heisenberg Lie algebra with basis X, Y, Z, such that $[X, Y] = Z$. Then the symmetrization $\sigma(X^k Y^k)$ can be written as a polynomial in $\sigma(XY)$ and Z, and this polynomial is identified as a Meixner-Pollaczek polynomial. This is an observation by Bender, Mead and Pinsky, a proof of which has been given by Koornwinder. We extend this result in the framework of Gelfand pairs associated with the Heisenberg group. This extension involves multivariable Meixner-Pollaczek polynomials.

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Contents

The starting point of this paper is an identity in the Heisenberg algebra which has been observed by Bender, Mead and Pinsky ([1986], 1987]), and revisited by Koornwinder who gave an alternative proof. Let X, Y, Z generate the three dimensional Heisenberg Lie algebra, with $[X, Y] = Z$. Then the symmetrization of $X^k Y^k$ can be written as a polynomial in the symmetrization of XY, and this polynomial is a Meixner-Pollaczek polynomial. We rephrase this question in the framework of the spherical analysis for a Gelfand pair. If (G, K) is a Gelfand pair with a Lie group G, the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on the quotinet space G/K is commutative. The spherical Fourier transform maps this algebra onto an algebra of continuous functions on the Gelfand spectrum Σ of the commutative Banach algebra $L^1(K\backslash G/K)$ of K-biinvariant integrable functions on G. For $D \in \mathbb{D}(G/K)$, the corresponding function is denoted by $\hat{D}$. Hence an identity in the algebra $\mathbb{D}(G/K)$ is equivalent to an identity for the functions $\hat{D}$. We consider Gelfand pairs associated to the Heisenberg groups. The unitary group $K = U(p)$ acts on the Heisenberg group $H = \mathbb{C}^p \times \mathbb{R}$. Let $G = K \ltimes H$ be the semi-direct product. Then (G, K) is a Gelfand pair. The functions $\widehat{\mathcal{L}_k}$, corresponding to a family $\mathcal{L}_k$ of invariant differential operators on the Heisenberg group, involve Meixner-Pollaczek polynomials, and give rise to identities in the algebra $\mathbb{D}(H)^K$ of differential operators on H which are left invariant by H and by the action of K. We extend this analysis to some Gelfand pairs associated to the Heisenberg group which have been considered by Benson, Jenkins, and Ratcliff [1992]. The Heisenberg group H is taken as $H = W \times \mathbb{R}$, with $W = M(n, p, \mathbb{C})$. The group $K = U(n) \times U(p)$ acts on W and (G, K) is a Gelfand pair, with $G = K \ltimes H$. We determine the functions $\hat{D}$ for families of differential operators on $D(H)^K$. These functions $\hat{D}$ involve multivariate Meixner-Pollaczek polynomials which have been introduced in [Faraut-Wakayama,2012]. The proofs use spherical Taylor expansions, and the connection between multivariate Laguerre polynomials and multivariate Meixner-Pollaczek polynomials. In the last section, the Heisenberg group is taken as $W \times \mathbb{R}$, where W is a simple complex Jordan algebra, and $K = \mathrm{Str}(W) \cap U(W)$, where $\mathrm{Str}(W)$ is the structure group of W, and $U(W)$ the unitary group.

1 Gelfand pairs

Let G be a locally compact group, and K a compact subgroup, and let $L^1(K\backslash G/K)$ denote the convolution algebra of K-invariant integrable functions on G. One says that (G,K) is a Gelfand pair if the algebra $L^1(K\backslash G/K)$ is commutative. From now on we assume that it is the case. A spherical function is a continuous function φ on G, K-biinvariant, with $\varphi(e)=1$, and

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y),$$

where α is the normalized Haar measure on K. The characters χ of the commutative Banach algebra $L^1(K\backslash G/K)$ are of the form

$$\chi(f) = \int_G f(x)\varphi(x)m(dx),$$

where φ is a bounded spherical function (m is a Haar measure on the unimodular group G). Hence the Gelfand spectrum Σ of the commutative Banach algebra $L^1(K\backslash G/K)$ can be identified with the set of bounded spherical functions. We denote by $\varphi(\sigma;x)$ the spherical function associated to $\sigma\in\Sigma$. The spherical Fourier transform of $f\in L^1(K\backslash G/K)$ is the function $\hat{f}$ defined on Σ by

$$\hat{f}(\sigma) = \int_G \varphi(\sigma;x)f(x)m(dx).$$

Assume now that G is a Lie group, and denote by $\mathbb{D}(G/K)$ the algebra of G-invariant differential operators on G/K. This algebra is commutative. A spherical function is $\mathcal{C}^\infty$ and eigenfunction of every $D\in\mathbb{D}(G/K)$:

$$D\varphi(\sigma;x) = \hat{D}(\sigma)\varphi(\sigma;x),$$

where $\hat{D}(\sigma)$ is a continuous function on Σ. The map

$$D\mapsto \hat{D},\quad \mathbb{D}(G/K)\to\mathcal{C}(\Sigma),$$

is an algebra morphism. Moreover the Gelfand topology of Σ is the initial topology associated to the functions $\sigma\mapsto\hat{D}(\sigma)$ $(D\in\mathbb{D}(G/K))$ ([Ferrari-Rufino,2007]).

We address the following questions:

- Given a differential operator $D \in \mathbb{D}(G/K)$, determine the function $\hat{D}$.

- Construct a linear basis $(D_\mu)_{\mu \in \mathfrak{M}}$ of $\mathbb{D}(G/K)$, and, for each μ, a K-invariant analytic function b_μ in a neighborhood of $o = eK \in G/K$ such that

$$D_\mu b_\nu(o) = \delta_{\mu\nu}.$$

- Establish a mean value formula: for an analytic function f on G/K, defined in a neighborhood of o,

$$\int_K f(xky)\alpha(dk) = \sum_{\mu \in \mathfrak{M}} (D_\mu f)(x) b_\mu(y).$$

Observe that it is enough to prove, for a K-invariant analytic function f, that

$$f(y) = \sum_{\mu \in \mathfrak{M}} (D_\mu f)(o) b_\mu(y).$$

In particular, for $f(x) = \varphi(\sigma; x)$, one gets a generalized Taylor expansion for the spherical functions

$$\varphi(\sigma; x) = \sum_{\mu \in \mathfrak{M}} \widehat{D_\mu}(\sigma) b_\mu(x).$$

Basic example

Take $G = \mathbb{R}$, $K = \{0\}$. Then $\Sigma = \mathbb{R}$, and

$$\varphi(\sigma; x) = e^{i\sigma x}.$$

We can take, with $\mathfrak{M} = \mathbb{N}$,

$$D_\mu = \Big(\frac{d}{dx}\Big)^\mu, \quad b_\mu(x) = \frac{x^\mu}{\mu!}.$$

Then the mean value formula is nothing but the Taylor formula

$$f(x+y) = \sum_{\mu=0}^{\infty} \Big(\big(\frac{d}{dx}\big)^\mu f\Big)(x) \frac{y^\mu}{\mu!},$$

and the Taylor formula for the spherical functions is the power expansion of the exponential:

$$e^{i\sigma x} = \sum_{\mu=0}^{\infty} (i\sigma)^\mu \frac{x^\mu}{\mu!}.$$

Historical example

Here $G = SO(n) \ltimes \mathbb{R}^n$, the motion group, $K = SO(n)$; then $G/K \simeq \mathbb{R}^n$. The spectrum Σ can be identified to the half-line, $\Sigma = [0, \infty[$. The spherical functions are given by

$$\varphi(\sigma; x) = \int_{S(\mathbb{R}^n)} e^{i\sigma(u|x)} \beta(du) \quad (\sigma \geq 0, x \in \mathbb{R}^n),$$

where β is the normalized uniform measure on the unit sphere $S(\mathbb{R}^n)$. (The function $\varphi(\sigma; x)$ can be written in terms of Bessel functions.) The algebra $\mathbb{D}(G/K)$ is generated by the Laplace operator Δ, and $\hat{\Delta}(\sigma) = -\sigma^2$. We can take, with $\mathfrak{M} = \mathbb{N}$,

$$D_\mu = \Delta^\mu, \; b_\mu = c_\mu \|x\|^{2\mu},$$

with

$$c_\mu = 2^{-2\mu} \frac{1}{\left(\frac{n}{2}\right)_\mu} \frac{1}{\mu!}.$$

Then the mean value formula can be written

$$\int_K f(x + k \cdot y) \alpha(dk) = \sum_\mu^\infty c_\mu (\Delta^\mu f)(x) \|y\|^{2\mu}.$$

In [Courant-Hilbert,1937], §3, Section 4, one finds the equivalent formula

$$\int_{S(\mathbb{R}^n)} f(x + ru) \beta(du) = \sum_{\mu=0}^\infty c_\mu (\Delta^\mu f)(x) r^{2\mu}.$$

The generalized Taylor expansion of the spherical functions

$$\varphi(\sigma; x) = \sum_{\mu=0}^\infty c_\mu (-1)^\mu \sigma^{2\mu} \|x\|^{2\mu},$$

is nothing but the power series expansion of the Bessel functions.

2 Gelfand pairs associated with the Heisenberg group

Let W be a complex Euclidean vector space. The set $H = W \times \mathbb{R}$, equipped with the product

$$(z, t)(z', t) = \big(z + z', t + t' + \operatorname{Im}(z'|z)\big),$$

is the Heisenberg group of dimension $2N+1$ ($N = \dim_{\mathbb{C}} W$). Relative to coordinates $z_1, \ldots, z_N$ with respect to a fixed orthogonal basis in W, consider the first order left-invariant differential operators on H:

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2i}\overline{z}_j\frac{\partial}{\partial t} \quad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{2i}z_j\frac{\partial}{\partial t} \quad (j = 1, \ldots, N).$$

Recall the notation

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big).$$

These operators form a basis of the Lie algebra $\mathfrak{h}$ of H. They satisfy

$$[Z_j, \overline{Z}_j] = iT,$$

and other brackets vanish.

Let K be a closed subgroup of the unitary group $U(W)$ of W, and G be the semi-direct product $G = K \ltimes H$. The pair (G, K) is a Gelfand pair if and only if the group K acts multiplicity free on the space $\mathcal{P}(W)$ of holomorphic polynomials on W. This result has been proven by Carcano [1987] (see also [Benson-Ratcliff-Ratcliff-Worku,2004], [Wolf,2007]). We assume that this condition holds. Hence the Banach algebra $L^1(H)^K$ of K-invariant integrable functions on H is isomorphic to $L^1(K\backslash G/K)$, hence commutative.

The Fock space $\mathcal{F}(W)$ is the space of holomorphic functions ψ on W such that

$$\|\psi\|^2 = \frac{1}{\pi^N}\int_W |\psi(z)|^2 e^{-\|z\|^2} m(dz) < \infty$$

(m denotes the Euclidean measure on W). The reproducing kernel of $\mathcal{F}(W)$ is

$$\mathcal{K}(z, w) = e^{(z|w)}.$$

The Fock space decomposes multiplicity free under K:

$$\mathcal{F}(W) = \widehat{\bigoplus_{m\in\mathcal{M}}} \mathcal{H}_m.$$

Let $\mathcal{K}_m$ denotes the reproducing kernel of $\mathcal{H}_m$. Then

$$e^{(z|w)} = \sum_{m\in\mathcal{M}} \mathcal{K}_m(z, w).$$

The algebra $\mathbb{D}(H)^K$ of differential operators on H which are invariant with respect to the left action of H and the action of K is isomorphic to the algebra $\mathbb{D}(G/K)$, hence commutative. To the polynomial $\mathcal{K}_m(z,w)$ one associates the invariant differential operators $\mathcal{D}_m$ and $\mathcal{L}_m$ in $\mathbb{D}(H)^K$. Let $\tilde{\mathcal{K}}_m$ be the polynomial in the $2N$ variables $z_1,\ldots,z_N,\bar{z}_1,\ldots,\bar{z}_N$ such that

$$\mathcal{K}_m(z,z)=\tilde{\mathcal{K}}_m(z,\bar{z}).$$

The operator $\mathcal{D}_m$ is defined by

$$\mathcal{D}_m=\tilde{\mathcal{K}}_m(\bar{Z}_1,\ldots,\bar{Z}_N;Z_1,\ldots Z_N).$$

The operators Z_j are applied first, then the operators $\bar{Z}_j$.

The operator $\mathcal{L}_m$ is defined by symmetrization from the K-invariant (non holomorphic) polynomial $\mathcal{K}_m(z,z)$: for a smooth function f on H,

$$(\mathcal{L}_m f)(z,t)=\mathcal{K}_m\big(\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\zeta}\big)f\big(z+\zeta,t+\mathrm{Im}(\zeta|z)\big)\big|_{\zeta=0}.$$

The eigenvalues $\widehat{\mathcal{D}_m}(\sigma)$ and $\widehat{\mathcal{L}_m}(\sigma)$ have gotten general formulas in terms of generalized binomial coefficients by Benson and Ratcliff [1998]. In the sequel we will consider some special cases and give explicit formulas for these eigenvalues in terms of classical polynomials.

For $\mu=(m,\ell)\in\mathfrak{M}=\mathcal{M}\times\mathbb{N}$, define $D_\mu=\mathcal{L}_m T^\ell$. Then the operators D_μ form a linear basis of the vector space $\mathbb{D}(H)^K$. Define

$$b_\mu(z,t)=\frac{1}{\mathrm{dim}\mathcal{H}_m}\mathcal{K}_m(z,z)\frac{1}{\ell!}t^\ell.$$

Proposition 2.1.

$$D_\mu b_\nu=\delta_{\mu\nu}.$$

This follows from

$$\mathcal{L}_k\mathcal{K}_m=\delta_{k,m}\mathrm{dim}\mathcal{H}_k\quad(k,m\in\mathcal{M}).$$

Theorem 2.2. *If f is a K-invariant analytic function on H in a neighborhood of 0, then*

$$\begin{aligned}f(z,t)&=\sum_{\mu\in\mathfrak{M}}(D_\mu f)(0,0)b_\mu(z,t)\\&=\sum_{m\in\mathcal{M}}\sum_{\ell=0}^{\infty}\frac{1}{\dim\mathcal{H}_m}\frac{1}{\ell!}(\mathcal{L}_m T^\ell f)(0,0)\mathcal{K}_m(z,z)t^\ell.\end{aligned}$$

This implies the following mean value formula: for an analytic function f on H,

$$\begin{aligned}
&\int_K f\big(z+k\cdot w, s+t+\mathrm{Im}(k\cdot w|z)\big)\alpha(dk)\\
=\ &\sum_{\mu\in\mathcal{M}} (D_\mu f)(z,s)b_\mu(w,t)\\
=\ &\sum_{m\in\mathcal{M}}\sum_{\ell=0}^{\infty} \frac{1}{\dim\mathcal{H}_m}\frac{1}{\ell!}(\mathcal{L}_m T^\ell)f(z,s)\mathcal{K}_m(w,w)t^\ell.
\end{aligned}$$

Corollary 2.3. *As a special case one obtains the following expansion for the spherical functions:*

$$\varphi(\sigma;z,t) = e^{i\lambda t}\sum_{m\in\mathcal{M}} \frac{1}{\dim\mathcal{H}_m}\widehat{\mathcal{L}_m}(\sigma)\mathcal{K}_m(z,z).$$

(Observe that $\varphi(\sigma;0,t)$ is an exponential, $= e^{i\lambda t}$, where λ depends on σ.) This formula will give a way for evaluating the eigenvalues $\widehat{\mathcal{L}_m}(\sigma)$.

The Bergmann representation π_λ is defined on the Fock space $\mathcal{F}_\lambda(W)$ ($\lambda\in\mathbb{R}^*$) of the holomorphic functions ψ on W such that

$$\|\psi\|_\lambda^2 = \Big(\frac{|\lambda|}{\pi}\Big)^N \int_W |\psi(\zeta)|^2 e^{-|\lambda|\|\zeta\|^2} m(d\zeta) < \infty.$$

For $\lambda > 0$,

$$\big(\pi_\lambda(z,t)\psi\big)(\zeta) = e^{\lambda\left(it-\frac{1}{2}\|z\|^2-(\zeta|z)\right)}\psi(\zeta+z).$$

For $\lambda < 0$, let $\pi_\lambda(z,t) = \pi_{-\lambda}(\bar{z},-t)$. Because of this simple relation we may assume that $\lambda > 0$, and will do most of the time in the sequel. The representation π_λ is irreducible. If $f\in L^1(H)^K$, then the operator $\pi_\lambda(f)$ commutes with the action of K on $\mathcal{F}_\lambda(W)$. By Schur's Lemma the subspace $\mathcal{H}_m$ is an eigenspace of $\pi_\lambda(f)$:

$$\pi_\lambda(f)\psi = \hat{f}(\lambda,m)\psi \quad (\psi\in\mathcal{H}_m),$$

and the eigenvalue can be written

$$\hat{f}(\lambda,m) = \int_H f(z,t)\varphi(\lambda,m;z,t)m(dz)dt.$$

The functions $\varphi(\lambda, m; z, t)$ are the bounded spherical functions of the first kind ($\lambda \in \mathbb{R}^*$, $m \in \mathcal{M}$).

The bounded spherical functions of the second kind are associated to the one-dimensional representations η_w of H:

$$\eta_w(z,t) = e^{2i\mathrm{Im}(z|w)} \quad (w \in W).$$

They are given by

$$\varphi(\dot{w}; z, t) = \int_K e^{2i\mathrm{Im}(z|k\cdot w)} \alpha(dk).$$

The Gelfand spectrum Σ can be described as the union $\Sigma = \Sigma_1 \cup \Sigma_2$. The part Σ_1 corresponds to the bounded spherical functions of the first kind, parametrized by the pairs (λ, m), with $\lambda \in \mathbb{R}^*$, $m \in \mathcal{M}$, and the part Σ_2 to the bounded spherical functions of the second kind, parametrized by $K\backslash W$, the set of K-orbits in W.

Recall that

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2i}\overline{z}_j\frac{\partial}{\partial t}, \quad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{2i}z_j\frac{\partial}{\partial t} \quad (j = 1, \ldots, N).$$

For the derived representations one obtains

$$\begin{aligned} d\pi_\lambda(T) = i\lambda, \quad d\pi_\lambda(Z_j) = \frac{\partial}{\partial \zeta_j}, \quad d\pi_\lambda(\overline{Z}_j) = -\lambda\zeta_j, \\ d\eta_w(T) = 0, \quad d\eta_w(Z_j) = \bar{w}_j, \quad d\eta_w(\overline{Z}_j) = -w_j. \end{aligned}$$

From the definition of $\mathcal{D}_p$ ($p \in \mathcal{M}$) it follows that

$$d\pi_\lambda(\mathcal{D}_p) = \tilde{\mathcal{K}}_p\big(-\lambda\zeta, \frac{\partial}{\partial\zeta}\big), \quad d\eta_w(\mathcal{D}_p) = \tilde{\mathcal{K}}_p(-w, \bar{w}).$$

The subspace $\mathcal{H}_m$ is an eigenspace of the operator $d\pi_\lambda(\mathcal{D}_p)$:

$$d\pi_\lambda(\mathcal{D}_p)\psi = \widehat{\mathcal{D}_p}(\lambda, m)\psi \quad (\psi \in \mathcal{H}_m).$$

This will give a way for evaluating $\widehat{\mathcal{D}_p}$.

3 The case $W = \mathbb{C}^p$, $K = U(p)$

We consider the Heisenberg group $H = \mathbb{C}^p \times \mathbb{R}$, with the action of $K = U(p)$. Then (G, K) with $G = U(p) \times \mathbb{C}^p$ is a Gelfand pair. It has been first observed by Korányi [1980] (see also [Faraut-1984]). In this case $\mathcal{M} \simeq \mathbb{N}$, $\mathcal{H}_m$ is the space of homogeneous polynomials of degree m, and

$$\mathcal{K}_m(z, w) = \frac{1}{m!}(z|w)^m, \quad \dim \mathcal{H}_m = \frac{(p)_m}{m!}.$$

Furthermore

$$\mathcal{L}_m f(z, t) = \frac{1}{m!}\Big(\sum_{j=1}^{p} \frac{\partial^2}{\partial \zeta_j \partial \overline{\zeta}_j}\Big)^m f\big(z + \zeta, t + \mathrm{Im}\,(\zeta|z)\big)\big|_{\zeta=0}.$$

For $m = 1$,

$$\mathcal{L}_1 = \sum_{j=1}^{p} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} + \frac{i}{2}\Big(z_j \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial \overline{z}_j}\Big)\frac{\partial}{\partial t} + \frac{1}{4}\|z\|^2 \frac{\partial^2}{\partial t^2}.$$

Up to a factor $\mathcal{L}_1$ is the sublaplacian Δ_0: $\mathcal{L}_1 = \frac{1}{4}\Delta_0$. The operator $\mathcal{L}_1$ can be obtained by symmetrization:

$$\mathcal{L}_1 = \frac{1}{2}\sum_{j=1}^{n}(Z_j \overline{Z}_j + \overline{Z}_j Z_j).$$

The algebra $\mathbb{D}(H)^K$ is generated by the two operators T and $\mathcal{L}_1$.

The spectrum of the Gelfand pair (G, K) is the union $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is parametrized by the set of pairs (λ, m), with $\lambda \in \mathbb{R}^*$, $m \in \mathbb{N}$, and $\Sigma_2 \simeq [0, \infty[$. The bounded spherical functions of the first kind are expressed in terms of the ordinary Laguerre polynomials $L_m^{(\nu)}$: for $\sigma = (\lambda, m) \in \Sigma_1$,

$$\varphi(\lambda, m; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda|\|z\|^2} \frac{L_m^{(p-1)}(|\lambda|\|z\|^2)}{L_m^{(p-1)}(0)}.$$

This function admits the following expansion

$$\varphi(\lambda, m; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda|\|z\|^2} \sum_{k=0}^{m} (-1)^k \frac{1}{(p)_k}\frac{1}{k!}|\lambda|^k [m]_k \|z\|^{2k}.$$

We recall the Pochhammer symbols:

$$[x]_k = x(x-1)\dots(x-k+1), \quad (x)_k = x(x+1)\cdots(x+k-1).$$

The bounded spherical functions of the second kind are expressed in terms of the Bessel functions: for $\sigma = \tau \in \Sigma_2$,

$$\varphi(\tau; z, t) = j_{n-1}(2\sqrt{\tau}\|z\|),$$

where $j_\nu(r) = \Gamma(\nu+1)\big(\frac{r}{2}\big)^{-\nu} J_\nu(r)$. These functions admits the following expansion

$$\varphi(\tau; z, t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(p)_k} \frac{1}{k!} \tau^k \|z\|^{2k}.$$

Proposition 3.1. *The eigenvalues of the differential operator $\mathcal{D}_k$ are given, for $(\lambda, m) \in \Sigma_1$, $\lambda > 0$, by*

$$\widehat{\mathcal{D}_k}(\lambda, m) = \frac{(-1)^k}{k!} \lambda^k [m]_k,$$

and, for $(\tau) \in \Sigma_2$, by

$$\widehat{\mathcal{D}_k}(\tau) = \frac{(-1)^k}{k!} \tau^k.$$

Proof.

We saw that

$$d\pi_\lambda(\mathcal{D}_k) = \tilde{\mathcal{K}}_k\big(-\lambda\zeta, \frac{\partial}{\partial\zeta}\big).$$

Since

$$\tilde{\mathcal{K}}_k(z, w) = \frac{1}{k!}\Big(\sum_{j=1}^{p} z_j w_j\Big)^k,$$

this means that $\pi_\lambda(\mathcal{D}_k)$ is the differential operator with symbol

$$\sigma(\zeta, \xi) = \frac{(-1)^k}{k!} \lambda^k \Big(\sum_{j=1}^{p} \zeta_j \xi_j\Big)^k = \frac{(-1)^k}{k!} \lambda^k \sum_{|\alpha|=p} \frac{k!}{\alpha!} \zeta^\alpha \xi^\alpha,$$

where

$$\alpha = (\alpha_1, \dots, \alpha_p),\ \alpha_j \in \mathbb{N},\ \alpha! = \alpha_1! \dots \alpha_p!,\ \zeta^\alpha = \zeta_1^{\alpha_1} \dots \zeta_p^{\alpha_p}.$$

The operator $\pi_\lambda(\mathcal{D}_k)$ is given explicitely as follows

$$\pi_\lambda(\mathcal{D}_k) = \frac{(-1)^k}{k!}\lambda^k \sum_{|\alpha|=p} \frac{k!}{\alpha!}\zeta^\alpha\big(\frac{\partial}{\partial\zeta}\big)^\alpha.$$

Let us apply this operator to the polynomial $\psi(\zeta) = \zeta_1^m$ which belongs to $\mathcal{H}_m$:

$$\pi_\lambda(\mathcal{D}_k)\psi(\zeta) = \frac{(-1)^k}{k!}\lambda^k\zeta_1^k\big(\frac{\partial}{\partial\zeta_1}\big)^k\zeta_1^m = \frac{(-1)^k}{k!}\lambda^k[m]_k\psi(\zeta).$$

Therefore

$$\widehat{\mathcal{D}_k}(\lambda, m) = \frac{(-1)^k}{k!}\lambda^k[m]_k.$$

In case of the one-dimensional representation η_w,

$$\eta_w(\mathcal{D}_k) = \frac{(-1)^k}{k!}|w|^{2k} = \frac{(-1)^k}{k!}\tau^k.$$

∎

It follows that the expansion of the spherical functions can be written

$$\varphi(\sigma; z, t) = e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2}\sum_{k=0}^m \frac{1}{(p)_k}\widehat{\mathcal{D}_k}(\sigma)\|z\|^{2k}.$$

Corollary 3.2. *For every D in $\mathbb{D}(H)^K$ there is a polynomial F_D in two variables such that, for $(\lambda, m) \in \Sigma_1$, $\lambda > 0$,*

$$\widehat{D}(\lambda, m) = F_D(\lambda, \lambda m),$$

and, for $(\tau) \in \Sigma_2$,

$$\widehat{D}(\tau) = F_D(0, \tau).$$

The map $D \mapsto F_D$, $\mathbb{D}(H)^K \to \mathrm{Pol}(\mathbb{C}^2)$ is an algebra isomorphism.

In particular, for $D = \mathcal{D}_k$,

$$F_{\mathcal{D}_k}(u, v) = \frac{(-1)^k}{k!}v(v-u)\dots(v-(k-1)u).$$

Let us embed Σ into $\mathbb{R}^2$ by the map

$$(\lambda, m) \in \Sigma_1 \mapsto (\lambda, \lambda m),\ (\tau) \in \Sigma_2 \mapsto (0, \tau).$$

Then, according to [Ferrari-Rufino,2007], the Gelfand topology of Σ is induced by the topology of $\mathbb{R}^2$. This means in particular that, for $D \in \mathbb{D}(H)^K$,

$$\lim \widehat{D}(\lambda, m) = \widehat{D}(\tau),$$

as $\lambda \to 0$, $\lambda m \to \tau$.

We will evaluate the eigenvalues $\widehat{\mathcal{L}_m}(\sigma)$ in terms of the Meixner-Pollaczek polynomials. We introduce the one variable polynomials $q_k^{(\nu)}(s)$ as defined by the generating formula:

$$\sum_{k=0}^{\infty} q_k^{(\nu)}(s) w^k = (1-w)^{s-\frac{\nu}{2}}(1+w)^{-s-\frac{\nu}{2}}.$$

The relation to the classical Meixner-Pollaczek polynomials is as folllows

$$q_k^{(\nu)}(i\lambda) = (-i)^k P_k^{\frac{\nu}{2}}\big(\lambda; \frac{\pi}{2}\big).$$

Observe that

$$q_0^{(\nu)}(s) = 1, \quad q_1^{(\nu)}(s) = -2s.$$

These polynomias admit the following hypergeometric representation

$$q_m^{(\nu)}(s) = \frac{(\nu)_m}{m!} {}_2F_1\big(-m, s + \frac{\nu}{2}; \nu; 2\big) = \frac{(\nu)_m}{m!} \sum_{k=0}^{m} \frac{[m]_k [-s - \frac{\nu}{2}]_k}{(\nu)_k} \frac{1}{k!} 2^k.$$

One checks that

$$q_m^{(\nu)}(s) = \frac{1}{m!}(-2)^m s^m + \text{lower order terms.}$$

For $\nu = 1$, the polynomials $q_k^{(1)}(i\lambda)$ are orthogonal with respect to the weight

$$\frac{1}{\cosh \pi\lambda}.$$

More generally, for $\nu > 0$, the polynomials $q_k^{(\nu)}(i\lambda)$ are orthogonal with respect to the weight

$$\big|\Gamma\big(i\lambda + \frac{\nu}{2}\big)\big|^2.$$

Theorem 3.3. *The eigenvalues of the differential operator* $\mathcal{L}_k$ *are given, for* $(\lambda, m) \in \Sigma_1,\ \lambda > 0,$ *by*

$$\widehat{\mathcal{L}_k}(\lambda, m) = \big(\frac{1}{2}|\lambda|\big)^k q_k^{(p)}\big(m + \frac{p}{2}\big),$$

and, for $(\tau) \in \Sigma_1,$ *by*

$$\widehat{\mathcal{L}_k}(\tau) = (-1)^k \frac{\tau^k}{k!}.$$

It follows that $\mathcal{L}_k = Q_k(T, \mathcal{L}_1)$ *with*

$$Q_k(t, s) = \big(\frac{t}{2}\big)^k q_k^{(p)}\big(-\frac{1}{t}s\big).$$

For $p = 1$ this result has been established by Koornwinder [1988]. The proof we give below is different.

Since

$$q_k^{(\nu)}(s) = \frac{1}{k!}(-2)^k s^k + \text{lower order terms},$$

one checks that

$$\lim \widehat{\mathcal{L}_k}(\lambda, m) = \widehat{\mathcal{L}_k}(\tau),$$

as $\lambda \to 0$, $\lambda m \to \tau$.

Proof.

We start from a generating formula for the polynomials $q_k^{(\nu)}$ related to the confluent hypergeometric function

$${}_1F_1(\alpha, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k} \frac{1}{k!} z^k.$$

This generating formula can be written:

$$e^{-u}{}_1F_1\big(s + \frac{\nu}{2}; \nu; 2u\big) = \sum_{k=0}^{\infty} q_k^{(\nu)}(-s) \frac{1}{(\nu)_k} u^k.$$

(see for instance [Andrews-Askey-Roy,1999], p.349). For $\alpha = -m$, the hypergeometric series terminates and reduces to a Laguerre polynomial:

$$L_m^{(\nu-1)}(z) = \frac{(\nu)_m}{m!}{}_1F_1(-m, \nu; z) = \frac{(\nu)_m}{m!} \sum_{k=0}^{m} (-1)^k \frac{[m]_k}{(\nu)_k} \frac{1}{k!} z^k,$$

and, for for $s+\frac{\nu}{2}=-m$ ($m\in\mathbb{N}$), one gets

$$e^{-u}L_m^{(\nu-1)}(2u)=\frac{(\nu)_m}{m!}\sum_{k=0}^{\infty}q_k^{(\nu)}\big(m+\frac{\nu}{2}\big)\frac{1}{(\nu)_k}u^k.$$

Hence the bounded spherical function of the first kind can be written

$$\varphi(\lambda,m;z,t)=e^{i\lambda t}\sum_{k=0}^{\infty}\frac{1}{(p)_k}q_k^{(p)}\big(m+\frac{p}{2}\big)\big(\frac{1}{2}|\lambda|\|z\|^2\big)^k.$$

On the other hand, by Corollary 2.3,

$$\varphi(\lambda,m;z,t)=e^{i\lambda t}\sum_{k=0}^{\infty}\frac{1}{(p)_k}\widehat{\mathcal{L}_k}(\lambda,m)\|z\|^{2k}.$$

Therefore

$$\widehat{\mathcal{L}_k}(\lambda,m)=\big(\frac{1}{2}|\lambda|\big)^k q_k^{(p)}\big(m+\frac{p}{2}\big).$$

From the expansion

$$\varphi(\tau;z,t)=\sum_{k=0}^{\infty}(-1)^k\frac{1}{(p)_k}\frac{1}{k!}\tau^k\|z\|^{2k},$$

it follows that

$$\widehat{\mathcal{L}_k}(\tau)=(-1)^k\frac{\tau^k}{k!}.$$

□

In Section 6 we will consider a multivariate analogue of the case we have seen in this section. For that we will introduce in Sections 4 and 5 certain multivariate functions associated to symmetric cones.

4 Symmetric cones and spherical expansions

We consider an irreducible symmetric cone Ω in a simple Euclidean Jordan algebra V, with rank n, multipllicity d, and dimension

$$N=n+\frac{d}{2}n(n-1).$$

Let L be the identity component in the group $G(\Omega)$ of linear automorphisms of Ω, and $K_0\subset L$ the isotropy subgroup of the unit element $e\in V$. Then

(L, K_0) is a Gelfand pair. The spherical function $\varphi_{\mathbf{s}}$, for $\mathbf{s} \in \mathbb{C}^n$, is defined on Ω by

$$\varphi_{\mathbf{s}}(x) = \int_{K_0} \Delta_{\mathbf{s}+\rho}(k \cdot x)\alpha(dk),$$

where $\Delta_{\mathbf{s}}$ is the power function, $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_j = \frac{d}{4}(2j - n - 1)$. The algebra $\mathbb{D}(\Omega)$ of L-invariant differential operators on Ω is commutative, the spherical function $\varphi_{\mathbf{s}}$ is an eigenfunction of every $D \in \mathbb{D}(\Omega)$:

$$D\varphi_{\mathbf{s}} = \gamma_D(\mathbf{s})\varphi_{\mathbf{s}},$$

and γ_D is a symmetric polynomial function in n variables. (See [Faraut-Korányi,1994].) The Gelfand spectrum Σ can be seen as a closed subset of $\mathbb{C}^n/\mathfrak{S}_n$, and $\hat{D}(\mathbf{s})$ can be identified to $\gamma_D(\mathbf{s})$. The space $\mathcal{P}(V)$ of polynomial functions on V decomposes multiplicity free under L as

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}},$$

where $\mathcal{P}_{\mathbf{m}}$ is a subspace of finite dimension $d_{\mathbf{m}}$, irreducible under L. The parameter $\mathbf{m}$ is a partition: $\mathbf{m} = (m_1, \ldots, m_n)$, $m_j \in \mathbb{N}$, $m_1 \geq \cdots \geq m_n \geq 0$. The subspace $\mathcal{P}_{\mathbf{m}}^{K_0}$ of K_0-invariant polynomial functions is one-dimensional, generated by the spherical polynomial $\Phi_{\mathbf{m}}$, normalized by the condition $\Phi_{\mathbf{m}}(e) = 1$. The polynomials $\Phi_{\mathbf{m}}$ form a basis of the space $\mathcal{P}(V)^{K_0}$ of K_0-invariant polynomials. Let $D^{\mathbf{m}}$ be the invariant differential operator determined by the condition

$$D^{\mathbf{m}} f(e) = \Big(\Phi_{\mathbf{m}}\big(\frac{\partial}{\partial x}\big) f\Big)(e).$$

Then the operators $D^{\mathbf{m}}$ form a linear basis of $\mathbb{D}(\Omega)$. The generalized Pochhammer symbol $(\alpha)_{\mathbf{m}}$ is defined by

$$(\alpha)_{\mathbf{m}} = \prod_{j=1}^{n} \big(\alpha - (j-1)\frac{d}{2}\big)_{m_j}.$$

A K_0-invariant function f, analytic in a neighborhood of 0, admits a spherical Taylor expansion:

$$f(x) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Big(\Phi_{\mathbf{m}}\big(\frac{\partial}{\partial x}\big) f\Big)(0)\Phi_{\mathbf{m}}(x).$$

For $D = D^{\mathbf{m}}$, we will write $\gamma_{D^{\mathbf{m}}}(\mathbf{s}) = \gamma_{\mathbf{m}}(\mathbf{s})$. The function $\gamma_{\mathbf{m}}$ can be seen as a multivariate analogue of the Pochhammer symbol $[s]_m$. In fact, for $n = 1$ ($s \in \mathbb{C},\ m \in \mathbb{N}$),

$$\gamma_m(s) = [s]_m = (-1)^m (s)_m.$$

Observe that

$$\gamma_{\mathbf{m}}(\alpha, \dots, \alpha) = (\alpha - \rho)_{\mathbf{m}}.$$

With this notation we can write a multivariate binomial formula.

Proposition 4.1. (i) *For $z \in \mathcal{D}$, the unit ball in $V_{\mathbb{C}}$ centered at 0, relatively to the spectral norm,*

$$\varphi_{\mathbf{s}}(e+z) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \gamma_{\mathbf{m}}(\mathbf{s}) \Phi_{\mathbf{m}}(z).$$

The convergence is uniform on compact sets in $\mathcal{D}$.

(ii) *For $\mathbf{s} \in \mathbb{C}^n$, and r, $0 < r < 1$, there is a constant $A(\mathbf{s}, r) > 0$ such that, for every $\mathbf{m}$,*

$$|\gamma_{\mathbf{m}}(\mathbf{s})| \leq A(\mathbf{s}, r) \frac{\left(\frac{N}{n}\right)_{\mathbf{m}}}{r^{|\mathbf{m}|}}.$$

Proof.

(i) Observe first that

$$\Phi_{\mathbf{m}}\Big(\frac{\partial}{\partial x}\Big) \varphi_{\mathbf{s}}(e+z)\Big|_{z=0} = D^{\mathbf{m}} \varphi_{\mathbf{s}}(e) = \gamma_{\mathbf{m}}(\mathbf{s}).$$

We will see that the function $\varphi_{\mathbf{s}}$ has a holomorphic continuation to $e + \mathcal{D}$. By Theorem XII.3.1 in [Faraut-Korányi,1994], it will follow that the Taylor expansion of $\varphi_{\mathbf{s}}(e+z)$ converges uniformly on compact sets in $\mathcal{D}$. From the integral representation of the spherical functions $\varphi_{\mathbf{s}}$, it follows that these functions admit a holomorphic continuation to the tube $\Omega + iV$. Let us prove the inclusion $e + \mathcal{D} \subset \Omega + iV$. To prove this it suffices to show that $e + \mathcal{D} \cap V \subset \Omega$. In fact, to see that, consider the conjugation $z \mapsto \bar{z}$ of $V_{\mathbb{C}} = V + iV$ with respect to the Euclidean real form V. For $z \in \mathcal{D}$, we will show that $e + \frac{1}{2}(z + \bar{z}) \in \Omega$. Since $\mathcal{D}$ is invariant under this conjugation and convex, $\frac{1}{2}(z + \bar{z}) \in \mathcal{D} \cap V$. Moreover

$$\mathcal{D} \cap V = (e - \Omega) \cap (-e + \Omega),$$

therefore

$$e + \mathcal{D} \cap V = \Omega \cap (2e - \Omega) \subset \Omega.$$

(ii) Let

$$f(z) = \sum_{\mathbf{m}} d_{\mathbf{m}} a_{\mathbf{m}} \Phi_{\mathbf{m}}(z)$$

be the spherical Taylor expansion of a K_0-invariant analytic function in $\mathcal{D}$. Then the coefficients $a_{\mathbf{m}}$ are given, for $0 < r < 1$, by

$$a_{\mathbf{m}} = \frac{1}{r^{|\mathbf{m}|}} \int_K f(rk \cdot e) \overline{\Phi_{\mathbf{m}}(k \cdot e)} \alpha(dk),$$

where $K = \mathrm{Str}(V_{\mathbb{C}}) \cap U(V_{\mathbb{C}})$, hence satisfy the following Cauchy inequality: for $0 < r < 1$,

$$|a_{\mathbf{m}}| \leq \frac{1}{r^{|\mathbf{m}|}} \sup_{k \in K} |f(rk \cdot e)|,$$

It follows that, for $\mathbf{s} \in \mathbb{C}^n$, and r, $0 < r < 1$, there is a constant $A(\mathbf{s}, r)$ such that

$$|\gamma_{\mathbf{m}}(\mathbf{s})| \leq A(\mathbf{s}, r) \frac{\left(\frac{N}{n}\right)_{\mathbf{m}}}{r^{|\mathbf{m}|}}. \qquad \square$$

For $\mathbf{s} = \mathbf{m} - \rho$, $\varphi_{\mathbf{m}-\rho}(z) = \Phi_{\mathbf{m}}(z)$, and the binomial formula can be written in that case

$$\Phi_{\mathbf{m}}(e + z) = \sum_{\mathbf{k} \subset \mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(z).$$

In fact the generalized binomial coefficient

$$\binom{\mathbf{m}}{\mathbf{k}} = d_{\mathbf{k}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{m} - \rho)$$

vanishes if $\mathbf{k} \not\subset \mathbf{m}$.

In the case of the cone Ω of $n \times n$ Hermitian matrices of positive type, $\Omega \subset V = Herm(n, \mathbb{C})$, i.e. $d = 2$, the spherical polynomials can be expressed in terms of the Schur functions $s_{\mathbf{m}}$:

$$\Phi_{\mathbf{m}}\big(\mathrm{diag}(a_1, \ldots, a_n)\big) = \frac{s_{\mathbf{m}}(a_1, \ldots, a_n)}{s_{\mathbf{m}}(1^n)}.$$

The spherical expansion of the exponential of the trace can be written

$$e^{\operatorname{tr} x} = \sum_{\mathbf{m}} \frac{1}{h(\mathbf{m})} \chi_{\mathbf{m}}(x),$$

where $h(\mathbf{m})$ is the product of the hook-lengths of the partition $\mathbf{m}$, and $\chi_{\mathbf{m}}$ is the character of the representation of $GL(n, \mathbb{C})$ with highest weight $\mathbf{m}$. Equivalently

$$e^{a_1+\cdots+a_n} = \sum_{\mathbf{m}} \frac{1}{h(\mathbf{m})} s_{\mathbf{m}}(a_1, \ldots, a_n)$$

(see [Macdonald, 1995], p.66). Furthermore

$$d_{\mathbf{m}} = \big(s_{\mathbf{m}}(1^n)\big)^2, \text{ therefore } \frac{1}{h(\mathbf{m})} = \frac{s_{\mathbf{m}}(1^n)}{(n)_{\mathbf{m}}}.$$

The binomial formula for the Schur functions is written as

$$\frac{s_{\mathbf{m}}(1+a_1, \ldots, 1+a_n)}{s_{\mathbf{m}}(1^n)} = \sum_{\mathbf{k} \subset \mathbf{m}} \frac{1}{(n)_{\mathbf{k}}} s^*_{\mathbf{k}}(\mathbf{m}) s_{\mathbf{k}}(a_1, \ldots, a_n),$$

where $s^*_{\mathbf{k}}(\mathbf{m})$ is a the shifted Schur function ([Okounkov-Olshanski,1997]. The following relations follow

$$\binom{\mathbf{m}}{\mathbf{k}} = \frac{1}{h(\mathbf{k})} s^*_{\mathbf{k}}(\mathbf{m}), \quad \gamma_{\mathbf{k}}(\mathbf{m} - \rho) = \frac{s^*_{\mathbf{k}}(\mathbf{m})}{s_{\mathbf{k}}(1^n)}.$$

5 A generating formula for multivariate Meixner-Pollaczek polynomials

The multivariate Meixner-Pollaczek polynomials $Q^{(\nu)}_{\mathbf{m}}(\mathbf{s})$ can be defined by the generating formula

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q^{(\nu)}_{\mathbf{m}}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta(e - w^2)^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}\big((e-w)(e+w)^{-1}\big)$$

([Faraut-Wakayama,2012]). The polynomial $Q^{(\nu)}_{\mathbf{m}}(\mathbf{s})$ admits the following "hypergeometric representation"

$$Q^{(\nu)}_{\mathbf{m}}(\mathbf{s}) = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m} - \rho) \gamma_{\mathbf{k}}\left(-\mathbf{s} - \frac{\nu}{2}\right)}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} 2^{|\mathbf{k}|}.$$

The polynomials $Q^{(\nu)}_{\mathbf{m}}(i\lambda)$ are orthogonal with respect to the measure $M_\nu(d\lambda)$ on $\mathbb{R}^n$ given by

$$M_\nu(d\lambda) = \prod_{j=1}^{n} \left|\Gamma(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\right|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda),$$

where m is the Lebesgue measure, and c is the Harish-Chandra c-function of the symmetric cone Ω:

$$c(\mathbf{s}) = c_0 \prod_{j<k} B\big(s_j - s_k, \frac{d}{2}\big)$$

(B is the Euler beta function). One can see that

$$Q^{(\nu)}_{\mathbf{m}}(\mathbf{s}) = \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} (-2)^{|\mathbf{m}|} \Phi_{\mathbf{m}}(\mathbf{s}) + \text{lower order terms}.$$

We consider a multivariate analogue of the confluent hypergeometric functions ${}_1F_1$:

$$F(\mathbf{s}, \nu; x) = \sum_{\mathbf{k}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(-\mathbf{s})}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x),$$

for $\mathbf{s} \in \mathbb{C}^n$, $\nu > \frac{d}{2}(n-1)$, $x \in V_{\mathbb{C}}$.

Proposition 5.1. *The series converges for every $x \in V_{\mathbb{C}}$.*

Proof.
This follows from the Cauchy inequalities: part (ii) in Proposition 4.1, and the fact that, for $\nu > \frac{d}{2}(n-1)$, and every $R > 0$,

$$\sum_{\mathbf{k}} d_{\mathbf{k}} \frac{1}{(\nu)_{\mathbf{k}}} R^{|\mathbf{k}|} < \infty.$$

∎

For $\mathbf{s} = \rho - \mathbf{m}$, $\mathbf{m}$ a partition, the function $F(\rho - \mathbf{m}, \nu; x)$ is essentially a multivariate Laguerre polynomial:

$$\begin{aligned} L^{(\nu-1)}_{\mathbf{m}}(x) &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} F(\rho - \mathbf{m}, \nu; x) \\ &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m} - \rho)}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x). \end{aligned}$$

Theorem 5.2. *The multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{k}}^{(\nu)}$ admit the following generating formula*

$$e^{-\operatorname{tr} u} F\big(\mathbf{s} + \frac{\nu}{2}; \nu; 2u\big) = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} d_{\mathbf{k}} \frac{1}{(\nu)_{\mathbf{k}}} Q_{\mathbf{k}}^{(\nu)}(\mathbf{s}) \Phi_{\mathbf{k}}(u).$$

For $\mathbf{s} = \rho - \mathbf{m} - \frac{\nu}{2}$, one obtains

$$e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\nu-1)}(2u) = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k}} d_{\mathbf{k}} \frac{1}{(\nu)_{\mathbf{k}}} Q_{\mathbf{k}}^{(\nu)}\big(\mathbf{m} + \frac{\nu}{2} - \rho\big) \Phi_{\mathbf{k}}(u).$$

Lemma 5.3. *(Bingham identity)*

$$e^{\operatorname{tr} x} \Phi_{\mathbf{m}}(x) = \sum_{\mathbf{k} \supset \mathbf{m}} d_{\mathbf{k}} \gamma_{\mathbf{m}}(\mathbf{k} - \rho) \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}(x).$$

This formula, which has been established by Bingham [1974] in case of $V = Sym(n, \mathbb{R})$, generalizes the formula

$$e^x x^m = \sum_{k=m}^{\infty} [k]_m \frac{1}{k!} x^k.$$

We will give a different proof.

Proof.

The symbol $\sigma_D(x, \xi)$ of a differential operator D is defined by the relation

$$D e^{(x|\xi)} = \sigma_D(x, \xi) e^{(x|\xi)}.$$

If D is invariant, $D \in \mathbb{D}(\Omega)$, then its symbol is invariant in the following sense: for $g \in G$,

$$\sigma_D(gx, \xi) = \sigma_D(x, g^* \xi).$$

For $x = \xi = e$, one gets

$$\sigma_D(ge, e) = \sigma_D(e, g^* e),$$

and taking g selfadjoint, it follows that, for $x \in \Omega$, $\sigma_D(x, e) = \sigma_D(e, x)$, and

$$D e^{\operatorname{tr} x} = \sigma_D(x, e) e^{\operatorname{tr} x}.$$

For $D = D^{\mathbf{m}}$,

$$\sigma_D(x,e) = \sigma_D(e,x) = \Phi_{\mathbf{m}}(x),$$

and

$$D^{\mathbf{m}} e^{\operatorname{tr} x} = \Phi_{\mathbf{m}}(x) e^{\operatorname{tr} x}.$$

On the other hand

$$\begin{aligned} D^{\mathbf{m}} e^{\operatorname{tr} x} &= D^{\mathbf{m}} \Big(\sum_{\mathbf{k}} d_{\mathbf{k}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{k}}(x) \Big) \\ &= \sum_{\mathbf{k}} d_{\mathbf{k}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} D^{\mathbf{m}} \Phi_{\mathbf{k}}(x) \\ &= \sum_{\mathbf{k}} d_{\mathbf{k}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \gamma_{\mathbf{m}}(\mathbf{k} - \rho) \Phi_{\mathbf{k}}(x). \end{aligned}$$

Furthermore we know that $\gamma_{\mathbf{m}}(\mathbf{k} - \rho) = 0$ if $\mathbf{m} \not\subset \mathbf{k}$. We obtain finally

$$e^{\operatorname{tr} x} \Phi_{\mathbf{m}}(x) = \sum_{\mathbf{k} \supset \mathbf{m}} d_{\mathbf{k}} \gamma_{\mathbf{m}}(\mathbf{k} - \rho) \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}(x). \qquad \square$$

In case of the cone Ω of $n \times n$ Hermitian matrices of positive type, $\Omega \subset V = Herm(n, \mathbb{C})$, i.e. $d = 2$, we get the following Schur expansion

$$e^{a_1 + \cdots + a_n} s_{\mathbf{m}}(a) = \sum_{\mathbf{k} \supset \mathbf{m}} \frac{1}{h(\mathbf{k})} s^*_{\mathbf{m}}(\mathbf{k}) s_{\mathbf{k}}(a).$$

Proof of Theorem 5.2

By using the Bingham identity (Lemma 5.3) we get

$$\begin{aligned} e^{-\operatorname{tr} u} F\big(\mathbf{s} + \frac{\nu}{2}; \nu; 2u\big) &= \sum_{\mathbf{k}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}\big(-\mathbf{s} - \frac{\nu}{2}\big)}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} 2^{|\mathbf{k}|} e^{-\operatorname{tr} u} \Phi_{\mathbf{k}}(-u) \\ = \sum_{\mathbf{k}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}\big(-\mathbf{s} - \frac{\nu}{2}\big)}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} 2^{|\mathbf{k}|} & \Big(\sum_{\mathbf{j} \supset \mathbf{k}} d_{\mathbf{j}} \gamma_{\mathbf{k}}(\mathbf{j} - \rho) \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{j}}} \Phi_{\mathbf{j}}(-u) \Big) \\ = \sum_{\mathbf{j}} d_{\mathbf{j}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{j}}} \Big(\sum_{\mathbf{k} \subset \mathbf{j}} d_{\mathbf{k}} & \frac{\gamma_{\mathbf{k}}\big(-\mathbf{s} - \frac{\nu}{2}\big) \gamma_{\mathbf{k}}(\mathbf{j} - \rho)}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} 2^{|\mathbf{k}|} \Big) \Phi_{\mathbf{j}}(-u) \\ = \sum_{\mathbf{j}} (-1)^{|\mathbf{j}|} d_{\mathbf{j}} \frac{1}{(\nu)_{\mathbf{j}}} Q^{(\nu)}_{\mathbf{j}}(\mathbf{s}) \Phi_{\mathbf{j}}(u). & \end{aligned}$$

$\square$

6 The case $W = M(n,p;\mathbb{C})$, $K = U(n) \times U(p)$

The group $K = U(n) \times U(p)$ acts on the space $W = M(n,p;\mathbb{C})$ $(n \leq p)$ of $n \times p$ matrices by the transformations

$$z \mapsto uzv^* \quad \big(u \in U(n),\ v \in U(p)\big).$$

Its action on the space $\mathcal{P}(W)$ of holomorphic polynomials on W is multiplicity free and the parameter set $\mathcal{M}$ is the set of partitions $\mathbf{m}$ of lenghts $\ell(\mathbf{m}) \leq n$: $\mathbf{m} = (m_1, \ldots, m_n)$ with $m_i \in \mathbb{N}$, $m_1 \geq \cdots m_n \geq 0$. The subspace $\mathcal{H}_{\mathbf{m}} \subset \mathcal{P}(W)$ corresponding to the partition $\mathbf{m}$ is generated by the polynomials

$$\Delta_{\mathbf{m}}(uzv) \quad (u \in U(n), v \in U(p)),$$

where

$$\Delta_{\mathbf{m}}(z) = \Delta_1(z)^{m_1 - m_2} \ldots \Delta_n(z)^{m_n},$$

with

$$\Delta_k(z) = \det\big((z_{ij})_{1 \leq i \leq j \leq k}\big),$$

the principal minor of order k $(k \leq n)$. The character $\chi_{\mathbf{m}}$ of the representation of $U(n)$ with highest weight $\mathbf{m}$ can be expressed in terms of the Schur functions $s_{\mathbf{m}}$:

$$\chi_{\mathbf{m}}\big(\mathrm{diag}(t_1, \ldots, t_n)\big) = s_{\mathbf{m}}(t_1, \ldots, t_n),$$

and $\chi_{\mathbf{m}}$ extends as a polynomial on $M(n,\mathbb{C})$ of degree $|\mathbf{m}|$. The reproducing kernel $\mathcal{K}_{\mathbf{m}}$ of the subspace $\mathcal{H}_{\mathbf{m}}$ is given by

$$\mathcal{K}_{\mathbf{m}}(z,w) = \frac{1}{h(\mathbf{m})} \chi_{\mathbf{m}}(zw^*).$$

The Heisenberg group H of dimension $2np+1$ is seen as $H = W \times \mathbb{R}$, and the group $K = U(n) \times U(p)$ acts on H. With $G = K \ltimes W$, (G,K) is a Gelfand pair, and its Gelfand spectrum can be described as the union $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is the set of pairs $(\lambda, \mathbf{m})$ with $\lambda \in \mathbb{R}^*$, $\mathbf{m}$ is a partition with $\ell(\mathbf{m}) \leq n$, and

$$\Sigma_2 = \{\tau \in \mathbb{R}^n \mid \tau_1 \geq \cdots \geq \tau_n \geq 0\}.$$

The bounded spherical functions of the first kind are expressed in terms of multivariate Laguerre polynomials associated to the Jordan algebra $Herm(n,\mathbb{C})$:

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2} \frac{L_{\mathbf{m}}^{(p-1)}(|\lambda| zz^*)}{L_{\mathbf{m}}^{(p-1)}(0)}.$$

This function admits the following expansion

$$\varphi(\lambda, \mathbf{m}; z, t)$$
$$= e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2} \sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{|\mathbf{k}|} |\lambda|^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} \frac{1}{(p)_{\mathbf{k}}} s^*_{\mathbf{k}}(\mathbf{m}) \chi_{\mathbf{k}}(zz^*).$$

The bounded spherical functions of the second kind are given by

$$\varphi(\tau; z) = \int_{U(n)\times U(p)} e^{2i\mathrm{Re}\,\mathrm{tr}\,(uzv^*w^*)} \beta_n(du)\beta_p(dv),$$

where $\tau = (\tau_1, \ldots, \tau_n)$, and $\tau_1 \geq \cdots \geq \tau_n \geq 0$ are the eigenvalues of ww^*. This function admits the following expansion

$$\varphi(\tau; z, t) = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} \frac{1}{(p)_{\mathbf{k}}} s_{\mathbf{k}}(\tau) \chi_{\mathbf{k}}(zz^*).$$

(See [Faraut,2010a])

We will give formulas for the eigenvalues $\widehat{\mathcal{D}_{\mathbf{k}}}(\sigma)$ and $\widehat{\mathcal{L}_{\mathbf{k}}}(\sigma)$ of the operators $\mathcal{D}_{\mathbf{k}}$ and $\mathcal{L}_{\mathbf{k}}$ we have introduced in Section 3 associated to a partition $\mathbf{k}$.

Theorem 6.1.

$$\widehat{\mathcal{D}_{\mathbf{k}}}(\lambda, \mathbf{m}) = \frac{(-1)^{|\mathbf{k}|}}{h(\mathbf{k})} \lambda^{|\mathbf{k}|} s^*_{\mathbf{k}}(\mathbf{m}), \quad \widehat{\mathcal{D}_{\mathbf{k}}}(\tau) = \frac{(-1)^{|\mathbf{k}|}}{h(\mathbf{k})} s_{\mathbf{k}}(\tau).$$

Proof.

From the definition of the operator $\mathcal{D}_{\mathbf{k}}$, one obtains

$$d\pi_\lambda(\mathcal{D}_{\mathbf{k}}) = \frac{(-1)^{|\mathbf{k}|}}{h(\mathbf{k})} \lambda^{|\mathbf{k}|} s_{\mathbf{k}}(1^n) \tilde{D}^{\mathbf{k}},$$

where $\tilde{D}^{\mathbf{k}}$ is a differential operator whose restriction to the subspace $W_0 = M(n;\mathbb{C}) \subset W = M(n,p;\mathbb{C})$ is equal to the operator $D^{\mathbf{k}}$ introduced in Section 4. For $\psi \in \mathcal{H}_{\mathbf{m}}$,

$$d\pi_\lambda(\mathcal{D}_{\mathbf{k}})\psi = \widehat{\mathcal{D}_{\mathbf{k}}}(\lambda, \mathbf{m})\psi.$$

Choosing $\psi(\zeta) = \Phi_{\mathbf{m}}(\zeta_0)$, where ζ_0 is the projection of ζ on W_0, we get

$$\tilde{D}^{\mathbf{k}}\psi = \gamma_{\mathbf{k}}(\mathbf{m} - \rho)\psi.$$

Since $s_{\mathbf{k}}(1^n)\,\gamma_{\mathbf{k}}(\mathbf{m}-\rho)=s^*_{\mathbf{k}}(\mathbf{m})$, we obtain

$$\widehat{\mathcal{D}_{\mathbf{k}}}(\lambda,\mathbf{m})=\frac{(-1)^{|\mathbf{k}|}}{h(\mathbf{k})}\lambda^{|\mathbf{k}|}s^*_{\mathbf{k}}(\mathbf{m}).$$

Furthermore

$$d\eta_w(\mathcal{D}_{\mathbf{k}})=\mathcal{K}_{\mathbf{k}}(-w,w)=\frac{(-1)^{|\mathbf{k}|}}{h(\mathbf{k})}\chi_{\mathbf{k}}(ww^*)=\frac{(-1)^{|\mathbf{k}|}}{h(\mathbf{k})}s_{\mathbf{k}}(\tau). \qquad \square$$

Corollary 6.2. *For every $D\in\mathbb{D}(H)^K$ there is a polynomial F_D in $n+1$ variables $u,v_1,\ldots,v_n$, symmetric in the variables $v_1,\ldots,v_n$, such that*

$$\widehat{\mathcal{D}}(\lambda,\mathbf{m})=F_D\big(\lambda,\lambda(m_1-\rho_1),\ldots,\lambda(m_n-\rho_n)\big).$$

The map $D\mapsto F_D$, $\mathbb{D}(H)^K\to\mathcal{P}(\mathbb{C})\otimes\mathcal{P}(\mathbb{C}^n)^{\mathfrak{S}_n}$ is an algebra isomorphism.

Let us embed the Gelfand spectrum Σ into $\mathbb{R}^{n+1}$ by the map

$$(\lambda,\mathbf{m})\in\Sigma_1\mapsto(\lambda,\lambda m_1,\ldots,\lambda m_n),\quad (\tau)\in\Sigma_1\mapsto(0,\tau_1,\ldots,\tau_n).$$

As in Section 3, according to [Ferrari-Rufino,2007], the Gelfand topology of Σ is induced by the topology of $\mathbb{R}^{n+1}$. This implies in particular that

$$\lim\widehat{\mathcal{D}_{\mathbf{k}}}(\lambda,\mathbf{m})=\widehat{\mathcal{D}_{\mathbf{k}}}(\tau),$$

as $\lambda\to 0$, $\lambda m_j\to\tau_j$. In fact

$$s^*_{\mathbf{k}}(\mathbf{m})=s_{\mathbf{k}}(\mathbf{m})+\text{ lower order terms.}$$

Recall that the differential operator $\mathcal{L}_{\mathbf{m}}\in\mathbb{D}(H)^K$ has been defined by

$$\mathcal{L}_{\mathbf{m}}=\mathcal{K}_{\mathbf{m}}\Big(\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\zeta}\Big)f(z+\zeta,t+\operatorname{Im}(\zeta|z))\big|_{\zeta=0}.$$

Theorem 6.3.

$$\begin{aligned}\widehat{\mathcal{L}_{\mathbf{k}}}(\lambda,\mathbf{m}) &= d_{\mathbf{k}}\Big(\frac{1}{2}|\lambda|\Big)^{|\mathbf{k}|}Q^{(p)}_{\mathbf{k}}\big(\mathbf{m}+\frac{p}{2}-\rho\big).\\ \widehat{\mathcal{L}_{\mathbf{k}}}(\tau) &= (-1)^{|\mathbf{k}|}\frac{1}{h(\mathbf{k})}s_{\mathbf{k}}(\tau).\end{aligned}$$

Proof.

By Corollary 2.3, the spherical functions admit the following expansion:

$$\varphi(\sigma; z, t) = e^{i\lambda t} \sum_{\mathbf{k}} \frac{1}{\dim \mathcal{H}_{\mathbf{k}}} \widehat{L_{\mathbf{k}}}(\sigma) \mathcal{K}_{\mathbf{k}}(z, z),$$

where the summation is over all partitions $\mathbf{k}$ with $\ell(\mathbf{k}) \leq n$. By using the formulas

$$\begin{aligned} \dim \mathcal{H}_{\mathbf{k}} &= s_{\mathbf{k}}(1^n) s_{\mathbf{k}}(1^p) = \frac{(n)_{\mathbf{k}}}{h(\mathbf{k})} \frac{(p)_{\mathbf{k}}}{h(\mathbf{k})}, \\ \mathcal{K}_{\mathbf{k}}(z, w) &= \frac{1}{h(\mathbf{k})} \chi_{\mathbf{k}}(zw^*) = \frac{s_{\mathbf{k}}(1^n)}{h(\mathbf{k})} \Phi_{\mathbf{k}}(zw^*), \end{aligned}$$

we get

$$\varphi(\sigma; z, t) = e^{i\lambda t} \sum_{\mathbf{k}} \frac{1}{(p)_{\mathbf{k}}} \widehat{\mathcal{L}_{\mathbf{k}}}(\sigma) \Phi_{\mathbf{k}}(zz^*).$$

On the other hand, by Theorem 5.2, with $\mathbf{s} = \rho - \mathbf{m} - \frac{p}{2}$, $\nu = p$, we obtain for $\sigma = (\lambda, \mathbf{m}) \in \Sigma_1$,

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} \sum_{\mathbf{k}} d_{\mathbf{k}} \frac{1}{(p)_{\mathbf{k}}} Q_{\mathbf{k}}^{(p)} \big(\mathbf{m} + \frac{p}{2} - \rho\big) \big(\frac{1}{2}|\lambda|\big)^{|\mathbf{k}|} \Phi_{\mathbf{k}}(zz^*).$$

Therefore,

$$\widehat{\mathcal{L}_{\mathbf{k}}}(\lambda, \mathbf{m}) = d_{\mathbf{k}} \big(\frac{1}{2}|\lambda|\big)^{|\mathbf{k}|} Q_{\mathbf{k}}^{(p)} \big(\mathbf{m} + \frac{p}{2} - \rho\big).$$

For $\sigma = (\tau) \in \Sigma_2$,

$$\begin{aligned} \varphi(\tau; z, t) &= \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} \frac{1}{(p)_{\mathbf{k}}} s_{\mathbf{k}}(r) \chi_{\mathbf{k}}(zz^*) \\ &= \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(p)_{\mathbf{k}}} \frac{1}{h(\mathbf{k})} s_{\mathbf{k}}(r) \Phi_{\mathbf{k}}(zz^*). \end{aligned}$$

Therefore

$$\widehat{\mathcal{L}_{\mathbf{k}}}(\tau) = (-1)^{|\mu|} \frac{1}{h(\mathbf{k})} s_{\mathbf{k}}(\tau).$$

□

7 W is a simple complex Jordan algebra

For a simple complex Jordan algebra W we consider the Heisenberg group $H = W \times \mathbb{R}$. Let $\mathcal{D}$ be the bounded symmetric domain in W, which is the unit ball with respect to the spectral norm, and $K = \mathrm{Str}(W) \cap U(W)$. The group K acts multiplicity free on the space $\mathcal{P}(W)$ of holomorphic polynomials on W. Let n be the rank and d the multiplicity.

W	K	d	rank
$Sym(n, \mathbb{C})$	$U(n)$	1	n
$M(n, \mathbb{C})$	$U(n) \times U(n)$	2	n
$Skew(2n, \mathbb{C})$	$U(2n)$	4	n
$Herm(3, \mathbb{O})_{\mathbb{C}}$	$E_6 \times \mathbb{T}$	8	3
$\mathbb{C}^\ell$	$SO(\ell) \times \mathbb{T}$	$\ell - 2$	2

Let V be a Euclidean real form of W, and $c_1, \ldots, c_n$ a Jordan frame in V. An element $z \in W$ can be written

$$z = k \sum_{j=1}^{n} a_j c_j \quad (a_j \in \mathbb{R},\ k \in K).$$

We will denote by $r_j = r_j(z)$ the numbers a_j^2 assume to satisfy $r_1 \geq \cdots \geq r_n \geq 0$, and put $\mathbf{r} = \mathbf{r}(z) = r_1 c_1 + \cdots + r_n c_n$.

The Fock space decomposes multiplicity free into the subspaces $\mathcal{P}_{\mathbf{m}}$ ($\mathbf{m}$ is a partition). The dimension of $\mathcal{P}_{\mathbf{m}}$ is denoted by $d_{\mathbf{m}}$. The reproducing kernel $K^{\mathbf{m}}$ of $\mathcal{P}_{\mathbf{m}}$ is determined by the conditions

$$\begin{aligned} K^{\mathbf{m}}(gz, w) &= K^{\mathbf{m}}(z, g^* w) \quad (g \in L), \\ K^{\mathbf{m}}(z, e) &= d_{\mathbf{m}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z). \end{aligned}$$

(See [Faraut-Korányi,1994], Section XI.3.)

We consider in this section the Gelfand pair (G, K), where $G = K \ltimes H$. The bounded spherical functions of the first kind are given by, for $\lambda > 0$, and $\mathbf{m}$ is a partition

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda \|z\|^2} \frac{L_{\mathbf{m}}^{(\nu-1)}\big(-\lambda \mathbf{r}(z)\big)}{L_{\mathbf{m}}^{(\nu-1)}(0)},$$

with $\nu = \frac{N}{n}$. This spherical function admits the following expansion

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} e^{-\frac{1}{2}\|z\|^2} \sum_{\mathbf{k}} \frac{d_{\mathbf{k}}}{\Big(\big(\frac{N}{n}\big)_{\mathbf{k}}\Big)^2} (-1)^{|\mathbf{k}|} \lambda^{|\mathbf{k}|} \gamma_{\mathbf{k}}(\mathbf{m} - \rho) \Phi_{\mathbf{k}}\big(\mathbf{r}(z)\big).$$

The bounded spherical functions of the second kind are given by the expansion

$$\varphi(\tau; z, t) = \sum_{\mathbf{k}} \frac{d_{\mathbf{k}}}{\Big(\big(\frac{N}{n}\big)_{\mathbf{k}}\Big)^2} (-1)^{|\mathbf{k}|} \Phi_{\mathbf{k}}(\tau) \Phi\big(\mathbf{r}(z)\big),$$

where $\tau = \tau_1 c_1 + \cdots + \tau_n c_n$, $\tau_1 \geq \cdots \geq \tau_n \geq 0$. As in the case considered in Section 6, the Gelfand spectrum is a union $\Sigma = \Sigma_1 \cup \Sigma_2$. The part Σ_1 is parametrized by pairs $(\lambda, \mathbf{m})$, with $\lambda \in \mathbb{R}^*$, and $\mathbf{m}$ is a partition with $\ell(\mathbf{m}) \leq n$, and Σ_2 by points $\tau \in \mathbb{R}^n$, $\tau_1 \geq \cdots \geq \tau_n \geq 0$. (See [Dib,1990], [Faraut,2010b]).

Theorem 7.1. (i) *The eigenvalues of the differential operator $\mathcal{D}_{\mathbf{k}}$ associated to the partition $\mathbf{k}$ are given, for $(\lambda, \mathbf{m}) \in \Sigma_1$, $\lambda > 0$, by*

$$\widehat{\mathcal{D}_{\mathbf{k}}}(\lambda, \mathbf{m}) = \frac{d_{\mathbf{k}}}{\big(\frac{N}{n}\big)_{\mathbf{k}}} (-1)^{|\mathbf{k}|} \lambda^{|\mathbf{k}|} \gamma_{\mathbf{k}}(\mathbf{m} - \rho),$$

and, for $\tau \in \Sigma_2$, by

$$\widehat{\mathcal{D}_{\mathbf{k}}}(\tau) = \frac{d_{\mathbf{k}}}{\big(\frac{N}{n}\big)_{\mathbf{k}}} (-1)^{|\mathbf{k}|} \Phi_{\mathbf{k}}(\tau).$$

(ii) *The eigenvalues of the operator $\mathcal{L}_{\mathbf{k}}$ are given, for $(\lambda, \mathbf{m}) \in \Sigma_1$, $\lambda > 0$, by*

$$\widehat{\mathcal{L}_{\mathbf{k}}}(\lambda, \mathbf{m}) = d_{\mathbf{k}} \big(\frac{1}{2}\lambda\big)^{|\mathbf{k}|} Q_{\mathbf{k}}^{\nu}\big(\mathbf{m} + \frac{N}{2n} - \rho\big),$$

with $\nu = \frac{N}{n}$, and, for $\tau \in \Sigma_2$, by

$$\widehat{\mathcal{L}_{\mathbf{k}}}(\tau) = \frac{d_{\mathbf{k}}}{\big(\frac{N}{n}\big)_{\mathbf{k}}} (-1)^{|\mathbf{k}|} \Phi_{\mathbf{k}}(\tau).$$

The proofs are similar to the ones which are given in Section 6.

ANDREWS, G. E., R. ASKEY, & R. ROY (1999). Special functions. *Cambridge.*

BENDER C. M., L. R. LEAD, & S. S. PINSKY (1986). Resolution of the operator-ordering problem by the method of finite elements, *Phys. Rev. Lett.*, **56**, 2445–2448.

BENDER, C. M., L. R. MEAD, & S. S. PINSKY (1987). Continuous Hahn polynomials and the Heisenberg algebra, *J. Math. Phys.*, **28**, 509-613.

BENSON C., J. JENKINS & G. RATCLIFF (1992). Bounded K-spherical functions on Heisenberg groups, *J. Funct. Anal.*, **105**, 409–443.

BENSON C., J. JENKINS, G. RATCLIFF & T. WORKU (1996). Spectra for Gelfand pairs associated with the Heisenberg group, *Colloquium Math.*, **71**, 305–328.

BENSON, C. & G. RATCLIFF (1998). Combinatorics and spherical functions on the Heisenberg group, *Representation theory*, **2**, 79–105.

BINGHAM, C. (1974). An identity involving partitional generalized binomial coefficients, *J. Multivariate Anal.*, **4**, 210–233.

CARCANO, G. (1987). A commutativity condition for algebras of invariant functions, *Boll. Un. Mat. Ital.*, **7**, 1091–1105.

COURANT, R. & D. HILBERT (1937). Math ???. *Springer.*

DIB, H. (1990). Fonctions de Bessel sur une algèbre de Jordan, *J. Math. Pures Appl.*, **69**, 403–448.

FARAUT, J. (1987). Analyse harmonique et fonctions spéciales, in *Deux cours d'analyse harmonique. Birkhäuser.*

FARAUT, J. (2010a). Asymptotic spherical analysis on the Heisenberg, *Colloquium Math.*, **118**, 233–258.

FARAUT, J. (2010b). Olshanski spherical pairs related to the Heisenberg group. *Preprint.*

FARAUT, J. & A. KORÁNYI (1994). Analysis on symmetric cones. *Oxford University Press.*

FARAUT, J.& M. WAKAYAMA (2012). Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials. *Preprint.*

FERRARI-RUFINO, F. (2007). The topology of the spectrum for Gelfand pairs on Lie groups, *Boll. Un. Mat. Ital.*, **10**, 569–579.

KOORNWINDER, T. H. (1988). Meixner-Pollaczek polynomials and the Heisenberg algebra, *J. Math. Phys.*, **30**, 767–769.

KORÁNYI, A. (1980). Some applications of Gelfand pairs in classical analysis, in *Harmonic Analysis and Group Representations. Liguori, Napoli, 333-348.*

MACDONALD, I. G. (1995). Symmetric functions and Hall polynomials. *Oxford University Press.*

OKOUNKOV, A. & G. OLSHANSKI (1997). Shifted Jack polynomials, binomial formula, and applications, *Math. Res. Letters*, **4**, 69–78.

WOLF, J. A. (2007). Harmonic Analysis on Commutative Spaces. *Amer. Math. Soc.*.

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