INVARIANT DIFFERENTIAL OPERATORS
ON THE HEISENBERG GROUP
AND MEIXNER-POLLACZEK POLYNOMIALS

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Abstract Consider the Heisenberg Lie algebra with basis $X, Y, Z$, such that $[X, Y] = Z$. Then the symmetrization $\sigma(X^k Y^k)$ can be written as a polynomial in $\sigma(XY)$ and $Z$, and this polynomial is identified as a Meixner-Pollaczek polynomial. This is an observation by Bender, Mead and Pinsky, a proof of which has been given by Koornwinder. We extend this result in the framework of Gelfand pairs associated with the Heisenberg group. This extension involves multivariable Meixner-Pollaczek polynomials.

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7. $W$ is a simple complex Jordan algebra
The starting point of this paper is an identity in the Heisenberg algebra which has been observed by Bender, Mead and Pinsky ([1986], [1987]), and revisited by Koornwinder who gave an alternative proof. Let $X, Y, Z$ generate the three dimensional Heisenberg Lie algebra, with $[X, Y] = Z$. Then the symmetrization of $X^k Y^k$ can be written as a polynomial in the symmetrization of $XY$, and this polynomial is a Meixner-Pollaczek polynomial. We rephrase this question in the framework of the spherical analysis for a Gelfand pair. If $(G, K)$ is a Gelfand pair with a Lie group $G$, the algebra $\mathbb{D}(G/K)$ of $G$-invariant differential operators on the quotient space $G/K$ is commutative. The spherical Fourier transform maps this algebra onto an algebra of continuous functions on the Gelfand spectrum $\Sigma$ of the commutative Banach algebra $L^1(K\backslash G/K)$ of $K$-biinvariant integrable functions on $G$. For $D \in \mathbb{D}(G/K)$, the corresponding function is denoted by $\hat{D}$. Hence an identity in the algebra $\mathbb{D}(G/K)$ is equivalent to an identity for the functions $\hat{D}$. We consider Gelfand pairs associated to the Heisenberg groups. The unitary group $K = U(p)$ acts on the Heisenberg group $H = \mathbb{C}^p \times \mathbb{R}$. Let $G = K \ltimes H$ be the semi-direct product. Then $(G, K)$ is a Gelfand pair. The functions $\hat{L}_k$, corresponding to a family $L_k$ of invariant differential operators on the Heisenberg group, involve Meixner-Pollaczek polynomials, and give rise to identities in the algebra $\mathbb{D}(H^K)$ of differential operators on $H$ which are left invariant by $H$ and by the action of $K$. We extend this analysis to some Gelfand pairs associated to the Heisenberg group which have been considered by Benson, Jenkins, and Ratcliff [1992]. The Heisenberg group $H$ is taken as $H = W \times \mathbb{R}$, with $W = M(n, p, \mathbb{C})$. The group $K = U(n) \times U(p)$ acts on $W$ and $(G, K)$ is a Gelfand pair, with $G = K \ltimes H$. We determine the functions $\hat{D}$ for families of differential operators on $D(H^K)$. These functions $\hat{D}$ involve multivariate Meixner-Pollaczek polynomials which have been introduced in [Faraut-Wakayama, 2012]. The proofs use spherical Taylor expansions, and the connection between multivariate Laguerre polynomials and multivariate Meixner-Pollaczek polynomials. In the last section, the Heisenberg group is taken as $W \times \mathbb{R}$, where $W$ is a simple complex Jordan algebra, and $K = \text{Str}(W) \cap U(W)$, where $\text{Str}(W)$ is the structure group of $W$, and $U(W)$ the unitary group.
1 Gelfand pairs

Let $G$ be a locally compact group, and $K$ a compact subgroup, and let $L^1(K \setminus G/K)$ denote the convolution algebra of $K$-invariant integrable functions on $G$. One says that $(G, K)$ is a Gelfand pair if the algebra $L^1(K \setminus G/K)$ is commutative. From now on we assume that it is the case. A spherical function is a continuous function $\varphi$ on $G$, $K$-biinvariant, with $\varphi(e) = 1$, and

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y),$$

where $\alpha$ is the normalized Haar measure on $K$. The characters $\chi$ of the commutative Banach algebra $L^1(K \setminus G/K)$ are of the form

$$\chi(f) = \int_G f(x)\varphi(x)m(dx),$$

where $\varphi$ is a bounded spherical function ($m$ is a Haar measure on the unimodular group $G$). Hence the Gelfand spectrum $\Sigma$ of the commutative Banach algebra $L^1(K \setminus G/K)$ can be identified with the set of bounded spherical functions. We denote by $\varphi(\sigma; x)$ the spherical function associated to $\sigma \in \Sigma$. The spherical Fourier transform of $f \in L^1(K \setminus G/K)$ is the function $\hat{f}$ defined on $\Sigma$ by

$$\hat{f}(\sigma) = \int_G \varphi(\sigma; x)f(x)m(dx).$$

Assume now that $G$ is a Lie group, and denote by $\mathbb{D}(G/K)$ the algebra of $G$-invariant differential operators on $G/K$. This algebra is commutative. A spherical function is $C^\infty$ and eigenfunction of every $D \in \mathbb{D}(G/K)$:

$$D\varphi(\sigma; x) = \hat{D}(\sigma)\varphi(\sigma; x),$$

where $\hat{D}(\sigma)$ is a continuous function on $\Sigma$. The map

$$D \mapsto \hat{D}, \quad \mathbb{D}(G/K) \rightarrow \mathcal{C}(\Sigma),$$

is an algebra morphism. Moreover the Gelfand topology of $\Sigma$ is the initial topology associated to the functions $\sigma \mapsto \hat{D}(\sigma)$ ($D \in \mathbb{D}(G/K)$) ([Ferrari-Rufino, 2007]).

We address the following questions:
- Given a differential operator \( D \in \mathcal{D}(G/K) \), determine the function \( \hat{D} \).

- Construct a linear basis \( (D_\mu)_{\mu \in \mathfrak{M}} \) of \( \mathcal{D}(G/K) \), and, for each \( \mu \), a \( K \)-invariant analytic function \( b_\mu \) in a neighborhood of \( o = eK \in G/K \) such that
  \[
  D_\mu b_\nu (o) = \delta_{\mu\nu}.
  \]

- Establish a mean value formula: for an analytic function \( f \) on \( G/K \), defined in a neighborhood of \( o \),
  \[
  \int_K f(xky)\alpha(dk) = \sum_{\mu \in \mathfrak{M}} (D_\mu f)(x)b_\mu(y).
  \]

Observe that it is enough to prove, for a \( K \)-invariant analytic function \( f \), that
  \[
  f(y) = \sum_{\mu \in \mathfrak{M}} (D_\mu f)(o)b_\mu(y).
  \]

In particular, for \( f(x) = \varphi(\sigma; x) \), one gets a generalized Taylor expansion for the spherical functions
  \[
  \varphi(\sigma; x) = \sum_{\mu \in \mathfrak{M}} \hat{D}_\mu(\sigma)b_\mu(x).
  \]

**Basic example**

Take \( G = \mathbb{R}, K = \{0\} \). Then \( \Sigma = \mathbb{R}, \) and
  \[
  \varphi(\sigma; x) = e^{i\sigma x}.
  \]

We can take, with \( \mathfrak{M} = \mathbb{N} \),
  \[
  D_\mu = \left( \frac{d}{dx} \right)^\mu, \quad b_\mu(x) = \frac{x^\mu}{\mu!}.
  \]

Then the mean value formula is nothing but the Taylor formula
  \[
  f(x + y) = \sum_{\mu=0}^\infty \left( \left( \frac{d}{dx} \right)^\mu f(x) \right) \frac{y^\mu}{\mu!},
  \]

and the Taylor formula for the spherical functions is the power expansion of the exponential:
  \[
  e^{i\sigma x} = \sum_{\mu=0}^\infty (i\sigma)^\mu \frac{x^\mu}{\mu!}.
  \]
Historical example

Here $G = SO(n) \times \mathbb{R}^n$, the motion group, $K = SO(n)$; then $G/K \simeq \mathbb{R}^n$. The spectrum $\Sigma$ can be identified to the half-line, $\Sigma = [0, \infty[$. The spherical functions are given by

$$\varphi(\sigma; x) = \int_{S(\mathbb{R}^n)} e^{\sigma(u|x)} \beta(du) \quad (\sigma \geq 0, x \in \mathbb{R}^n),$$

where $\beta$ is the normalized uniform measure on the unit sphere $S(\mathbb{R}^n)$. (The function $\varphi(\sigma; x)$ can be written in terms of Bessel functions.) The algebra $\mathcal{D}(G/K)$ is generated by the Laplace operator $\Delta$, and $\Delta(\sigma) = -\sigma^2$. We can take, with $\mathfrak{M} = \mathbb{N}$,

$$D_\mu = \Delta^\mu, \quad b_\mu = c_\mu \|x\|^{2\mu},$$

with

$$c_\mu = 2^{-2\mu} \frac{1}{\left(\frac{n}{2}\right)_\mu} \frac{1}{\mu!}.$$ 

Then the mean value formula can be written

$$\int_K f(x + k \cdot y) \alpha(dk) = \sum_{\mu=0}^\infty c_\mu(\Delta^\mu f)(x)\|y\|^{2\mu}. $$

In [Courant-Hilbert,1937], §3, Section 4, one finds the equivalent formula

$$\int_{S(\mathbb{R}^n)} f(x + ru) \beta(du) = \sum_{\mu=0}^\infty c_\mu(\Delta^\mu f)(x)r^{2\mu}. $$

The generalized Taylor expansion of the spherical functions

$$\varphi(\sigma; x) = \sum_{\mu=0}^\infty c_\mu(-1)^\mu \sigma^{2\mu} \|x\|^{2\mu},$$

is nothing but the power series expansion of the Bessel functions.

2 Gelfand pairs associated with the Heisenberg group

Let $W$ be a complex Euclidean vector space. The set $H = W \times \mathbb{R}$, equipped with the product

$$(z, t)(z', t') = (z + z', t + t' + \text{Im}(z'|z)), $$

is a Gelfand pair.
is the Heisenberg group of dimension $2N + 1$ ($N = \dim_{\mathbb{C}} W$). Relative to coordinates $z_1, \ldots, z_N$ with respect to a fixed orthogonal basis in $W$, consider the first order left-invariant differential operators on $H$:

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2i} z_j \frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{2i} z_j \frac{\partial}{\partial t} \quad (j = 1, \ldots, N).$$

Recall the notation

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

These operators form a basis of the Lie algebra $\mathfrak{h}$ of $H$. They satisfy

$$[Z_j, \bar{Z}_j] = iT,$$

and other brackets vanish.

Let $K$ be a closed subgroup of the unitary group $U(W)$ of $W$, and $G$ be the semi-direct product $G = K \ltimes H$. The pair $(G, K)$ is a Gelfand pair if and only if the group $K$ acts multiplicity free on the space $\mathcal{P}(W)$ of holomorphic polynomials on $W$. This result has been proven by Carcano [1987] (see also [Benson-Ratcliff-Ratcliff-Worku, 2004], [Wolf, 2007]). We assume that this condition holds. Hence the Banach algebra $L^1(H)^K$ of $K$-invariant integrable functions on $H$ is isomorphic to $L^1(K \backslash G / K)$, hence commutative.

The Fock space $\mathcal{F}(W)$ is the space of holomorphic functions $\psi$ on $W$ such that

$$||\psi||^2 = \frac{1}{\pi^N} \int_W |\psi(z)|^2 e^{-||z||^2} m(dz) < \infty$$

($m$ denotes the Euclidean measure on $W$). The reproducing kernel of $\mathcal{F}(W)$ is

$$\mathcal{K}(z, w) = e^{(z|w)}.$$

The Fock space decomposes multiplicity free under $K$:

$$\mathcal{F}(W) = \bigoplus_{m \in \mathcal{M}} \mathcal{H}_m.$$

Let $\mathcal{K}_m$ denotes the reproducing kernel of $\mathcal{H}_m$. Then

$$e^{(z|w)} = \sum_{m \in \mathcal{M}} \mathcal{K}_m(z, w).$$
The algebra $\mathbb{D}(H)^K$ of differential operators on $H$ which are invariant with respect to the left action of $H$ and the action of $K$ is isomorphic to the algebra $\mathbb{D}(G/K)$, hence commutative. To the polynomial $K_m(z, w)$ one associates the invariant differential operators $D_m$ and $L_m$ in $\mathbb{D}(H)^K$. Let $\tilde{K}_m$ be the polynomial in the $2N$ variables $z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N$ such that
\[
K_m(z, z) = \tilde{K}_m(z, \bar{z}).
\]
The operator $D_m$ is defined by
\[
D_m = \tilde{K}_m(\bar{Z}_1, \ldots, \bar{Z}_N; Z_1, \ldots Z_N).
\]
The operators $Z_j$ are applied first, then the operators $\bar{Z}_j$.

The operator $L_m$ is defined by symmetrization from the $K$-invariant (non holomorphic) polynomial $K_m(z, z)$: for a smooth function $f$ on $H$,
\[
(L_m f)(z, t) = K_m(\frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \bar{\zeta}}) f(z + \zeta, t + \text{Im}(\zeta | z)) \big|_{\zeta = 0}.
\]

The eigenvalues $\hat{D}_m(\sigma)$ and $\hat{L}_m(\sigma)$ have gotten general formulas in terms of generalized binomial coefficients by Benson and Ratcliff [1998]. In the sequel we will consider some special cases and give explicit formulas for these eigenvalues in terms of classical polynomials.

For $\mu = (m, \ell) \in \mathcal{M} = \mathcal{M} \times \mathbb{N}$, define $D_\mu = L_m T^\ell$. Then the operators $D_\mu$ form a linear basis of the vector space $\mathbb{D}(H)^K$. Define
\[
b_\mu(z, t) = \frac{1}{\dim H_m} K_m(z, z) \frac{1}{\ell!} t^\ell.
\]

**Proposition 2.1.**
\[
D_\mu b_\nu = \delta_{\mu\nu}.
\]

This follows from
\[
L_k K_m = \delta_{k, m} \dim H_k \quad (k, m \in \mathcal{M}).
\]

**Theorem 2.2.** If $f$ is a $K$-invariant analytic function on $H$ in a neighborhood of 0, then
\[
f(z, t) = \sum_{\mu \in \mathcal{M}} (D_\mu f)(0, 0) b_\mu(z, t)
\]
\[
= \sum_{m \in \mathcal{M}} \sum_{\ell = 0}^{\infty} \frac{1}{\dim H_m} \frac{1}{\ell!} (L_m T^\ell f)(0, 0) K_m(z, z) t^\ell.
\]
This implies the following mean value formula: for an analytic function \( f \) on \( H \),

\[
\int_{K} f(z + k \cdot w, s + t + \text{Im}(k \cdot w|z)) \alpha(dk) \\
= \sum_{\mu \in \mathcal{M}} (D_{\mu} f)(z, s)b_{\mu}(w, t) \\
= \sum_{m \in \mathcal{M}} \sum_{\ell = 0}^{\infty} \frac{1}{\dim \mathcal{H}_m \ell!} (L_m T^{\lambda}) f(z, s) K_m(w, w) t^{\ell}.
\]

**Corollary 2.3.** As a special case one obtains the following expansion for the spherical functions:

\[
\varphi(\sigma; z, t) = e^{i\lambda t} \sum_{m \in \mathcal{M}} \frac{1}{\dim \mathcal{H}_m} \hat{L}_m(\sigma) K_m(z, z).
\]

(Observe that \( \varphi(\sigma; 0, t) \) is an exponential, \( = e^{i\lambda t} \), where \( \lambda \) depends on \( \sigma \).) This formula will give a way for evaluating the eigenvalues \( \hat{L}_m(\sigma) \).

The Bergmann representation \( \pi_\lambda \) is defined on the Fock space \( \mathcal{F}_\lambda(W) \) (\( \lambda \in \mathbb{R}^* \)) of the holomorphic functions \( \psi \) on \( W \) such that

\[
\| \psi \|_{\lambda}^2 = \left( \frac{|\lambda|}{\pi} \right)^N \int_W |\psi(\zeta)|^2 e^{-|\lambda||\zeta||^2} m(d\zeta) < \infty.
\]

For \( \lambda > 0 \),

\[
(\pi_\lambda(z, t)\psi)(\zeta) = e^{\lambda(z - \frac{1}{2}\|\zeta\|^2 - (\zeta|z))} \psi(\zeta + z).
\]

For \( \lambda < 0 \), let \( \pi_\lambda(z, t) = \pi_{-\lambda}(\bar{z}, -t) \). Because of this simple relation we may assume that \( \lambda > 0 \), and will do most of the time in the sequel. The representation \( \pi_\lambda \) is irreducible. If \( f \in L^1(H)^K \), then the operator \( \pi_\lambda(f) \) commutes with the action of \( K \) on \( \mathcal{F}_\lambda(W) \). By Schur’s Lemma the subspace \( \mathcal{H}_m \) is an eigenspace of \( \pi_\lambda(f) \):

\[
\pi_\lambda(f)\psi = \hat{f}(\lambda, m)\psi \quad (\psi \in \mathcal{H}_m),
\]

and the eigenvalue can be written

\[
\hat{f}(\lambda, m) = \int_H f(z, t) \varphi(\lambda, m; z, t)m(dz)dt.
\]
The functions $\varphi(\lambda, m; z, t)$ are the bounded spherical functions of the first kind ($\lambda \in \mathbb{R}^*, m \in \mathcal{M}$).

The bounded spherical functions of the second kind are associated to the one-dimensional representations $\eta_w$ of $H$:

$$\eta_w(z, t) = e^{2i\text{Im}(z|w)} \quad (w \in W).$$

They are given by

$$\varphi(\dot{w}; z, t) = \int_K e^{2i\text{Im}(z|k\cdot w)} \alpha(dk).$$

The Gelfand spectrum $\Sigma$ can be described as the union $\Sigma = \Sigma_1 \cup \Sigma_2$. The part $\Sigma_1$ corresponds to the bounded spherical functions of the first kind, parametrized by the pairs $(\lambda, m)$, with $\lambda \in \mathbb{R}^*, m \in \mathcal{M}$, and the part $\Sigma_2$ to the bounded spherical functions of the second kind, parametrized by $K\backslash W$, the set of $K$-orbits in $W$.

Recall that

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2i} \frac{\partial}{\partial \bar{z}_j}, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{2i} \frac{\partial}{\partial t} \quad (j = 1, \ldots, N).$$

For the derived representations one obtains

$$d\pi_{\lambda}(T) = i\lambda, \quad d\pi_{\lambda}(Z_j) = \frac{\partial}{\partial \zeta_j}, \quad d\pi_{\lambda}(\bar{Z}_j) = -\lambda \zeta_j,$$

$$d\eta_w(T) = 0, \quad d\eta_w(Z_j) = \bar{w}_j, \quad d\eta_w(\bar{Z}_j) = -w_j.$$

From the definition of $D_p$ ($p \in \mathcal{M}$) it follows that

$$d\pi_{\lambda}(D_p) = \tilde{K}_p(-\lambda \zeta, \frac{\partial}{\partial \zeta}), \quad d\eta_w(D_p) = \tilde{K}_p(-w, \bar{w}).$$

The subspace $\mathcal{H}_m$ is an eigenspace of the operator $d\pi_{\lambda}(D_p)$:

$$d\pi_{\lambda}(D_p)\psi = \tilde{D}_p(\lambda, m)\psi \quad (\psi \in \mathcal{H}_m).$$

This will give a way for evaluating $\tilde{D}_p$. 
3 The case $W = \mathbb{C}^p$, $K = U(p)$

We consider the Heisenberg group $H = \mathbb{C}^p \times \mathbb{R}$, with the action of $K = U(p)$. Then $(G, K)$ with $G = U(p) \times \mathbb{C}^p$ is a Gelfand pair. It has been first observed by Korányi [1980] (see also [Faraut-1984]). In this case $M \simeq N$, $\mathcal{H}_m$ is the space of homogeneous polynomials of degree $m$, and

$$\mathcal{K}_m(z, w) = \frac{1}{m!} (z|w)^m, \quad \dim \mathcal{H}_m = \frac{(p)_m}{m!}.$$ 

Furthermore

$$\mathcal{L}_m f(z, t) = \frac{1}{m!} \left( \sum_{j=1}^{p} \frac{\partial^2}{\partial \zeta_j \partial \bar{\zeta}_j} \right)^m f(z + \zeta, t + \text{Im} (\zeta|z) )|_{\zeta=0}.$$ 

For $m = 1$,

$$\mathcal{L}_1 = \sum_{j=1}^{p} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + \frac{i}{2} \left( z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \frac{\partial}{\partial t} + \frac{1}{4} \|z\|^2 \frac{\partial^2}{\partial t^2}.$$ 

Up to a factor $\mathcal{L}_1$ is the sublaplacian $\Delta_0$: $\mathcal{L}_1 = \frac{1}{4} \Delta_0$. The operator $\mathcal{L}_1$ can be obtained by symmetrization:

$$\mathcal{L}_1 = \frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j).$$

The algebra $\mathbb{D}(H)^K$ is generated by the two operators $T$ and $\mathcal{L}_1$.

The spectrum of the Gelfand pair $(G, K)$ is the union $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ is parametrized by the set of pairs $(\lambda, m)$, with $\lambda \in \mathbb{R}^*$, $m \in \mathbb{N}$, and $\Sigma_2 \simeq [0, \infty]$. The bounded spherical functions of the first kind are expressed in terms of the ordinary Laguerre polynomials $L_m^{(\nu)}$: for $\sigma = (\lambda, m) \in \Sigma_1$,

$$\varphi(\lambda, m; z, t) = e^{i \lambda t} e^{-\frac{1}{2} \|z\|^2} \frac{L_m^{(p-1)}((\lambda\|z\|^2)}{L_m^{(p-1)}(0)}.$$ 

This function admits the following expansion

$$\varphi(\lambda, m; z, t) = e^{i \lambda t} e^{-\frac{1}{2} \|z\|^2} \sum_{k=0}^{m} (-1)^k \frac{1}{(p)_k} \frac{1}{k! |\lambda|^{[k]} |z|^{2k}}.$$
We recall the Pochhammer symbols:

\[ [x]_k = x(x - 1) \ldots (x - k + 1), \quad (x)_k = x(x + 1) \ldots (x + k - 1). \]

The bounded spherical functions of the second kind are expressed in terms of the Bessel functions: for \( \sigma = \tau \in \Sigma_2 \),

\[ \varphi(\tau; z, t) = j_{n-1}(2 \sqrt{\tau} \|z\|), \]

where \( j_\nu(r) = \Gamma(\nu + 1) \left( \frac{r}{2} \right)^{-\nu} J_\nu(r) \). These functions admits the following expansion

\[ \varphi(\tau; z, t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(p)_k k!} \|z\|^{2k}. \]

Proposition 3.1. The eigenvalues of the differential operator \( D_k \) are given, for \((\lambda, m) \in \Sigma_1, \lambda > 0\), by

\[ \hat{D}_k(\lambda, m) = \frac{(-1)^k}{k!} \lambda^k [m]_k, \]

and, for \((\tau) \in \Sigma_2\), by

\[ \hat{D}_k(\tau) = \frac{(-1)^k}{k!} \tau^k. \]

Proof.
We saw that

\[ d\pi_\lambda(D_k) = \tilde{K}_k(-\lambda \zeta, \frac{\partial}{\partial \zeta}). \]

Since

\[ \tilde{K}_k(z, w) = \frac{1}{k!} \left( \sum_{j=1}^{p} z_j w_j \right)^k, \]

this means that \( \pi_\lambda(D_k) \) is the differential operator with symbol

\[ \sigma(\zeta, \xi) = \frac{(-1)^k}{k!} \lambda^k \left( \sum_{j=1}^{p} \zeta_j \xi_j \right)^k = \frac{(-1)^k}{k!} \lambda^k \sum_{|\alpha|=p} \frac{k!}{\alpha!} \zeta^\alpha \xi^\alpha, \]

where

\[ \alpha = (\alpha_1, \ldots, \alpha_p), \quad \alpha_j \in \mathbb{N}, \quad \alpha! = \alpha_1! \ldots \alpha_p!, \quad \zeta^\alpha = \zeta_1^{\alpha_1} \ldots \zeta_p^{\alpha_p}. \]
The operator $\pi_{\lambda}(D_k)$ is given explicitly as follows

$$\pi_{\lambda}(D_k) = \frac{(-1)^k}{k!} \lambda^k \sum_{|\alpha|=p} \frac{k!}{\alpha!} \zeta^\alpha \left( \frac{\partial}{\partial \zeta} \right)^\alpha.$$

Let us apply this operator to the polynomial $\psi(\zeta) = \zeta^m$ which belongs to $\mathcal{H}_m$:

$$\pi_{\lambda}(D_k)\psi(\zeta) = \frac{(-1)^k}{k!} \lambda^k \zeta^k \sum_{|\alpha|=p} \frac{k!}{\alpha!} \zeta^\alpha \left( \frac{\partial}{\partial \zeta} \right)^\alpha \zeta^m = \frac{(-1)^k}{k!} \lambda^k [m]_{(k)} \psi(\zeta).$$

Therefore

$$\widehat{D}_k(\lambda, m) = \frac{(-1)^k}{k!} \lambda^k [m]_{(k)}.$$

In case of the one-dimensional representation $\eta_w$,

$$\eta_w(D_k) = \frac{(-1)^k}{k!} |w|^{2k} = \frac{(-1)^k}{k!} \tau^k.$$

It follows that the expansion of the spherical functions can be written

$$\varphi(\sigma; z, t) = e^{i\lambda t} e^{-\frac{1}{2} \lambda \|z\|^2} \sum_{k=0}^{m} \frac{1}{(p)_k} \widehat{D}_k(\sigma) \|z\|^{2k}.$$  

**Corollary 3.2.** For every $D$ in $\mathbb{D}(H)^K$ there is a polynomial $F_D$ in two variables such that, for $(\lambda, m) \in \Sigma_1$, $\lambda > 0$,

$$\widehat{D}(\lambda, m) = F_D(\lambda, \lambda m),$$

and, for $(\tau) \in \Sigma_2$,

$$\widehat{D}(\tau) = F_D(0, \tau).$$

The map $D \mapsto F_D$, $\mathbb{D}(H)^K \rightarrow \text{Pol}(\mathbb{C}^2)$ is an algebra isomorphism.

In particular, for $D = D_k$,

$$F_{D_k}(u, v) = \frac{(-1)^k}{k!} v(v - u) \ldots (v - (k - 1)u).$$

Let us embed $\Sigma$ into $\mathbb{R}^2$ by the map

$$(\lambda, m) \in \Sigma_1 \mapsto (\lambda, \lambda m), \quad (\tau) \in \Sigma_2 \mapsto (0, \tau).$$
Then, according to [Ferrari-Rufino, 2007], the Gelfand topology of $\Sigma$ is induced by the topology of $\mathbb{R}^2$. This means in particular that, for $D \in \mathbb{D}(H)^K,$

$$\lim \hat{D}(\lambda, m) = \hat{D}(\tau),$$

as $\lambda \to 0$, $\lambda m \to \tau$.

We will evaluate the eigenvalues $\hat{E}_m(\sigma)$ in terms of the Meixner-Pollaczek polynomials. We introduce the one variable polynomials $q^{(\nu)}_k(s)$ as defined by the generating formula:

$$\sum_{k=0}^{\infty} q^{(\nu)}_k(s) w^k = (1 - w)^{s-\frac{\nu}{2}} (1 + w)^{-s-\frac{\nu}{2}}.$$

The relation to the classical Meixner-Pollaczek polynomials is as follows

$$q^{(\nu)}_k(i\lambda) = (-i)^k P^\nu_{\frac{\nu}{2}}(\lambda; \frac{\pi}{2}).$$

Observe that

$$q^{(\nu)}_0(s) = 1, \quad q^{(\nu)}_1(s) = -2s.$$

These polynomials also admit the following hypergeometric representation

$$q^{(\nu)}_m(s) = \frac{(\nu)_m}{m!} 2F_1(-m, s + \frac{\nu}{2}; \nu; 2) = \frac{(\nu)_m}{m!} \sum_{k=0}^{m} \frac{[m]_k[-s-\frac{\nu}{2}]_k}{(\nu)_k} \frac{1}{k!} 2^k.$$  

One checks that

$$q^{(\nu)}_m(s) = \frac{1}{m!} (-2)^m s^m + \text{lower order terms}.$$

For $\nu = 1$, the polynomials $q^{(1)}_k(i\lambda)$ are orthogonal with respect to the weight

$$\frac{1}{\cosh \pi \lambda}.$$  

More generally, for $\nu > 0$, the polynomials $q^{(\nu)}_k(i\lambda)$ are orthogonal with respect to the weight

$$|\Gamma(i\lambda + \frac{\nu}{2})|^2.$$
Theorem 3.3. The eigenvalues of the differential operator $L_k$ are given, for $(\lambda, m) \in \Sigma_1$, $\lambda > 0$, by

$$\hat{L}_k(\lambda, m) = \left(\frac{1}{2} |\lambda|\right)^k q_k^{(\nu)}(m + \frac{p}{2}),$$

and, for $(\tau) \in \Sigma_1$, by

$$\hat{L}_k(\tau) = (-1)^k \tau^k k!.$$

It follows that $L_k = Q_k(T, L_1)$ with

$$Q_k(t, s) = \left(\frac{t}{2}\right)^k q_k^{(\nu)}(-s).$$

For $p = 1$ this result has been established by Koornwinder [1988]. The proof we give below is different.

Since

$$q_k^{(\nu)}(s) = \frac{1}{k!} (-2)^k s^k + \text{lower order terms},$$

one checks that

$$\lim \hat{L}_k(\lambda, m) = \hat{L}_k(\tau),$$

as $\lambda \to 0, \lambda m \to \tau$.

Proof.

We start from a generating formula for the polynomials $q_k^{(\nu)}$ related to the confluent hypergeometric function

$$1_F(\alpha, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k} \frac{1}{k!} z^k.$$

This generating formula can be written:

$$e^{-u} 1_F\left(s + \frac{\nu}{2}; \nu; 2u\right) = \sum_{k=0}^{\infty} q_k^{(\nu)}(-s) \frac{1}{(\nu)_k} u^k.$$

(see for instance [Andrews-Askey-Roy,1999], p.349). For $\alpha = -m$, the hypergeometric series terminates and reduces to a Laguerre polynomial:

$$L_m^{(\nu-1)}(z) = \frac{(\nu)_m}{m!} 1_F(-m, \nu; z) = \frac{(\nu)_m}{m!} \sum_{k=0}^{m} (-1)^k [m]_k \frac{1}{(\nu)_k} k! z^k,$$
and, for $s + \frac{\nu}{2} = -m \ (m \in \mathbb{N})$, one gets
\[ e^{-u L_{m}^{(\nu-1)}(2u)} = \frac{(\nu)_m}{m!} \sum_{k=0}^{\infty} q_k^{(\nu)}(m + \frac{\nu}{2}) \frac{1}{(\nu)_k} u^k. \]

Hence the bounded spherical function of the first kind can be written
\[ \varphi(\lambda, m; z, t) = e^{i\lambda t} \sum_{k=0}^{\infty} \frac{1}{(p)_k} q_k^{(\nu)}(m + \frac{p}{2}) \left( \frac{1}{2} |\lambda||z|^{2} \right)^k. \]

On the other hand, by Corollary 2.3,
\[ \varphi(\lambda, m; z, t) = e^{i\lambda t} \sum_{k=0}^{\infty} \frac{1}{(p)_k} \mathcal{L}_k(\lambda, m) ||z||^{2k}. \]

Therefore
\[ \mathcal{L}_k(\lambda, m) = \left( \frac{1}{2} |\lambda| \right)^k q_k^{(\nu)}(m + \frac{p}{2}). \]

From the expansion
\[ \varphi(\tau; z, t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(p)_k} \frac{1}{k!} \tau^k ||z||^{2k}, \]

it follows that
\[ \mathcal{L}_k(\tau) = (-1)^k \frac{\tau^k}{k!}. \]

In Section 6 we will consider a multivariate analogue of the case we have seen in this section. For that we will introduce in Sections 4 and 5 certain multivariate functions associated to symmetric cones.

4 Symmetric cones and spherical expansions

We consider an irreducible symmetric cone $\Omega$ in a simple Euclidean Jordan algebra $V$, with rank $n$, multiplicity $d$, and dimension
\[ N = n + \frac{d}{2} n(n - 1). \]

Let $L$ be the identity component in the group $G(\Omega)$ of linear automorphisms of $\Omega$, and $K_0 \subset L$ the isotropy subgroup of the unit element $e \in V$. Then
\((L, K_0)\) is a Gelfand pair. The spherical function \(\varphi_s\), for \(s \in \mathbb{C}^n\), is defined on \(\Omega\) by
\[
\varphi_s(x) = \int_{K_0} \Delta_{s+\rho}(k \cdot x) \alpha(dk),
\]
where \(\Delta_s\) is the power function, \(\rho = (\rho_1, \ldots, \rho_n)\), \(\rho_j = \frac{d}{2}(2j - n - 1)\). The algebra \(\mathbb{D}(\Omega)\) of \(L\)-invariant differential operators on \(\Omega\) is commutative, the spherical function \(\varphi_s\) is an eigenfunction of every \(D \in \mathbb{D}(\Omega)\):
\[
D \varphi_s = \gamma_D(s) \varphi_s,
\]
and \(\gamma_D\) is a symmetric polynomial function in \(n\) variables. (See [Faraut-Korányi,1994].) The Gelfand spectrum \(\Sigma\) can be seen as a closed subset of \(\mathbb{C}^n/\mathfrak{S}_n\), and \(\hat{D}(s)\) can be identified to \(\gamma_D(s)\). The space \(P(V)\) of polynomial functions on \(V\) decomposes multiplicity free under \(L\) as
\[
P(V) = \bigoplus_m P_m,
\]
where \(P_m\) is a subspace of finite dimension \(d_m\), irreducible under \(L\). The parameter \(m\) is a partition: \(m = (m_1, \ldots, m_n)\), \(m_j \in \mathbb{N}\), \(m_1 \geq \cdots \geq m_n \geq 0\). The subspace \(P^K_{m_0}\) of \(K_0\)-invariant polynomial functions is one-dimensional, generated by the spherical polynomial \(\Phi_m\), normalized by the condition \(\Phi_m(e) = 1\). The polynomials \(\Phi_m\) form a basis of the space \(P(V)^{K_0}\) of \(K_0\)-invariant polynomials. Let \(D^m\) be the invariant differential operator determined by the condition
\[
D^m f(e) = \left( \Phi_m \left( \frac{\partial}{\partial x} \right) f \right)(e).
\]
Then the operators \(D^m\) form a linear basis of \(\mathbb{D}(\Omega)\). The generalized Pochhammer symbol \((\alpha)_m\) is defined by
\[
(\alpha)_m = \prod_{j=1}^n (\alpha - (j - 1)\frac{d}{2})_{m_j},
\]
A \(K_0\)-invariant function \(f\), analytic in a neighborhood of 0, admits a spherical Taylor expansion:
\[
f(x) = \sum_m d_m \left( \frac{1}{\binom{n}{m}} \right) \Phi_m \left( \frac{\partial}{\partial x} \right) f(0) \Phi_m(x).
\]
For $D = D^m$, we will write $\gamma_{D^m}(s) = \gamma_m(s)$. The function $\gamma_m$ can be seen as a multivariate analogue of the Pochhammer symbol $[s]_m$. In fact, for $n = 1$ ($s \in \mathbb{C}$, $m \in \mathbb{N}$),

$$\gamma_m(s) = [s]_m = (-1)^m(s)_m.$$ 

Observe that

$$\gamma_m(\alpha, \ldots, \alpha) = (\alpha - \rho)_m.$$ 

With this notation we can write a multivariate binomial formula.

**Proposition 4.1.** (i) For $z \in D$, the unit ball in $V_{\mathbb{C}}$ centered at 0, relatively to the spectral norm,

$$\varphi_s(e + z) = \sum_m d_m \frac{1}{(N_m)_m} \gamma_m(s) \Phi_m(z).$$

The convergence is uniform on compact sets in $D$.

(ii) For $s \in \mathbb{C}^n$, and $r$, $0 < r < 1$, there is a constant $A(s, r) > 0$ such that, for every $m$,

$$|\gamma_m(s)| \leq A(s, r) \frac{(N_m)_m}{r^{|s|}}.$$ 

**Proof.**

(i) Observe first that

$$\Phi_m \left( \frac{\partial}{\partial \xi} \right) \varphi_s(e + z) \bigl|_{z=0} = D^m \varphi_s(e) = \gamma_m(s).$$

We will see that the function $\varphi_s$ has a holomorphic continuation to $e + D$. By Theorem XII.3.1 in [Faraut-Korányi, 1994], it will follow that the Taylor expansion of $\varphi_s(e + z)$ converges uniformly on compact sets in $D$. From the integral representation of the spherical functions $\varphi_s$, it follows that these functions admit a holomorphic continuation to the tube $\Omega + iV$. Let us prove the inclusion $e + D \subset \Omega + iV$. To prove this it suffices to show that $e + D \cap V \subset \Omega$. In fact, to see that, consider the conjugation $z \mapsto \bar{z}$ of $V_{\mathbb{C}} = V + iV$ with respect to the Euclidean real form $V$. For $z \in D$, we will show that $e + \frac{1}{2}(z + \bar{z}) \in \Omega$. Since $D$ is invariant under this conjugation and convex, $\frac{1}{2}(z + \bar{z}) \in D \cap V$. Moreover

$$D \cap V = (e - \Omega) \cap (-e + \Omega),$$
therefore 

\[ e + \mathcal{D} \cap V = \Omega \cap (2e - \Omega) \subset \Omega. \]

(ii) Let

\[ f(z) = \sum_m d_m a_m \Phi_m(z) \]

be the spherical Taylor expansion of a \( K_0 \)-invariant analytic function in \( \mathcal{D} \). Then the coefficients \( a_m \) are given, for \( 0 < r < 1 \), by

\[ a_m = \frac{1}{r^{|m|}} \int_K f(rk \cdot e) \Phi_m(k \cdot e) \alpha(dk), \]

where \( K = \text{Str}(V) \cap U(V) \), hence satisfy the following Cauchy inequality: for \( 0 < r < 1 \),

\[ |a_m| \leq \frac{1}{r^{|m|}} \sup_{k \in K} |f(rk \cdot e)|, \]

It follows that, for \( s \in \mathbb{C}^n \), and \( r, 0 < r < 1 \), there is a constant \( A(s, r) \) such that

\[ |\gamma_m(s)| \leq A(s, r) \left( \frac{N}{n} \right)^m r^{|m|}. \]

For \( s = m - \rho \), \( \varphi_{m-\rho}(z) = \Phi_m(z) \), and the binomial formula can be written in that case

\[ \Phi_m(e + z) = \sum_{k \subseteq m} \binom{m}{k} \Phi_k(z). \]

In fact the generalized binomial coefficient

\[ \binom{m}{k} = d_k \frac{1}{\left( \frac{N}{n} \right)_k} \gamma_k(m - \rho) \]

vanishes if \( k \not\subset m \).

In the case of the cone \( \Omega \) of \( n \times n \) Hermitian matrices of positive type, \( \Omega \subset V = \text{Herm}(n, \mathbb{C}) \), i.e. \( d = 2 \), the spherical polynomials can be expressed in terms of the Schur functions \( s_m \):

\[ \Phi_m(\text{diag}(a_1, \ldots, a_n)) = \frac{s_m(a_1, \ldots, a_n)}{s_m(1^n)}. \]
The spherical expansion of the exponential of the trace can be written

\[ e^{\text{tr}x} = \sum_m \frac{1}{h(m)} \chi_m(x), \]

where \( h(m) \) is the product of the hook-lengths of the partition \( m \), and \( \chi_m \) is the character of the representation of \( GL(n, \mathbb{C}) \) with highest weight \( m \). Equivalently

\[ e^{a_1 + \ldots + a_n} = \sum_m \frac{1}{h(m)} s_m(a_1, \ldots, a_n) \]

(see [Macdonald, 1995], p.66). Furthermore

\[ d_m = (s_m(1^n))^2, \text{ therefore } \frac{1}{h(m)} = \frac{s_m(1^n)}{(n)_m}. \]

The binomial formula for the Schur functions is written as

\[ \frac{s_m(1 + a_1, \ldots, 1 + a_n)}{s_m(1^n)} = \sum_{k \subset m} \frac{1}{(n)_k} s^*_k(m) s_k(a_1, \ldots, a_n), \]

where \( s^*_k(m) \) is a the shifted Schur function ([Okounkov-Olshanski, 1997]). The following relations follow

\[ \binom{m}{k} = \frac{1}{h(k)} s^*_k(m), \quad \gamma_k(m - \rho) = \frac{s^*_k(m)}{s_k(1^n)}. \]

5 A generating formula for multivariate Meixner-Pollaczek polynomials

The multivariate Meixner-Pollaczek polynomials \( Q^{(\nu)}_m(s) \) can be defined by the generating formula

\[ \sum_m d_m Q^{(\nu)}_m(s) \Phi_m(w) = \Delta(e - w^2)^{-\frac{\nu}{2}} \varphi_s((e - w)(e + w)^{-1}) \]

([Faraut-Wakayama, 2012]). The polynomial \( Q^{(\nu)}_m(s) \) admits the following "hypergeometric representation"

\[ Q^{(\nu)}_m(s) = \binom{\nu}{\frac{n}{2}} \sum_{k \subset m} d_k \gamma_k(m - \rho) \gamma_k(-s - \frac{\nu}{2}) \frac{1}{\binom{\nu}{\frac{n}{2}}} 2^{|k|}. \]
The polynomials \( Q^{(\nu)}_m(i\lambda) \) are orthogonal with respect to the measure \( M_\nu(d\lambda) \) on \( \mathbb{R}^n \) given by

\[
M_\nu(d\lambda) = \prod_{j=1}^{n} |\Gamma(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1))|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda),
\]

where \( m \) is the Lebesgue measure, and \( c \) is the Harish-Chandra \( c \)-function of the symmetric cone \( \Omega \):

\[
c(s) = c_0 \prod_{j<k} B(s_j - s_k, \frac{d}{2})
\]

\((B \text{ is the Euler beta function}). \) One can see that

\[
Q^{(\nu)}_m(s) = \frac{1}{\left(\frac{n}{2}\right)_m} (-2)^{|m|} \Phi_m(s) + \text{lower order terms}.
\]

We consider a multivariate analogue of the confluent hypergeometric functions \( \text{}_{1}F_{1} \):

\[
F(s, \nu; x) = \sum_k d_k \gamma_k(-s) \frac{1}{(\nu)_k} \frac{1}{(\frac{n}{2})_k} \Phi_k(-x),
\]

for \( s \in \mathbb{C}^n, \nu > \frac{d}{2}(n-1), \) \( x \in V_C \).

**Proposition 5.1.** The series converges for every \( x \in V_C \).

**Proof.**

This follows from the Cauchy inequalities: part (ii) in Proposition 4.1, and the fact that, for \( \nu > \frac{d}{2}(n-1) \), and every \( R > 0 \),

\[
\sum_k d_k \frac{1}{(\nu)_k} R^{|k|} < \infty.
\]

For \( s = \rho - m, \) \( m \) a partition, the function \( F(\rho - m, \nu; x) \) is essentially a multivariate Laguerre polynomial:

\[
I^{(\nu-1)}_m(x) = \frac{\left(\frac{n}{2}\right)_m}{(\frac{n}{2})_m} F(\rho - m, \nu; x)
= \frac{(\nu)_m}{(\frac{n}{2})_m} \sum_{m \subseteq k \subseteq m} d_k \frac{\gamma_k(m - \rho)}{(\nu)_k} \frac{1}{(\frac{n}{2})_k} \Phi_k(-x).
\]
Theorem 5.2. The multivariate Meixner-Pollaczek polynomials $Q^{(\nu)}_k$ admit the following generating formula

$$e^{-\text{tr} u} F(s + \frac{\nu}{2}; \nu; 2u) = \sum_k (-1)^{|k|} d_k \frac{1}{\binom{\nu}{k}} Q^{(\nu)}_k(s) \Phi_k(u).$$

For $s = \rho - m - \frac{\nu}{2}$, one obtains

$$e^{-\text{tr} u} F^{(\nu-1)}_m (2u) = \binom{\nu}{m} \sum_k d_k \frac{1}{\binom{\nu}{k}} Q^{(\nu)}_k(m + \frac{\nu}{2} - \rho) \Phi_k(u).$$

Lemma 5.3. (Bingham identity)

$$e^{\text{tr} x} \Phi_m(x) = \sum_{k \geq m} d_k \gamma_m(k - \rho) \frac{1}{\binom{\nu}{k}} \Phi_k(x).$$

This formula, which has been established by Bingham [1974] in case of $V = Sym(n, \mathbb{R})$, generalizes the formula

$$e^x x^m = \sum_{k=m}^{\infty} [k]_m \frac{1}{k!} x^k.$$

We will give a different proof.

Proof.

The symbol $\sigma_D(x, \xi)$ of a differential operator $D$ is defined by the relation

$$De^{[x][\xi]} = \sigma_D(x, \xi) e^{[x][\xi]}.$$

If $D$ is invariant, $D \in \mathbb{D}(\Omega)$, then its symbol is invariant in the following sense: for $g \in G$,

$$\sigma_D(gx, \xi) = \sigma_D(x, g^* \xi).$$

For $x = \xi = e$, one gets

$$\sigma_D(ge, e) = \sigma_D(e, g^* e),$$

and taking $g$ selfadjoint, it follows that, for $x \in \Omega$, $\sigma_D(x, e) = \sigma_D(e, x)$, and

$$De^{\text{tr} x} = \sigma_D(x, e) e^{\text{tr} x}.$$
For $D = D^m$,
\[
\sigma_D(x, e) = \sigma_D(e, x) = \Phi_m(x),
\]
and
\[
D^m e^{trx} = \Phi_m(x)e^{trx}.
\]

On the other hand
\[
D^m e^{trx} = D^m \left( \sum_k d_k \left( \frac{n}{n} \right)_m^k \Phi_k(x) \right)
= \sum_k d_k \left( \frac{n}{n} \right)_k^m D^m \Phi_k(x)
= \sum_k d_k \left( \frac{n}{n} \right)_k^m \gamma_m(k - \rho) \Phi_k(x).
\]

Furthermore we know that $\gamma_m(k - \rho) = 0$ if $m \not\subset k$. We obtain finally
\[
e^{trx} \Phi_m(x) = \sum_{k \supset k} d_k \gamma_m(k - \rho) \left( \frac{n}{n} \right)_k^m \Phi_k(x).
\]

In case of the cone $\Omega$ of $n \times n$ Hermitian matrices of positive type, $\Omega \subset V = \text{Herm}(n, \mathbb{C})$, i.e. $d = 2$, we get the following Schur expansion
\[
e^{a_1 + \cdots + a_n} s_m(a) = \sum_{k \supset m} \frac{1}{h(k)} s_m^*(k) s_k(a).
\]

**Proof of Theorem 5.2**

By using the Bingham identity (Lemma 5.3) we get
\[
e^{-tru} F(s + \frac{\nu}{2}; \nu; 2u) = \sum_k d_k \frac{\gamma_k(-s - \frac{\nu}{2})}{(\nu)_k} \left( \frac{n}{n} \right)_k^m 2^{|k|} e^{-tru} \Phi_k(-u)
= \sum_k d_k \frac{\gamma_k(-s - \frac{\nu}{2})}{(\nu)_k} \left( \frac{n}{n} \right)_k^m 2^{|k|} \left( \sum_{j \supset k} d_j \gamma_j(j - \rho) \left( \frac{n}{n} \right)_j^m \Phi_j(-u) \right)
= \sum_j d_j \left( \frac{n}{n} \right)_j^m \left( \sum_{k \subset j} d_k \frac{\gamma_k(-s - \frac{\nu}{2}) \gamma_j(j - \rho)}{(\nu)_k} \left( \frac{n}{n} \right)_k^m 2^{|k|} \right) \Phi_j(-u)
= \sum_j (-1)^j d_j \left( \frac{n}{n} \right)_j^m Q_j^{(\nu)}(s) \Phi_j(u).
\]
6 The case $W = M(n, p; \mathbb{C})$, $K = U(n) \times U(p)$

The group $K = U(n) \times U(p)$ acts on the space $W = M(n, p; \mathbb{C})$ ($n \leq p$) of $n \times p$ matrices by the transformations

$$z \mapsto uzv^* \quad (u \in U(n), \ v \in U(p)).$$

Its action on the space $\mathcal{P}(W)$ of holomorphic polynomials on $W$ is multiplicity free and the parameter set $\mathcal{M}$ is the set of partitions $m$ of lengths $\ell(m) \leq n$: $m = (m_1, \ldots, m_n)$ with $m_i \in \mathbb{N}$, $m_1 \geq \cdots \geq m_n \geq 0$. The subspace $\mathcal{H}_m \subset \mathcal{P}(W)$ corresponding to the partition $m$ is generated by the polynomials

$$\Delta_m(uzv) \quad (u \in U(n), \ v \in U(p)),$$

where

$$\Delta_m(z) = \Delta_1(z)^{m_1-m_2} \cdots \Delta_n(z)^{m_n},$$

with

$$\Delta_k(z) = \det((z_{ij})_{1 \leq i \leq j \leq k}),$$

the principal minor of order $k$ ($k \leq n$). The character $\chi_m$ of the representation of $U(n)$ with highest weight $m$ can be expressed in terms of the Schur functions $s_m$:

$$\chi_m(\text{diag}(t_1, \ldots, t_n)) = s_m(t_1, \ldots, t_n),$$

and $\chi_m$ extends as a polynomial on $M(n, \mathbb{C})$ of degree $|m|$. The reproducing kernel $K_m$ of the subspace $\mathcal{H}_m$ is given by

$$K_m(z, w) = \frac{1}{\hbar(m)} \chi_m(zw^*).$$

The Heisenberg group $H$ of dimension $2np + 1$ is seen as $H = W \times \mathbb{R}$, and the group $K = U(n) \times U(p)$ acts on $H$. With $G = K \rtimes W$, $(G, K)$ is a Gelfand pair, and its Gelfand spectrum can be described as the union $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ is the set of pairs $(\lambda, m)$ with $\lambda \in \mathbb{R}^*$, $m$ is a partition with $\ell(m) \leq n$, and

$$\Sigma_2 = \{ \tau \in \mathbb{R}^n \mid \tau_1 \geq \cdots \geq \tau_n \geq 0 \}.$$

The bounded spherical functions of the first kind are expressed in terms of multivariate Laguerre polynomials associated to the Jordan algebra $\text{Herm}(n, \mathbb{C})$:

$$\varphi(\lambda, m; z, t) = e^{i\lambda t} e^{-\frac{1}{2} \|zz^*\|^2} \frac{L_m^{(p-1)}(|\lambda|zz^*)}{L_m^{(p-1)}(0)}.$$
This function admits the following expansion

\[ \varphi(\lambda, m; z, t) = e^{i\lambda t} e^{-\frac{1}{2}||\lambda||z||^2} \sum_{k \subset m} (-1)^{|k|}|\lambda||^{|k|} \frac{1}{(n)_{k}} \frac{1}{(p)_{k}} s_{k}^{*}(m) \chi_{k}(zz^{*}). \]

The bounded spherical functions of the second kind are given by

\[ \varphi(\tau; z) = \int_{U(n) \times U(p)} e^{2i\text{Re} \text{tr}(uze^{*}w^{*})} \beta_{n}(du)\beta_{p}(dv), \]

where \( \tau = (\tau_{1}, \ldots, \tau_{n}) \), and \( \tau_{1} \geq \cdots \geq \tau_{n} \geq 0 \) are the eigenvalues of \( ww^{*} \).

This function admits the following expansion

\[ \varphi(\tau; z, t) = \sum_{k} (-1)^{|k|} \frac{1}{(n)_{k}} \frac{1}{(p)_{k}} s_{k}(\tau) \chi_{k}(zz^{*}). \]

(See [Faraut, 2010a])

We will give formulas for the eigenvalues \( \widehat{D}_{k}(\sigma) \) and \( \widehat{L}_{k}(\sigma) \) of the operators \( D_{k} \) and \( L_{k} \) we have introduced in Section 3 associated to a partition \( k \).

**Theorem 6.1.**

\[ \widehat{D}_{k}(\lambda, m) = \frac{(-1)^{|k|}}{h(k)} \lambda^{|k|} \widehat{s}_{k}^{*}(m), \quad \widehat{D}_{k}(\tau) = \frac{(-1)^{|k|}}{h(k)} s_{k}(\tau). \]

**Proof.**

From the definition of the operator \( D_{k} \), one obtains

\[ d\pi_{\lambda}(D_{k}) = \frac{(-1)^{|k|}}{h(k)} \lambda^{|k|} s_{k}(1^{n}) \tilde{D}^{k}, \]

where \( \tilde{D}^{k} \) is a differential operator whose restriction to the subspace \( W_{0} = M(n; \mathbb{C}) \subset W = M(n, p; \mathbb{C}) \) is equal to the operator \( D^{k} \) introduced in Section 4. For \( \psi \in \mathcal{H}_{m} \),

\[ d\pi_{\lambda}(D_{k}) \psi = \widehat{D}_{k}(\lambda, m) \psi. \]

Choosing \( \psi(\zeta) = \Phi_{m}(\zeta_{0}) \), where \( \zeta_{0} \) is the projection of \( \zeta \) on \( W_{0} \), we get

\[ \tilde{D}^{k} \psi = \gamma_{k}(m - \rho) \psi. \]
Since $s_k(1^n) \gamma_k(m - \rho) = s_k^*(m)$, we obtain
\[ \hat{D}_k(\lambda, m) = \frac{(-1)^{|k|}}{h(k)} \lambda^{|k|} s_k^*(m). \]

Furthermore
\[ d\eta_w(\hat{D}_k) = K_k(-w, w) = \frac{(-1)^{|k|}}{h(k)} \chi_k(ww^*) = \frac{(-1)^{|k|}}{h(k)} s_k(\tau). \]

**Corollary 6.2.** For every $D \in \mathbb{D}(H)^K$ there is a polynomial $F_D$ in $n + 1$ variables $u, v_1, \ldots, v_n$, symmetric in the variables $v_1, \ldots, v_n$, such that
\[ \hat{D}(\lambda, m) = F_D(\lambda, \lambda(m_1 - \rho_1), \ldots, \lambda(m_n - \rho_n)). \]
The map $D \mapsto F_D$, $\mathbb{D}(H)^K \to \mathcal{P}(\mathbb{C}) \otimes \mathcal{P}(\mathbb{C}^n)^{\mathbb{S}_n}$ is an algebra isomorphism.

Let us embed the Gelfand spectrum $\Sigma$ into $\mathbb{R}^{n+1}$ by the map
\[ (\lambda, m) \in \Sigma_1 \mapsto (\lambda, \lambda m_1, \ldots, \lambda m_n), \quad (\tau) \in \Sigma_1 \mapsto (0, \tau_1, \ldots, \tau_n). \]
As in Section 3, according to [Ferrari-Rufino, 2007], the Gelfand topology of $\Sigma$ is induced by the topology of $\mathbb{R}^{n+1}$. This implies in particular that
\[ \lim \hat{D}_k(\lambda, m) = \hat{D}_k(\tau), \]
as $\lambda \to 0$, $\lambda m_j \to \tau_j$. In fact
\[ s_k^*(m) = s_k(m) + \text{ lower order terms}. \]

Recall that the differential operator $L_m \in \mathbb{D}(H)^K$ has been defined by
\[ L_m = \mathcal{K}_m \left( \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \bar{\zeta}} \right) f(z + \zeta, t + \text{Im} (\zeta|z)) \big|_{\zeta=0}. \]

**Theorem 6.3.**
\[ \hat{L}_k(\lambda, m) = \frac{1}{2} |\lambda| \frac{|k|}{Q_k^{(p)}} (m + \frac{p}{2} - \rho). \]
\[ \hat{L}_k(\tau) = (-1)^{|k|} \frac{1}{h(k)} s_k(\tau). \]
Proof.

By Corollary 2.3, the spherical functions admit the following expansion:

\[ \varphi(\sigma; z, t) = e^{i\lambda t} \sum_k \frac{1}{\dim H_k} \widehat{L}_k(\sigma)K_k(z, z), \]

where the summation is over all partitions \( k \) with \( \ell(k) \leq n \). By using the formulas

\[ \dim H_k = s_k(1^n)s_k(1^p) = \frac{(n)_k (p)_k}{h(k) h(k)}, \]
\[ K_k(z,w) = \frac{1}{h(k)} \chi_k(zw^*) = \frac{s_k(1^n)}{h(k)} \Phi_k(zw^*), \]

we get

\[ \varphi(\sigma; z, t) = e^{i\lambda t} \sum_k \frac{1}{(p)_k} \widehat{L}_k(\sigma)\Phi_k(zz^*). \]

On the other hand, by Theorem 5.2, with \( s = \rho - m - \frac{p}{2}, \nu = p \), we obtain for \( \sigma = (\lambda, m) \in \Sigma_1 \),

\[ \varphi(\lambda, m; z, t) = e^{i\lambda t} \sum_k d_k \frac{1}{(p)_k} Q_k^{(p)}(m + \frac{p}{2} - \rho) \left( \frac{1}{2} |\lambda| \right)^{|k|} \Phi_k(zz^*). \]

Therefore,

\[ \widehat{L}_k(\lambda, m) = d_k \left( \frac{1}{2} |\lambda| \right)^{|k|} Q_k^{(p)}(m + \frac{p}{2} - \rho). \]

For \( \sigma = (\tau) \in \Sigma_2 \),

\[ \varphi(\tau; z, t) = \sum_k (-1)^{|k|} \frac{1}{(n)_k \lambda_k} \frac{1}{(p)_k} s_k(r) \chi_k(zz^*) \]
\[ = \sum_k (-1)^{|k|} \frac{1}{(p)_k \lambda_k} \frac{1}{h(k)} s_k(r) \Phi_k(zz^*). \]

Therefore

\[ \widehat{L}_k(\tau) = (-1)^{|\mu|} \frac{1}{h(k)} s_k(\tau). \]
7 W is a simple complex Jordan algebra

For a simple complex Jordan algebra \( W \) we consider the Heisenberg group
\[ H = W \times \mathbb{R}. \]
Let \( \mathcal{D} \) be the bounded symmetric domain in \( W \), which is the unit
ball with respect to the spectral norm, and \( K = \text{Str}(W) \cap U(W) \). The group
\( K \) acts multiplicity free on the space \( \mathcal{P}(W) \) of holomorphic polynomials on
\( W \). Let \( n \) be the rank and \( d \) the multiplicity.

\[
\begin{array}{|c|c|c|}
\hline
\text{W} & \text{K} & d \\
\hline
\text{Sym}(n, \mathbb{C}) & U(n) & n \\
\text{M}(n, \mathbb{C}) & U(n) \times U(n) & 2n \\
\text{Skew}(2n, \mathbb{C}) & \text{U}(2n) & 4n \\
\text{Herm}(3, \mathbb{C}) & E_6 \times \mathbb{T} & 8 \times 3 \\
\mathbb{C}^\ell & \text{SO}(\ell) \times \mathbb{T} & \ell - 2 \times 2 \\
\hline
\end{array}
\]

Let \( V \) be a Euclidean real form of \( W \), and \( c_1, \ldots, c_n \) a Jordan frame in
\( V \). An element \( z \in W \) can be written
\[ z = k \sum_{j=1}^{n} a_j c_j \quad (a_j \in \mathbb{R}, \; k \in K). \]

We will denote by \( r_j = r_j(z) \) the numbers \( a_j^2 \) assume to satisfy
\( r_1 \geq \cdots \geq r_n \geq 0 \), and put \( r = r(z) = r_1 c_1 + \cdots + r_n c_n \).

The Fock space decomposes multiplicity free into the subspaces \( \mathcal{P}_m \) (\( m \)
is a partition). The dimension of \( \mathcal{P}_m \) is denoted by \( d_m \). The reproducing
kernel \( K^m \) of \( \mathcal{P}_m \) is determined by the conditions
\[
K^m(gz, w) = K^m(z, g^* w) \quad (g \in L), \\
K^m(z, e) = d_m \left( \frac{1}{\Phi_m(z)} \right).
\]

(See [Faraut-Korányi,1994], Section XI.3.)

We consider in this section the Gelfand pair \((G, K)\), where \( G = K \ltimes H \).
The bounded spherical functions of the first kind are given by, for \( \lambda > 0 \), and
\( m \) is a partition
\[
\varphi(\lambda, m; z, t) = e^{i\lambda t} e^{-\frac{1}{2} \lambda \| z \|^2} \frac{L^{(\nu-1)}(\lambda r(z))}{L^{(\nu-1)}(0)},
\]

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with $\nu = \frac{N}{n}$. This spherical function admits the following expansion

$$\varphi(\lambda, m; z, t) = e^{i\lambda t}e^{-\frac{1}{2}\|z\|^2}\sum_{k} \frac{d_k}{(\frac{N}{n})_k} (-1)^{|k|}\lambda^{\gamma_k}(m - \rho)\Phi_k(r(z)).$$

The bounded spherical functions of the second kind are given by the expansion

$$\varphi(\tau; z, t) = \sum_{k} \frac{d_k}{(\frac{N}{n})_k} (-1)^{|k|}\Phi_k(\tau)\Phi(r(z)),$$

where $\tau = \tau_1c_1 + \cdots + \tau_nc_n$, $\tau_1 \geq \cdots \geq \tau_n \geq 0$. As in the case considered in Section 6, the Gelfand spectrum is a union $\Sigma = \Sigma_1 \cup \Sigma_2$. The part $\Sigma_1$ is parametrized by pairs $(\lambda, m)$, with $\lambda \in \mathbb{R}^*$, and $m$ is a partition with $\ell(m) \leq n$, and $\Sigma_2$ by points $\tau \in \mathbb{R}^n$, $\tau_1 \geq \cdots \geq \tau_n \geq 0$. (See [Dib, 1990], [Faraut, 2010b]).

**Theorem 7.1.** (i) The eigenvalues of the differential operator $D_k$ associated to the partition $k$ are given, for $(\lambda, m) \in \Sigma_1$, $\lambda > 0$, by

$$\widehat{D}_k(\lambda, m) = \frac{d_k}{(\frac{N}{n})_k} (-1)^{|k|}\lambda^{\gamma_k}(m - \rho),$$

and, for $\tau \in \Sigma_2$, by

$$\widehat{D}_k(\tau) = \frac{d_k}{(\frac{N}{n})_k} (-1)^{|k|}\Phi_k(\tau).$$

(ii) The eigenvalues of the operator $L_k$ are given, for $(\lambda, m) \in \Sigma_1$, $\lambda > 0$, by

$$\widehat{L}_k(\lambda, m) = d_k \left(\frac{1}{2}\lambda\right)^{|k|}Q_\nu^{\prime}(m + \frac{N}{2n} - \rho),$$

with $\nu = \frac{N}{n}$, and, for $\tau \in \Sigma_2$, by

$$\widehat{L}_k(\tau) = \frac{d_k}{(\frac{N}{n})_k} (-1)^{|k|}\Phi_k(\tau).$$

The proofs are similar to the ones which are given in Section 6.


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